Reading course in Tropical Geometry – Problem set 4
Matroids and tropical linear spaces

Problem 1. Let L_1 and L_2 be two tropical linear spaces in $\mathbb{R}^2/\mathbb{R}\,\mathbf{1}$.

(i) Show that their stable intersection $L_1 \cap_{st} L_2$ is a tropical linear space.

(ii) Express the tropical Pücker coordinates of $L_1 \cap_{st} L_2$ in terms of those of L_1 and L_2.

Problem 2. Let $d = 3$, $n = 6$ and let $\mathcal{A}$ be the arrangement in $\mathbb{P}^3$ consisting of the planes spanned by the facets of a regular octahedron $\operatorname{conv}(\{e_i + e_j : 0 \leq i < j \leq 3\})$. Write $\mathbb{P}^3 \smallsetminus \cup\mathcal{A}$ as a linear subvariety $V(I)$ in a torus and determine $\operatorname{Trop}(V(I))$.

Problem 3. Given a classical constant-coefficient linear space X, prove that the Bergman fan of its matroid agrees with the Gröbner fan structure on $\operatorname{Trop}(X)$ that is defined by the homogeneous ideal $I(X)_{\text{proj}}$.

Problem 4. (Tropicalization of linear spaces) Let V be the row space of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

defined over $K = \mathbb{C}\{\{t\}\}$. Let M be the matroid of columns of A with groundset $[6] := \{1, 2, \ldots, 6\}$.

(i) Let $I(V)$ be the linear ideal in $K[x_1, \ldots, x_6]$ defining $V \subset \mathbb{P}^5$. Compute $I(V)$, list the circuits of $I(V)$ and M, respectively.

(ii) Draw the Hasse diagram of the lattice of flats of M and show that the flats of M are in correspondence with partitions of the set $\{2, 3, 4, 5\}$.

(iii) Compute the tropical variety $\operatorname{Trop} V/\mathbb{R}\cdot\mathbf{1} \subset \mathbb{R}^5/\mathbb{R}\cdot\mathbf{1}$. Prove that it is homeomorphic to a cone over the Petersen graph.

(iv) For each $i = 1, \ldots, 6$ let H_i be the hyperplane in $\mathbb{P}^3_K$ with normal vector $a_i = i^{\text{th}}$. column of A. Let $X = \mathbb{P}^3_K \smallsetminus \bigcup_{i=1}^{6} H_i$ be the hyperplane complement. Show that the map

$$\varphi\colon X \to \mathbb{P}^5_K \qquad \mathbf{x} = [x_1 : \ldots : x_6] \mapsto [a_1 \cdot \mathbf{x} : \ldots : a_6 \cdot \mathbf{x}] \in (K^*)^6/K^* \simeq (K^*)^5$$

is injective and identifies its image with V.

(v) Check that the hyperplanes $\{H_i\}_{i=1}^6$ from item (iv), ordered by inclusion, form a partially ordered set that is dual to the lattice of flats of the matroid M. Conclude that the tropical variety records the information of "what is missing" from X.

Problem 5. Given the lattice of flats of a matroid M (not necessarily realizable), describe a method to recover the circuits of M, the independent sets of M and the bases of M. Illustrate this method with the matroid from Problem 4.

Problem 6. Let M be a matroid on $[n] = \{0, 1, \dots, n\}$ with circuits $\mathcal{C}$. For any $\omega \in \mathbb{R}^{n+1}$ we define the *initial matroid* M_ω as the matroid on $[n]$ with circuits:

$$\bigcup_{\substack{C \in \mathcal{C} \\ C \text{ minimal}}} \{j \in C : \omega_j = \min_{i \in C} \omega_i\}.$$

(i) Show that M_ω is a matroid. (*Hint:* You can look at "The Bergman complex of a matroid and phylogenetic trees" by Ardila and Klivans for inspiration).

(ii) Determine the six initial matroids M_ω of the uniform matroid $M = U_{3,6}$ given by $\omega = (1,1,1,1,1,2)$, $\omega = (1,1,1,1,2,2)$, $\omega = (1,1,1,2,2,2)$, $\omega = (1,1,2,2,2,2)$, $\omega = (1,2,2,2,2,2)$, and $\omega = (2,2,2,2,2,2)$.

Problem 7. Characteristic dependence of TropGr$(\mathbf{3}, \mathbf{7})$.

Consider the weight vector $\omega = -(e_{124} + e_{235} + e_{346} + e_{457} + e_{156} + e_{267} + e_{137}) \in \mathbb{R}^{\binom{7}{3}}$ corresponding to the negative incidence vector for the lines in the *Fano plane.* Let

$$\begin{aligned} f = {} & 2\, p_{123}p_{467}p_{567} - p_{367}p_{567}p_{124} - p_{167}p_{467}p_{235} - p_{127}p_{567}p_{346} - p_{126}p_{367}p_{457} - p_{237}p_{467}p_{156} \\ & + p_{134}p_{567}p_{267} + p_{246}p_{567}p_{137} + p_{136}p_{267}p_{457}. \end{aligned}$$

(i) Show that $f \in I_{3,7}$ (the Plücker ideal defining $\mathbb{G}\mathrm{r}(3,7)$).

(ii) Compute $\mathrm{in}_\omega(f)$ and conclude that $\omega \notin \mathrm{Trop}\,\mathbb{G}\mathrm{r}(3,7)$ if $\mathrm{char}(\Bbbk) \neq 2$.

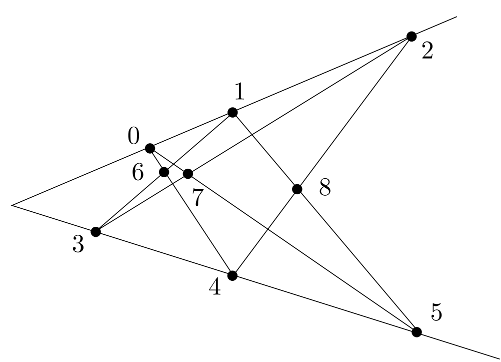
(iii) Show that if $\Bbbk = \mathbf{F}_2$ (for example for $\mathbf{K} = \mathbb{Q}_2$) then $\omega \in \mathrm{Trop}\,\mathbb{G}\mathrm{r}(3,7)$.

(iv) Consider the weight vector $\omega' = \omega + e_{124}$. Then show that $\mathrm{in}_{\omega'}(f)$ is a monomial if $\mathrm{char}(\Bbbk) = 2$.

(v) Show that if $\mathrm{char}(\Bbbk) = 0$, $\mathrm{in}_{\omega'}(I_{3,7})$ does not contain a monomial (*Hint:* You might want to explore how to do this using `Gfan` or the `Macaulay2` build-in command `leadTerm`*).

(vi) Show that both ω and ω' lie in Dr_M where M is the Fano matroid.

Problem 8. (The non-Pappus matroid). The *non-Pappus* matroid is the rank 3 matroid on $\{0, \dots, 8\}$ with circuits 012, 046, 057, 136, 158, 237, 248, 345 plus every subset of size four not containing one of these triples.



(i) Show that this matroid is not realizable over any field, as Pappus' Theorem implies that any realizable matroid with these circuits also has the circuit 678, i.e. the points labeled $6, 7, 8$ are collinear.

(ii) Compute the f-vector of the matroid polytope of the non-Pappus matroid.

(iii) Describe the tropical linear space $\mathrm{Trop}(M) \subseteq \mathbb{R}^8$, i.e. the Bergman fan of the non-Pappus matroid M.

(iv) Show directly that there is no variety $X \subseteq (K^*)^8$ with $\mathrm{Trop}(X) = \mathrm{Trop}(M)$.

Problem 9. Compute the Dressian Dr_M for the non-Pappus matroid M from Problem 8.

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*`http://www.math.uiuc.edu/Macaulay2/doc/Macaulay2-1.9/share/doc/Macaulay2/Macaulay2Doc/html/___Weights.html`