Lecture XXVIII: Maps to $\mathbb{P}^N$, base points, canonical maps

Recall: Last time we defined maps $F: X \longrightarrow \mathbb{P}^N$ induced by a tuple in $\mathcal{M}(X)$

(1) Fix $f_0, \dots, f_N \in \mathcal{M}(X)$ which do not vanish identically on X. We define
(e.g. if we want F non-deg)

$$\boxed{F = [f_0 : \cdots : f_N] : X \longrightarrow \mathbb{P}^N}$$

as follows. Given $x \in X$, pick a local chart (V, z) with $V \underset{z}{\overset{\sim}{\longrightarrow}} \mathbb{D}$, $x \longmapsto 0$ & set

$k = \min_j \mathrm{ord}_x(f_j)$ Write $f_j|_V = z^k g_{j}(z)$ g_j holo near x

$\exists\, j$ with $g_j(x) \neq 0 \Rightarrow F|_V(z) = [g_0(z) : \cdots : g_N(z)] \in \mathbb{P}^N$

- Definition is chart independent.
- Main examples of non-deg maps = those induced by $|D|$
- $\{f_0, \dots, f_N\}$ basis of $H^0(X, \mathcal{O}_D) \rightsquigarrow F = \phi_D : X \xrightarrow{|D|} \mathbb{P}^N$.

Lemma: Up to $\mathrm{Aut}(\mathbb{P}^N)$, ϕ_D is independent of the basis
$\updownarrow$
matrix in $PGL_N(\mathbb{C})$

Proof:

$$\begin{tikzcd} X \arrow[r, "{(f_0,\dots,f_N)}"] \arrow[dr, "{(\tilde{f}_0:\cdots:\tilde{f}_N)}"'] & \mathbb{P}^N \arrow[d, "\wr"] \arrow[dl, phantom, "\text{(C)}"] \\ & \mathbb{P}^N \end{tikzcd}$$

Automorphism induced by $(N+1)\times(N+1)$ change of basis matrix in $H^0(X, \mathcal{O}_D) \simeq \mathbb{C}^{N+1}$

Definition:
- $F: X \longrightarrow \mathbb{P}^N$ is immersion $\Leftrightarrow$ F separates tangents, i.e. $\forall x \in X\ \exists\, j$ with $dg_j(x) \neq 0$
- embedding = injective + immersion

(2) $\mathcal{O}_D$ globally generated $\Leftrightarrow$ $\forall x\ \exists f \in \mathcal{O}_D(X)$ with $\mathrm{ord}_x f = -D(x)$

Lemma: $\mathcal{O}_D$ is globally generated $\Leftrightarrow$ $\dim H^0(X, \mathcal{O}_{D-[p]}) = \dim H^0(X, \mathcal{O}_D) - 1$ $\forall p \in X$.

$\Leftrightarrow \exists\, f \in H^0(X, \mathcal{O}_D) \setminus H^0(X, \mathcal{O}_{D-[p]})$

Theorem: Fix X compact R.S. of genus g & $D \in \mathrm{Div}(X)$ with $\deg D \geq 2g$

Then, $\mathcal{O}_D$ is globally generated.

§28.1. Immersions vs Embeddings

Theorem 1: Fix X compact R.S. of genus g & $D \in \mathrm{Div}(X)$. Fix $\{f_0, \ldots, f_N\}$ basis for $H^0(X, \mathcal{O}_D)$ & $\Phi_D : X \xrightarrow{|D|} \mathbb{P}^N$. If $\deg D \geq 2g+1$, then Φ_D is an embedding.

Proof • Φ_D is injective: we saw this last time (used $\mathcal{O}_D$ & $\mathcal{O}_{D-[x_2]}$ are globally gen for degree reasons)

• Φ_D separates points: Fix $x_0 \in X$ & set $D' = D - [x_0]$ $\rightsquigarrow$ $\mathcal{O}_{D'} \subseteq \mathcal{O}_D$

$\deg D' = \deg D - 1 \geq 2g \underset{\text{Thm}}{\Longrightarrow} \mathcal{O}_{D'}$ is globally generated

• Pick $f \in H^0(X, \mathcal{O}_{D'})$ with $\mathrm{ord}_{x_0}(f) = -D'(x_0) = -D(x_0) + 1$

• $f \in H^0(X, \mathcal{O}_D) = \langle f_0, \ldots, f_N \rangle$ so we write $f = \sum_{j=0}^{N} \lambda_j f_j$

$\mathcal{O}_D$ is globally generated $\Longrightarrow$ $\boxed{k = \min_j \mathrm{ord}_{x_0} f_j = -D(x_0)}$ (last time)
$\{f_0, \ldots, f_N\}$ basis of $\mathcal{O}_D(X)$

Pick (V, z) local chart around x_0 with $V \xrightarrow[z]{\simeq} \mathbb{D}$, $x_0 \longmapsto 0$

Write $f_j = z^k g_j$ & $f = z^k g$ (with $\mathrm{ord}_{x_0} f = k+1$)
$g_j \in \mathcal{O}(V)$ $\qquad g \in \mathcal{O}(V)$ $g(x_0) = 0$ & $g'(x_0) \neq 0$

Pick j_0 realizing k so $g_{j_0}(x_0) \neq 0$. After permutations, we assume $j_0 = 0$

CASE 1: $dg_{j_0}(x_0) \neq 0 \Longrightarrow \Phi_D$ separates tangents

CASE 2: $dg_{j_0}(x_0) = 0$

$g_0(x_0) \neq 0 \Longrightarrow \Phi_D|_V = [g_0 : \cdots : g_N] = \left[1 : \frac{g_1}{g_0} : \cdots : \frac{g_N}{g_0}\right]$

$$\Rightarrow \quad g = \sum_{j=0}^{N} \lambda_j g_j \Rightarrow \quad \frac{g}{g_0} - \lambda_0 = \sum_{j=1}^{N} \lambda_j \frac{g_j}{g_0}$$

$$\Rightarrow \quad \sum_{j=1}^{N} \lambda_j \, d\left(\frac{g_j}{g_0}\right) = d\left(\frac{g}{g_0}\right) = \frac{g' g_0}{g_0^2} - \frac{g g_0'}{g_0^2} \qquad (*)$$

Evaluate at x_0 to get: $d\left(\frac{g}{g_0}\right)(x_0) = g'(x_0) - 0 = g'(x_0) \neq 0$

By $(*)$ $\exists\ 1 \leq j \leq N$ with $d\left(\frac{g_j}{g_0}\right)(x_0) \neq 0$

Our assumption on g_0 gives

$$0 \neq d\left(\frac{g_j}{g_0}\right)(x_0) = \frac{dg_j(x_0)}{g_0(x_0)} - \frac{g_j(x_0)}{g_0^2(x_0)} \overbrace{dg_0(x_0)}^{=0} = dg_j(x_0)/g_0(x_0)$$

$\Rightarrow dg_j(x_0) \neq 0$ ie Φ_D separates tangents. $\square$

__Remark 1:__ This result can be strengthen. $\exists \{\varphi_0, \ldots, \varphi_3\} \subseteq H^0(X, \mathcal{O}_D)$ where $F = [\varphi_0 : \ldots : \varphi_3] : X \longrightarrow \mathbb{P}^3$ is an embedding

(Comes from projecting $\Phi_D(X) \subseteq \mathbb{P}^n$ to some $\mathbb{P}^3$ (4 coords on $\mathbb{P}^n$))

⚠ We can't do better! $F(X) \subseteq \mathbb{P}^2$ will be nodal.

§28.2. Base points of linear systems

⚠ $\deg D \geq 2g+1$ condition is sufficient but not necessary. We only used it to ensure $\mathcal{O}_D, \mathcal{O}_{D-[p]}$ were globally generated. We only needed

$\forall x$: $\min_j \operatorname{ord}_x f_j = -D(x)$ & $\exists f$ with $\operatorname{ord}_x f = -D(x)+1$

$(\mathcal{O}_D$ is gl. gen.$)$ $\qquad (\mathcal{O}_{D-[p]}$ is gl. gen.$)$

This condition has a name:

__Definition.__ Fix $\Lambda \subseteq |D|$ projective subspace. We say $p \in X$ is a __base pt__ of Λ if for every $E \in \Lambda$ (ie $E \sim D$ $E \geq 0$) we have $p \in \operatorname{Supp} E$

• Λ is base point free if it has no base points

Recall Λ is determined by a subspace V of $\mathbb{P}(H^0(X,\mathcal{O}_D))$

$E \in \Lambda \rightsquigarrow (f_E) = E - D$ for $f_E \in \mathcal{O}_D(X) \Rightarrow V = \mathbb{P}(\langle f_E : E \in \Lambda \rangle)$

Lemma : $p \in \Lambda \subseteq |D|$ is a base pt $\iff V \subseteq \mathbb{P}(H^0(X,\mathcal{O}_{D-p}))$

Proof V corresponding to Λ has a finite basis $\{f_{E_1}, \dots, f_{E_s}\}$

Claim: $p \in \operatorname{Supp} E \ \forall E \in \Lambda \iff p \in \operatorname{Supp} E_1 \cap \dots \cap \operatorname{Supp} E_s$

Pf/ $(\Rightarrow)$ ✓ $\quad (\Leftarrow)$ Write $f_E = \sum_{i=1}^{s} \lambda_i f_{E_i}$

$$E[p] - D[p] = \operatorname{ord}_p f_E \geq \underbrace{\min_{1 \le j \le s} \operatorname{ord}_p f_{E_j}}_{= \operatorname{ord}_p f_{E_{j_0}}} \quad \& \quad \operatorname{ord}_p f_{E_j} = E_j[p] - D[p]$$

$\Rightarrow$ Comparing orders we get $E[p] \geq E_{j_0}[p] > 0$ because $p \in \operatorname{Supp} E_{j_0}$, so $p \in \operatorname{Supp} E$ as well.

• Using this claim we can prove the statement

p is a base point of $\Lambda \overset{\text{Claim}}{\iff} p \in \operatorname{Supp} E_1 \cap \dots \cap \operatorname{Supp} f_{E_s}$

$$\iff \min_{1 \le j \le s} \operatorname{ord}_p f_{E_j} = E_{j_0}[p] - D[p] > 0 - D[p] \iff$$

$$\operatorname{ord}_p f_{E_j} \geq -D[p] + 1 = -(D - [p])_{[p]} \underset{\{f_{E_1} \dots f_{E_s}\} \text{ basis}}{\iff} V \subseteq \mathcal{O}_{D-[p]}. \qquad \square$$

Recall (last time) : Two useful bounds on $\dim H^0(X,\mathcal{O}_D)$:

Proposition : (1) $H^0(X, \mathcal{O}_{[p]}) = \mathbb{C}$ if $X \not\cong \mathbb{P}^1$

(2) $\dim H^0(X, \mathcal{O}_{D-[p]}) \geq \dim H^0(X,\mathcal{O}_D) - 1$. $\quad \forall p \in X$

The lemma has a key corollary:

Corollary: $|D|$ is base pt free $\iff \mathcal{O}_D(X) \neq \mathcal{O}_{D-p}(X)$ $\forall p \in X$.

$\iff \dim H^0(X, \mathcal{O}_{D-\{p\}}) = \dim H^0(X, \mathcal{O}_D) - 1 \quad \forall p \in X$

($\iff \mathcal{O}_D$ is globally generated)

Note: We can always get rid of base pts by removing them!

Set $F = \min\{E \mid E \in |D|\}$ (ie $F(x) = \min\{E(x) : E \in |D|\}$)

$\Rightarrow |D-F|$ has no base pts & $|D| = F + |D-F|$

corresponds to z^k in our definition of $\phi_{D|V}$

Name: F = "fixed part of D" & $|D-F|$ = "moving part of D"

Proposition $\mathcal{O}_{D-F}(X) \simeq \mathcal{O}_D(X)$ ($\Rightarrow$ can always assume $|D|$ is base-pt free)

Proof: ($\subseteq$) $F \geq 0 \Rightarrow \mathcal{O}_{D-F} \subseteq \mathcal{O}_D$ (induct on $\deg F$ & use $\mathcal{O}_{D'-p} \subseteq \mathcal{O}_{D'}$ $\forall D' \in \mathrm{Div}(X)$)

$\Rightarrow \mathcal{O}_{D-F}(X) \subseteq \mathcal{O}_D(X)$

($\supseteq$) Assume $H^0(X, \mathcal{O}_D) \neq 0$ (otherwise $|D| = \emptyset$, $F=0$ & $|D-F| = \emptyset$ so $H^0(X, \mathcal{O}_{D-F}) = 0$)

Since $|D| \neq \emptyset$, pick $f \in \mathcal{O}_D(X)$ with $(f) + D = E \geq 0$ E in $|D|$.

By definition of F, we write $(f) + D = F + E'$ with $E' \geq 0$

$\Rightarrow (f) + (D-F) = E' \geq 0$, so $E' \in |D-F|$ & $f \in \mathcal{O}_{D-F}(X)$.

Conclude: $\mathcal{O}_D(X) \subseteq \mathcal{O}_{D-F}(X)$. $\square$

Remark: If $|D|$ is bpf, then given any $p \in X$ we can pick a basis of $H^0(X, \mathcal{O}_D)$ "adapted to p", ie a basis $\{f_0, \ldots, f_N\}$ of $H^0(X, \mathcal{O}_D)$

with $\min_j \operatorname{ord}_p(f_j) = -D(p)$ & minimum is attained exactly once.

(if achieve at $f_{j_1}, \dots, f_{j_r}$, take $\tilde{f}_{j_k} = f_{j_k} - \underset{\mathbb{C}}{\underset{\cap}{\lambda_k}} f_{j_1}$ so that

$\operatorname{ord}_p \tilde{f}_{j_k} > \operatorname{ord}_p f_{j_1}$ (cancel initial terms!))

Q Why is this useful?

A: Φ_D is independent of choices of bases of $H^0(X, \mathcal{O}_D)$! So this choice can simplify proofs & expressions for $\Phi_D(p) = [1:0:\dots:0]$ or a permutation of it!.

Theorem: [Criteria for embedding / immersion for base pt free $|D|$]

Fix X compact RS & $D \in \operatorname{Div}(X)$. Assume $|D|$ is base pt free Then

(1) $\Phi_D(p) = \Phi_D(q) \iff \mathcal{O}_{D-p-q} = \mathcal{O}_{D-p} = \mathcal{O}_{D-q}$

(2) Φ_D is injective $\iff \dim H^0(X, \mathcal{O}_{D-p-q}) = \dim H^0(X, \mathcal{O}_D) - 2 \quad \forall p \neq q$ in X

(3) Φ_D separates tangent spaces $\iff \forall p: \mathcal{O}_{D-2p} \neq \mathcal{O}_{D-p}$

$\iff \forall p \ \dim H^0(X, \mathcal{O}_{D-2p}) = \dim H^0(X, \mathcal{O}_{D-p}) - 1.$

(4) Φ_D is an embedding $\iff \forall p, q \in X$ (potentially equal) we have

$\dim H^0(X, \mathcal{O}_{D-p-q}) = \dim H^0(X, \mathcal{O}_D) - 2$

__Proof:__ (1) Writing Φ_D using the basis $\{f_0, \dots, f_N\}$ adapted to p with

$\operatorname{ord}_p f_0 = -D(p) \implies \{f_1, \dots, f_N\}$ is a basis of $\mathcal{O}_{D-p}$ $(\dim \mathcal{O}_{D-p}(X) = N)$
↳ p is not a base pt of $|D|$, so use Corollary

$\Phi_D(p) = [1:0:\dots\dots:0] = \Phi_D(q) \iff \operatorname{ord}_q f_0 = -D(q)$ & $\operatorname{ord}_q f_j > -D(q) \ \forall j > 0$
$(\therefore \mathcal{O}_{D-p} \subseteq \mathcal{O}_{D-q})$

This means $\{f_1, \dots, f_N\}$ is a basis for $\mathcal{O}_{D-q}$ (use Corollary above)

__Conclude:__ $\Phi_D(p) = \Phi_D(q) \iff \mathcal{O}_{D-q} = \mathcal{O}_{D-p} \overset{(*)}{\underset{\substack{\downarrow \\ p \neq q}}{=}} \mathcal{O}_{D-p-q}$

Proof of (*):

$(\subseteq)$ We saw if $f \in \mathcal{O}_D$ satisfies $\operatorname{ord}_p(f) > -D(p)$ $\Rightarrow$ $\operatorname{ord}_q(f) > -D(q)$ ($f \in \mathcal{O}_{D-p}$) $f \in \mathcal{O}_{D-q}$

Hence $\mathcal{O}_{D-p} \subseteq \mathcal{O}_{D-q}$ & $-(D-p)_{(q)} = -D(q)$ so $f \in \mathcal{O}_{D-p-q}$

$\Rightarrow \mathcal{O}_{D-p} \subseteq \mathcal{O}_{D-p-q}$

By symmetry between p & q we get $\mathcal{O}_{D-q} \subseteq \mathcal{O}_{D-p-q}$.

($\mathcal{O}_{D-p-q} \subseteq \mathcal{O}_{D-p}$ is always true)

$(\supseteq)$ $\mathcal{O}_{D-p-q} \subseteq \mathcal{O}_{D-p}$ is always true

(2) $\dim H^0(X, \mathcal{O}_{D-p}) = \dim H^0(X, \mathcal{O}_D) - 1 = \dim H^0(X, \mathcal{O}_{D-q})$

$\Rightarrow \dim H^0(X, \mathcal{O}_{D-p-q}) = \dim H^0(X, \mathcal{O}_D) - 1$ or $\dim H^0(X, \mathcal{O}_D) - 2$

Proposition (2) §28.1

But Φ_D is 1-1 $\Leftrightarrow$ $\mathcal{O}_{D-p} \neq \mathcal{O}_{D-p-q}$ by (1), so (2) follows.

(3) We know $\dim H^0(X, \mathcal{O}_{D-p}) = \dim H^0(X, \mathcal{O}_D) - 1$ &

$\dim H^0(X, \mathcal{O}_{D-2p}) = \dim H^0(X, \mathcal{O}_{D-p})$ or $\dim H^0(X, \mathcal{O}_{D-p}) - 1$

$(\Leftarrow)$ Assume $\dim H^0(X, \mathcal{O}_{D-2p}) = \dim H^0(X, \mathcal{O}_{D-p}) - 1 = \dim H^0(X, \mathcal{O}_D) - 2$

Then $\exists f \in H^0(X, \mathcal{O}_{D-p}) \setminus H^0(X, \mathcal{O}_{D-2p})$ ie $\mathcal{O}_{D-p}$ is globally gen,

so $\operatorname{ord}_p(f) = -(D - [p])_{(p)} = -D(p) + 1$

Following the proof of Theorem §28.1, we see that Φ_D is an immersion.

$(\Rightarrow)$ Assume Φ_D is an immersion. Fix p & pick a basis $\{f_0, \dots, f_N\}$ of $\mathcal{O}_K(X)$ $(N = g-1)$

adapted to p. Up to permutation, assume $\operatorname{ord}_p f_0 = -D(p)$

$\operatorname{ord}_p f_i > -D(p)$ for $i = 1, \dots, N$.

So $\Phi_D(p) = [1 : 0 : \dots : 0]$

Since $|D|$ is bpf $\dim H^0(X, \mathcal{O}_{D-p}) = \dim H^0(X, \mathcal{O}_D) - 1 = N$

$\{f_1, \dots, f_N\} \subseteq \mathcal{O}_{D-p}(X)$ & all i, so they form a basis of $H^0(X, \mathcal{O}_{D-p})$

Pick a chart (V, z) around p with $V \xrightarrow[z]{\sim} \mathbb{D}$, $p \longmapsto 0$

Write $f_0 = z^{-D(p)} g_0(z)$, $f_i = z^{-D(p)} g_i(z)$ for $i = 1, \dots, N$ $\quad g_i \in \mathcal{O}(V)$
$g_0 \in \mathcal{O}(V)$ $\quad g_0(p) \neq 0$ $\qquad g_i(p) = 0$

$$\Phi_{D|V}(z) = [g_0(z) : g_1(z) : \cdots : g_N(z)]$$

$$g_0(x_0) \neq 0 \Rightarrow \Phi_{D|_V} = [g_0 : \cdots : g_N] = \left[1 : \frac{g_1}{g_0} : \cdots : \frac{g_N}{g_0}\right]$$

$\Phi_{D|V}$ is an immersion. This forces $d\left(\frac{g_i}{g_0}\right)(p) \neq 0$ for some $i = 1, \dots, N$

$$0 \neq d\left(\frac{g_i}{g_0}\right)(p) = \frac{dg_i(p)}{g_0(p)} - \underset{0}{\underset{\shortparallel}{g_i(p)}} \frac{dg_0(p)}{g_0^2(p)} = \frac{dg_i(p)}{g_0(p)} \Rightarrow dg_i(p) \neq 0$$

Conclusion $f_i \in H^0(X, \mathcal{O}_{D-p}) \setminus H^0(X, \mathcal{O}_{D-2p})$. Conclusion: (3) holds $\square$

(4) is a direct combination of (2) & (3).

Definition: D is very ample if $|D|$ is bpf & Φ_D is an embedding.

§28.3 Canonical maps for compact R.S.:

Recall: $H^0(X, \mathcal{O}_K) \simeq H^0(X, \Omega) \simeq H^1(X, \mathcal{O})$ has dimension g

Define $\Phi_K : X \longrightarrow \mathbb{P}^{g-1}$ to be the canonical map

For $g \geq 2$ this has a chance of being an embedding.

⚠ $\deg D = 2g - 2 \not\geq 2g+1$ so we can't use Thm 1 §27.3 to decide this

• We can bypass this if $|K|$ is base point free & we have info on X.

Lemma: $|K|$ is base point free if $g \geq 1$

Proof: By Corollary §28.2, we have to show $\mathcal{O}_{K-p} \neq \mathcal{O}_K \; \forall p \in X$

Equivalently, $\dim H^0(X, \mathcal{O}_{K-p}) = \dim H^0(X, \mathcal{O}_K) - 1 = g-1$

Prop §27.1, $\dim H^0(X, \mathcal{O}_{[p]}) \leq 1$ because $X \not\cong \mathbb{P}^1$ $(g \geq 1)$

Since $1 \in H^0(X, \mathcal{O}_{[p]})$, we get $\dim H^0(X, \mathcal{O}_{[p]}) = 1$.

Riemann Roch $\Rightarrow 1 = \dim H^0(X, \mathcal{O}_{[p]}) = \dim H^1(X, \mathcal{O}_{[p]}) + 1 - g + \deg([p])$

$\Leftrightarrow \quad \dim H^1(X, \mathcal{O}_{[p]}) = g-1$

$$\dim \overset{\overset{SD}{\parallel}}{H^0}(X, \Omega_{-[p]}) = \dim H^0(X, \mathcal{O}_{K-[p]})$$

✓

Q: When is ϕ_K injective? We have a partial answer.

Proposition 1: If $\exists\, p, q \in X$ with $p \neq q$ & $\phi_K(p) = \phi_K(q)$ then $\exists\, f: X \longrightarrow \mathbb{P}^1$ holomorphic deg 2 map, i.e. X is hyperelliptic.

Remark: Contrapositive statement is the one most often used
"X is <u>not</u> hyperelliptic $\Rightarrow$ ϕ_K is injective"

Theorem 1: If X is not hyperelliptic & $g \geq 2$, then $\phi_K: X \longrightarrow \mathbb{P}^{g-1}$ is an embedding. (K is very ample)

Theorem 2: All genus 2 compact R.S are hyperelliptic ($\Rightarrow$ can put $g \geq 3$ in Thm)

Q: What does ϕ_K look like if X is hyperelliptic?

Proposition 2: Fix X hyperelliptic of genus $g \geq 2$, then ϕ_K is the composition of the double cover $X \longrightarrow \mathbb{P}^1$ and the Veronese map. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$
$(T, z) \longmapsto z$ $\qquad$ $z \longmapsto [1 : z : \cdots : z^{g-1}]$

Proofs: Next time!