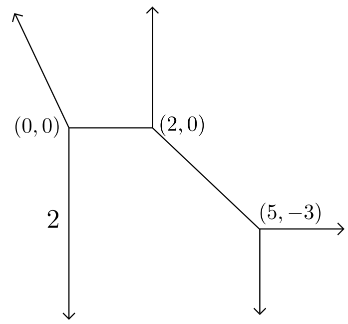
Reading course in Tropical Geometry – Problem set 3
Tropical hypersurfaces, tropical varieties, recession fans and stable intersections

Problem 1. Find a polynomial $f \in \mathbb{C}\{\{t\}\}[x,y]$ giving rise to the (max) tropical plane curve below. Unless otherwise indicated, the multiplicity of an edge is assumed to be 1. The upper left ray has direction $(-1,2)$.



Problem 2. Draw the *tropical hypersurface* $\operatorname{Trop}(V(f_i))$ and the recession fan for each of the following Laurent polynomials over the field $\mathbb{C}\{\{t\}\}$.

(i) $f_1(x,y) = t^3\,y^3 + y^2 - xy^2 - y - t^{-1}\,xy + x^2y + t^2 + x + x^2 + t^2\,x^3$;

(ii) $f_2(x,y) = xy + 5\,xy^2 - xy^3 + t\,x^2y + 3t^2\,x^2y^2 - 7t^2x^3y$;

(iii) $f_3(x,y) = t + xy + x^{-1}y + xy^{-1} + x^{-1}y^{-1}$;

(iv) $f_4(x,y,z) = 1 + 2x + 3y + 4z$;

(v) $f_5(x,y,z) = t\,x + y + z$.

Repeat the calculation for $f_4(x,y,z)$ over $\overline{\mathbb{Q}}_2$ and $\overline{\mathbb{Q}}_3$.
(*Useful Hint:* You might want to investigate how to do some of the previous examples using the software `Gfan`, or the packages `tropical.lib` (for `Singular`) and `tropical` (for `Macaulay2`).)

Problem 3. Compute the tropical hypersurface in $\mathbb{R}^9$ defined by the tropically singular 3×3-matrices (see Problem 3 of Problem Set 1). What is it's f-vector? Can you determine its lineality space and how many cones up to $\mathbb{S}_3$-symmetry does it have? (*Hint:* You might want to try using `gfan` or `Macaulay2` for your computations)

Problem 4. Consider the ideal $I = \langle f, g\rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm}, y^{\pm}]$ where

$$f = t^2\,x^2 + xy + t^2\,y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6t\,x + 17t\,y - 4t^3\,xy.$$

(i) For each $\omega \in \operatorname{Trop}(V(f)) \cap \operatorname{Trop}(V(g))$, compute $\operatorname{in}_\omega I$. Is $\{f, g\}$ a tropical basis for I?

(ii) There are four points in the variety $V(I) \subset (\mathbb{C}\{\{t\}\}^*)^2$. Compute the leading term of each point.

Problem 5. Consider the polynomial $f := t^2\,x^2 + x\,y + (t^2+t^3)\,x^2 + (1+t^3)x + t^{-1}y + t^3$ in $\mathbb{C}\{\{t\}\}[x^{\pm}, y^{\pm}]$.

1. Compute the (max) tropical hypersurface $\operatorname{Trop}(V(f))$ and show that $\omega = (1,0)$ is a vertex of it.
2. Describe all points $(x,y) \in V(f)$ with $(-\operatorname{val}(x), -\operatorname{val}(y)) = \omega$, and verify that this set is Zariski dense in $V(f)$.

Problem 6. Consider the linear ideal $I = \langle x_1 + x_2 + x_3 + x_4 + x_5, 3x_2 + 5x_3 + 7x_4 + 11x_5 \rangle \subset \mathbb{C}[x_1^{\pm}, \ldots, x_5^{\pm}]$. The tropical variety $\mathrm{Trop}(V(I))$ is a three-dimensional fan with a one-dimensional lineality space. It is a fan over the complete graph K_5. The fan has ten maximal cones and five codimensional-one cones. The following shows that a change of coordinates in $T = (\mathbf{K}^*)^5$ might change the structure of the tropical variety.

Consider the automorphism $\varphi^* \colon T \to T$ defined by $x_1 \mapsto x_1$, $x_2 \mapsto x_2x_3$, $x_3 \mapsto x_3x_4$, $x_4 \mapsto x_4x_5$, $x_5 \mapsto x_5$ and let $J = (\varphi^*)^{-1}(I) \subset \mathbb{C}[x_1^{\pm}, \ldots, x_5^{\pm}]$.

(i) Show that $\mathrm{Trop}(V(J))$ has the same support as $\mathrm{Trop}(V(I))$.

(ii) Show that the Gröbner structure of $\mathrm{Trop}(V(J))$ has 12 maximal cones, obtained as the cone over a subdivision of K_5 where 2 edges are subdivided. (*Hint:* You can use `Gfan` to verify this.)

Problem 7. Let I be the ideal in $\mathbb{C}[x_1^{\pm}, \ldots, x_4^{\pm}]$ generated by the five polynomials:

$\{(x_1+x_3)^2(x_3+x_4), (x_1+x_2)(x_1+x_4)^2, (x_1+x_3)^2(x_1+x_4), (x_1+x_2)(x_1+x_3)(x_1+x_4), (x_1+x_2)(x_1+x_3)(x_3+x_4)^2\}$.

(i) Find all associated primes of I and an explicit primary decomposition of I.

(ii) Compute the tropical variety $\mathrm{Trop}(V(I))$ with its multiplicities.

Problem 8. Let X and Y be subvarieties of $(K^*)^n$, and let $\Sigma = \mathrm{Trop}(X) + \mathrm{Trop}(Y)$ be the Minkowski sum of their tropicalizations in $\mathbb{R}^n$.

(i) Show that Σ is a tropical variety.

(ii) Explain how to construct a subvariety Z of $(K^*)^n$ with $\Sigma = \mathrm{Trop}(Z)$.

(*Hint:* Check out "An implicitization challenge for binary factor analysis" by Cueto, Tobis and Yu.)

Problem 9. Show that the k-skeleton of any n-dimensional polytope is connected through codimension 1. (*Hint:* Get started wtih $k = 1$ and $n = 3$.)

Problem 10. Let $X \subset (K^*)^{12}$ be the variety of 3×4-matrices of rank at most 2.

(i) Determine a fan structure on $\mathrm{Trop}(X)$.

(ii) Draw the adjacency graph of the maximal cones and verify that $\mathrm{Trop}(X)$ is connected through codimension 1.

Problem 11. Show that if Σ is an n-dimensional weighted balanced Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$, then the support $|\Sigma|$ equals $\mathbb{R}^n$, and the weight of each n-dimensional polyhedron in Σ is the same.

Problem 12. Recession fans and Stable intersection

(i) Given two polyhedral complexes Σ, Σ' in $\mathbb{R}^n$, show that the recession cones of $\Sigma \cap \Sigma'$, Σ and Σ' are related as follows:

$$\mathrm{rec}(\Sigma \cap \Sigma') = \mathrm{rec}(\Sigma) \wedge \mathrm{rec}(\Sigma').$$

(ii) Show that if L is a classical linear space in $\mathbb{R}^n$ contained in the lineality space of two weighted balanced Γ_{val}-rational polyhedral complexes $\Sigma, \Sigma' \subseteq \mathbb{R}^n$, then L is contained in the lineality space of the stable intersection $\Sigma \cap_{st} \Sigma'$. In addition, prove the following identity:

$$(\Sigma/L) \cap_{st} (\Sigma'/L) = (\Sigma \cap_{st} \Sigma')/L.$$

Problem 13. Determinantal varieties and stable intersections

Consider the 3×3-matrix of unknowns $A = (x_{ij})_{1 \leq i,j \leq 3}$ and its associated determinant f.

(i) Show that the tropical 3×3 determinant $X := \mathrm{Trop}(V(f))$ is an eight-dimensional fan in $\mathbb{R}^9$.

(ii) Compute the fans $X \cap_{st} X$ and $X \cap_{st} X \cap_{st} X$.

(iii) Realize these two fans as the tropicalization of two explicit varieties in the torus of 3×3-matrices.