Reading course in Tropical Geometry – Problem set 1
The Tropical semiring, valuations, tropical plane curves

Problem 1. The Tropical Fundamental Theorem of Algebra

(i) Prove the tropical fundamental theorem of algebra: every non-constant tropical polynomial

$$F(x) = x^{\dot{d}} \oplus c_{d-1} \odot x^{\odot(d-1)} \oplus \ldots \oplus c_0 \in \overline{\mathbb{R}}_{\text{trop}}[x]$$

when viewed as a function $F\colon \mathbb{R} \to \mathbb{R}$ factorizes as a product of linear forms $\bigodot_{k=1}^{d} (x \oplus \lambda_k)$.

(ii) Define the *multiplicity* of a root λ of a non-constant tropical polynomial to be the number of times it appears in the factorization from (i), that is $\text{mult}(\lambda, F) = \#\{k \in \{1, \ldots, d\} : \lambda_k = \lambda\}$.

Prove that for any $f \in \mathbb{C}\{\{t\}\}[x]$, the multiplicity of a root ω in $\text{trop}(f)$ equals the number of roots in $\text{in}_\omega(f) \in \mathbb{C}^*$ (counted with multiplicity).

(iii) How many roots in $\mathbb{C}\{\{t\}\}$ does the polynomial $t^3x^5 - x^2 + t^4$ have? Describe the first few terms of each series.

Problem 2. Valued field extensions through tropicalization Consider a valued field (K, val) and a finite field extension L of K. It is known that the valuation val on K can be extended (in at most $[L : K]^{\text{sep}}$ ways) to a valuation on L.

(i) Assume $L{=}K(\alpha)$ for some α and prove that for any extension of val to L, $-\text{val}(\alpha)$ is a zero of the tropical polynomial $\text{trop}(\min(\alpha, K))$, where $\min(\alpha, K)$ is the minimal polynomial of α over K.

(ii) [Exercise 2.7.3 in [MS]]† The quotient ring $L = \mathbb{Q}[s]/\langle 3s^3 + s^2 + 36s + 162\rangle$ is a field. Describe all valuations on this field that extend the 3-adic valuation on $\mathbb{Q}$.

Problem 3. Tropical plane quadrics

(i) How many combinatorial types of plane quadrics (degree two) are there? For a few cases, find an example of $f \in \mathbb{C}\{\{t\}\}[x, y]$ or $\overline{\mathbb{Q}_p}[x, y]$, draw the corresponding Newton subdivision and the tropical curves $\mathcal{T}f$.

(ii) Given five general points in $\mathbb{R}^2$, there exists a unique tropical quadric passing through these points. Compute and draw the tropical quadric through $(0, 5)$, $(1, 0)$, $(4, 2)$, $(7, 3)$ and $(9, 4)$.

Problem 4. Prove **tropical Bézout's theorem** for transverse intersetions: two plane tropical curves C and D of degrees c and d (so they are dual to subdivisions of the 2-simplex dilated by c and d, respectively) that meet transversely have exactly $c \cdot d$ intersection pooints (counted with multiplicity). The multiplicity at a point p is given by the formula:

$$\text{mult}(e)\ \text{mult}(f) \left| \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|,$$

where e and f are the edges of C and D containing p, and $e' = (u_1, v_1)$, $f' = (u_2, v_2)$ are the primitive vectors in $\mathbb{Z}^2$ in the directions of e and f, respectively. (*Hint:* Consider $C \cup D$ as a tropical plane curve.)

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† *Optional (for extra credit)*