Reading course in Tropical Geometry Sp 2022 – Problem set 1
The Tropical semiring and tropical plane curves

Problem 1. The Tropical Fundamental Theorem of Algebra

(i) Prove the tropical fundamental theorem of algebra: every non-constant tropical polynomial

$$F(x) = x^{\dot{d}} \oplus c_{d-1} \odot x^{\odot(d-1)} \oplus \ldots \oplus c_0 \in \overline{\mathbb{R}}_{\text{trop}}[x]$$

when viewed as a function $F\colon \mathbb{R} \to \mathbb{R}$ factorizes as a product of linear forms $\bigodot_{k=1}^{d} (x \oplus \lambda_k)$.

(ii) Define the *multiplicity* of a root λ of a non-constant tropical polynomial to be the number of times λ appears in the factorization from (i), that is $\operatorname{mult}(\lambda, F) = \#\{k \in \{1, \ldots, d\} : \lambda_k = \lambda\}$. How do you determine the multiplicity of λ from the graph of F?

(iii) Find all roots of the tropical quintic $x^5 \oplus 1 \odot x^4 \oplus 3 \odot x^3 \oplus 6 \odot x^2 \oplus 10 \odot x \oplus 15$.

Problem 2. Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a function that is continuous, concave, and piecewise linear with finitely many pieces that are linear functions with integer coefficients. Show that p can be represented by a tropical polynomial in $x_1, \ldots, x_n$ (with the min convention).

Problem 3. (Tropical singular matrices) The tropical 3×3-determinat is a piecewise linear real-valued function $\mathbb{R}^{3\times 3} \to \mathbb{R}$ on the nine-dimensional space of 3×3-matirces.

(i) Describe all the regions of linearity of this function and their boundaries.

(ii) What does it mean for a matrix to be tropically singular? (*Aside:* There are various notions of rank for tropical matrices as seen in "On the rank of a tropical matrix" by Develin, Santos and Sturmfels.)

(iii) For which values of x is the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & x \end{pmatrix}$ tropically singular? What about $\begin{pmatrix} x & 2 & 3 \\ 2 & x & 6 \\ 3 & 6 & x \end{pmatrix}$?

Problem 4. Realizations of tropical plane curves
Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ be vectors in $\mathbb{Z}^n$ with zero-sum, i.e. $\sum_{i=1}^{m} \mathbf{v_i} = \mathbf{0}$. Show that there exists an algebraic curve in $(\mathbb{C}^n)$ whose tropical curve in $\mathbb{R}^n$ consists of the rays spanned by the m vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$.

Problem 5. Prove that the self stable-intersection of a tropical plane curve equals its set of vertices.

Problem 6. (Tropical plane quadrics) For this problem, use the **max-convention** for tropicalization.

(i) How many combinatorial types of plane quadrics (degree two) are there? For a few cases, draw the corresponding Newton subdivision of the triangle with vertices $(0,0), (2,0)$ and $(0,2)$ and its dual tropical curve.
(**Hint:** The count can be done using the number of maximal cones in the *secondary fan* of this triangle with its six lattice points. Read about this *before* you attempt to find the correct number, but draw examples to familiarize yourself with the duality between Newton subdivisions and tropical curves.)

(ii) Given five general points in $\mathbb{R}^2$, there exists a unique tropical quadric passing through these points. Compute and draw the tropical quadric through $(0,5)$, $(1,0)$, $(4,2)$, $(7,3)$ and $(9,4)$.

Problem 7. Effective tropical interpolation

In Problem 6, you were asked to find a tropical quadric through five points in $\mathbb{R}^2$. The objective of this exercise is to reflect on this fact and see what is true in general. We fix $\mathbb{K}$ to be a field of characteristic zero.

(i) Given five generic points in $\mathbb{K}^2$, show that exists a unique plane quadric through them. Give an algorithm to construct its defining equation (up to scalar).

(ii) Derive a tropical analog of the algorithm in the previous item (in particular, you will have to decide what "tropically generic points" means). Conclude from this that for any multiset of five generic points in $\mathbb{R}^2$ there exists a unique tropical quadric passing through them. What happens for special configurations?

Problem 8. Smooth tropical plane cubics

A tropical plane cubic curve is *smooth* if it has precisely nine vertices.

(i) Show that a tropical plane cubic is smooth if and only if it is dual to a unimodular triangulation of the triangle with vertices $(0,0)$, $(3,0)$ and $(0,3)$.

(ii) Prove that every tropical smooth plane cubic curve has a unique bounded region and that this region can have either three, four, five, six, seven, eight or nine edges. Draw examples for all seven cases. (*Aside:* If the tropical plane cubic is the tropicalization of a smooth cubic curve defined over a field with valuation, such as $\mathbb{Q}_p$ or $\mathbb{C}\{\{t\}\}$, the lattice length of the boundary of this unique bounded region is bounded above by $\max\{0, -\operatorname{val}(j)\}$ where j is the j-invariant of the algebraic cubic curve. This was used to define the "tropical j-invariant".)

Problem 9. (Implicitization via tropical geometry) Consider the parameterized plane curve given by

$$x = (t-1)^{13}\, t^{19}\, (t+1)^{29} \quad \text{and} \quad y = (t-1)^{31}\, t^{23}\, (t+1)^{17}.$$

Find the Newton polygon of its implicit equation $f(x,y) = 0$. How many terms do you expect the polynomial $f(x,y)$ to have? (*Hint:* Use tropical implicitization. You can learn about this on the paper "The Newton polytope of the implicit equation" by Sturmfels, Tevelev and Yu)

Problem 10. Given 8 general points in the plane $\mathbb{C}^2$, what is the number $N_{0,3}$ of rational cubic curves that pass through these 14 points? In particular, can you draw all tropical plane cubics (counted with multiplicity) through $(1,0)$, $(-1,1)$, $(-8,2)$, $(-4,3)$, $(-7,4)$, $(5,-1)$, $(7,-2)$ and $(9,-3)$?

Problem 11. Tropical compactifications

The set X of singular 3×3-matrices with non-zero complex entries is a hypersurface in the torus $(\mathbb{C}^*)^{3\times 3}$.

(i) Determine its tropical compactification X^{trop}.

(ii) How many irreducible components does the boundary $X^{\text{trop}} \smallsetminus X$ have? How do they intersect?

Problem 12. Prove **tropical Bézout's theorem** for transverse intersetions: two plane tropical curves C and D of degrees c and d (so they are dual to subdivisions of the 2-simplex dilated by c and d, respectively) that meet transversely have exactly $c \cdot d$ intersection pooints (counted with multiplicity). The multiplicity at a point p is given by the formula:

$$\operatorname{mult}(e)\ \operatorname{mult}(f) \left| \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|,$$

where e and f are the edges of C and D containing p, and $e' = (u_1, v_1)$, $f' = (u_2, v_2)$ are the primitive vectors in $\mathbb{Z}^2$ in the directions of e and f, respectively. (*Hint:* Consider $C \cup D$ as a tropical plane curve.)