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REGULAR O_{n}-MANIFOLDS AND THE TWIST SUSPENSION OF KNOTS*

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1. Introduction.

This paper contains the details of the results announced in [3]. Let Σ^k be an oriented smooth submanifold of the oriented sphere S^{k+2}. We use the term "knot" for this general situation (Σ^k need not even be connected), while "spherical knot" describes the case in which Σ^k is a homotopy sphere. We always assume that $k > 0$.

Our purpose is to describe and study the construction of a knot $\omega(S^{k+2}, \Sigma^k) = (S^{k+4}, \Sigma^{k+2})$ called the "twist suspension of (S^{k+2}, Σ^k)". (This was simply called "suspension" in [3].) It is a certain canonically defined embedding in S^{k+4} of the cyclic double cover Σ^{k+2} of S^{k+2} branched at Σ^k. From this it is clear that the twist suspension of a spherical knot need not be a spherical knot. It turns out, however, that the double twist suspension $\omega^2 = \omega\circ\omega$ takes spherical knots to spherical knots (of dimension 4 greater). Moreover, this double suspension produces the periodicity of knot cobordism groups [12] and of isotopy classes of simple knots [11]. This is the first explicit geometric description of these periodicities. (In the case of knot cobordism, Cappell and Shaneson [4] have, concurrently with and independently from the present work, provided a geometric description of the periodicity in a completely different manner.)

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Our construction of the twist suspension is based on the study of "regular O_n-manifolds" which are O_n-manifolds modeled on twice the standard representation of the orthogonal group O_n plus some trivial representations. Specifically, we make use of a well known theorem of Jänich [1,2,9,10] on "O_n-knot manifolds". (Actually we use this theorem rather minimally. The philosophy behind the theorem has more importance to us.)

It would be possible to construct the twist suspension and prove its properties without mentioning O_n-manifolds. In fact, one can (but we will not) give a rather simple minded cut and paste description of the twist suspension and prove its purely knot theoretic properties. However, the relationship with O_n-manifolds is what gives our results a major part of their interest. Applications to transformation groups have already been made in [1] and [14]. Indeed we believe that the theory of O_n-knot manifolds has its proper setting in the ideas of this paper. Also the definition of ω is quite natural from the point of view of O_n-manifolds, and the proofs of certain properties are simplified by the use of related O_n-actions. We do attempt, as much as is possible, to make the paper readable to people who have little knowledge of the literature on O_n-actions.

We begin, in section 2, with some elementary remarks about actions modeled on a representation. In section 3 we give an explicit construction of the "O_n-manifold $M^{2n+k}(\Sigma^k)$ over a knot $(D^{k+3}, S^{k+2}, \Sigma^k)$". Most of the technical details in that section have nothing to do with the O_n-manifold $M^{2n+k}(\Sigma^k)$ itself, or with the twist suspension, but are important for the elegance of the later proofs of certain properties of these things. In section 4 the twist suspension is defined. In section 5 a Seifert surface $V^{k+3} = \omega(V^{k+1})$ cobounding the twist suspended knot is constructed from a given Seifert surface V^{k+1} cobounding the original knot Σ^k and this is studied in sections 6 and 7 leading to the main theorem in section 8 that ω preserves Seifert linking invariants. Some applications are detailed in section 9.

2. G-manifolds modeled on a representation.

In this section we present some elementary background material on smooth G-manifolds that will be useful in understanding some of the details of our constructions.

Let G be a compact Lie group and let $\rho : G \longrightarrow Gl(V)$ be a representation of G on the real vector space V. Then by a G-manifold "modeled on ρ" we mean a smooth G-manifold M such that each orbit in M has an open invariant neighborhood which is equivariantly diffeomorphic to an open invariant set in the representation space V of ρ.

In this paper we shall be concerned only with the case in which $G = O_n$, $V = R^n \times R^n \times R^k$ and $\rho = 2\rho_n + \theta_k$ is twice the standard representation plus a trivial k-dimensional representation. An O_n-manifold M modeled on this representation will simply be called a "regular O_n-manifold" in this paper. (In [1] the further restriction, that the bundle of principal orbits is trivial, is imposed. This is not necessary for our present discussion, but will be the case for the O_n-manifolds we shall be concerned with in this paper since this bundle will have a contractible base space.)

The restriction of the O_n-representation $2\rho_n + \theta_k$ on $R^n \times R^n \times R^k$ to the standardly embedded subgroup $O_r \subset O_n$ is the representation $2\rho_r + \theta_{k+2(n-r)}$ of O_r on $R^r \times R^r \times R^{k+2(n-r)}$. Thus the restriction of a regular O_n-manifold to the action of O_r is a regular O_r-manifold.

Suppose that M^{2n+k} is a regular O_n-manifold and that $N \subset M$ is a connected invariant smooth submanifold. The representation of O_n on the tangent space to N about a stationary point (if one exists) is a subrepresentation of $2\rho_n + \theta_k$. Thus it must be one of the three possibilities: $2\rho_n + \theta_i$; $\rho_n + \theta_j$; or θ_ℓ. These are distinguished by the principal isotropy type: O_{n-2}; O_{n-1}; or O_n respectively. Similar remarks apply to the slice representations (on the normal space to an orbit), and it is easily concluded that the O_n-manifold N is modeled on $2\rho_n + \theta_i$ (regular); $\rho_n + \theta_j$; or θ_ℓ (trivial action). In particular, a connected invariant submanifold of the representation space of $2\rho_n + \theta_k$ must be one of these three types.

The representation $2\rho_n + \theta_k$ may be viewed as the representation of O_n on $C^n \times R^k$ as a subgroup of U_n. With $z = (z_1, \ldots, z_n) \in C^n$ and $x \in R^k$, the orbit map of this O_n-action may be identified with the map

$$\bar{\pi}: C^n \times R^k \longrightarrow K \times R^k \subset (R \times C) \times R^k$$

$$\bar{\pi}(z, x) = (\|z\|^2, \; z_1^2 + \ldots + z_n^2, \; x)$$

where K is the positive cone

$$K = \left\{ (y, v) \in R \times C \;\middle|\; y \geqslant |v| \right\}.$$

See [2; p.273] and [1; §9]. It follows, moreover, from a theorem of Glaeser [7] that $\overline{\pi}$ gives the <u>induced</u> functional structure; that is, a real valued map f defined on an open set in $K \times \mathbb{R}^k$ is differentiable iff $f \circ \overline{\pi}$ is differentiable (C^∞). Note that points on orbits of type O_n/O_{n-1} correspond to points $(y,v,x) \in K \times \mathbb{R}^k \subset \mathbb{R} \times \mathbb{C} \times \mathbb{R}^k$ with $0 \neq y = |v|$, and fixed points correspond to points of $\{0\} \times \mathbb{R}^k$.

If M^{2n+k} is a regular O_n-manifold then it is clear that its orbit space M^*, <u>with the induced functional structure</u>, is modeled, in the obvious sense, on $K \times \mathbb{R}^k$. Thus M^* has the structure of a manifold with boundary ∂M^* but with a "corner" along M^G which is a submanifold of ∂M^* with codimension two. Principal orbits correspond precisely to interior points of M^*.

We shall often wish to straighten the angle along M^G in M^*. This is done as follows. The cone K is straightened by means of the homeomorphism

$$K \longrightarrow \mathbb{R}^+ \times \mathbb{C}$$

taking (y,v) to $(y^2 - |v|^2, v)$. Note that this is a diffeomorphism outside the vertex 0 of the cone K. The orbit map of O_n on $\mathbb{C}^n$ then becomes the map

$$\pi: \mathbb{C}^n \longrightarrow \mathbb{R}^+ \times \mathbb{C}$$

$$\pi(z) = (\|z\|^4 - |\sum z_i^2|^2, \quad \sum z_i^2).$$

For a regular O_n-manifold M, we select an invariant tubular neighborhood of M^G in M and note that this has the form of a $\mathbb{C}^n$-bundle over M^G. This induces a tubular neighborhood of M^G

in M^* which is a bundle over M^G with fiber K, and structure group O_2 acting in the obvious way on C. The above angle straightening of K then induces an angle straightening of M^*; see [1] and [2; p.334] for more details of this. This produces a differentiable structure on M^* making it into a smooth $(k+3)$-manifold with boundary ∂M^* and with M^G a smooth submanifold of ∂M^* of codimension two. This structure is the <u>induced</u> structure on the complement of M^G in M^*. This structure is not natural (along M^G) since it definitely depends on the choice of the tubular neighborhood. However, the uniqueness theorem for invariant tubular neighborhoods [2; VI.2.6] implies that this structure is well defined <u>up to diffeomorphism</u>, and this suffices for our purposes.

3. Construction of O_n-knot manifolds.

In this section we recall a construction given in [1; §5] of the O_n-manifold associated with a knot $\Sigma^k \subset S^{k+2} \subset D^{k+3}$, and discuss some properties of importance to us. This material is of primary importance to this paper, but we should mention that many of the details of the construction have no importance for the desired O_n-manifold or the definition of the twist suspension of a knot, but are specified in order to simplify some details of later proofs of the properties of these things.

Let Σ^k be an oriented closed submanifold of S^{k+2} (oriented). We do not assume that Σ^k is connected. It is well known that Σ^k bounds an oriented (hence normally framed) manifold $V^{k+1} \subset S^{k+2}$, called a Seifert surface, and that V^{k+1} is determined up to cobordism relative to Σ^k. (See, for example, [2; p.335] and [1; §4].) Note that Σ^k has trivial normal bundle.

Let $\underset{\sim}{R}^+$ denote the nonnegative reals. As in [1] we shall use the following notation:

$$D_+^3 = \{(x,y) \in \underset{\sim}{R}^+ \times \underset{\sim}{C} \mid 0 \leq x \leq 1 - |y|^2\}$$
$$D^2 = \{(x,y) \in \underset{\sim}{R}^+ \times \underset{\sim}{C} \mid 0 \leq x = 1 - |y|^2\}$$
$$E^2 = \{(0,y) \in \underset{\sim}{R}^+ \times \underset{\sim}{C} \mid |y| < 1\}.$$

Let

$$\theta_\Sigma : D_+^3 \times \Sigma^k \longrightarrow D^{k+3}$$

be the restriction to $D_+^3 \times \Sigma^k$ of a tubular neighborhood

$$(\underset{\sim}{R}^+ \times \underset{\sim}{C}) \times \Sigma^k \longrightarrow D^{k+3}$$

of Σ^k in $D^{k+3} \supset S^{k+2}$, which we shall regard as an <u>inclusion</u>. Assume, as we may, that in this tubular neighborhood, V^{k+1} has the form

$$V^{k+1} \cap (\underset{\sim}{R}^+ \times \underset{\sim}{C} \times \Sigma^k) = \{0\} \times \underset{\sim}{R}^+ \times \Sigma^k \subset \underset{\sim}{R}^+ \times \underset{\sim}{C} \times \Sigma^k.$$

We may regard points of D^{k+3} as having the cone (inverted polar) coordinates (w,t) where $w \in S^{k+2}$, $0 \leq t \leq 1$, with $t = 0$ corresponding to the boundary S^{k+2} and $t = 1$ corresponding to the origin. We may assume that the tubular neighborhood θ_Σ is "nice" with respect to these coordinates, by which we mean the following two things:

(i) Some collar $[0, \epsilon] \times S^1 \times \Sigma^k \longrightarrow D^2 \times \Sigma^k$ of $S^1 \times \Sigma^k$ in $D^2 \times \Sigma^k$ is the restriction of the collar $[0, \epsilon] \times S^{k+2} \longrightarrow D^{k+3}$ of S^{k+2} in D^{k+3} given by the cone coordinates; that is, $D^2 \times \Sigma^k$ is "vertical" near S^{k+2} in D^{k+3}.

(ii) The projection $(w, t) \longmapsto w$ of $D^{k+3} - \{0\} \longrightarrow S^{k+2}$ takes $(\underset{\sim}{R}^+ \times \underset{\sim}{R}^+ z \times \{y\}) \cap (D^3_+ \times \Sigma^k)$ to $\{0\} \times [0,1] \cdot z \times \{y\}$ for each $z \in S^1$ and $y \in \Sigma^k$.

We shall now recall the construction given in [1; §5] of the $\underset{\sim}{O}_n$-manifold over $(D^{k+3}, S^{k+2}, \Sigma^k)$; $n \geqslant 2$.

Consider the action of $\underset{\sim}{O}_n$, for $n \geqslant 2$, on $\underset{\sim}{C}^n$ via the inclusion $\underset{\sim}{O}_n \subset \underset{\sim}{U}_n$. As in section 2, the orbit map $\underset{\sim}{C}^n \longrightarrow \underset{\sim}{C}^n / \underset{\sim}{O}_n$ will be identified with the map

$$\pi: \underset{\sim}{C}^n \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$$

$$\pi(z_1, \ldots, z_n) = \left(\|z\|^4 - \left|\sum z_i^2\right|^2, \ \sum z_i^2\right).$$

It is easy to construct a smooth map (multiplication by a function of the radius) from $\underset{\sim}{C}^n$ to itself which is $\underset{\sim}{O}_n$-equivariant, equal to the identity near the origin, has image the unit disk D^{2n} and which collapses a collar of S^{2n-1} in D^{2n} to S^{2n-1}. This then induces a map

$$\text{stretch}: \underset{\sim}{R}^+ \times \underset{\sim}{C} \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$$

which has image D^3_+, is the identity near the origin, and collapses a collar of D^2 in D^3_+ to D^2. It takes the quadrant $\underset{\sim}{R}^+ \times \underset{\sim}{R}^+ z$ into itself for each $z \in S^1 \subset \underset{\sim}{C}$. It is also smooth, which follows from general principles (or by inspection of an explicit construction). See [1; § 9] for more details.

By the Thom-Pontriagin construction, the framed manifold V^{k+1} coincides outside the tube $D^3_+ \times \Sigma^k$ with $\tau^{-1}(1)$ where $\tau: S^{k+2} - (E^2 \times \Sigma^k) \longrightarrow S^1$ is a smooth map with $1 \in S^1 \subset \underset{\sim}{C}$ as regular value and with $\tau(z,y) = z$ for $(z,y) \in S^1 \times \Sigma^k$. Let us extend τ to a smooth map

$$\bar{\tau}: D^{k+3} \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$$

as follows.

First we define an extension

$$\tau': D^{k+3} \longrightarrow D^3_+ \subset \underset{\sim}{R}^+ \times \underset{\sim}{C}$$

of τ. In the tube $D^3_+ \times \Sigma^k$ define

$$\tau'(x,y) = \text{stretch}(x).$$

On a collar $[0,\epsilon] \times (S^{k+2} - (E^2 \times \Sigma^k))$ of the pair $(S^{k+2} - (E^2 \times \Sigma^k), S^1 \times \Sigma^k)$ in $(D^{k+3} - \text{int}(D^3_+ \times \Sigma^k), D^2 \times \Sigma^k)$ define τ' to be the collar map

$$\tau' = 1 \times \tau: [0,\epsilon] \times (S^{k+2} - (E^2 \times \Sigma^k)) \longrightarrow [0,\epsilon] \times S^1 \subset D^2 \subset D^3_+.$$

On the remainder of D^{k+3} we extend the definition of τ' arbitrarily such that the remainder is taken into the complement of the collar $[0,\epsilon] \times S^1$ in D^2. (This extension exists by the Tietze theorem and smoothing.) This construction of τ' is also given in [1; § 5] and this is the extension of τ we defined as $\bar{\tau}$ there. This, and indeed a much less detailed construction, suffices for use in the definition of the $\underset{\sim}{O}_n$-manifold over $(D^{k+3}, S^{k+2}, \Sigma^k)$ given below. However, we shall now proceed to modify the extension τ' of τ (into what we shall call $\bar{\tau}$ here) in order to achieve a property that will make later investigation of certain things much easier.

Note that the orbit map $\pi: \underset{\sim}{C}^n \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$ is a bundle map over $(0, \infty) \times \underset{\sim}{C}$. Thus there is an equivariant trivialization $(\underset{\sim}{O}_n/\underset{\sim}{O}_{n-2}) \times (0, \infty) \times \underset{\sim}{C}$ of $\pi^{-1}((0, \infty) \times \underset{\sim}{C})$ which can be regarded as an equivariant tubular neighborhood of the orbit over the point $(1,0) \in \underset{\sim}{R}^+ \times \underset{\sim}{C}$. Thus we can clearly define an equivariant smooth map

$$f: \underset{\sim}{C}^n \longrightarrow \underset{\sim}{C}^n$$

which collapses a smaller tubular neighborhood to this orbit and which equals the identity over a neighborhood of $\{0\} \times \underset{\sim}{C}$ in $\underset{\sim}{R}^+ \times \underset{\sim}{C}$. This induces a smooth map

$$f^*: \underset{\sim}{R}^+ \times \underset{\sim}{C} \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$$

on the orbit space and it is clear that we may arrange that f^* has the following three properties:

(1) The complement of any preassigned collar neighborhood of $(\{0\} \times \underset{\sim}{C}) \cap D^3_+$ in D^3_+ is collapsed to the point $(1,0)$ by f^*.

(2) f^* moves each point $x \in \underset{\sim}{R}^+ \times \underset{\sim}{C}$ to a point on the line segment between x and $(1,0)$.

(3) f^* is the identity in a neighborhood of $\{0\} \times \underset{\sim}{C}$.

Then we define

$$\bar{\tau}: D^{k+3} \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$$

to be $\bar{\tau} = f^* \circ \tau'$. The purpose of f^* is to achieve the following property of $\bar{\tau}$ which follows from (ii) above by taking the neighborhood in (1) sufficiently small:

(*) For each $z \in S^1$ we have that $\bar{\tau}^{-1}(\underset{\sim}{R}^+ \times \underset{\sim}{R}^+ z)$ is the cone on $\bar{\tau}^{-1}(\{0\} \times \underset{\sim}{R}^+ z)$, where D^{k+3} is regarded as the cone on S^{k+2} via the given cone coordinates.

(We remark that this property is used only in the proof of 5.4.)

Note that

$$V^{k+1} = \bar{\tau}^{-1}(\{0\} \times \underset{\sim}{R}^+).$$

We now define

$$M^{2n+k}(\Sigma^k) = \left\{(w,z) \in D^{k+3} \times \underset{\sim}{C}^n \;\middle|\; \bar{\tau}(w) = \pi(z)\right\}$$

which is the pull-back

$$\begin{array}{ccc} M^{2n+k}(\Sigma^k) & \longrightarrow & \underset{\sim}{C}^n \\ \downarrow & & \downarrow \pi \\ D^{k+3} & \xrightarrow{\;\bar{\tau}\;} & \underset{\sim}{R}^+ \times \underset{\sim}{C}. \end{array}$$

The $\underset{\sim}{O}_n$-action on $\underset{\sim}{C}^n$ induces an $\underset{\sim}{O}_n$-action on $M^{2n+k}(\Sigma^k)$. The orbit map of this action is just the projection to D^{k+3}. Note that the orbits of type $\underset{\sim}{O}_n/\underset{\sim}{O}_{n-2}$ correspond to interior points of D^{k+3}; the fixed point set corresponds to Σ^k; and orbits of type $\underset{\sim}{O}_n/\underset{\sim}{O}_{n-1}$ correspond to points of $S^{k+2} - \Sigma^k$.

It is shown in [1; 5.1] that the map $\varphi\colon D^{k+3} \times \underset{\sim}{C}^n \longrightarrow \underset{\sim}{R} \times \underset{\sim}{C}$ given by $\varphi(w,z) = \pi(z) - \bar{\tau}(w)$ has the origin as a regular value, and hence $M^{2n+k}(\Sigma^k)$ is a smooth submanifold of $D^{k+3} \times \underset{\sim}{C}^n$. It follows from the remarks in section 2 that $M^{2n+k}(\Sigma^k)$ is a <u>regular</u> $\underset{\sim}{O}_n$-manifold and it is clear that the projection to D^{k+3} is the "straightened" orbit map.

[The fact that 0 is a regular value of φ follows from the following three elementary facts:

(1) If $\bar{\tau}(w) = 0$ then the differential $\bar{\tau}_*$ is onto at w.

(2) If $\pi(z) \notin \{0\} \times \underset{\sim}{C}$ then π_* is onto at z.

(3) If $0 \neq \bar{\tau}(w) = \pi(z) \in \{0\} \times \underset{\sim}{C}$ then $\operatorname{Im} \bar{\tau}_*$ contains a nonzero normal vector to $\{0\} \times \underset{\sim}{C}$ and $\operatorname{Im} \pi_*$ contains the tangent space to $\{0\} \times \underset{\sim}{C}$.]

Note that this construction also works for $n = 1$. In this case $\pi: \underset{\sim}{C} \longrightarrow \underset{\sim}{R}^+ \times \underset{\sim}{C}$ has image $\{0\} \times \underset{\sim}{C}$. Thus $M^{2+k}(\Sigma^k)$ projects only to the boundary S^{k+2} of D^{k+3} and is clearly the cyclic double cover of S^{k+2} branched along Σ^k.

We now note some elementary lemmas concerning the $\underset{\sim}{O}_n$-manifold $M^{2n+k}(\Sigma^k)$.

3.1 <u>Lemma</u>. <u>For</u> $n \geq 2$, $M^{2n+k}(\Sigma^k)$ <u>is simply connected</u>. <u>For</u> n <u>even</u>, Σ^k <u>is a homology sphere if and only if</u> $M^{2n+k}(\Sigma^k)$ <u>is a homotopy sphere</u>.

<u>Proof</u>. This is a straightforward computation, the details of which are given in V.11.1, V.11.2, and V.11.3 of [2].

3.2 <u>Lemma</u>. <u>For the standard inclusion</u> $\underset{\sim}{O}_n \times \underset{\sim}{O}_r \subset \underset{\sim}{O}_{n+r}$ <u>we have a canonical equivariant diffeomorphism</u>

$$M^{2(n+r)+k}(\Sigma^k)^{\underset{\sim}{O}_r} \approx M^{2n+k}(\Sigma^k)$$

<u>of</u> $\underset{\sim}{O}_n$<u>-manifolds</u>.

<u>Proof</u>. This is an obvious consequence of the observation that $(\underset{\sim}{C}^{n+r})^{\underset{\sim}{O}_r} = \underset{\sim}{C}^n \times \{0\}$.

For the standard inclusion $\underset{\sim}{O}_n \times \underset{\sim}{O}_r \times \underset{\sim}{O}_s \subset \underset{\sim}{O}_{n+r+s}$ we have, from 3.2, the diffeomorphism

$$\frac{M^{2(n+r)+k}(\Sigma^k)}{\underset{\sim}{O}_r} \approx \frac{M^{2(n+r+s)+k}(\Sigma^k)^{\underset{\sim}{O}_s}}{\underset{\sim}{O}_r} .$$

We also have the map (inclusion of orbits)

$$\frac{M^{2(n+r+s)+k}(\Sigma^k)^{O_s}}{O_r} \longrightarrow \frac{M^{2(n+r+s)+k}(\Sigma^k)}{O_{r+s}}$$

which is "modeled" on the obvious map

$$\frac{\mathbb{C}^{n+r} \times \{0\}}{O_r} \longrightarrow \frac{\mathbb{C}^{n+r+s}}{O_{r+s}}.$$

The latter map is easily seen to be a diffeomorphism when $r > 1$ and is an embedding onto the boundary (the singular orbits) for $r = 1$. This implies, in general, the following lemma.

3.3 Lemma. The canonical map

$$\frac{M^{2(n+r)+k}(\Sigma^k)}{O_r} \approx \frac{M^{2(n+r+s)+k}(\Sigma^k)^{O_s}}{O_r} \longrightarrow \frac{M^{2(n+r+s)+k}(\Sigma^k)}{O_{r+s}}$$

is a diffeomorphism for $r > 1$ and is an embedding onto the boundary for $r = 1$.

We now apply these lemmas to prove the following basic fact.

3.4 Theorem. For the standard embedding $O_n \times O_r \subset O_{n+r}$ we have the diffeomorphisms

$$\frac{M^{2(n+r)+k}(\Sigma^k)}{O_r} \approx \begin{cases} S^{2n+k+2} & \text{for } r = 1 \\ D^{2n+k+3} & \text{for } r \geqslant 2. \end{cases}$$

Proof. Application of 3.3 shows that this manifold for $r \geqslant 2$ is independent of r and that for $r = 1$ is the boundary of that for $r \geqslant 2$. Thus it suffices to prove the second diffeomorphism for r arbitrarily large. The boundary of this manifold (which is $M^{2(n+1)+k}(\Sigma^k)/\underset{\sim}{O}_1$) is simply connected by 3.1. Since it suffices to consider only the case $n \geqslant 1$ and $k \geqslant 1$, this manifold has dimension $2n+k+3 \geqslant 6$ and thus it suffices to prove that it is contractible. Since the orbits of $\underset{\sim}{O}_{n+r}$ on $M = M^{2(n+r)+k}(\Sigma^k)$ are all $(n+r-3)$-connected, the Vietoris Mapping Theorem, applied to the orbit map $M \longrightarrow M/\underset{\sim}{O}_{n+r} \approx D^{k+3}$, implies that M can be taken as highly connected as we please by taking r sufficiently large. The orbits of $\underset{\sim}{O}_r$ on M are $(r-3)$-connected, and thus the Vietoris Mapping Theorem applied to the orbit map $M \longrightarrow M/\underset{\sim}{O}_r$ shows that $M/\underset{\sim}{O}_r$ (the manifold in question) is as highly connected as we please. Since this manifold is <u>independent of</u> $r \geqslant 2$, it is contractible as claimed.

It follows from a theorem of Jänich (see [1, 9]) that the $\underset{\sim}{O}_n$-manifold $M^{2n+k}(\Sigma^k)$ depends, up to orientation preserving diffeomorphism, only on the oriented knot $\Sigma^k \subset S^{k+2}$ and not on the choice of a Seifert surface V^{k+1}, or on the choices of such things as the tubular neighborhoods, the extension $\bar{\tau}$, and so on. Most of the details of our construction have the sole purpose of simplifying the proofs of certain <u>properties</u> of $M^{2n+k}(\Sigma^k)$ and of the twist suspension to be defined.

4. The twist suspension.

Recall that by a "knot" $\Sigma^k \subset S^{k+2}$ we simply mean an oriented submanifold of codimension two in the sphere, not necessarily connected.

Given the knot (S^{k+2}, Σ^k), we have constructed an $\underset{\sim}{O}_n$-manifold $M^{2n+k}(\Sigma^k)$. For the special case $n=2$, we have the $\underset{\sim}{O}_2$-manifold $M^{4+k}(\Sigma^k)$ over $(D^{k+3}, S^{k+2}, \Sigma^k)$. By restricting the $\underset{\sim}{O}_2$-action to $\underset{\sim}{O}_1$ and using 3.2 and 3.4 we have that

$$\left(\frac{M^{4+k}(\Sigma^k)}{\underset{\sim}{O}_1}, \; M^{4+k}(\Sigma^k)^{\underset{\sim}{O}_1} \right) \approx \left(S^{k+4}, \; M^{2+k}(\Sigma^k) \right)$$

where the $\underset{\sim}{O}_1$-manifold $M^{2+k}(\Sigma^k) = M^{4+k}(\Sigma^k)^{\underset{\sim}{O}_1}$ is just the cyclic double cover of S^{k+2} branched at Σ^k.

Let us briefly discuss orientation conventions. A cyclic branched cover inherits a canonical orientation from that of its base space. Thus $M^{2+k}(\Sigma^k)$ is oriented from $S^{k+2} = M^{2+k}(\Sigma^k)/\underset{\sim}{O}_1$ in this way. Similarly, orientations of $M^{4+k}(\Sigma^k)$ and $M^{4+k}(\Sigma^k)/\underset{\sim}{O}_1$ are canonically related to one another and it remains to specify an orientation on the $\underset{\sim}{O}_2$-manifold $M^{4+k}(\Sigma^k)$. This is done by taking it to coincide with the orientation on an invariant tubular neighborhood $(\underset{\sim}{R}^2 \times \underset{\sim}{R}^2) \times \Sigma^k$ of the fixed point set Σ^k in $M^{4+k}(\Sigma^k)$. This is canonical since the structure group $\underset{\sim}{O}_2$ of this bundle preserves the fiber orientation. The reader is free to take the opposite orientation convention here since we will not attempt the difficult and pointless task of tracing these conventions through our constructions.

Thus we have produced the oriented codimension two submanifold $M^{2+k}(\Sigma^k) \subset S^{k+4}$. We define this to be the "twist suspension" of (S^{k+2},Σ^k) and denote it by

$$\omega(S^{k+2},\Sigma^k) = \left(\frac{M^{4+k}(\Sigma^k)}{\underset{\sim}{O}_1} , M^{4+k}(\Sigma^k)^{\underset{\sim}{O}_1} \right) \approx (S^{k+4}, M^{2+k}(\Sigma^k)).$$

More generally, using the $\underset{\sim}{O}_{n+1}$-manifold $M^{2(n+1)+k}(\Sigma^k)$ and the standard inclusion $\underset{\sim}{O}_1 \times \underset{\sim}{O}_n \subset \underset{\sim}{O}_{n+1}$ we define

$$\omega^n(S^{k+2},\Sigma^k) = \left(\frac{M^{2(n+1)+k}(\Sigma^k)}{\underset{\sim}{O}_1} , M^{2(n+1)+k}(\Sigma^k)^{\underset{\sim}{O}_1} \right) \approx (S^{2n+k+2}, M^{2n+k}(\Sigma^k)).$$

By 3.3 we see that for $r \geqslant 1$ this is the boundary of

$$\left(\frac{M^{2(n+r+1)+k}(\Sigma^k)}{\underset{\sim}{O}_{r+1}} , M^{2(n+r+1)+k}(\Sigma^k)^{\underset{\sim}{O}_{r+1}} \right) \approx (D^{2n+k+3}, M^{2n+k}(\Sigma^k)).$$

Thus $M^{2(n+r+1)+k}(\Sigma^k)$ is a regular $\underset{\sim}{O}_{r+1}$-manifold whose orbit knot is $\omega^n(S^{k+2},\Sigma^k)$. The latter knot (i.e., $M^{2n+k}(\Sigma^k)$) is connected for $n \geqslant 1$ and thus it follows from a theorem of Jänich (see [2; VI.7.2] or [1]) that there is a unique such $\underset{\sim}{O}_{r+1}$-manifold. Consequently we conclude that

$$\left(\frac{M^{2(n+r+1)+k}(\Sigma^k)}{\underset{\sim}{O}_1} , M^{2(n+r+1)+k}(\Sigma^k)^{\underset{\sim}{O}_1} \right) \approx \omega^r(\omega^n(S^{k+2},\Sigma^k)).$$

But the left hand side of this equation is $\omega^{n+r}(S^{k+2},\Sigma^k)$ by definition. Thus $\omega^r \circ \omega^n = \omega^{n+r}$ and consequently ω^n is simply the twist suspension ω iterated n times; justifying the notation. We <u>orient</u> ω^n as the n-fold iterate of ω.

The main interest in this fact results from the case of the double twist suspension ω^2. Note that the single twist suspension (branched double cover) of a spherical knot need not be a spherical knot. For its iterate ω^2, however, we have the following simple, but basic, fact.

4.1 Theorem. The double twist suspension ω^2 takes homology spherical k-knots to spherical (k+4)-knots. It also takes h-cobordant knots to h-cobordant knots.

Proof. The first fact follows directly from 3.1; that is, from the calculations of [2; V.11.1] and [2; V.11.2]. The statement about knot cobordism can also be derived as in these references as follows. Let $(S^{k+2} \times I, W^{k+1})$ be an h-cobordism between the knots (S^{k+2}, Σ_0^k) and (S^{k+2}, Σ_1^k). Using a cobounding manifold $V^{k+2} \subset S^{k+2} \times I$ of W^{k+1} we can construct, as in section 3, an $\underset{\sim}{O}_n$-manifold $M^{2n+k+1}(W^{k+1})$ over $(D^{k+3} \times I, S^{k+2} \times I, W^{k+1})$ which has boundary $M^{2n+k}(\Sigma_1^k) - M^{2n+k}(\Sigma_0^k)$. Just as above we can define the n-fold twist suspended cobordism

$$\omega^n(S^{k+2} \times I,\ W^{k+1}) = \left(\frac{M^{2(n+1)+k+1}(W^{k+1})}{\underset{\sim}{O}_1},\ M^{2(n+1)+k+1}(W^{k+1})^{\underset{\sim}{O}_1} \right)$$

between $\omega^n(S^{k+2}, \Sigma_0^k)$ and $\omega^n(S^{k+2}, \Sigma_1^k)$. One can show, just as in section 3, that this is diffeomorphic to a cobordism of the form

$$(S^{2n+k+2} \times I,\ M^{2n+k+1}(W^{k+1})).$$

Thus it suffices to show that $M = M^{2n+k+1}(W^{k+1})$ is an h-cobordism between $M_0 = M^{2n+k+1}(\Sigma_0^k)$ and $M_1 = M^{2n+k+1}(\Sigma_1^k)$ for $n = 2$ (or generally for n <u>even</u>).

Just as in [2; V.11.1] it can be seen that M, M_0 and M_1 are all simply connected. The pair (M, M_0) is a stratified bundle over the pair $(D^{k+3} \times I, D^{k+3} \times \{0\})$ and the three strata pairs in the base are all acyclic. A straightforward computation, using homology sequences of pairs of strata, shows that (M, M_0) is acyclic and hence that $M_0 \subset M$ is a homotopy equivalence. (The crucial fact in this computation which differs from the case of odd n is that the bundle over $(S^{k+2} \times I - W^{k+1}, (S^{k+2} - \Sigma_0^k) \times \{0\})$ is orientable for n even.) Similarly $M_1 \subset M$ is a homotopy equivalence, as was to be shown.

We remark that similar considerations show that the double twist suspension ω^2 takes doubly null-cobordant spherical knots [16] to doubly null-cobordant spherical knots.

It is clear that ω commutes with connected sums.

5. <u>The suspended Seifert surface</u>.

We wish to obtain information about the knot invariants of the twist suspended knot $\omega(S^{k+2}, \Sigma^k)$ from those of the original knot. Towards this aim we shall construct, in this section, a Seifert surface $V^{k+3} = \omega(V^{k+1})$ cobounding $\omega(S^{k+2}, \Sigma^k)$ in S^{k+4} and canonically related to a given Seifert surface $V^{k+1} \subset S^{k+2}$ cobounding Σ^k.

Recall that the $\underset{\sim}{O}_2$-manifold $M^{4+k}(\Sigma^k)$ is defined by the pull-back diagram

$$\begin{array}{ccc} M^{4+k}(\Sigma^k) & \xrightarrow{\ \psi\ } & \underset{\sim}{C}^2 \\ \mu \downarrow & & \downarrow \pi \\ D^{k+3} & \xrightarrow{\ \bar{\tau}\ } & \underset{\sim}{R}^+ \times \underset{\sim}{C} \end{array}$$

where $\pi(a,b) = ((|a|^2 + |b|^2)^2 - |a^2+b^2|^2,\ a^2+b^2)$ and $\bar{\tau}^{-1}(\{0\} \times \underset{\sim}{R}^+) = V^{k+1}$ is a given Seifert surface cobounding Σ^k with $1 \in S^1 \subset \underset{\sim}{C}$ being a regular value of τ as in section 3. We shall simply use M to denote the $\underset{\sim}{O}_2$-manifold $M^{4+k}(\Sigma^k)$ in this section.

5.1 <u>Lemma</u>. <u>The map ψ is transverse regular on</u> $\{0\} \times \underset{\sim}{C} = (\underset{\sim}{C}^2)^{\underset{\sim}{O}_1} \subset \underset{\sim}{C}^2$.

<u>Proof</u>. Note that $\psi^{-1}(\{0\} \times \underset{\sim}{C}) = M^{\underset{\sim}{O}_1}$ and that $\mu(M^{\underset{\sim}{O}_1}) = S^{k+2}$. Let g denote the nontrivial element of $\underset{\sim}{O}_1$ and note that the normal space to $M^{\underset{\sim}{O}_1}$ in M may be identified with the -1 eigenspace of g. The differential μ_* annihilates this since for $g_*(X) = -X$ we have $\mu_*(X) = \mu_* g_*(X) = \mu_*(-X) = -\mu_*(X)$. Since $\mu \times \psi$ is an embedding we conclude that ψ_* is a monomorphism on the -1 eigenspace of g at a point of $M^{\underset{\sim}{O}_1}$ to the -1 eigenspace of g at the image point in $\{0\} \times \underset{\sim}{C}$. Both of these vector spaces are two-dimensional so that the conclusion follows for dimensional reasons.

Because of this lemma, there is a tubular neighborhood of M^{O_1} in M and an O_1-equivariant bundle map to a neighborhood of $\{0\} \times C$ in C^2. We can use this tubular neighborhood to specify the straightening of the corner of M/O_1 at M^{O_1}. Note that the (unstraightened) orbit map of the O_1-action on C^2 is the map $(a,b) \longmapsto (|a^2|, a^2, b)$ of $C^2 \longrightarrow (\partial K) \times C$ where $\partial K \subset R^+ \times C$ is the positive cone $\{(r,z) \in R^+ \times C \mid r = |z|\}$. Now one can straighten the corner of this cone (and hence of M/O_1 along M^{O_1}) as specified in section 2. However this would result in some rather ungainly formulas and we shall use the alternative approach of straightening the corner simply by ignoring the R^+ coordinate in ∂K. This results in a different differentiable structure on M/O_1 at M^{O_1} but, since the two types of corner straightening are based on canonical constructions involving the same tubular neighborhood of M^{O_1}, the resulting manifolds M/O_1 (and knots M^{O_1}) are clearly diffeomorphic.

With this understanding, the equivariant map ψ and the orbit maps of the O_1-actions give the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \psi\ } & C^2 \\ \downarrow & & \downarrow \gamma \\ S^{k+4} \approx M/O_1 & \xrightarrow{\ \varphi\ } & C^2 \end{array}$$

where the right hand vertical map γ is $(a,b) \longmapsto (u,v) = (a^2, b)$. Inclusion of O_1-orbits in O_2-orbits gives the pull-back diagram

$$\begin{array}{ccc} S^{k+4} \approx M/\underset{\sim}{O}_1 & \xrightarrow{\ \varphi\ } & \underset{\sim}{C}^2 \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ D^{k+3} & \xrightarrow{\ \overline{\tau}\ } & \underset{\sim}{R}^+ \times \underset{\sim}{C} \end{array}$$

where a short calculation shows that

$$\beta(u,v) = ((|u| + |v^2|)^2 - |u+v^2|^2, \quad u+v^2)$$
$$= (2(|\overline{u}v^2| - \mathrm{Re}(\overline{u}v^2)), \quad u+v^2).$$

(Note that the present choice of the structure results in β being nondifferentiable along $\{0\} \times \underset{\sim}{C}$, and α along $M^{\underset{\sim}{O}_1}$, but the entire structure is understood completely along this set by means of the tubular neighborhood resulting from 5.1.)

We may now take this diagram as our <u>definition</u> of the twist suspended knot

$$\omega(S^{k+2}, \Sigma^k) = (S^{k+4}, \Sigma^{k+2})$$

where

$$\Sigma^{k+2} = M^{\underset{\sim}{O}_1} = \varphi^{-1}(\{0\} \times \underset{\sim}{C}) \subset M/\underset{\sim}{O}_1 = \dot{S}^{k+4}.$$

We will now find a nice Seifert surface for Σ^{k+2} in S^{k+4}. The following lemma is basic for this.

5.2 <u>Lemma</u>. <u>The map</u> φ <u>is transverse regular on</u> $\underset{\sim}{R}^+ \times \underset{\sim}{C} \subset \underset{\sim}{C}^2$.

<u>Proof</u>. Transverse regularity is clear at points of $\{0\} \times \underset{\sim}{C}$ so that we may confine our attention to points $(u,v) \in \underset{\sim}{C}^2$ with u real and positive. Since $\underset{\sim}{R}^+ \times \underset{\sim}{C}$ has codimension one in $\underset{\sim}{C}^2$,

we must show that if $y \in S^{k+4} = M/\underset{\sim}{O}_1$ and $\varphi(y) = (u,v)$, $u > 0$, then φ_* takes some tangent vector to $M/\underset{\sim}{O}_1$ at y to a vector in $\underset{\sim}{C}^2$ whose first component is not real. Now y is the $\underset{\sim}{O}_1$-orbit of a point $x \in M$ with $\psi(x) = (a,b)$ where $a^2 = u$ and $b = v$. We first look at the tangent vector to the $\underset{\sim}{SO}_2$-orbit through x. Since ψ is $\underset{\sim}{O}_2$-equivariant, ψ_* takes this to the tangent vector to the $\underset{\sim}{SO}_2$-orbit through (a,b). The rotation through an angle θ in $\underset{\sim}{SO}_2$ takes (a,b) to

$$(a\cos\theta + b\sin\theta,\ -a\sin\theta + b\cos\theta).$$

Squaring the first component of this (that is, applying the $\underset{\sim}{O}_1$-orbit map γ) and taking $d/d\theta$ at $\theta = 0$ gives $2ab$. This is real only when $v = b$ is real. Thus φ is transverse regular at points (u,v) with $u > 0$ and v <u>not</u> real.

Now consider the remaining case for which $u > 0$ and v is real. Then $\beta(u,v) = (0, u + v^2) \in \underset{\sim}{R}^+ \times \underset{\sim}{C}$ and, since $u + v^2 > 0$, $\alpha(y) \in \overline{\tau}^{-1}(\{0\} \times \underset{\sim}{R}^+) = V^{k+1}$. <u>Since</u> $\overline{\tau}$ <u>is transverse regular on</u> $\{0\} \times \underset{\sim}{R}^+$ there is a tangent vector X to D^{k+3} (actually to S^{k+2}) at the point $\alpha(y)$ such that the vector $\overline{\tau}_*(X)$ at $(0, u + v^2)$ has coordinates $(0,z)$ for some <u>nonreal</u> z. Since β is a submersion over each stratum (and hence over $\{0\} \times (\underset{\sim}{C} - \{0\})$) there is a tangent vector Y at (u,v) in $\underset{\sim}{C}^2$ with $\beta_*(Y) = \overline{\tau}_*(X)$. (The reader may verify that the vector Y with coordinates $(z,0)$ works.) By the pull-back interpretation of $M/\underset{\sim}{O}_1 = S^{k+4}$, it follows that there is a tangent vector Z to $M/\underset{\sim}{O}_1$ at y with $\alpha_*(Z) = X$ and $\varphi_*(Z) = Y$. Since the first coordinate of Y clearly cannot be real, since z is not real, the equation $\varphi_*(Z) = Y$ is our desired conclusion and finishes the proof.

Because of 5.2 we can now define the "suspended Seifert surface"

$$V^{k+3} = \omega(V^{k+1}) = \varphi^{-1}(\mathbb{R}^+ \times \mathbb{C})$$

which cobounds the twist suspended knot

$$\Sigma^{k+2} = \varphi^{-1}(\{0\} \times \mathbb{C}) \subset S^{k+4}.$$

It remains to study this manifold V^{k+3}.

Note that it follows from the pull-back description (or from a look at the O_1-action on an O_2-orbit) that for a point $w \in D^{k+3}$, the fiber $\alpha^{-1}(w)$ is a circle for $w \in \operatorname{int}(D^{k+3})$, it is an arc for $w \in S^{k+2} - \Sigma^k$ and a point for $w \in \Sigma^k$.

5.3 __Proposition.__ _For a point_ $w \in D^{k+3}$ _we have that_

$$\alpha^{-1}(w) \cap V^{k+3} \approx \begin{cases} \text{two points} & \text{for } w \in D^{k+3} - V^{k+1} \\ \text{an arc} & \text{for } w \in V^{k+1} - \Sigma^k \\ \text{a point} & \text{for } w \in \Sigma^k. \end{cases}$$

Proof. First let us consider a _given orbit_ of the O_2-action on $\mathbb{C}^2$ and ask for all those points (a,b) in this orbit such that $a^2 \geq 0$; that is, such that a is real. For a given point $(a,b) \in \mathbb{C}^2$ the points on its orbit have the form

$$(\pm(a\cos\theta + b\sin\theta),\ -a\sin\theta + b\cos\theta).$$

It is clear that the first component is real for an appropriate value of θ. (Set the imaginary part equal to zero and solve for θ.) Thus we may as well assume, for simplicity, that a is real. Then $a\cos\theta + b\sin\theta$ is real _only_ for $\theta = 0, \pi$ unless

b is also real. Thus for a real and b not real, the four points $(\pm a, \pm b)$ are the only points on the orbit of (a,b) meeting this criterion. If a and b are both real then <u>all</u> points on the orbit meet the criterion. Moreover, if a and b are both real then $\pi(a,b) = (0, a^2 + b^2) \in \{0\} \times \mathbb{R}^+$ and clearly all points in $\{0\} \times \mathbb{R}^+$ correspond to such orbits. Now by passing to $(u,v) = (a^2, b)$ it follows that for $(t,z) \in \mathbb{R}^+ \times \mathbb{C}$ we have

$$\beta^{-1}(t,z) \cap (\mathbb{R}^+ \times \mathbb{C}) = \begin{cases} \text{two points} & \text{for } (t,z) \notin \{0\} \times \mathbb{R}^+ \\ \text{an arc} & \text{for } t = 0,\ z > 0 \\ \text{a point} & \text{for } t = 0,\ z = 0. \end{cases}$$

The desired result follows from the pull-back definition.

Recall the coordinatization of the unit disk D^{k+3} by cone coordinates (w,t) where $w \in S^{k+2}$ and $1-t$ is the radius. Thus $t = 0$ describes the boundary S^{k+2}. Let us denote by CV^{k+1}, the cone $\{(w,t) \in D^{k+3} \mid w \in V^{k+1}\}$ on V^{k+1}; and by C_0V^{k+1}, the open cone $t \neq 0$ in this. Let SV^{k+1} denote the <u>closure</u> in S^{k+4} of the set

$$\alpha^{-1}(C_0V^{k+1}) \cap V^{k+3}.$$

5.4 <u>Lemma</u>. <u>The space</u> SV^{k+1} <u>is the pull-back</u>

$$\begin{array}{ccc} SV^{k+1} & \longrightarrow & \mathbb{R}^+ \times i\mathbb{R} \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ CV^{k+1} & \xrightarrow{\ \bar{\tau}\ } & \mathbb{R}^+ \times \mathbb{R} \end{array}$$

<u>and hence is a suspension of</u> V^{k+1}. (<u>That is, it is the union</u>

of two sets each mapping homeomorphically by α onto CV^{k+1} with their intersection mapping homeomorphically to V^{k+1}.)

Proof. Note that $\bar{\tau}(CV^{k+1}) \subset \mathbb{R}^+ \times \mathbb{R}^+$ by property (*) of section 3. By 5.3 it follows that the set $A = \alpha^{-1}(C_0 V^{k+1}) \cap V^{k+3}$ is just a trivial double cover of the open cone $C_0 V^{k+1}$. Let B denote the indicated pull-back, which is compact. Clearly $A \subset B$. Since $\beta: \mathbb{R}^+ \times i\mathbb{R} \longrightarrow \mathbb{R}^+ \times \mathbb{R}$ is a double cover over $(0, \infty) \times \mathbb{R}^+$ we see that B is a double cover over $C_0 V^{k+1}$ and hence coincides with A there. It follows that $B = \bar{A} = SV^{k+1}$, and the other statements in the lemma are clear.

6. Further study of the Seifert surface.

We wish to study the linking of the suspended Seifert surface $V^{k+3} = \omega(V^{k+1})$ with a displacement V_+^{k+3} of it in the positive normal direction. To do this efficiently we shall have to consider a slightly altered version of V^{k+3} (and of V^{k+1}).

First let us note that the use of $1 \in S^1$ in the definition $V^{k+1} = \bar{\tau}^{-1}(\{0\} \times \mathbb{R}^+)$, and in the consequent definition $V^{k+3} = \varphi^{-1}(\mathbb{R}^+ \times \mathbb{C})$, makes use of no properties of 1 other than that it is a regular value of τ. The proofs of some things such as 5.2 were notationally simplified by using 1 as a regular value of τ, but the reader can easily check that these proofs go through in more generality for an arbitrary regular value $z \in S^1$ of τ and allow the definition of the surfaces

$$V_z^{k+1} = \bar{\tau}^{-1}(\{0\} \times \mathbb{R}^+ z)$$

$$V_z^{k+3} = \varphi^{-1}(\mathbb{R}^+ z \times \mathbb{C})$$

and the verification of the same properties as those for the special case $z=1$. We are mainly interested in those values of z near 1 and having positive imaginary part. Note that V_z^{k+1} (resp. V_z^{k+3}) intersects V^{k+1} (resp. V^{k+3}) in Σ^k (resp. Σ^{k+2}) and, except for this, is a positive displacement of V^{k+1} (resp. V^{k+3}).
In order to get an actual displacement we can shrink these surfaces to the surfaces

$$U_z^{k+1} = \bar{\tau}^{-1}(\{0\}\times[\epsilon,\infty)\cdot z)$$

$$U_z^{k+3} = \varphi^{-1}([\epsilon,\infty)\cdot z \times \underset{\sim}{C})$$

where $\epsilon > 0$ is sufficiently small. A transversality argument shows that the U_z are the complements of open boundary collars of the V_z.

Note that U_z^{k+3} contains the suspension

$$SU_z^{k+1} = SV_z^{k+1} \cap \alpha^{-1}(CU_z^{k+1}).$$

(For $z=1$, this follows from 5.4.)

Let $Q = D^{k+3} \cup_A D^{k+3}$ where $A = V_z^{k+1}$. There is a map $f: V_z^{k+3} \to Q$ which shrinks each of the arcs $\alpha^{-1}(w) \cap V_z^{k+3}$, for $w \in V_z^{k+1}$ to points (see 5.3) and which is thus a homotopy equivalence. The restriction of f to $SV_z^{k+1} \to Q$ is also clearly a homotopy equivalence and we conclude that the inclusion $SV_z^{k+1} \subset V_z^{k+3}$ is a homotopy equivalence. The inclusions $SU_z^{k+1} \subset SV_z^{k+1}$ and $U_z^{k+3} \subset V_z^{k+3}$ are homotopy equivalences, so that <u>all</u> the following inclusions are homotopy equivalences:

$$\begin{array}{ccc} SU_z^{k+1} & \subset & SV_z^{k+1} \\ \cap & & \cap \\ U_z^{k+3} & \subset & V_z^{k+3}. \end{array}$$

To make the computation of linking numbers feasible we now consider the map

$$h: S^{k+4} = M/\underset{\sim}{O}_1 \longrightarrow S^{k+2} * S^1$$

defined by $h((w,t),(u,v)) = (w, t, v/|v|)$ where a point in $S^{k+4} = M/\underset{\sim}{O}_1$ is coordinatized by its pull-back definition as a subspace of $D^{k+3} \times \underset{\sim}{C}^2$ (the coordinates (w,t) in D^{k+3} being the cone coordinates) and a point of the join $S^{k+2} * S^1$ is coordinatized by its join coordinates (x,t,z) where $x \in S^{k+2}$, $z \in S^1$, $t \in [0,1]$ and $t = 0$ describes S^{k+2} while $t = 1$ describes S^1.

To show that h is well defined, one must verify that $t = 0$ when $v = 0$. But this is true since $\beta(u,0) \in \{0\} \times \underset{\sim}{C}$ and $\bar{\tau}^{-1}(\{0\} \times \underset{\sim}{C}) = \partial D^{k+3} = S^{k+2}$ which is the set where $t = 0$.

6.1 <u>Lemma</u>. <u>For</u> $w \in S^{k+2} - \Sigma^k$, h <u>collapses the arc</u> $\alpha^{-1}(w) \subset S^{k+4}$ <u>to a point</u>. <u>These are the only identifications resulting from</u> h. <u>In particular</u>, h <u>is a homotopy equivalence of</u> $h^{-1}(A) \longrightarrow A$ <u>for any nice subspace</u> $A \subset S^{k+2} * S^1$.

<u>Proof</u>. For $t = 0$ the coordinate $v/|v|$ is immaterial and $h^{-1}(w,0,\cdot) = \alpha^{-1}(w)$ which is as claimed. Thus we must show that $h^{-1}(w,t,z)$ is a point for $t > 0$. Note that the coordinates (w,t) specify an orbit of the $\underset{\sim}{O}_2$-action on M. Thus it suffices to find all points $(a,b) \in \underset{\sim}{C}^2$ on a <u>given orbit</u> of $\underset{\sim}{O}_2$ on $\underset{\sim}{C}^2$ for which $b/|b| = z$. If

$$(\underline{+}(a\cos\theta + b\sin\theta), -a\sin\theta + b\cos\theta)$$

is another such point then

$$-a\sin\theta + b\cos\theta = \lambda b$$

for some real $\lambda > 0$. If $\sin\theta \neq 0$ then $a = \mu b$ for some real μ. But then $\pi(a,b) \in \{0\} \times \mathbb{C}$ and thus $t = 0$. Thus $\sin\theta = 0$, $\cos\theta = 1$ and $(-a,b)$ is the only other point on the orbit of (a,b) with this value of $b/|b|$. Passing to $(u,v) = (a^2,b)$, this shows that for $t > 0$ the map $((w,t),(u,v)) \longmapsto v/|v|$ of $\alpha^{-1}(w,t)$ into S^1 is an injection. Since both of these are circles, the map is a homeomorphism onto. In particular, $h^{-1}(w,t,z)$ is a point when $t > 0$, as claimed.

For $z \in S^1$, let us use the notation

$$S^0_z \subset S^1$$

for the two point set $\sqrt{z}$. For example, $S^0_1 = \{1,-1\}$ and $S^0_{-1} = \{i,-i\}$.

6.2 <u>Lemma</u>. <u>The map</u> h <u>carries</u> SU^{k+1}_z <u>homeomorphically onto the join</u> $U^{k+1}_z * S^0_{-z} \subset S^{k+2} * S^1$.

<u>Proof</u>. For the case $z = 1$ this follows immediately from 5.4 which implies that SU^{k+1} is the pull-back

$$\begin{array}{ccc} SU^{k+1} & \longrightarrow & \mathbb{R}^+ \times i\mathbb{R} \\ \downarrow & & \downarrow \\ CU^{k+1} & \longrightarrow & \mathbb{R}^+ \times \mathbb{R}. \end{array}$$

The general case follows in the same way from the obvious generalization of 5.4 or on purely theological grounds.

7. Suspension of linking.

Consider the sphere S^m and its double suspension $S^m * S^1$. (The dimension m corresponds to $k+2$ of section 6.) Let $S^0 = \{i,-i\} \subset S^1$ and $S^0_+ = \{z,-z\} \subset S^1$ where $z \in S^1$ has negative real part. (It would not be any loss of generality to take $z = -1$ in this section.)

For a point $w \in S^1$ recall the definition of the cone map

$$c_w \colon \Delta_p(S^m) \longrightarrow \Delta_{p+1}(S^m * \{w\}) \subset \Delta_{p+1}(S^m * S^1)$$

of augmented singular complexes. If $\sigma\colon \Delta_p \longrightarrow S^m$ then $c_w(\sigma)\colon \Delta_{p+1} \longrightarrow S^m * \{w\}$ is the singular simplex taking the point $\sum_{i=0}^{p+1} \lambda_i e_i$ (where $e_0, \ldots, e_{p+1}$ are the vertices of the standard simplex Δ_{p+1}) to the point of $S^m * \{w\}$ given by the join coordinates

$$(\sigma(\sum_{i=0}^{p} (\frac{\lambda_{i+1}}{1-\lambda_0}) e_i), \lambda_0, w).$$

Also c_w takes the standard augmentation generator of $\Delta_{-1}(S^m) = \underset{\sim}{Z}$ to $w \in \Delta_0(S^m * \{w\})$. Then

$$\partial c_w = 1 - c_w \partial .$$

Define the suspension maps (of augmented complexes)

$$\varphi\colon \Delta_p(S^m) \longrightarrow \Delta_{p+1}(S^m * S^0) \subset \Delta_{p+1}(S^m * S^1)$$

$$\varphi^+\colon \Delta_p(S^m) \longrightarrow \Delta_{p+1}(S^m * S^0_+) \subset \Delta_{p+1}(S^m * S^1)$$

by

$$\varphi = c_i - c_{-i} \quad \text{and} \quad \varphi^+ = c_z - c_{-z}.$$

Then φ and φ^+ are chain maps $(\partial\varphi = -\varphi\partial)$. For subsets $A \subset S^m$ and $B \subset S^m$ we have that

$$\varphi: \Delta_p(A) \longrightarrow \Delta_{p+1}(A * S^0)$$

$$\varphi^+: \Delta_p(B) \longrightarrow \Delta_{p+1}(B * S^0_+)$$

and induce, as is well known, <u>isomorphisms</u>

$$\varphi_* : \tilde{H}_p(A) \xrightarrow{\approx} \tilde{H}_{p+1}(A * S^0)$$

$$\varphi^+_* : \tilde{H}_p(B) \xrightarrow{\approx} \tilde{H}_{p+1}(B * S^0_+),$$

Assume now that $A \cap B = \emptyset$ and let

$$Lk: \tilde{H}_p(A) \times \tilde{H}_{n-p-1}(B) \longrightarrow \underset{\sim}{Z}$$

be the linking pairing in S^m and

$$Lk: \tilde{H}_{p+1}(A * S^0) \times \tilde{H}_{n-p}(B * S^0_+) \longrightarrow \underset{\sim}{Z}$$

be the linking pairing in $S^m * S^1 \approx S^{m+2}$.

7.1 <u>Proposition</u>. <u>Up to a constant sign depending on orientation conventions, the suspension isomorphisms</u> φ_*, φ^+_* <u>preserve linking numbers; that is, for</u> $\alpha \in \tilde{H}_p(A)$ <u>and</u> $\beta \in \tilde{H}_{n-p-1}(B)$ <u>we have</u>

$$Lk(\alpha, \beta) = Lk(\varphi_*(\alpha), \varphi^+_*(\beta)).$$

<u>Proof</u>. The classes α and β may be replaced by their images in the homology of open disjoint neighborhoods of A and B in S^m without changing either of these linking numbers.

Thus we may as well assume that α and β are represented by singular chains a and b based on smooth singular simplices of S^m. Let $b = \partial c$ where c is also such a chain and assume, as we may, that the simplices in c meet those of a in general position (transversely). Then $Lk(\alpha,\beta)$ is the intersection number $a \cdot c$ and can be evaluated by counting points of intersection with appropriate signs. Now $\partial\varphi^+(c) = -\varphi^+(\partial c) = -\varphi^+(b)$ and thus $Lk(\varphi_*(\alpha), \varphi_*^+(\beta)) = -\varphi(a)\cdot\varphi^+(c)$ (intersection number in $S^m * S^1 = S^{m+2}$). Now the simplices in $\varphi(a)$ and $\varphi^+(c)$ are not quite in general position, but it is still geometrically clear how to calculate $\varphi(a)\cdot\varphi^+(c)$ and to see that, up to orientation sign, it coincides with $a\cdot c$.

8. The main theorem.

As before, let $\Sigma^k \subset S^{k+2}$ be a knot (i.e., an oriented submanifold of codimension two) and V^{k+1} a Seifert surface cobounding Σ^k. We have the twist suspended knot $\omega(S^{k+2},\Sigma^k) = (S^{k+4},\Sigma^{k+2})$ and the induced Seifert surface $V^{k+3} = \omega(V^{k+1})$.

Let V_+^{k+1} be a displacement of V^{k+1} in the positive normal direction, and similarly for V_+^{k+3}. The displacement gives a canonical isomorphism $\tilde{H}_*(V^{k+1}) \longrightarrow \tilde{H}_*(V_+^{k+1})$ which we denote by $\alpha \longmapsto \alpha^+$.

8.1 Theorem. There is a canonical "suspension" isomorphism $\omega: \tilde{H}_p(V^{k+1}) \xrightarrow{\approx} \tilde{H}_{p+1}(\omega(V^{k+1})) = \tilde{H}_{p+1}(V^{k+3})$ which preserves Seifert linking (up to constant sign depending only on orientation conventions). That is, for $\alpha \in \tilde{H}_p(V^{k+1})$, $\beta \in \tilde{H}_{k-p+1}(V^{k+1})$ we have

$$Lk(\alpha,\beta^+) = Lk(\omega(\alpha), \omega(\beta)^+).$$

<u>Proof</u>. The surface V^{k+1} can be replaced by U_1^{k+1} of section 6 and V_+^{k+1} by U_z^{k+1} for $z \in S^1$ of small positive argument. Similarly $V^{k+3} = \omega(V^{k+1})$ and V_+^{k+3} can be replaced by U_1^{k+3} and U_z^{k+3} and these in turn by SU_1^{k+1} and SU_z^{k+1}. Now the map $h: S^{k+4} \longrightarrow S^{k+2} * S^1$ of section 6 preserves linking by 6.1. (This can be deduced from the properties of intersection number given in [5; pp.335-338]. We caution the reader that a general homotopy equivalence, even of triples, does not preserve linking, but a "local" homotopy equivalence does.) Thus by 6.2, SU_1^{k+1} and SU_z^{k+1} can, in turn, be replaced by $U_1^{k+1} * S_{-1}^0$ and $U_z^{k+1} * S_{-z}^0$ respectively. The result then follows from 7.1.

When $k = 2m-1$ is odd and when $\alpha_1, \alpha_2, \dots$ is a basis of $H_m(V^{2m}; \mathbb{Z})$, modulo torsion, the matrix whose entries are

$$Lk(\alpha_i, \alpha_j^+)$$

is called the Seifert matrix of the Seifert surface V^{2m} of the knot Σ^{2m-1}. Thus we have

8.2 <u>Corollary</u>. <u>Let</u> $\Sigma^{2m-1} \subset S^{2m+1}$ <u>be a knot with Seifert surface</u> V^{2m}. <u>Then, with respect to a given basis of</u> $H_m(V^{2m}; \mathbb{Z})$/torsion, <u>and the induced basis of</u> $H_{m+1}(\omega(V^{2m}); \mathbb{Z})$/torsion, <u>the Seifert matrix of</u> $\omega(V^{2m}) \subset S^{2m+3}$ <u>is identical to that of</u> $V^{2m} \subset S^{2m+1}$ (<u>up to constant sign depending on orientation conventions</u>).

9. Applications.

Let C_n denote the knot cobordism group in dimension n. That is, C_n is the group under connected sum of h-cobordism classes of knots $\Sigma^n \subset S^{n+2}$ where Σ^n is a homotopy sphere. Recall that $C_n = 0$ for n even.

By 4.1 the double suspension ω^2 induces a homomorphism

$$\omega^2 : C_n \longrightarrow C_{n+4}.$$

For $n > 1$, Levine [12,13] has shown that these groups are isomorphic to certain "cobordism" groups of Seifert matrices. His results, together with 8.2, immediately imply the following theorem.

9.1 Theorem. *The homomorphism $\omega^2 : C_n \longrightarrow C_{n+4}$ is an isomorphism for $n \neq 1, 3$, an epimorphism for $n = 1$, and a monomorphism onto a subgroup of index two for $n = 3$.*

A knot $(S^{2m+1}, \Sigma^{2m-1})$ is called *simple* [11] if it has a Seifert surface V^{2m} which is $(m-1)$-connected. (For a homology spherical knot $\Sigma^{2m-1} \subset S^{2m+1}$ this is equivalent to the statement that $\pi_i(S^{2m+1} - \Sigma^{2m-1}) \approx \pi_i(S^1)$ for $i \leqslant m-1$; see [15].) In this case the Seifert surface V^{2m} will be called simple. Clearly this implies that $\omega(V^{2m})$ is simple since, by section 6, it is homotopy equivalent to $V^{2m} * S^0$. Thus the twist suspension takes simple knots to simple knots. Levine [11] has classified simple spherical knots in terms of their Seifert matrices and the following theorem is immediate from his results and 8.2.

9.2 *Theorem.* *The double suspension* ω^2 *gives an isomorphism from the semigroup of isotopy classes of simple spherical* $(2m-1)$-*knots to that of simple spherical* $(2m+3)$-*knots for* $m \neq 1, 2$. *It is a surjection for* $m = 1$ *and an injection for* $m = 2$.

It should be stressed that the periodicities in 9.1 and 9.2 are due to Levine. It is only the explicit geometric construction of it that is novel here. For 9.1 a very different geometric description was independently discovered by Cappell and Shaneson [4].

A knot $\Sigma^k \subset S^{k+2}$ is called a *fibered knot* if it has a tubular neighborhood $D^2 \times \Sigma^k \subset S^{k+2}$ such that the projection $S^1 \times \Sigma^k \longrightarrow S^1$ extends to a submersion $S^{k+2} - (E^2 \times \Sigma^k) \longrightarrow S^1$ (hence a fibration). This may be taken to be the map τ of section 3 and thus every $z \in S^1$ is a regular value of τ. By 5.2 and the remarks in section 6 it follows that the map $\varphi: S^{k+4} \longrightarrow \underset{\sim}{C}^2$ of section 5 is transverse regular on $\underset{\sim}{R}^+ z \times \underset{\sim}{C}$ for *each* $z \in S^1$. The map $S^{k+4} - \Sigma^{k+2} = S^{k+4} - \varphi^{-1}(\{0\} \times \underset{\sim}{C}) \longrightarrow S^1$, which is φ followed by the projection $(u,v) \longmapsto u/|u|$ of $(\underset{\sim}{C} - \{0\}) \times \underset{\sim}{C} \longrightarrow S^1$, is then a fibering for the twist suspended knot. This proves the following fact.

9.3 *Theorem.* *The twist suspension* ω *takes fibered knots to fibered knots.*

We believe that these results clearly indicate that the twist suspension operation should be of interest in pure higher dimensional knot theory. The more immediate application of our ideas, however, is to the determination (below) of the differentiable structures of $\underset{\sim}{O}_n$-knot manifolds. This information has already been used in [1] and [14].

Let us begin by recalling some definitions. Let $\Sigma^{2m-1} \subset S^{2m+1}$ be a knot with Seifert surface V^{2m} and let B be the Seifert form

$$B(\alpha,\beta) = Lk(\alpha,\beta^+)$$

on $H_m(V^{2m};\underset{\sim}{Z})$/torsion. Then $B + (-1)^m B'$ is the intersection form of V^{2m}.

The signature of the knot $\Sigma^{2m-1} \subset S^{2m+1}$ is defined to be the signature of the symmetric form $B + B'$ where B is a Seifert form for Σ^{2m-1}. Since we don't know of a reference for the fact that the signature is a knot invariant when Σ^{2m-1} may not be a homology sphere, we shall give a simple argument for this. Since, by 8.2, the twist suspension ω preserves the signature, it clearly suffices to prove invariance for the case m even. In this case $B + B'$ is the intersection form of V^{2m} so that the signature is just the index of V^{2m}. If V_1^{2m} is another Seifert surface, consider the closed manifold $W^{2m} \subset S^{2m+2}$ which is the union of V^{2m} pushed into the hemisphere D_+^{2m+2} and V_1^{2m} pushed into the hemisphere D_-^{2m+2}. It suffices to show that the index of W^{2m} is zero. But W^{2m}, being a codimension two submanifold of S^{2m+2}, bounds a (2m+1)-manifold, and hence has index zero, proving our contention.

Let us now turn to the Arf invariant of $\Sigma^{2m-1} \subset S^{2m+1}$. For this we assume either that m is odd and Σ^{2m-1} is a homology sphere, or that m is even and the cyclic double cover of S^{2m+1} branched at Σ^{2m-1} (i.e., the twist suspension Σ^{2m+1}) is a homology sphere. In these cases the skew symmetric form $B - B'$ is unimodular, since it is the intersection form of V^{2m} or $\omega(V^{2m})$ according as m is odd or even. Thus $\det(B+B') \equiv \det(B-B') \equiv 1 \pmod 2$ and we can define the Arf invariant to be 0 if $\det(B+B') \equiv \pm 1 \pmod 8$ and to be 1 if $\det(B+B') \equiv \pm 3 \pmod 8$. For m odd this is a well defined knot invariant by [13] Poly inv. and for m even the same fact follows from the case m odd by application of the twist suspension and 8.2.

Now recall that the O_n-manifold $M^{2n+k}(\Sigma^k)$ is embedded in S^{2n+k+2} as the n-fold twist suspension $\omega^n(S^{k+2}, \Sigma^k)$. When n is even then, by 3.1, $M^{2n+k}(\Sigma^k)$ is a homotopy sphere if and only if Σ^k is a homology sphere. It also follows that if $n > 1$ is odd, then $M^{2n+k}(\Sigma^k)$ is a homotopy sphere if and only if the branched double cover $M^{2+k}(\Sigma^k)$ is a homology sphere. These considerations immediately imply the following result, using 8.2.

9.4 Theorem. *Suppose the underlying manifold of the* $O_{\underset{\sim}{m}n}$*-manifold* $M^{2n+k}(\Sigma^k)$ *is a homotopy sphere* Σ^{2n+k}. *If* k *is even then* Σ^{2n+k} *is a standard sphere. If* $2n+k \equiv 3 \pmod 4$, *then* Σ^{2n+k} *is* $\sigma/8$ *times the standard Milnor generator of* bP_{2n+k+1}, *where* σ *is the signature of the knot* (S^{k+2}, Σ^k). *If* $2n+k \equiv 1 \pmod 4$, *then* Σ^{2n+k} *is the Kervaire sphere or the standard sphere according as the Arf invariant of* (S^{k+2}, Σ^k) *is* 1 *or* 0.

We remark that in the case $k = 1$ of classical knots this result is due to Hirzebruch [8] and Erle [6]. Their proof may generalize to the case of simple knots, but, in any case, it now seems clear that the present viewpoint is the proper context for this result.

There are some other applications of our results to the theory of $O_{\underset{\sim}{m}n}$-manifolds which are briefly mentioned in [3]. We will not repeat these here.

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