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Lecture V: Gröbner bases over valued fields I

Notation: (K, val) a field with valn & a splitting homomorphism $\Gamma_{\mathrm{val}} \to K$, $\gamma \mapsto t^{\gamma}$ (value group Γ_{val})

$S := S_K = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. $\overline{} : R_K \to \mathcal{O}_K/m_K$ is ring homom.

$f \in S$ $\quad f = \sum_u c_u X^u$ $\quad \mathrm{trop}(f)(w) = \max_u (-\mathrm{val}(c_u) + w \cdot u)$

$W := \mathrm{trop}(f)(w)$

Def $\mathrm{in}_w(f) = \sum_u \overline{t^{\mathrm{trop}(f)(w) - \langle w, u\rangle} c_u} \; X^u$

Note $\mathrm{val}(t^{\mathrm{trop}(f)(w) - \langle w,u\rangle} c_u) = \mathrm{trop}(f)(w) - \langle w,u\rangle + \mathrm{val}(c_u)$
$= \mathrm{trop}(f)(w) - (-\mathrm{val}(c_u) + \langle w,u\rangle)$

≥ 0 & $=$ exactly for those u where max is achieved.

Rephrase: $\mathrm{in}_w(f) = \sum_{u \in I_{f,w}} \overline{c_u t^{-\mathrm{val}(c_u)}} \; x^u$

where $I_{f,w} = \{u \mid \mathrm{trop}(f)(w) = -\mathrm{val}(c_u) + \langle w,u\rangle\}$

Lemma 0. Given $f, g \in S$, $\mathrm{trop}(fg) = \mathrm{trop}(f) + \mathrm{trop}\, g$ as functions

Pf/ Write $f = \sum c_u x^u$, $g = \sum d_v X^v$

$\mathrm{trop}(fg)(w) = \max_a \left(-\mathrm{val}\left(\sum_{u+v=a} c_u d_v\right) + \langle w, a\rangle\right)$

Claim: $= \max_u (-\mathrm{val}(c_u) + \langle u, w\rangle) + \max_v (-\mathrm{val}(d_v) + \langle v, w\rangle)$.

$(= \mathrm{trop}(f) + \mathrm{trop}(g))$

Prove $\leq$ & $\geq$.

$(\leq)$ $\mathrm{val}\left(\sum_{u+v=a} c_u d_v\right) \geq \min_{u+v=a} \{\mathrm{val}(c_u) + \mathrm{val}(d_v)\}$

$-\mathrm{val}(\sum c_u d_v) \leq \max_{u+v} \{-(\mathrm{val}(c_u) + \mathrm{val}(d_v))\}$

So $\langle w, a\rangle - \mathrm{val}(\sum c_u d_v) \leq \langle w, \overset{\shortparallel}{a}\rangle - \min_{u+v=a} \{\mathrm{val}(c_u) + \mathrm{val}(d_v)\} =$

$$= \max_{u+v=a} \{ \langle w, u \rangle - \mathrm{val}(c_u) + \langle w, v \rangle - \mathrm{val}(d_v) \}$$

$$\leq \max_{u,v} (\langle w, u \rangle - \mathrm{val}(c_u) + \langle w, v \rangle - \mathrm{val}(d_v) \}$$

$$\leq \max_{u} (\langle w, u \rangle - \mathrm{val}(c_u)) + \max_{v} \{ \langle w, v \rangle - \mathrm{val}(d_v) \}$$

$$= \mathrm{trop}(f) + \mathrm{trop}(g)$$

($\geq$) If $w \notin \mathcal{T}(V(f)) \cup \mathcal{T}(V(g))$ we will prove $\leq$. Since the functions are continuous and this set is dense in $\mathbb{R}^n$, we are done!

$$\mathcal{T}(V(f)) = \{ w \mid \mathrm{trop}(f)_{(w)} \text{ is attained twice} \}$$

Subclaim: $c_u d_v x^{u+v}$ is a term in (fg) ~~given~~ if max for $\mathrm{trop}(f)_{(w)}$ is attained at u & max for $\mathrm{trop}(g)(w)$ is attained at v.
i.e. $(\text{Term} = (c_u d_v + \text{Tail}) x^{u+v}$

$$\Rightarrow \mathrm{trop}(fg)(w) \geq \mathrm{trop}(c_u d_v x^{u+v})_{(w)} = \mathrm{trop}(f)(w) + \mathrm{trop}(g)$$

Write $f = c_u x^u + \tilde{f}$
$g = c_v x^v + \tilde{g}$

$$fg = c_u d_v x^{u+v} + c_u x^u \tilde{g} + d_v x^v \tilde{f} + \tilde{f}\tilde{g}$$

$$\mathrm{trop}(c_u d_v x^{u+v}) > \mathrm{trop}(c_u x^u \tilde{g}), \mathrm{trop}(d_v x^v \tilde{f}), \underbrace{\mathrm{trop}(\tilde{f}) + \mathrm{trop}(\tilde{g})}_{\vee\ \mathrm{trop}(\tilde{f}\tilde{g})}.$$

So no cancellation!

Lemma 1: Given $f, g \in S \setminus \{0\}$, $\mathrm{in}_w(fg) = \mathrm{in}_w(f) + \mathrm{in}_w(g)$

Proof: By Lemma 0, $\mathrm{trop}(fg)(w) = \mathrm{trop}(f)_{(w)} + \mathrm{trop}(g)(w)$

$$I_{f,w} + I_{g,w} := \left\{ u+v \;\middle|\; \begin{array}{l} \mathrm{trop}(f)(w) = -\mathrm{val}(c_u) + \langle u, w \rangle \\ \mathrm{trop}(g)(w) = -\mathrm{val}(d_v) + \langle v, w \rangle \end{array} \right\}$$

$$\underset{\substack{\uparrow \\ \text{Lemma 0}}}{=} \{ a \mid \mathrm{trop}(fg)(w) = \underbrace{-\mathrm{val}(\mathrm{coef}_{x^a}(fg)) + \langle w, a \rangle \}}_{\overset{\shortparallel}{I_{fg,w}}}.$$

$$\mathrm{in}_\omega(fg) = \sum_{a \in I_{fg,\omega}} \overline{t^{-\mathrm{val}(\mathrm{coeff}_{x^a}(fg))}\,\mathrm{coeff}_{x^a} fg}\; x^a$$

But

$$\mathrm{coeff}_{x^a}(fg) = \sum_{u+v=a} c_u d_v \,, \quad -\mathrm{val}(\mathrm{coeff}_{x^a} fg) = -\mathrm{val}(c_u) - \mathrm{val}(d_v)$$

for some $u+v=a$, $u \in I_{f,\omega}$, $v \in I_{g,\omega}$.

so
$$\mathrm{in}_\omega(fg) = \sum_{a \in I_{fg,\omega}} \overline{\Big(\sum_{\substack{u+v=a \\ u \in I_{f,\omega} \\ v \in I_{g,\omega}}} c_u t^{-\mathrm{val}(c_u)} d_v t^{-\mathrm{val}(d_v)} \Big)} x^a$$

$$= \sum_a \sum_{\substack{u+v=a \\ \cdots}} \overline{c_u t^{-\mathrm{val}(c_u)}}\; \overline{d_v t^{-\mathrm{val}(c_v)}}\; x^u x^v$$

$$= \Big(\sum_{u \in I_{f,\omega}} \overline{c_u t^{-\mathrm{val}(c_u)}}\, x^u \Big) \Big(\sum_{v \in I_{g,\omega}} \overline{d_v t^{-\mathrm{val}(d_v)}}\, x^v \Big) \qquad \square$$

$$= \mathrm{in}_\omega(f)\, \mathrm{in}_\omega(g)$$

Classical Gröbner basis

Def $I \subseteq R$ graded <u>homogeneous</u> if $f \in I$, $f = \sum_{i \in \text{grading}(R)} f_i$ then $f_i \in I$.

Eg $\deg(x^\alpha) = \sum_{i=1}^n \alpha_i$ $\quad R = S = k[x_1, \ldots, x_n]$

<u>Extension</u>: Multigrading induced by $\mathbb{Z}^{m \times n}$ $\quad \deg(x^\alpha) = A\alpha$ in $\mathbb{Z}^m$

<u>Eg</u>: $f = x^3 - yz^2$ is A-homog with resp to $A = \begin{bmatrix} \boxed{1\;1\;1} \\ 1\;3\;0 \end{bmatrix}$ standard grading

Why? f defines a proj curve set $z=1 \rightsquigarrow x^3 - y$ yields $[1\;3]$ & set weight of $z = 0$.

Def $I \subseteq S$ homogeneous, $\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(f) \mid f \in I \rangle$

Def $\mathcal{G} = \{g_1, \dots, g_n\} \subset I$ is a Gröbner basis for I wrt w if

$$\mathrm{in}_w I = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_n) \rangle$$

Traditional GB Theory:

- total order on monomials, $1 < x^\alpha$
- $\mathrm{in}_<(f)$ is a monomial
- $x^\alpha < x^\beta \Rightarrow x^{\alpha+\delta} < x^{\beta+\delta}$

Ex: • $\leq_{lex}$ $x^\alpha < x^\beta$ if first non-zero term of $\beta - \alpha$ is > 0.

• $\leq_{glex}$ graded lex $x^\alpha < x^\beta$ if $|\alpha| < |\beta|$ or if $|\alpha| = |\beta|$ term $x^\alpha <_{lex} x^\beta$.

• $\leq_{degrlex}$ reverse lex. $x^\alpha < x^\beta$ if $|\alpha| < |\beta|$ or if $|\alpha| = |\beta|$ terms $x^\alpha <_{rlex} x^\beta$ (~~last~~ first nonzero term of $\beta - \alpha$ is < 0)

~~• Weighted order~~

• Block orders: ① Partition variables into blocks & assign weight w_B to all vars in ~~that~~ block B

Eg: $\underbrace{x_1 \; x_2}_{w_1}$ $\underbrace{x_3 \; x_4}_{w_2}$ $\underbrace{x_5}_{w_3}$

② compute $\langle \alpha_{|_B}, w_B 1_B \rangle$ ⇝ vector in $\mathbb{R}^{\#\text{blocks}}$, $\tilde{\alpha}$

③ use lex/revlex etc to compare vectors $\tilde{\alpha}$.

$x_1 \; x_2 \; x_3$

$(2 \; 1) \; (3)$

$x_1^2 \rightsquigarrow (4,0)$

$x_2 x_3 \rightsquigarrow (1,3)$ ∧ ← compare lex / revlex / etc.