Lecture VI Gröbner Bases over valued fields II: Homogeneity, perturbations, Hilbert functions & initial degenerations

Def A weight ~~ord~~ (partial) order $\leq_w$, $w \in \mathbb{R}^n$

$x^\alpha \leq_w x^\beta$ if $\langle \alpha, w\rangle \leq \langle \beta, w\rangle$

Not a Total order: $w = (1,2)$ $\alpha = (2,0)$, $\beta = (0,1)$

Desirable properties
- $x^\alpha \leq x^\beta \Rightarrow x^{\alpha+\gamma} \leq x^{\beta+\gamma}$ always.
- Well-order: quite enough $1 \leq_w x^\alpha$ (Need $w_i > 0 \;\forall i$)
- TOTAL order $\leq_w$: generic (ratios $\frac{w_i}{w_j}$ irrational)

Def Block Orders:

A block order is given by partitioning variables using a $\leq_{w_i}$ on each part & then using a lex order

Example: $K[x_1, x_2, x_3]$

(x_1, x_2), x_3 $\qquad x_1 x_2 x_3 \rightsquigarrow (3,2)$

$(2,1)$ $\quad 2$

Prop 1 (Robbiano 1986) Every mon order is a block order.

Prop 2 (Bayer 1982) Every monomial order is approximated by a weight order

Say $\leq$ is a mon order. & S finite set, then ordering on S given by $\leq$ $(x^{u_1} \leq \cdots \leq x^{u_n})$ is realized by some $\leq_w$ $(x^{u_1} \leq_w \cdots \leq_w x^{u_n})$

Example $\leq = \text{lex}$. Pick $\delta_1 \gg \cdots \gg \delta_n$. $\rightsquigarrow w = (\delta_1, \ldots, \delta_n)$

$\leq$ grlex Pick ______ $\rightsquigarrow w = (1+\delta_1, \ldots, 1+\delta_n)$

Q Why monomial orders? A We have a division algorithm!

Can compute
- Gröbner basis for each term order.
- Intersections, Radicals, inclusion of ideals

Initial forms:

Lemma 1: $\{f_\alpha\}$ finite collection of nonzero Laurent polynomials with pairwise disjoint support. Let $W_\alpha = \operatorname{trop}(f_\alpha)(\omega)$, $f = \sum f_\alpha$, $W = \operatorname{trop}(f)(\omega)$. Then $W = \max(W_\alpha)$ & $\operatorname{in}_W f = \sum'_{W_\alpha = W} \operatorname{in}_{W_\alpha}(f_\alpha)$.

Lemma 2: $I \subset K[x_1, \ldots, x_n]$ homog, $w \in \mathbb{R}^n$. Then: $\operatorname{in}_\omega(I)$ is homogeneous in $\tilde{K}[x_1, \ldots, x_n]$ & it has a homogeneous Gröbner basis for $\operatorname{in}_\omega(I)$.

Def Homogeneity: ~~$g \in \operatorname{in}_\omega(I)$~~ $f \in I$ so $f = \sum_i f_i$ $f_i \in I$ disjoint support, so $\operatorname{in}_\omega(f) = \sum \operatorname{in}_\omega(f_i)$. So $\operatorname{in}_\omega I$ is homogeneous (it's gen by homog. elem)

$\operatorname{trop}(f_i)_{(\omega)} = \operatorname{trop}(f)_{(\omega)}$

- GB argument follows by construction, once we know one exists.

Def: A Gröbner basis for I w.r.t. ω is $\{g_1, \ldots, g_n\} \subseteq I$ s.t.

$$\langle \operatorname{in}_\omega(g_1), \ldots, \operatorname{in}_\omega(g_n) \rangle = \operatorname{in}_\omega I$$

- $\exists$ follows by Noetherianness.

Prop $I \in K[x_1, \ldots, x_n]$ I homogeneous, $w \in \Gamma^n_{val}$, $g \in \operatorname{in}_w I$ homogeneous then $g = \operatorname{in}_\omega(f)$ for some $f \in I$

2 questions for initial ideals:

① How many $\operatorname{in}_\omega I$ do we have for a fixed I?

② How to determine $\operatorname{in}_\omega I = \operatorname{in}_{\omega'} I$ given ω, ω'?

I principal: $\# \operatorname{in}_\omega I$ = regular subdivisions of NP induced by ω.

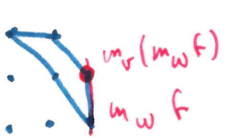
Prop 1: $f \in K[x_1, \ldots, x_n]$ $\omega, v \in \mathbb{R}^n$, $\exists\, \varepsilon > 0$ such that $\forall\, 0 < \varepsilon' < \varepsilon$

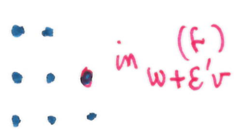
$$\operatorname{in}_v(\operatorname{in}_\omega f) = \operatorname{in}_{\omega + \varepsilon' v} f.$$

Ex $f = x^2y + y^2x + x^2 + y^2$

$\omega \rightarrow$

$v: \uparrow$





$\varepsilon' \gg 0$ will fail, since $\operatorname{in}_{\omega + \varepsilon' v}(f) = x^2y + \underline{xy^2}$

Prop 2: $\operatorname{in}_v(\operatorname{in}_w I) = \operatorname{in}_{w+\varepsilon' v} I$ for I homog ideal.
$\exists\, \varepsilon>0$ st $\forall\, 0<\varepsilon'<\varepsilon$.

Pf/ (1) Show ($\subseteq$) $\operatorname{in}_v(\operatorname{in}_w I) \subseteq \operatorname{in}_{w+\varepsilon' v}(I)$
[Use Prop 0 to get ε from $\operatorname{in}_w I = \langle \operatorname{in}_w(g_1), \ldots, \operatorname{in}_w(g_s)\rangle$ & pick.
$\varepsilon = \min\{\varepsilon(g_1), \ldots, \varepsilon(g_s)\} > 0$.]

(2) $\boxed{J \subseteq K[x_1, \ldots, x_n] \quad HF(J) = HF(\operatorname{in}_w J)}$

(3) RHS & LHS have the same HF . .

(4) ($\subseteq$) + (2), (3) $\Rightarrow$ $=$.

Def The Hilbert function HF for $I \subseteq k[x_1, \ldots, x_n]$

$$HF(I)(d) = \dim_k I_d$$

$I = k[x_1, \ldots, x_n]$

$HF(I)(d) = ?$

$d =$	0	1	2
I_d	$\langle 1\rangle$	$\langle x_1 \ldots x_n\rangle$	$\langle x_i x_j\rangle$
dim	1	n	$\binom{n}{2}+n$.

$$H.F(I)(d) = \binom{n+d}{d}$$