# Reading course in Tropical Geometry – Problem set 3
# Gröbner bases over valued fields, Gröbner complex, and tropical varieties

**Problem 1. Gröbner basis computation.** Given a valued field $\mathbf{K}$ with a splitting of val $a \mapsto t^a$, a *homogeneous* polynomial $f = \sum_u c_u x^u \in K[x_1, \ldots, x_n]$ and $\omega \in \mathbb{R}^n$, recall that $\mathrm{in}_\omega(f) = \sum_u \overline{c_u t^{\mathrm{trop}(f)(\omega) - \langle u, \omega \rangle}} x^u \in \Bbbk[x_1, \ldots, x_n]$. The following outlines how to compute an initial ideal $\mathrm{in}_\omega(I)$ when the valuation is trivial and $I$ is homogeneous.

(i) Given any monomial ordering $\prec$, show that $\mathrm{in}_\prec \mathrm{in}_\omega I = \mathrm{in}_{\prec_\omega} I$ where $\prec_\omega$ is the monomial order refining the $\omega$-order by $\prec$, i.e. $x^\alpha \prec_\omega x^\beta$ if $\langle \alpha, \omega \rangle < \langle \beta, \omega \rangle$ or $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle$ and $x^\alpha \prec x^\beta$.

(ii) Show that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for $I$ with respect to $\prec_\omega$, then $\mathrm{in}_\omega(g_i)$ is a Gröbner basis for $\mathrm{in}_\omega I$ with respect to $\prec$. (*Hint:* A Gröbner basis for a monomial order generates the ideal.) Conclude that $\mathrm{in}_\omega I = \langle \mathrm{in}_\omega g_1, \ldots, \mathrm{in}_\omega g_s \rangle$.

(iii) [Optional] What happens if the valuation on $K$ is non-trivial?

**Problem 2.** Consider the ideal $I = \langle f, g \rangle \subset \mathbb{C}\{\{t\}\}[x^\pm, y^\pm]$ where

$$f = t^2 x^2 + xy + t^2 y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6t\, x + 17t\, y - 4t^3\, xy.$$

(i) For each $\omega \in \mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$, compute $\mathrm{in}_\omega I$. Is $\{f, g\}$ a tropical basis for $I$?

(ii) There are four points in the variety $V(I) \subset (\mathbb{C}\{\{t\}\}^*)^2$. Compute the leading term of each point.

**Problem 3.** Consider the linear ideal $I = \langle x_1 + x_2 + x_3 + x_4 + x_5, 3x_2 + 5x_3 + 7x_4 + 11x_5 \rangle \subset \mathbb{C}[x_1^\pm, \ldots, x_5^\pm]$. The tropical variety $\mathrm{Trop}(I)$ is a three-dimensional fan with a one-dimensional lineality space. It is a fan over the complete graph $K_5$. The fan has ten maximal cones and five codimensional-one cones. The following shows that a change of coordinates in $T = (\mathbf{K}^*)^5$ might change the structure of the tropical variety.

Consider the automorphism $\varphi^* \colon T \to T$ defined by $x_1 \mapsto x_1$, $x_2 \mapsto x_2 x_3$, $x_3 \mapsto x_3 x_4$, $x_4 \mapsto x_4 x_5$, $x_5 \mapsto x_5$ and let $J = (\varphi^*)^{-1}(I) \subset \mathbb{C}[x_1^\pm, \ldots, x_5^\pm]$.

(i) Show that $\mathrm{Trop}(J)$ has the same support as $\mathrm{Trop}(I)$.

(ii) Show that the Gröbner structure of $\mathrm{Trop}(J)$ has 12 maximal cones, obtained as the cone over a subdivision of $K_5$ where 2 edges are subdivided. (*Hint:* You can use `Gfan` to verify this.)