Lecture XXIX: Canonical maps, Abel-Jacobi theory

Last Time: base point free complete linear systems on compact R.S X

Def: $|D|$ is base pt free $\iff$ $\mathcal{O}_D(X) \neq \mathcal{O}_{D-p}(X)$ $\forall p \in X$.

$\iff$ $\dim H^0(X, \mathcal{O}_{D-p}) = \dim H^0(X, \mathcal{O}_D) - 1$ $\forall p \in X$

Theorem: Assume $|D|$ is bpf. Then:

ϕ_D is an embedding $\iff$ $\forall p, q \in X$ (potentially equal) we have

$\dim H^0(X, \mathcal{O}_{D-p-q}) = \dim H^0(X, \mathcal{O}_D) - 2$

(injective: $p \neq q$ case; immersion: $p = q$)

§29.1 Canonical maps for compact R.S.:

Fix K canonical divisor on a compact R.S X of genus $g \geq 1$. $K = (\omega)$ for some meromorphic 1-form ω on X (any other choice gives a linearly equiv divisor)

Lemma: $|K|$ is base point free if $g \geq 1$

Proof: By Corollary §28.2, we have to show $\mathcal{O}_{K-p}(X) \neq \mathcal{O}_K(X)$ $\forall p \in X$

Equivalently, $\dim H^0(X, \mathcal{O}_{K-p}) = \dim H^0(X, \mathcal{O}_K) - 1 = g - 1$

Prop §27.1, $\dim H^0(X, \mathcal{O}_{[p]}) \leq 1$ because $X \not\cong \mathbb{P}^1$ $(g \geq 1)$

Since $1 \in H^0(X, \mathcal{O}_{[p]})$, we get $\dim H^0(X, \mathcal{O}_{[p]}) = 1$.

Riemann Roch $\Rightarrow 1 = \dim H^0(X, \mathcal{O}_{[p]}) = \dim H^1(X, \mathcal{O}_{[p]}) + 1 - g + \deg([p])$

$\iff$ $\dim H^1(X, \mathcal{O}_{[p]}) = g - 1$

$\quad\quad\quad \overset{\text{SD}}{\|}$

$\dim H^0(X, \Omega_{-[p]}) = \dim H^0(X, \mathcal{O}_{K-[p]})$ ✓

Q: When is ϕ_K injective? We have a partial answer.

Proposition 1: If $\exists\, p, q \in X$ with $p \neq q$ & $\phi_K(p) = \phi_K(q)$ then $\exists\, f: X \longrightarrow \mathbb{P}^1$ holomorphic deg 2 map, i.e. X is hyperelliptic.

Remark: Contrapositive statement is the one most often used
" X is <u>not</u> hyperelliptic $\Longrightarrow$ ϕ_K is injective "

Proof: By Lemma, & Theorem 1 §28.2, we have $\mathcal{O}_{K-p} = \mathcal{O}_{K-q} = \mathcal{O}_{K-p-q}$

$$\Longleftrightarrow \dim H^0(X, \mathcal{O}_{K-p-q}) = \dim H^0(X, \mathcal{O}_{K-p}) \underset{|K| \text{ bpf}}{=} \dim H^0(X, \mathcal{O}_K) - 1 = g-1$$

Using Riemann-Roch, we get

$$\dim H^0(X, \mathcal{O}_{p+q}) = \dim H^1(X, \mathcal{O}_{p+q}) + 1 - g + \deg(p+q)$$
$$= \dim H^0(X, \mathcal{O}_{K-p-q}) + 1 - g + 2$$
$$= \dim H^0(X, \mathcal{O}_K) - 1 + 1 - g + 2 = g - g + 2 = 2$$

$\Rightarrow \exists\, f : X \longrightarrow \mathbb{P}^1$ mero $f \in \mathcal{O}_{p+q}$ non-constant

Since $|K|$ is bpf, the same calculation says:

$$\dim H^0(X, \mathcal{O}_p) = \dim H^0(X, \mathcal{O}_{K-p}) + 1 - g + \deg([p]) = g - 1 + 1 - g + 1 = 1$$
$$\& \dim H^0(X, \mathcal{O}_q) = 1$$

$\Longrightarrow$ f has poles at both p & q of order 1, so $\deg f = 2$ & $f : X \longrightarrow \mathbb{P}^1$ Turns X into a hyperelliptic curve.

Theorem 1: If X is not hyperelliptic & $g \geq 2$, then $\phi_K : X \longrightarrow \mathbb{P}^{g-1}$ is an embedding. ($\Rightarrow K$ is very ample)

Proof Since $|K|$ is bpf & ϕ_K is injective by Proposition, we need to check

$$\dim H^0(X, \mathcal{O}_{K-2p}) = \underbrace{\dim H^0(X, \mathcal{O}_K)}_{=g} - 2 \quad [\text{ie. Theorem } §28.2(4)]$$

Again, by Riemann-Roch & Serre duality, it's enough to check:

$$\dim H^0(X, \mathcal{O}_{2p}) = \dim H^1(X, \mathcal{O}_{2p}) + 1 - g + \deg(2[p])$$
$$= \dim H^0(X, \mathcal{O}_{K-2p}) + 1 - g + 2$$
$$\overset{?}{=} g - 2 + 1 - g + 2 = 1 \quad (*)$$

But $\dim H^0(X,\mathcal{O}_{2p}) \leq 2$ & $\dim H^0(X,\mathcal{O}_p) = 1$ because $g \neq 0$.

If $\dim H^0(X,\mathcal{O}_{2p}) = 2$, then $\exists f \in H^0(X,\mathcal{O}_{2p})$ holo non-constant

$\Rightarrow$ f has to have a pole at p of order ≤ 2.

- If order 2, then $f: X \to \mathbb{P}^1$ is a hyperelliptic cover. Contr!
- If order 1, then $f: X \to \mathbb{P}^1$ has degree 1 ie f is a covering & $X \simeq \mathbb{P}^1$ Contr! ($g \geq 2$)

Conclude $\dim H^0(X,\mathcal{O}_{2p}) \leq 1$ &

But $\dim H^0(X,\mathcal{O}_{2p}) \geq 1$ because $\mathcal{O}_{2p} \supseteq \mathcal{O}_p$, so (*) is confirmed. $\square$

Theorem 2: All genus 2 compact R.S are hyperelliptic ($\Rightarrow$ can put $g \geq 3$ in Thm)

Proof Since $|K|$ is bpf it's enough to show $\Phi_K : X \to \mathbb{P}^1$ is not inj

If Φ_K is injective, then its degree is 1 (#zeroes $= 1$ and simple, otherwise Φ_K is branched, & so Φ_K is not injective), hence biholomorphic. This can't happen because genus $X = 2 \neq 0$. $\square$

Q2: What does $\Phi_K : X \to \mathbb{P}^{g-1}$ look like when X is hyperelliptic?

We can answer this by viewing X as the algebraic function of $Q(T) = T^2 - h(z)$ when $h \in \mathbb{C}[z]$ square free (degree $h = 2g+1$ or $2g+2$)

$2g+1$: ∞ = branch pt; $2g+2$: ∞: not a branch pt

Recall: $H^0(X,\Omega) = \mathrm{Span}\left\langle \frac{z^{j-1}\,dz}{\sqrt{h(z)}} : j = 1, \ldots, g \right\rangle$ (Proposition §26.1)

Proposition 2: Fix X hyperelliptic of genus $g \geq 2$, then Φ_K is the composition of the double cover $X \to \mathbb{P}^1$, $(T,z) \mapsto z$ and the Veronese map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$, $z \mapsto [1 : z : \cdots : z^{g-1}]$

Proof Use $K = \left(w = \frac{dz}{\sqrt{h(z)}} = \frac{dz}{T}\right)$ view $\mathcal{O}_K \simeq \Omega$ via $f \mapsto f \cdot w$

$\Rightarrow$ Basis for $H^0(X, \mathcal{O}_K)$ is $\{1, z, z^2, \dots, z^{g-1}\}$

$$\Rightarrow \quad \phi_K : X \longrightarrow \mathbb{P}^{g-1} \qquad (\pi, z) \longmapsto [1 : z : z^2 : \dots : z^{g-1}]$$

becomes

$$X \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^{g-1} \qquad (\pi, z) \to z \longmapsto [1 : \dots : z^{g-1}]$$

deg 2   deg 1

$\Rightarrow$ degree $\phi_K = 2$.

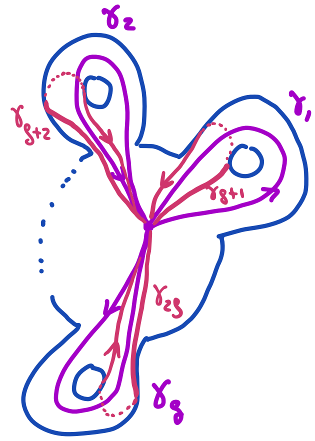
§ 29.2 Abel-Jacobi Theory

- Fix X compact R.S. of genus $g \geq 1$. Fix $p_0 \in X$ (base point for loops on X)
- Recall $H^0(X, \Omega) \cong H^1(X, \mathcal{O})^\vee$ so $\dim H^0(X, \Omega) = g$

Fix a basis $\{\omega_1, \dots, \omega_g\}$ for $H^0(X, \Omega)$

- Topologically we know $X \simeq$ Torus with g holes $\simeq \underbrace{\mathbb{T} \# \cdots \# \mathbb{T}}_{g \text{ times}}$

$\Rightarrow H_1(X, \mathbb{Z})$ is a free abelian group of order $2g$ with generators $\langle \gamma_1, \dots, \gamma_g, \gamma_{g+1}, \dots, \gamma_{2g} \rangle$



They satisfy $\gamma_i \cdot \gamma_{g+i} = 1$, $\gamma_{g+i} \cdot \gamma_i = -1$ $\quad i = 1, \dots, g$

& $\gamma_j \cdot \gamma_k = 0$ if $|k - j| \neq g$

Definition: Given γ_j $j = 1, \dots, 2g$, we set $\pi_j = \begin{bmatrix} \int_{\gamma_j} \omega_1 \\ \vdots \\ \int_{\gamma_j} \omega_g \end{bmatrix} \in \mathbb{C}^g$

Name: Period vector

Note: $d\omega_k = 0$ (because $\omega_k = f_k dz$ $f_k \in \mathcal{O}(V)$ so $\frac{\partial f_k}{\partial \bar{z}} = 0$)

$\Rightarrow$ Corollary 3 §15.2 gives $\int_{\gamma_j} \omega_k$ is independent on the homotopy class of γ_j.

Remark: $H^0(X, \Omega) \longrightarrow \mathrm{Hom}_{gp}(\pi_1(X, p_0), (\mathbb{C}, +))$ is a linear map

$\omega \longmapsto \rho_\omega := (\gamma \longmapsto \int_\gamma \omega)$ [period map]

Now: $(\mathbb{C},+)$ is an abelian group, and

(1) $\int_{\gamma * \gamma'} \omega = \int_{\gamma} \omega + \int_{\gamma'} \omega = \int_{\gamma' * \gamma} \omega$

(2) $\int_{\gamma^-} \omega = -\int_{\gamma} \omega$

So ρ_ω extends to $\pi_1(X,p_0) \big/ [\pi_1(X,p_0) : \pi_1(X,p_0)] = H_1(X,\mathbb{Z})$.

By Main Theorem §15.3 we know ω is exact $\iff \rho_\omega \equiv 0$.

Write $\Pi = (\pi_1, \dots \pi_{2g}) \in \mathbb{C}^{g \times 2g}$ & call it a <u>period matrix</u> for X

<u>Proposition</u>: The $2g$ period vectors are linearly independent over $\mathbb{R}$

<u>Proof</u>: We argue by contradiction & fix a dependency relation:

$$a_1 \pi_1 + \dots + a_{2g} \pi_{2g} = 0 \in \mathbb{C}^g$$

with $a_1, \dots, a_{2g} \in \mathbb{R}$ not all 0.

Taking complex conjugate gives $a_1 \overline{\pi_1} + \dots + a_{2g} \overline{\pi_{2g}} = 0 \in \mathbb{C}^g$

with $\overline{\pi_j} = \begin{bmatrix} \int_{\gamma_j} \overline{\omega_1} \\ \vdots \\ \int_{\gamma_j} \overline{\omega_g} \end{bmatrix}$

We write a square matrix of size $2g$ $\quad \Omega^* = \begin{bmatrix} \pi_1 & \dots & \pi_{2g} \\ \overline{\pi_1} & \dots & \overline{\pi_{2g}} \end{bmatrix}$

We know $\begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} \in \mathbb{R}^{2g} \setminus \{\underline{0}\}$ lies in $\operatorname{Ker} \Omega^*$, so $\operatorname{rk}(\Omega^*) < 2g$

In particular, the $2g$ rows of Ω^* are ld & we can find $[\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g]$ in $\mathbb{C}^{2g}$ with $\int_{\gamma_i} \sum_{j=1}^{g} \lambda_j \omega_j + \mu_j \overline{\omega_j} = 0 \quad \forall i = 1, \dots, 2g$. with $(\underline{\lambda}, \underline{\mu}) \neq \underline{0}$

$\Rightarrow \quad \omega = \sum_{j=1}^{g} \lambda_j w_j \quad \& \quad \varphi = \sum_{j=1}^{g} \overline{\mu_j}\, w_j \quad \in H^0(X, \Omega)$ satisfy

$\int_{\gamma_i} (\omega + \overline{\varphi}) = 0 \quad \forall i = 1, \ldots, 2g. \quad \Longrightarrow \rho_{\omega+\overline{\varphi}} = \int_{\gamma} \omega + \overline{\varphi} = 0 \quad \forall \gamma \in H_1(X,\mathbb{Z})$

Now: $\omega + \overline{\varphi}$ is closed, then $\omega + \overline{\varphi}$ is exact by Main Thm §15.3

Lemma below confirms $\omega = \overline{\varphi} = 0$. Since $\{w_1, \ldots, w_g\}$ is a $\mathbb{C}$-basis of $H^0(X, \Omega)$, we get $\lambda_1 = \cdots = \lambda_g = \overline{\mu_1} = \cdots = \overline{\mu_g} = 0$, so $(\underline{\lambda}, \underline{\mu}) = \underline{0}$

This can't happen by our choice of scalars $(\underline{\lambda}, \underline{\mu})$. □

Lemma: Fix X a compact R.S & ω, φ in $\Omega(X) \subseteq \mathcal{E}^{(1)}(X)$.

If $\omega + \overline{\varphi}$ is exact (ie $= df$ for $f \in \mathcal{E}(X)$), then $\omega = \varphi = 0$.

Proof: We work locally on a chart (V, z) of X & write $\omega = h(z)\, dz$
$\varphi = g(z)\, dz$

$\Rightarrow \varphi \wedge \omega = -\omega \wedge \varphi = 0 \in \mathcal{E}^{(2)}(X)$

Similarly $\frac{i}{2} \varphi \wedge \overline{\varphi} = |g(z)|^2 \frac{i}{2}\, dz \wedge d\overline{z} = |g(z)|^2\, du \wedge dv$ if $z = u + iv$

Claim: $\varphi = 0$.

Pf/ We argue by contradiction & assume $\varphi \neq 0$. Then

$$\frac{i}{2} \iint_X \varphi \wedge \overline{\varphi} = \iint_X |g(z)|^2\, du \wedge dv > 0$$

But $\varphi \wedge \overline{\varphi} = 0 + \varphi \wedge \overline{\varphi} = \varphi \wedge \omega + \varphi \wedge \overline{\varphi} = \varphi \wedge (\omega + \overline{\varphi}) = \varphi \wedge df$

$= -df \wedge \varphi = -d(f\varphi)$ because φ is closed.

$f\varphi \in \mathcal{E}^{(1)}(X)$ & has compact support (X comp.) so by Stokes' Thm $\iint_X d(f\varphi) = 0$.

$\Rightarrow \frac{i}{2} \iint_X \varphi \wedge \overline{\varphi} = -\frac{i}{2} \iint_X d(f\varphi) = 0$ Contr!

The same trick applied to ω gives $\omega = 0$

$\left(\frac{i}{2} \omega \wedge \bar{\omega} = |h(z)|^2 \, du \wedge dv \text{ on } V \right.$, $\bar{\varphi} \wedge \bar{\omega} = 0$

$\Rightarrow \omega \wedge \bar{\omega} = 0 + \omega \wedge \bar{\omega} = \bar{\varphi} \wedge \bar{\omega} + \omega \wedge \bar{\omega} = df \wedge \bar{\omega} = d(f\bar{\omega})$ ($\bar{\omega}$ is closed) ($d\bar{z} = \overline{dz}$)

$0 < \frac{i}{2} \iint_X \omega \wedge \bar{\omega} = \frac{i}{2} \iint_X d(f\omega) = 0$ (contr!) $\square$

Theorem: $\Gamma = \text{Per}(\omega_1, \dots, \omega_g) = \left\{ \begin{bmatrix} \int_\gamma \omega_1 \\ \vdots \\ \int_\gamma \omega_g \end{bmatrix} \quad \gamma \in \pi_1(X, p_0) \right\} \subseteq \mathbb{C}^g$

is a rank $2g$ lattice in $\mathbb{C}^g$. We call it the period lattice.

Definition: The Jacobian of a compact R.S X of genus $g \geq 1$ is defined as the quotient $\text{Jac}(X) := \mathbb{C}^g / \Gamma$ (ab group under $+$)

Note: Picking a basis for Γ (polarization), realizes $\text{Jac}(X)$ as a product of g copies of elliptic curves.

Q: How much $\text{Jac}(X)$ knows about X? What happens for elliptic curves?

- Next, we construct gp hom $\Phi : \text{Div}_0(X) \longrightarrow \text{Jac}(X)$ with $\text{Div}_0(X) = \{ D \in \text{Div}(X) \mid \deg(D) = 0 \}$

In order to do this, it's enough to define Φ on divisors of the form $p - p_0$.

Recall: 0-chains on X $= C_0(X) = \mathbb{Z}\langle x : x \in X \rangle \Rightarrow \text{Div}(X) = C_0(X)$

1-chains on X $= C_1(X) = \mathbb{Z}\langle \gamma : [0,1] \to X \rangle$ (paths) $\quad \gamma \overset{\partial}{\longmapsto} \gamma(1) - \gamma(0)$

Fix $p_0 \in X$. For each $p \in X$ fix a path $\gamma = \gamma_p : [0,1] \to X$ with $\gamma(0) = p_0$, $\gamma(1) = p$.

For $D = p - p_0$, write $\Phi(D) = \begin{bmatrix} \int_\gamma \omega_1 \\ \vdots \\ \int_\gamma \omega_g \end{bmatrix} \in \mathbb{C}^g / \Lambda$

If we pick another path γ', then $\alpha = \gamma' * \gamma^- \in \pi_1(X, p_0)$ so $\begin{bmatrix} \int_\alpha \omega_1 \\ \vdots \\ \int_\alpha \omega_g \end{bmatrix} \in \Gamma$

Since $\int_\alpha \omega_i = \int_{\gamma'} \omega_i + \int_{\gamma^-} \omega_i = \int_{\gamma'} \omega_i - \int_\gamma \omega_i \quad \forall i$,

we conclude $\left[\int_\gamma \omega_i\right]_i \equiv \left[\int_{\gamma'} \omega_i\right]_i \mod \Gamma$. Thus $\Phi(p - p_0)$ is well-defined.

We write $\int_\gamma \omega_i = \int_{p_0}^{p} \omega_i$

Set: $\Phi(p_0 - p) = \int_{\gamma^-} \omega = -\int_\gamma \omega = -\Phi(p - p_0)$

Now, if $D \in \mathrm{Div}_0(X)$, write $D = \sum_{i=1}^{m} a_i p_i - \sum_{j=1}^{n} b_j q_j$ with $\sum a_i = \sum b_j$

$$\Rightarrow D = \sum_{i=1}^{m} a_i (p_i - p_0) - \sum_{j=1}^{n} b_j (q_j - p_0)$$

We set
$$\Phi(D) = \sum_{i=1}^{m} a_i \Phi(p_i - p_0) - \sum_{j=1}^{n} b_j \Phi(q_j - p_0)$$
$$= \begin{bmatrix} \sum_{i=1}^{m} a_i \int_{p_0}^{p_i} \omega_1 - \sum_{j=1}^{n} b_j \int_{p_0}^{q_j} \omega_1 \\ \vdots \\ \sum_{i=1}^{m} a_i \int_{p_0}^{p_i} \omega_g - \sum_{j=1}^{n} b_j \int_{p_0}^{q_j} \omega_g \end{bmatrix} \in \mathrm{Jac}(X)$$

Name Φ = Abel-Jacobi map

Theorem (Abel-Jacobi) The homomorphism sequence

$$\mathcal{M}(X)^{*}/\{0\} = \mathrm{Ppal}(X) \longrightarrow \mathrm{Div}_0(X) \longrightarrow \mathrm{Jac}(X) \longrightarrow 0$$

is exact.

Equivalently the Abel Jacobi map induces an isomorphism

$$\mathrm{Pic}_0(X) := \mathrm{Div}_0 X / \mathrm{Ppal}(X) \xrightarrow{\sim} \mathrm{Jac}(X) \qquad \text{for } g \geq 1$$

[illegible] $X \to \mathrm{Div}(X) \xrightarrow{\Phi} \mathrm{Jac}(X)$

and hence, an inclusion $X \hookrightarrow \mathrm{Div}_0(X) \xrightarrow{\;\;} \mathrm{Jac}(X)$

$$p \longmapsto p-p_0 \longmapsto \Phi(p-p_0)$$

Remark 1: The construction depends on the choice of a base point p & a basis $\{\omega_1,\dots,\omega_g\}$

But the statements are independent of these choices.

Remark 2: Exactness at $\mathrm{Jac}(X)$ is known as the Jacobi inversion theorem.

Theorem (Torelli) Two compact R.S X, X' of genus $g \geq 1$ are isomorphic (ie $\exists\ \varphi: X \to X'$ biholomorphism) if & only if they have the same period matrix after conveniently picking (canonical) homology bases (Π becomes $[I_g\ Z]$)

§29.3 Elliptic curves:

Next: Understand Abel-Jacobi for elliptic curves. (= genus 1)

- 2 incarnations: ① $X = \mathbb{C}/\Lambda$

  ② X = smooth cubic curve in $\mathbb{P}^2$.

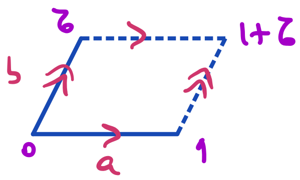
① ⇒ ② Weierstrass $\wp$-function

② ⇒ ① X = RS of $T^2 = h(z)$ $h(z)$ cubic with 3 simple roots & $\Lambda = \mathrm{Per}\left(\frac{dz}{T}\right)$

① $X = \mathbb{C}/\Lambda$

We normalize the lattice Λ & fix a basis $\{1, \tau\}$ with $\mathrm{Im}\,\tau > 0$.

Basis for $H^0(X,\Omega) = \{\omega = dz\}$

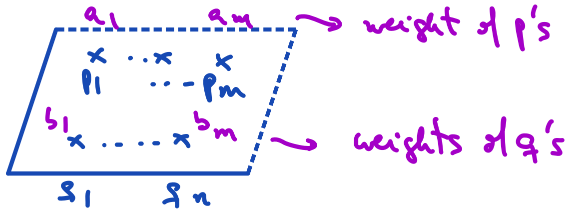


$$\int_a dz = z\Big|_0^1 = 1 \qquad \int_b dz = z\Big|_1^{1+\tau} = \tau$$

$$\Rightarrow \mathrm{Per}(\omega) = [1,\tau]\,\mathbb{Z}^2 = \mathbb{Z}\langle 1,\tau\rangle = \Lambda$$

& $\mathrm{Jac}(X) \cong \mathbb{C}/\Lambda = X$.

Can we write Φ explicitly? Fix $D \in \mathrm{Div}(X)$ & pick representatives of points in Supp(D) in the fundamental domain.

- By a generic perturbation ($\varepsilon \ll 1$) of ▱, we can assume Supp $D \subseteq$ Int(▱).
- 

Write $D = \sum_{i=1}^{n} a_i p_i - \sum_{j=1}^{m} b_j q_j$

$= p_{i_1} + p_{i_2} + \cdots + p_{i_d} - q_{j_1} - \cdots - q_{j_d}$

$d = \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$.

Fix paths γ_{i_k} joining p_0 to p_{i_k}

β_{j_k} ——— p_0 to q_{j_k}

$\Rightarrow C = \sum_{k=1}^{d} \gamma_{i_k} - \beta_{i_k} \in C_1(X)$

& $\partial C = D$

$$\Rightarrow \int_C dz = \sum_{k=1}^{d} \int_{\gamma_{i_k}} dz - \int_{\beta_{i_k}} dz = \sum_{k=1}^{d} (p_{i_k} - p_0) - (q_{i_k} - p_0)$$

$$= \sum_{k=1}^{d} p_{i_k} - \sum_{k=1}^{d} q_{i_k} = D.$$

$$\Rightarrow \Phi(D) = D \mod \Lambda.$$

- $X \hookrightarrow \mathrm{Div}_0(D) \xrightarrow{\Phi} \mathrm{Jac}(X)$ so $X \hookrightarrow \mathrm{Jac}(X)$ is

$x \longmapsto x - 0 \longmapsto \int_0^x dz = x \in \mathbb{C}/\Lambda$ the identity map!

Corollary: $\mathrm{Ker}(\Phi) = \mathrm{Ppal}(X)$ has the following interpretation:

$D = \sum_i a_i p_i - \sum_j b_j q_j$ is principal $\Longleftrightarrow \sum_i a_i p_i - \sum_j b_j q_j \in \Lambda$

This says when can we build meromorphic functions with prescribed 0's & poles in $\mathbb{C}/\Lambda$. The explicit function is constructed with Θ-functions (Jacobi)

Theorem (Abel) : ($\Rightarrow$) Part of Corollary (ie $\mathrm{Ppal}(X) \subseteq \mathrm{Ker}(\Phi)$)

Fix a doubly periodic meromorphic function $f : \mathbb{C} \longrightarrow \mathbb{P}^1$, with $f(z+1) = f(z)$, $f(z+\tau) = f(z)$

(so f determines a meromorphic function $f : X \longrightarrow \mathbb{P}^1$)

Assume f is non-constant. Then

(1) f has poles & zeroes $f^{-1}(0) = \{p_1, \dots, p_m\}$, mult $a_1 \dots a_m$; $f^{-1}(\infty) = \{q_1, \dots, q_n\}$, mult $= b_1 \dots b_m$

(2) $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$ $\quad (= \mathrm{degree}(f))$

(3) $\sum_i a_i p_i - \sum_j b_j q_j \in \Lambda$

Proof: Pick parallelogram so $f^{-1}(0) \cup f^{-1}(\infty) \subseteq \mathrm{Int}(\square)$, with vertices t, $1+t$, $1+\tau+t$, $t+\tau$.

(2) # zeroes $-$ # poles $= \sum_i a_i - \sum_j b_j = \frac{1}{2\pi i} \int_{\partial \square} \frac{f'(z)}{f(z)}\, dz = 0$ ($\uparrow$ f double periodic)

(3) $\sum a_i p_i - \sum b_j q_j = \frac{1}{2\pi i} \int_{\partial \square} z \frac{f'(z)}{f(z)}\, dz$

⚠ $g(z) = z \frac{f'(z)}{f(z)}$ is not doubly periodic, so we have a compute the integral explicitly.

We group the integral into $(\mathrm{I}) = \frac{1}{2\pi i} \left(\int_t^{1+t} g(z)\, dz + \int_{1+t+\tau}^{t+\tau} g(z)\, dz \right) = \frac{1}{2\pi i} \left(\int_t^{1+t} g(z)\, dz - \int_{t+\tau}^{1+t+\tau} g(z)\, dz \right)$

$$(\mathrm{II}) = \frac{1}{2\pi i} \left(\int_t^{t+\tau} g(z)\, dz + \int_{1+t}^{1+t+\tau} g(z)\, dz \right)$$

$$(\mathrm{I}) = \frac{1}{2\pi i} \left(\int_t^{1+t} z \frac{f'(z)}{f(z)}\, dz - \int_{t+\tau}^{1+t+\tau} z \frac{f'(z)}{f(z)}\, dz \right) = \frac{1}{2\pi i} \left(\int_t^{1+t} z \frac{f'(z)}{f(z)}\, dz - \int_t^{1+t} (w+\tau) \frac{f'(w)}{f(w)}\, dw \right)$$

↳ substitution $z = w + \tau$ $\quad$ $f(w+\tau) = f(w)$, $f'(w+\tau) = f'(w)$

$$= \frac{-\tau}{2\pi i} \int_t^{1+t} \frac{f'(z)}{f(z)}\, dz$$

$$(\mathrm{II}) = \frac{1}{2\pi i}\left(\int_t^{t+\tau} \frac{z f'(z)}{f(z)}\,dz + \int_{t+1}^{t+\tau+1} \frac{z f'(z)}{f(z)}\,dz\right) = \frac{1}{2\pi i}\int_t^{t+\tau} \frac{z f'(z)}{z}\,dz + \int_t (w+1)\frac{f'(w)}{f(w)}\,dw$$

↳ substitution $z = w+1$

$f(w+1) = f(w)$
$f'(w+1) = f'(w)$

$$= \frac{1}{2\pi i}\int_t^{t+\tau} \frac{f'(z)}{f(z)}\,dz$$

<u>Claim</u>: $\frac{1}{2\pi i}\int_t^{t+1} \frac{f'(z)}{f(z)}\,dz \;,\; \frac{1}{2\pi i}\int_t^{t+\tau} \frac{f'(z)}{f(z)}\,dz \in \mathbb{Z}$

Pf/ $\frac{1}{2\pi i}\int_t^{1+t} \frac{f'(w)}{f(w)}\,dw = \frac{1}{2\pi i}\int_\gamma \frac{du}{u}$ = Winding # of γ around 0. $\in \mathbb{Z}$

($u = f(w)$)

loop based at $f(t)$: $\gamma_{(s)} = f(s+t) : [0,1] \longrightarrow \mathbb{C}$

$0 \notin \gamma$ $\quad f^{-1}(0) \cap [t, 1+t] = \phi$

$\frac{1}{2\pi i}\int_t^{t+\tau} \frac{f'(w)}{f(w)}\,dw = \frac{1}{2\pi i}\int_{\gamma'} \frac{du}{u}$ = Winding # of γ' around 0 $\in \mathbb{Z}$

loop based at $f(t)$: $\gamma'_{(s)} = f(s+t) : [0,\tau] \longrightarrow \mathbb{C}$

$0 \notin \gamma$ $\quad f^{-1}(0) \cap [t, t+\tau] = \phi$.

$$\Rightarrow \sum a_i p_i - \sum b_j q_j = (\mathrm{I}) + (\mathrm{II}) \in -\tau\mathbb{Z} + \mathbb{Z} = \mathbb{Z} + \mathbb{Z}\tau = \Lambda.$$

<u>Jacobi's Inversion Thm</u>: If $D \in \mathrm{Div}_0(X)$ lies in $\mathrm{Ker}(\Phi)$, then $D \in \mathrm{Ppal}(X)$

To prove this statement, Jacobi used θ functions. Next, we describe this construction for the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}\,\tau > 0$.

<u>Definition</u>: A <u>theta function</u> is a holomorphic function $\theta : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying

(1) $\theta(z+1) = -\theta(z) \quad \forall z$

(2) $\theta(z+\tau) = e^{-\pi i \tau} e^{-2\pi i z}\theta(z) \quad \forall z$

(3) $\theta(z) = 0 \iff z \in \Lambda = \mathbb{Z} + \mathbb{Z}\tau$ & all it's zeroes are simple

(4) [Normalization condition] $\theta'(0) = 1$

<u>Lemma</u>: $\exists\,!\;\theta$ satisfying these conditions.

<u>Proof</u> <u>Uniqueness</u>: Assume we have 2 such functions θ_0 & θ_1. Then, their

ratio Θ_0/Θ_1 is holomorphic & doubly-periodic wrt the rank 2 lattice Λ.

Therefore, it is constant. The normalization condition says $\frac{\Theta_0}{\Theta_1} = 1$, so $\Theta_0 = \Theta_1$

Existence: Set $q = e^{2\pi i \tau}$. Since $\operatorname{Im}\tau > 0$ we have $|q| < 1$.

Set
$$\theta(z) = \frac{\sin(\pi z)}{\pi} \prod_{n \geq 1} \frac{(1 - q^n e^{2\pi i z})(1 + q^n e^{-2\pi i z})}{(1 - q^n)^2}$$

- We can check that the infinite product converges uniformly on discs $\overline{D}_{(0,R)}$ in $\mathbb{C}$, so Θ is holomorphic
- Conditions (1) through (4) follow by construction

Proof of Jacobi Inversion: Fix $D = \sum a_i p_i - \sum b_j q_j \in \operatorname{Div}_0(X)$ with $\phi(D) = 0$ in $\operatorname{Jac}(X)$. Writing D as $D = \sum_{i=1}^{d} \alpha_i - \sum_{j=1}^{d} \beta_j$ (allowing repetition)

(α_i: zeroes; β_j: poles)

We have $\sum_i \alpha_i - \sum_j \beta_j = \lambda \in \Lambda$

Now, change α_n to $\tilde{\alpha}_n = \alpha_n - \lambda$ & set $\tilde{\alpha}_i = \alpha_i$ for $i < n$

Now $\sum \tilde{\alpha}_i - \sum \beta_j = 0$

Check $f = \prod_{j=1}^{d} \frac{\Theta(z - \tilde{\alpha}_i)}{\Theta(z - \beta_i)}$ is a meromorphic function on $X = \mathbb{C}/\Lambda$

(doubly-periodic on $\mathbb{C}$) with zeroes at $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_d$ & poles at $\beta_1, \ldots, \beta_d$

ie $(f) = D$, as we wanted.

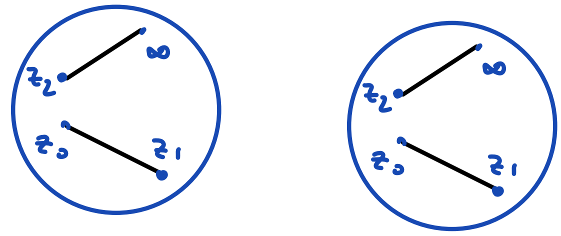
② Algebraic picture: X RS. associated to equation $T^2 - h(z) = 0$

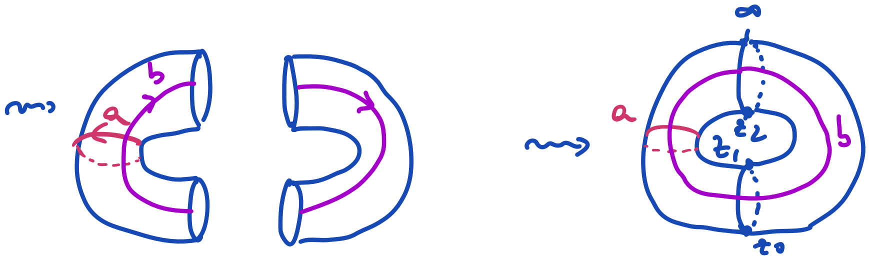
where $h(z)$ is a cubic with 3 distinct roots. $(z_0, z_1, z_2 \in \mathbb{C})$

Basis for $H^0(X, \Omega) = \{ \omega = \frac{dz}{\sqrt{h(z)}} = \frac{dz}{T} \}$

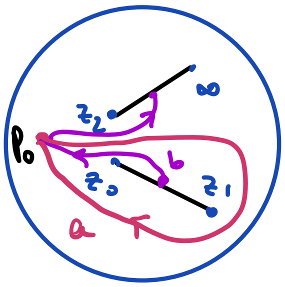
Q: How to pick loops on X?

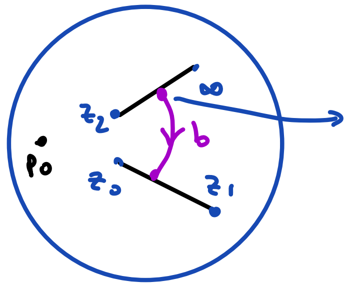
A: Use cuts & glue (Lecture 13)





We draw the paths in each copy of $\mathbb{P}^1$ cut





→ we need to add a jump in the value of ω ⇝ ω becomes $-\omega$ ($\sqrt{h(z)}$ changes sign)

Pick $p_0 \in X$ & assign $p \longmapsto \int_{p_0}^{p} \omega \in \mathbb{C}$ (Name = Elliptic integral!)

This integral is well-defined only up to $\mathbb{Z}$-combinations of $\int_a \omega$ & $\int_b \omega$

- Weierstrass $\wp$-function $\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$ for $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$
- $\wp$ meromorphic on $\mathbb{C}/\Lambda$ & with an order 2 pole at 0. $\Rightarrow$ 2 zeroes on $\mathbb{C}/\Lambda$.
- Weierstrass showed that $(\wp, \wp')$ lies on $T^2 - h(x) = 0$ for h cubic.

More precisely: $(\wp')^2 - 4\wp^3 - a\wp - b = 0$ for suitable $a, b \in \mathbb{C}$)

$\Rightarrow$ Define $\mathbb{C}/\Lambda \xrightarrow{(1, \wp, \wp')} \mathbb{P}^2$

- $\wp(z)$ gives a nice formula for these integrals

$$\int_{p_0}^{u} \frac{dt}{\sqrt{p(t)}} \underset{t = \wp(z)}{=} \int_{\wp(p_0)}^{\wp(u)} \frac{\wp'(z)}{\wp'(z)}\, dz = \wp(u) - \wp(p_0)$$

$\left(\begin{array}{l} \sqrt{P(\wp(z))} = y = \wp' \\ P_{(z)} = 4z^3 + az + b \end{array} \right)$

Catch: any $h(z)$ can be put in the form $p(z) = 4z^3 + az + b$ by linear coordinate changes in $\mathbb{P}^1$, so we can "compute" elliptic integrals with $\wp$.