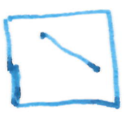
Lecture II: Polyhedra, subdivisions & Newton polytopes

GOALS: • Define polyhedra, collections thereof.
• Regular subdivisions.

Let V be a vector space over $\mathbb{F}$. field.

Def $L \subseteq V$ is convex if $x, y \in L$ then $\lambda x + (1-\lambda y) \in L$ for all $\lambda \in (0,1)$

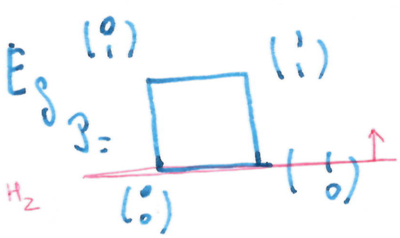
Ex.:  yes  NO

Def If $K \subset V$, then $\mathrm{conv}(K)$ is the smallest set containing K.
[Why? • $\cap$ of convex sets is convex
• V is convex.]

Def $C \subseteq V$ is a cone if $\forall r \in C$, $\lambda r \in C$ $\forall \lambda \geq 0$.

Polyhedra: 2 representations H- vs V-repn.

H-representation: A polyhedron in $\mathbb{R}^n$ is the intersection of finitely many half-spaces; ie $P = \{ \underline{x} \in \mathbb{R}^n : \underset{\mathbb{R}^{s\times n}}{A} \underline{x} \leq \underset{\mathbb{R}^s}{\underline{b}} \}$

Eg $P=$ 

$$P = \{ x \in \mathbb{R}^n \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \underline{x} \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \}$$

$H_2 = \{ (0,-1)\cdot(x,y) \leq 1 \}$

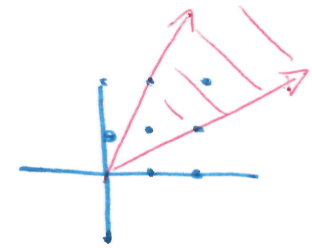
Eg:

A convex polyhedral cone is $P = \{ \lambda_1 v_1 + \dots + \lambda_m v_m \mid \lambda_i \geq 0 \} \subseteq \mathbb{R}^n$
$= \mathbb{R}_{\geq 0} \{ v_1, \dots, v_m \}$. = cone over a polyhedron

V-representation: Vertices of Polyhedra
$P = \mathrm{conv\ hull}\{ v_1, \dots, v_m \}$.

Properties: Let C be a polyhedral cone in $\mathbb{R}^d$:
(1) C is simplicial if $\dim(C) = n = \#$ gens.

2) C is <u>smooth</u> if all $\underline{x} \in C \cap \mathbb{Z}^n$ can be written as a $\mathbb{Z}$-linear combination of its primitive generators
↳ integer coefficients & gcd(coeffs) = 1.

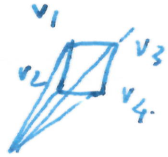


<u>Eg 1</u>: $\mathbb{R}_{\geq 0}\langle (1,2),(4,2)\rangle = \mathbb{R}\langle (1,2),(2,1)\rangle$

- simplicial
- not smooth : $(1,1) \in \mathbb{Z}\langle (1,2),(2,1)\rangle$

How to check this? $\det \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = 3 \neq \pm 1$.

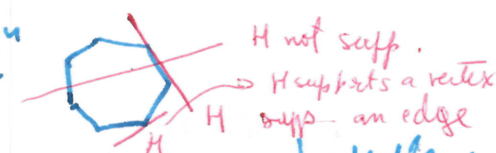
<u>Eg 2</u> : Cone over square :



dim = 3 minimal # gens = 4
not simplicial $\Rightarrow$ not smooth.

<u>Faces & supporting hyperplanes</u>:

<u>Def</u> K convex set $\subseteq \mathbb{R}^n$



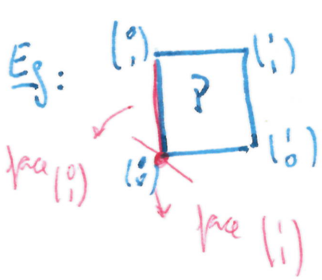
A supporting hyperplane H is a hyperplane st $H \cap K \neq \emptyset$ K is contained in one of the two halfspaces defined by H.

<u>Note</u>: If A is the matrix defining the H-rep. of K, then rows of A define supporting hyperplanes (not all!)

<u>Def</u> A face of a polyhedra is determined by $\underline{w} \in \mathbb{R}^n$ where $\underline{w}x = b$ is a supporting hyperplane.

$\mathrm{face}_w(P) = \{ \underline{x} \in P \mid wx \leq wy \ \forall y \in P \}$ = face supported by a Hyp with normal $\vec{w}$.

<u>Eg</u>:

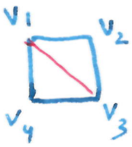


$\mathrm{face}_{\binom{0}{0}} P = P$.

<u>Note</u>: A face of a face of P is a face of P
- Faces of P are polyhedra.

<u>Notation</u>: If S is a face of a polyhedron σ, write $S \leq \sigma$ (no poset structure!)

Remark: If $C = \mathbb{R}_{\geq 0}\{v_1, \dots, v_m\}$, then a face of C is generated by a subset of $\{v_i\}_{i=1}^{n}$. But not every subset defines a face.

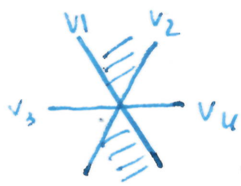
Eg:  $CH\{v_1, v_3\}$ is not a face

Q: What about collections of polyhedra?

Def A polyhedral fan is $\Sigma = \{\sigma \mid \sigma \text{ is a polyhedral cone}\}$ s.t. ($\{\sigma\}$ = cells of Σ)

① If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 & σ_2. (Write $\sigma_1 \cap \sigma_2 \leq \sigma_1$ & $\sigma_1 \cap \sigma_2 \leq \sigma_2$)

② If $\sigma \in \Sigma$ & $S \leq \sigma$, then $S \in \Sigma$.

Eg: 

YES

NOT pure


No.

Def A polyhedral complex is $\Sigma = \{\sigma \mid \sigma \text{ is a polyhedron}\}$ st ① & ② hold

Def The support of a polyhedral complex is $\operatorname{supp}(\Sigma) = |\Sigma| = \{\underline{x} \in \mathbb{R}^n \mid \underline{x} \in \sigma \text{ for some } \sigma \in \Sigma\}$

Properties $\dim(\Sigma) = \max_{\sigma \in \Sigma} \{\dim(\sigma)\}$

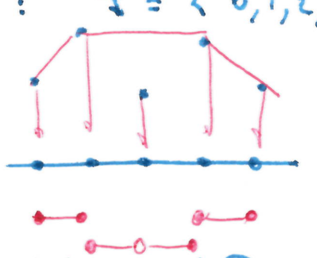
- $\dim(\sigma)$ = dimension of its affine span.

We say that Σ is pure if all maxl cells of Σ have the same dimension.

Regular subdivisions by example:

$\mathcal{P} = \operatorname{conv}(S)$ $\quad S = \mathcal{P} \cap \mathbb{Z}^n$

Def: $\mathcal{P} = \{0, 1, 2, 3, 4\} \subseteq \mathbb{R}^d$



$\rightsquigarrow$ Pick a ht for each pt in $S \subseteq \mathbb{R}^d \times \mathbb{R}$

$\rightsquigarrow$ $\tilde{\mathcal{P}} = \{(p, z)\,,\ z \leq h(p)\ p \in S\}$

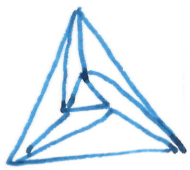
$\rightsquigarrow$ $\operatorname{Conv}(\tilde{\mathcal{P}})$ = upper envelope

Subdivision of $\mathcal{P}$ = project the facets of $\operatorname{conv}(\tilde{\mathcal{P}})$ down

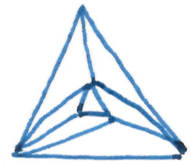
mark points in S if their lift $(p, h(p)) \in$ facet of $\operatorname{conv}(\tilde{\mathcal{P}})$

Not all polyhedral subdivisions are regular!

Eg:



NO but



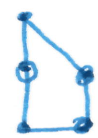
YES

Newton polygons:

Def Given $f = \sum_{\substack{u \in \mathbb{Z}^n \\ \text{finite}}} c_u x^u \in S = K[x_1^{\pm}, \dots, x_n^{\pm}]$.

The Newton polytope of $f = NP(f)$ is the convex hull $\{u \mid c_u \neq 0\}$

Eg: $K = \mathbb{C}\{\{t\}\}$. $\quad f = \underset{(0,0)}{\underset{\downarrow}{1}} + \underset{(1,0)}{\underset{\downarrow}{t^{-1}x}} + \underset{(0,2)}{\underset{\downarrow}{y^2}} - \underset{(1,1)}{\underset{\downarrow}{txy}}$



NEXT TIME: We'll see how to get a height for each monomial x^u via the valuation of the coefficient c_u.