Reading course in Tropical Geometry – Problem set 2
Gröbner bases over valued fields, Gröbner complex, Minkowski sums

Problem 1. (Valued field extensions through tropicalization) Consider a valued field (K, val) and a finite field extension L of K. It is known that the valuation on K can be extended (in at most $[L:K]^{\mathrm{sep}}$ ways) to a valuation on L.

(i) Assume that $\alpha \in L$ is algebraic over K, and let $\min(\alpha, K)$ be the minimal polynomial of α over K. If $L{=}K(\alpha)$, prove that for any extension of val to L, $-\mathrm{val}(\alpha)$ is a zero of the (max) tropical polynomial $\mathrm{trop}(\min(\alpha, K))$.

(ii) † The quotient ring $L = \mathbb{Q}[s]/\langle 3s^3 + s^2 + 36s + 162\rangle$ is a field. Describe all valuations on this field that extend the 3-adic valuation on $\mathbb{Q}$.

Problem 2. Solve the equation $x^5 + t\,x^4 + t^3\,x^3 + t^6\,x^2 + t^{10}x = t^{15}$ in the Puiseux field $\mathbb{C}\{\{t\}\}$ (*Hint:* Use Puiseux's method for solving equations, as in the proof of Theorem 2.1.5 of [MS]).

Problem 3. Solve the equation $x^5 + 2\,x^4 + 8\,x^3 + 64\,x^2 + 1024\,x = 32768$ over the field $\mathbb{Q}_2$. (*Hint:* Use Problem 1 (iii) in Problem set 1)

Problem 4. (Gröbner basis computations) Given a valued field K with a splitting of val $a \mapsto t^a$, a *homogeneous* polynomial $f = \sum_u c_u x^u \in K[x_1, \ldots, x_n]$ and $\omega \in \mathbb{R}^n$, recall that $\mathrm{in}_\omega(f) = \sum_u \overline{c_u t^{\mathrm{trop}(f)(\omega) - \langle u, \omega\rangle}} x^u \in \tilde{K}[x_1, \ldots, x_n]$, where $\tilde{K}$ is the residue field of K. The following outlines how to compute an initial ideal $\mathrm{in}_\omega(I)$ when the valuation is trivial and I is homogeneous.

(i) Given any monomial ordering $\prec$, show that $\mathrm{in}_\prec \mathrm{in}_\omega I = \mathrm{in}_{\prec_\omega} I$ where $\prec_\omega$ is the monomial order refining the ω-order by $\prec$, i.e. $x^\alpha \prec_\omega x^\beta$ if $\langle \alpha, \omega\rangle < \langle \beta, \omega\rangle$ or $\langle \alpha, \omega\rangle = \langle \beta, \omega\rangle$ and $x^\alpha \prec x^\beta$.

(ii) Show that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I with respect to $\prec_\omega$, then $\mathrm{in}_\omega(g_i)$ is a Gröbner basis for $\mathrm{in}_\omega I$ with respect to $\prec$. (*Hint:* A Gröbner basis for a monomial order generates the ideal.) Conclude that $\mathrm{in}_\omega I = \langle \mathrm{in}_\omega g_1, \ldots, \mathrm{in}_\omega g_s\rangle$.

(iii) [Optional] What happens if the valuation on K is non-trivial?

Problem 5. Let K be a valued field. Show that if $I \subset K[x_0, \ldots, x_n]$ is a principal homogeneous ideal generated by a homogeneous polynomial f, then $\{f\}$ is a universal Gröbner basis for I. (*Hint:* Look at the proof of Lemma 2.6.2 (3) of [MS]).

Problem 6. Let $I = \langle 7 + 8\,x_1 - x_1^2 + x_2 + 3\,x_2^2\rangle \subset \mathbb{C}[x_1^{\pm}, x_2^{\pm}]$.

(i) Compute all initial ideals of $I_{\mathrm{proj}} \subset \mathbb{C}[x_0, x_1, x_2]$ and draw the Gröbner complex of I_{proj}.

(ii) Draw $\{\omega \in \mathbb{R}^2 : \mathrm{in}_\omega(I) \neq \langle 1\rangle\}$.

(iii) Repeat (i) and (ii) for the ideal $J = \langle t\,x_1^2 + 3\,x_1\,x_2 - t\,x_2^2 + 5\,x_0\,x_1 - x_0\,x_2 + 2\,t\,x_0^2\rangle \subset \mathbb{C}\{\{t\}\}[x_0, x_1, x_2]$.

(*Hint:* Consider Theorem 2.5.7 in [MS].)

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† *Hard problem!*

Problem 7. Let I be the homogeneous ideal in $\mathbb{Q}[x, y, z]$ generated by the set

$$\mathcal{G} := \{x + y + z, x^2 y + x y^2, x^2 z + x z^2, y^2 z + y z^2\}.$$

(i) Show that $\mathcal{G}$ is a *universal Gröbner basis* (i.e., $\mathcal{G}$ is a Gröbner basis for I for all $\omega \in \mathbb{R}^3$).

(ii) Show that $\mathcal{G}$ is not a tropical basis.

Problem 8. Pick 2 triangles P and Q that lie in non-parallel planes in $\mathbb{R}^3$.

(i) Draw their Minkowski sum $P + Q$ and its normal fan.

(ii) Write down the f-vector of $P + Q$ (i.e., describe how many faces of each dimension does $P + Q$ have).

(iii) Verify that the normal fan of $P + Q$ is the *common refinement* of the normal fans of P and Q.

Problem 9. Let Σ_1 be the polyhedral complex consisting of all faces of the cube $[-1, 1]^3$, and let Σ_2 be the collection of all faces of the octahedron $\operatorname{conv}\{\pm e_1, \pm e_2, \pm e_3\}$. Determine the common refinement $\Sigma_1 \wedge \Sigma_2$.