CLASSIFICATION OF REGULAR ACTIONS OF CLASSICAL GROUPS WITH THREE ORBIT TYPES*

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1. Introduction.

By a regular action (with three orbit types including fixed points) of O_n, U_n or Sp_n we mean a smooth action for which the orbit types are those of twice the standard representation of these groups and for which the bundle of principal orbits is trivial. See section 3 for the precise definition. The purpose of this paper is to classify these actions.

Because of the fortuitous fact that the group O_2 and the real projective line $RP^1 \approx S^1$ have vanishing higher homotopy groups, the case of O_n-actions yields to relatively simple methods, and thus this case is well established; see [1,7,8,10,11]. These methods do not work well in the unitary and symplectic cases, and thus practically nothing is known in those cases. (The extent of previous knowledge in these cases is contained in [5].) In fact, the unitary and symplectic cases have been regarded, by this author as well as others, as being nearly intractible. Our present work shows that this is not the case, and, moreover, it shows that, given the proper viewpoint, the unitary and symplectic cases are quite analogous to the orthogonal case, contrary to some opinions expressed in the literature; see [1;p.337] and [5;1.4].

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The key to our classification is a new <u>relative</u> classification theorem for actions with two orbit types. This is developed in section 2.

In section 4, we classify regular actions "over a given base space". All such actions are shown, in section 5, to be constructible by an interesting pull-back construction, and this is used to obtain several facts about such actions, such as the connection between the O_n, U_n and Sp_n cases, and linear embedding results.

The classification of regular actions up to equivariant diffeomorphism (not "over" a base space) is accomplished in section 6 and this is applied to "knot-manifolds" (orbit space a disk and fixed set a homotopy sphere) in section 7. The knot-manifold case is surprisingly simple, and all such actions turn out to be well known actions on Brieskorn varieties.

In the appendix, section 9, we gather some information concerning the orbit map of twice the standard representation of O_n, U_n or Sp_n.

2. Relative classifcation for two orbit types.

Let $K \subset G$ be compact Lie groups, let $\beta : K \longrightarrow O_n$ be a representation which is transitive on the unit sphere S^{n-1} and let H be an isotropy group $H = \beta^{-1}(O_{n-1})$. We have the twisted product

$$G \times_K R^n ,$$

which is a smooth G-manifold with orbit types G/K and G/H and with orbit space diffeomorphic to R^+; see [1;VI.5].

The group $S = (N_H \cap N_K)/H$ acts on the right of $G \times_K R^n$ as a group of continuous self-equivalences (commuting with the orbit map) and we assume that G, K, H are such that this action is smooth. (This is probably always the case, and it is easily verified in the situations of interest to us in this paper; see the discussion of this point in [1; p. 368].) Then it is clear that S is precisely the group of orthogonal G-bundle equivalences of the G-bundle $G \times_K R^n$ over the G-space G/K (with fiber R^n).

We shall regard the above data G, K, H, β as fixed throughout this section.

Let X be a smooth $(m+1)$-manifold with boundary $\partial X = X^{\bullet} \cup B$ and with (acute) corner along the $(m-1)$-manifold $B^{\bullet} = X^{\bullet} \cap B$; see figure 1.

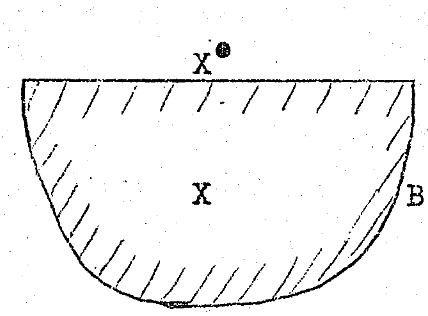

Figure 1.

By a <u>proper</u> <u>G-manifold</u> M <u>over</u> X we shall mean a smooth G-manifold M with boundary $M^{\bullet}$ together with a smooth map $\mu_M: M \longrightarrow X$ which induces a <u>diffeomorphism</u>

$$\mu_M^{*}: (M/G,\ M^{\bullet}/G) \xrightarrow{\approx} (X, X^{\bullet}),$$

where M/G has the functional structure induced from M, such that

(i) M has only the orbit types G/H and G/K.

(ii) B corresponds exactly to the set of singular orbits G/K.

(iii) Each singular orbit has a tubular neighborhood equivalent to $(G \times_K R^n) \times R^m$ (or R_+^m for a singular orbit in $M^{\bullet}$), where K acts on R^n via the given representation β.

(The map μ_M will usually be understood, and will not appear explicitly in our notation.)

Now suppose we are <u>given</u> a proper G-manifold $Y^{\bullet}$ over $X^{\bullet}$. Then by a <u>proper</u> G-<u>manifold</u> (M, φ_M) <u>over</u> X <u>and extending</u> $Y^{\bullet}$ we shall mean a proper G-manifold M over X together with a smooth equivalence

$$\varphi_M: M^{\bullet} \xrightarrow{\approx} Y^{\bullet}$$

commuting with the given maps of these to $X^{\bullet}$.

Two such pairs (M, φ_M) and (N, φ_N) will be called <u>equivalent</u> <u>over</u> X if there is an equivariant diffeomorphism $\psi: M \longrightarrow N$ such that

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ & {\scriptstyle\mu_M}\searrow \quad \swarrow{\scriptstyle\mu_N} & \\ & X & \end{array}$$

commutes <u>and</u> such that

$$\begin{array}{ccc} M^{\bullet} & \xrightarrow{\psi^{\bullet}} & N^{\bullet} \\ & {\scriptstyle \varphi_M} \searrow \quad \swarrow {\scriptstyle \varphi_N} & \\ & Y^{\bullet} & \end{array}$$

commutes, where $\psi^{\bullet}$ is the restriction of ψ to $M^{\bullet}$.

The purpose of this section is to give a classification of these equivalence classes. We shall follow the method and notation of the proof of the absolute case given in [1;V.6]. We assume familiarity with the basic facts developed there and in [1;VI.6]. The absolute casé is due, in a weaker form, to Jänich [11] and the Hsiangs [8].

The disk bundle $G \times_K D^n$ over G/K may be identified with the mapping cylinder M_π of the projection $\pi: G/H \longrightarrow G/K$. We put

$$N = N_H/H, \qquad S = (N_H \cap N_K)/H.$$

We select, once and for all, a smooth collar $I \times B \longrightarrow X$ of B in X (and regard it as inclusion) where $\{0\} \times B$ is identified with B. Let $B_1 = \{1\} \times B$, $X_1 = X - ([0,1) \times B)$, etc.

Let ρ be a given G/H-bundle over X_1 with structure group $N = N_H/H$, and let $P \to X_1$ be the associated principal bundle. We shall consider those proper G-manifolds over X whose principal orbit bundle is equivalent to ρ over X_1. In particular, we assume that the principal orbit bundle of the given proper G-manifold $Y^{\bullet}$ over $X^{\bullet}$ is equivalent to ρ over $X_1^{\bullet}$. We shall denote by P_1, $P^{\bullet}$, and $P_1^{\bullet}$ the restriction of P to B_1, $X^{\bullet}$, and $B_1^{\bullet} = B_1 \cap X^{\bullet}$, respectively.

As shown in [1], $Y^{\circledast}$ is equivalent over $X^{\circledast}$ to

$$Y^{\circledast}(\rho^{\circledast},(Q^{\circledast},f^{\circledast})) = (M_{\pi} \times_S Q^{\circledast}) \cup_{f^{\circledast}} (G/H \times_N P^{\circledast})$$

where $Q^{\circledast}$ is some principal S-bundle over $B_1^{\circledast}$ (and hence $M_{\pi} \times_S Q^{\circledast}$ is a G-manifold over $I \times B^{\circledast} \subset X^{\circledast}$) and

$$f^{\circledast}: G/H \times_S Q^{\circledast} \xrightarrow{\approx} G/H \times_N P_1^{\circledast}$$

is the equivalence canonically associated with a smooth S-reduction

$$f^{\circledast}: Q^{\circledast} \longrightarrow P_1^{\circledast}$$

(where the same notation $f^{\circledast}$ may be used since the correspondence is canonical; see [1;V.3.2]). It is no loss of generality to assume that, in fact,

$$Y^{\circledast} = Y^{\circledast}(\rho^{\circledast},(Q^{\circledast},f^{\circledast})).$$

Now suppose we have a proper G-manifold (M,φ_M) over X and extending $Y^{\circledast}$, and whose principal orbit bundle is equivalent to ρ over X_1. Then, in a similar manner, there is no difficulty in showing that (M,φ_M) is <u>equivalent</u> over X to

$$(\dot{Y}(\rho,(Q,f)),\ \varphi(Q,f,\sigma))$$

constructed as follows. As above, Q is a principal S-bundle over B_1 (or over B) and

$$f: Q \longrightarrow P_1$$

is an S-reduction; i.e.,

$$f: G/H \times_S Q \xrightarrow{\approx} G/H \times_N P_1$$

is an equivalence, and

$$Y(\rho,(Q,f)) = (M_{\pi} \times_S Q) \cup_f (G/H \times_N P).$$

(Also, for simplicity, we may, by canonical changes, assume that $Q^{\circledast}$ is just the restriction of Q to $B_1^{\circledast}$. Then $Y^{\circledast}(\rho,(Q,f))$ denotes the restriction of $Y(\rho,(Q,f))$ to $X^{\circledast}$.)

Furthermore, $\sigma \in \underset{\sim\sim}{\mathrm{Map}}^G(\rho^{\circledast},\rho^{\circledast})$ is some smooth self-equivalence over $X_1^{\circledast}$ of the restriction $\rho^{\circledast}$ of ρ to $X_1^{\circledast}$. (That is,

$$\sigma: G/H \times_N P^{\circledast} \longrightarrow G/H \times_N P^{\circledast}$$

is a G-equivariant smooth equivalence commuting with the projections to $X_1^{\circledast}$.) Then

$$\varphi(Q,f,\sigma): Y^{\circledast}(\rho,(Q,f)) \longrightarrow Y^{\circledast}(\rho^{\circledast},(Q^{\circledast},f\sigma^{-1})) \qquad (*)$$

is defined to be the equivalence which is the identity on $M_\pi \times_S Q^{\circledast}$ and σ^{-1} on $G/H \times_N P^{\circledast}$. (Here the composition $f\sigma^{-1}$ acts, as in [1;V.6], from left to right and is defined only over $B_1^{\circledast}$.) Since we wish to have $Y^{\circledast}$ on the right in (*), it is assumed that $f\sigma^{-1} = f^{\circledast}$; that is,

$$f = f^{\circledast}\sigma \quad \text{over } B_1^{\circledast}.$$

<u>Remark</u> 1. As stated above, we have assumed that $Q|B_1^{\circledast}$ is identified with $Q^{\circledast}$. This is done in order to conserve notational intricacy and simplify some remarks below. If this is not done then one should add to the data for φ an S-equivalence $g: Q|B_1^{\circledast} \longrightarrow Q^{\circledast}$ over $B_1^{\circledast}$. Then

$$\varphi(Q,f,\sigma,g): Y^{\circledast}(\rho,(Q,f)) \longrightarrow Y^{\circledast}(\rho^{\circledast},(Q^{\circledast},f^{\circledast}))$$

is the equivalence which is g on $M_\pi \times_S Q^{\circledast}$ and is σ^{-1} on $G/H \times_N P^{\circledast}$. Hence, it is demanded that

$$f = gf^{\circledast}\sigma \quad \text{over } B_1^{\circledast}.$$

$\sigma f g$

Note that the latter equation determines g <u>uniquely</u> from the other data $(\rho, Q, f, Q^{\circledast}, f^{\circledast}, \sigma)$. This formulation can be useful. For example, let

$$C = (\text{center } N) \cap S$$

and let

$$\gamma\colon X^{\circledast} \longrightarrow C$$

be a smooth map. Then right multiplication by γ determines <u>natural</u> self-equivalences $\gamma\colon Q^{\circledast} \longrightarrow Q^{\circledast}$ and $\gamma\colon P^{\circledast} \longrightarrow P^{\circledast}$. Since this commutes with $f^{\circledast}$, by naturality, it defines an equivalence

$$\gamma\colon Y^{\circledast}(\rho^{\circledast}, (Q^{\circledast}, f^{\circledast})) \longrightarrow Y^{\circledast}(\rho^{\circledast}, (Q^{\circledast}, f^{\circledast}))$$

over $X^{\circledast}$. Then, with composition from left to right,

$$(Y(\rho, (Q, f)),\ \varphi(Q, f, \sigma, g)\circ\gamma)$$

is a proper G-manifold over X extending $Y^{\circledast}$. Clearly there is an equivalence

$$(Y(\rho, (Q, f)),\ \varphi(Q, f, \sigma, g)\circ\gamma) \approx (Y(\rho, (Q, f)),\ \varphi(Q, f, \gamma^{-1}\sigma, g\gamma))$$

over X.

<u>Remark</u> 2. If the reader suspects that (*) may play havoc with differentiability, he is correct. This is so because we have suppressed, as is not unusual, the details of smooth patching. What one really does, for example, is to bicollar B_1 in X and to bicollar everything above it. One may then assume that all patchings are defined on these bicollars and constant in the collar direction. This, of course, will force σ to be constant in the collar direction. Once we show, below, the invariance under homotopies of the data, the necessity of all this disappears. We should have no need to comment on such well understood details further.

Thus, given the data ρ, Q, f, σ as above, with $Q|B_1^{\circ} = Q^{\circ}$ and $f\sigma = f^{\circ}$ over B_1°, we have constructed the proper G-manifold $(Y(\rho,(Q,f)), \varphi(Q,f,\sigma))$ over X extending Y°, and we have commented that it is easily seen, as in [1;p.254], that all such pairs can be constructed this way, up to equivalence over X. We now investigate some of the redundancies in the data.

First, suppose that $g: Q \longrightarrow Q'$ is an equivalence over B_1 of the S-reductions (Q,f) and (Q',f'); that is,

$$\begin{array}{ccc} Q & \xrightarrow{\;g\;} & Q' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & P_1 & \end{array}$$

commutes and g is S-equivariant. Then it is clear that there is an induced equivalence

$$(Y(\rho,(Q,f)), \varphi(Q,f,\sigma)) \approx (Y(\rho,(Q',f')), \varphi(Q',f',\sigma)),$$

given by g on $M_\pi \times_S Q$ and the identity on $G/H \times_N P$, and similarly with the maps φ. Note that g is uniquely determined by f and f', since these are injections, and hence this equivalence depends canonically on (Q,f) and (Q',f').

Next, suppose that $\omega \in \underset{\sim}{Map}^G(\rho,\rho)$; that is,

$$\omega: G/H \times_N P \xrightarrow{\;\approx\;} G/H \times_N P$$

is an equivalence over X_1. Then the identity on $M_\pi \times_S Q$ and ω on $G/H \times_N P$ give an equivalence (over X):

$$(Y(\rho,(Q,f)), \varphi(Q,f,\sigma)) \approx (Y(\rho,(Q,f\omega)), \varphi(Q,f\omega,\sigma\omega)).$$

(We remark again that the compositions $f\omega$ and $\sigma\omega$ operate from left to right.)

Finally we consider homotopies of the data. Thus suppose that

$$\{f_t\}: G/H \times_S Q \times I \longrightarrow G/H \times_N P_1$$

is a smooth homotopy over B_1 of S-reductions, and that

$$\sigma_t : G/H \times_N P^{\circledast} \times I \longrightarrow G/H \times_N P^{\circledast}$$

is a smooth homotopy of self equivalences over $X_1^{\circledast}$, such that

$$f_t = f^{\circledast}\sigma_t \quad \text{over} \quad B_1^{\circledast}.$$

(We may, of course, assume that the homotopies are constant for t near 0 and near 1.)

Consider the map

$$\omega': G/H \times_N P_1 \times I \longrightarrow G/H \times_N P_1 \times I$$

defined by

$$\omega'(x,t) = (x \cdot f_0^{-1} \cdot f_t,\ t).$$

Also let

$$\omega'': G/H \times_N P^{\circledast} \times I \longrightarrow G/H \times_N P^{\circledast} \times I$$

be

$$\omega''(x,t) = (x \cdot \sigma_0^{-1}\sigma_t,\ t).$$

It is immediate that $\omega' = \omega''$ on their common domain $G/H \times_N P_1^{\circledast} \times I$ and that they both equal the identity for t near 0. Together with the identity, these patch together as indicated in figure 2 to give a self equivalence $\omega: G/H \times_N P \longrightarrow G/H \times_N P$.

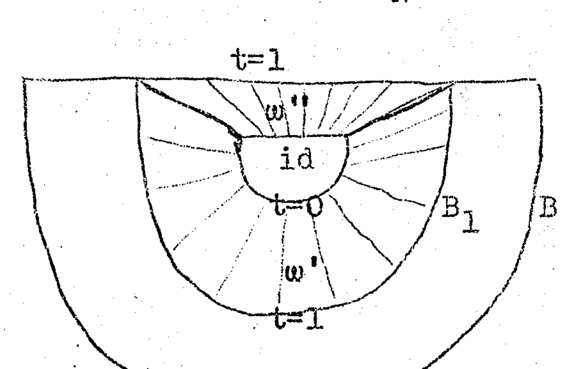

Figure 2.

On B_1, ω equals $f_0^{-1}f_1$ and on $X_1^{\bullet}$ it equals $\sigma_0^{-1}\sigma_1$. Hence the operation by ω defines an equivalence

$$(Y(\rho,(Q,f_0)),\ \varphi(Q,f_0,\sigma_0)) \approx (Y(\rho,(Q,f_1)),\ \varphi(Q,f_1,\sigma_1)).$$

This concludes our investigation of redundancies in the data. We shall now introduce some notation which will allow us to summarize our present knowledge in a convenient form.

Consider the bundle with fiber $S\backslash N$ associated with ρ. We shall denote this bundle over X_1 by

$$\mathcal{R} = S\text{-}\mathcal{R}ed(\rho) = S\backslash N \times_N P = S\backslash P.$$

We are mainly interested in the restriction $\mathcal{R}|B_1$ of $\mathcal{R}$ to B_1. Recall that equivalence classes of S-reductions (Q,f) of P_1 over B_1 are in one-one correspondence with the set of cross sections

$$\tau \in \Gamma(\mathcal{R}|B_1)$$

of $\mathcal{R}|B_1$. In particular, the <u>given</u> S-reduction $(Q^{\bullet}, f^{\bullet})$ corresponds to a section

$$\tau^{\bullet} \in \Gamma(\mathcal{R}|B_1^{\bullet}),$$

which is fixed once and for all. Now there is the right action

$$\Gamma(\mathcal{R}|B_1) \times \underset{\sim}{\mathrm{Map}}^G(\rho,\rho) \longrightarrow \Gamma(\mathcal{R}|B_1)$$

denoted by $(\tau,\omega) \longmapsto \tau\omega$, which can be regarded, as above, as composition of maps acting from left to right. Similarly we have

$$\Gamma(\mathcal{R}|B_1^{\bullet}) \times \underset{\sim}{\mathrm{Map}}^G(\rho^{\bullet},\rho^{\bullet}) \longrightarrow \Gamma(\mathcal{R}|B_1^{\bullet})$$

with $(\tau,\sigma) \longmapsto \tau\sigma$.

Suppose that we are given

$$\tau \in \Gamma(\mathcal{R}|B_1) \quad \text{and} \quad \sigma \in \mathrm{Map}^G(\rho^{\circledast}, \rho^{\circledast})$$

with

$$\tau = \tau^{\circledast}\sigma \quad \text{over} \quad B_1^{\circledast}.$$

Then τ corresponds (uniquely to S-equivalence) to an S-reduction (Q, f') of $\rho|B_1$ such that (Q, f') is S-equivalent to $(Q^{\circledast}, f^{\circledast}\sigma)$ over $B_1^{\circledast}$. This S-equivalence over $B_1^{\circledast}$ is uniquely determined by f' and $f^{\circledast}\sigma$ and can be used to <u>identify</u> $Q|B_1^{\circledast}$ with $Q^{\circledast}$. After this canonical identification, f' will be relabled f and we then have the S-reduction (Q, f) with

$$f = f^{\circledast}\sigma \quad \text{over} \quad B_1^{\circledast}.$$

Suppose now that (Q, f_0) and (Q, f) are S-reductions of $\rho|B_1$ with corresponding sections

$$\tau_0 \in \Gamma(\mathcal{R}|B_1) \quad \text{and} \quad \tau_1 \in \Gamma(\mathcal{R}|B_1)$$

and that $\sigma_0, \sigma_1 \in \mathrm{Map}^G(\rho^{\circledast}, \rho^{\circledast})$ are such that

$$\tau_0 = \tau^{\circledast}\sigma_0 \quad \text{and} \quad \tau_1 = \tau^{\circledast}\sigma_1 \quad \text{over} \quad B_1^{\circledast}.$$

Suppose further that these are connected by homotopies

$$\tau_t \in \Gamma(\mathcal{R}|B_1), \quad \sigma_t \in \mathrm{Map}^G(\rho^{\circledast}, \rho^{\circledast}), \quad \tau_t = \tau^{\circledast}\sigma_t \quad \text{over} \quad B_1^{\circledast}.$$

Then <u>define</u> $f_t = f^{\circledast}\sigma_t$ over $B_1^{\circledast}$. The covering homotopy theorem applied to the bundle $P_1 \longrightarrow S\backslash P_1$ then implies that the partial homotopy $\{f_t\}$ over $B_1^{\circledast}$ and the initial condition f_0 can be extended to a homotopy $\{f_t\}$ of S-reductions over B_1. Then (Q, f_1) is S-<u>equivalent</u> to (Q, f). Hence we have the equivalence

$$(Y(\rho, (Q, f_0)), \varphi(Q, f_0, \sigma_0)) \approx (Y(\rho, (Q, f)), \varphi(Q, f, \sigma_1)).$$

This shows that the equivalence class over X of $(Y(\rho,(Q,f)), \varphi(Q,f,\sigma))$ depends only on the homotopy class (relative to the relation $\tau = \tau^{\circledast}\sigma$ on $B_1^{\circledast}$) of the pair (τ,σ), where $\tau \in \Gamma(\mathcal{R}|B_1)$ is the cross section corresponding to (Q,f). Moreover (and, in fact, more generally) it is also dependent only on the orbit of (τ,σ) under the right action $((\tau,\sigma),\omega) \mapsto (\tau\omega,\sigma\omega)$ of $\underset{\sim}{\mathrm{Map}}^G(\rho,\rho)$ on these pairs.

Now

$$\underset{\sim}{\mathrm{Map}}^G(\rho,\rho) = \Gamma(\mathcal{N})$$

where

$$\mathcal{N} = \mathcal{M}ap^G(\rho,\rho)$$

is a bundle over X_1 with fiber N and structure group N acting by conjugation (and is associated with ρ). Also, of course,

$$\underset{\sim}{\mathrm{Map}}^G(\rho^{\circledast},\rho^{\circledast}) = \Gamma(\mathcal{N}|B_1^{\circledast}).$$

We shall denote the space of pairs (τ,σ), where

$$\tau \in \Gamma(\mathcal{R}|B_1), \quad \sigma \in \Gamma(\mathcal{N}|X_1^{\circledast}), \text{ and } \tau = \tau^{\circledast}\sigma \text{ over } B_1^{\circledast} = B_1 \cap X_1^{\circledast},$$

by

$$\Gamma_{\tau^{\circledast}}(\mathcal{R}|B_1, \mathcal{N}|X_1^{\circledast}).$$

The right action

$$\Phi: \Gamma_{\tau^{\circledast}}(\mathcal{R}|B_1, \mathcal{N}|X_1^{\circledast}) \times \Gamma(\mathcal{N}) \longrightarrow \Gamma_{\tau^{\circledast}}(\mathcal{R}|B_1, \mathcal{N}|X_1^{\circledast})$$

takes $((\tau,\sigma),\omega) \longmapsto (\tau\omega,\sigma\omega)$. The action on homotopy classes $[\tau,\sigma]$ induced by Φ will be denoted by

$$\Phi_0: \pi_0\Gamma_{\tau^{\circledast}}(\mathcal{R}|B_1, \mathcal{N}|X_1^{\circledast}) \times \pi_0\Gamma(\mathcal{N}) \longrightarrow \pi_0\Gamma_{\tau^{\circledast}}(\mathcal{R}|B_1, \mathcal{N}|X_1^{\circledast}).$$

The orbit of $[\tau,\sigma]$ under the action Φ_0 will be denoted by $[\tau,\sigma]^*$.

In summary then, we have shown that the equivalence class over X of the proper G-manifold $(Y(\rho,(Q,f)),\ \varphi(Q,f,\sigma))$ over X and extending $Y^{\circledast} = Y^{\circledast}(\rho^{\circledast},(Q^{\circledast},f^{\circledast}))$ depends only on the orbit $[\tau,\sigma]^*$ under Φ_0, where $\tau \in \Gamma(\mathcal{R}|B_1)$ corresponds to (Q,f). We denote this <u>equivalence class</u> over X of proper G-manifolds extending $Y^{\circledast}$ by

$$(Y(\rho,[\tau,\sigma]^*),\ \varphi([\tau,\sigma]^*).$$

<u>Remark</u>. 3. If $\gamma\colon X_1^{\circledast} \longrightarrow C = (\text{center } N) \cap S$ then, as in Remark 1, we have the proper G-manifold $(Y(\rho,(Q,f)),\ \varphi(Q,f,\sigma)\circ\gamma)$ over X. By Remark 1, we see that the equivalence class of this is just $(Y(\rho,[\tau,\gamma^{-1}\sigma]^*),\ \varphi([\tau,\gamma^{-1}\sigma]^*))$.

Our main theorem can now be stated as follows:

2.1 <u>Theorem</u>. <u>With the above notation, the assignment</u>

$$[\tau,\sigma]^* \longmapsto (Y(\rho,[\tau,\sigma]^*),\ \varphi([\tau,\sigma]^*))$$

<u>is a one-one correspondence between the set of orbits of the right action</u> Φ_0 of $\pi_0\Gamma(\mathcal{N})$ <u>on</u> $\pi_0\Gamma_{\tau^{\circledast}}(\mathcal{R}|B_1, \mathcal{N}|X_1^{\circledast})$, <u>and the set of smooth equivalence classes over</u> X <u>of smooth proper</u> G-<u>manifolds over</u> X <u>extending</u> $Y^{\circledast}$ <u>and having principal orbit bundle</u> ρ. <u>This remains true for topological equivalence classes of topological proper</u> G-<u>manifolds over</u> X <u>extending</u> $Y^{\circledast}$.

<u>Proof</u>. The main part of the proof has already been accomplished. Namely, we have shown that the correspondence is well defined and onto. This can also be proved, as in [1;V.6],

in exactly the same manner, for the topological case, given the background information provided by the "Tube Theorem" and "Straightening Lemma" of [1;V.4].

To complete the proof it remains to show that if $(Y(\rho,(Q_0,f_0)), \varphi(Q_0,f_0,\sigma_0))$ and $(Y(\rho,(Q_1,f_1)), \varphi(Q_1,f_1,\sigma_1))$ are equivalent over X, then the data define the same orbit $[\tau,\sigma]^*$. Since smooth equivalence implies topological equivalence, it suffices to do this in the topological case only. Thus we need not pay attention to smoothness in the remainder of the proof.

We are given an equivalence

$$\mu: (M_\pi \times_S Q_0) \cup_{f_0} (G/H \times_N P) \longrightarrow (M_\pi \times_S Q_1) \cup_{f_1} (G/H \times_N P)$$

over X, which, by assumption, commutes with the equivalences $\varphi(Q_i,f_i,\sigma_i)$ over $X^\circledast$. The $\varphi(Q_i,f_i,\sigma_i)$ are just the given identifications $M_\pi \times_S Q_i = M_\pi \times_S Q^\circledast$ over the collar $I \times B^\circledast$ and hence μ is also the identity there. Now the restriction μ' of μ to $M_\pi \times_S Q_0$ (the part over $I \times B$) is homotopic, relative to the part over $I \times B^\circledast$, to an S-equivalence $\bar{\mu}'$; that is, a map induced by some S-equivariant homeomorphism $Q_0 \longrightarrow Q_1$ over B. (This follows from the Straightening Lemma [1;V.4.1] and from [1;V.3.3(b)].) The restriction of this homotopy to $G/H \times_S Q_0 \approx G/H \times_N P$ can then be placed over a collar $[1,2] \times B$ of B_1 in X_1 and patched with $\bar{\mu}'$ on $[0,1] \times B$ and with the identity outside $[0,2] \times B$, to obtain a new equivalence

$$\bar{\mu} = \bar{\mu}' \cup \bar{\mu}'' : (M_\pi \times_S Q_0) \cup_{f_0} (G/H \times_N P) \longrightarrow (M_\pi \times_S Q_1) \cup_{f_1} (G/H \times_N P)$$

such that $\bar{\mu}'$ is an S-equivalence. (For more details of this see the proof of [1;V.5.1].) Moreover, $\bar{\mu} = \mu$ over $X^{\circledast}$ and hence also commutes with the equivalences $\varphi(Q_i, f_i, \sigma_i)$. Looking at this map over B_1, and putting $\omega = \bar{\mu}''$, we have the commutative diagram

$$\begin{array}{ccc} G/H \times_S Q_0 & \xrightarrow{\bar{\mu}'} & G/H \times_S Q_1 \\ \downarrow f_0 & & \downarrow f_1 \\ G/H \times_N P & \xrightarrow{\omega} & G/H \times_N P \end{array}$$

which shows that the S-reductions (Q_1, f_1) and $(Q_0, f_0\omega)$ are equivalent. Thus the sections $\tau_0, \tau_1 \in \Gamma(\mathcal{R}|B_1)$ corresponding to (Q_0, f_0) and (Q_1, f_1) satisfy

$$\tau_1 = \tau_0\omega.$$

Moreover, since $\bar{\mu}$ commutes with the $\varphi(Q_i, f_i, \sigma_i)$ over $X^{\circledast}$, the construction of these maps implies that

$$\sigma_1 = \sigma_0\omega.$$

Consequently $[\tau_1, \sigma_1] = [\tau_0, \sigma_0]\omega$, as was to be shown.

Note that since $(X_1, B_1, X_1^{\circledast}) \approx (X, B, X^{\circledast})$, all the bundles extend over X and one can restate the Theorem by dropping all the subscripts "1". We shall do this for our further remarks.

This Theorem takes a somewhat simpler form when $\mathcal{P}$ is trivial. In this case $\mathcal{R}$ and $\mathcal{N}$ are also trivial $S\backslash N$ and N-bundles respectively. In this case we shall denote $\pi_0 \Gamma_{\tau^{\circledast}}(\mathcal{R}|B, \mathcal{N}|X^{\circledast})$ by

$$[B, X^{\circledast}; S\backslash N, N]_{\tau^{\circledast}},$$

which thus stands for the homotopy classes $[\tau,\sigma]$ of pairs $\tau: B \longrightarrow S\setminus N$ and $\sigma: X^{\circledast} \longrightarrow N$ such that $\tau = \tau^{\circledast}\sigma$ on $B^{\circledast} = B \cap X^{\circledast}$, (where $\tau^{\circledast}: B^{\circledast} \longrightarrow S\setminus N$ is given and to which $Y^{\circledast}$ over $X^{\circledast}$ is associated via the classification theorem in the absolute case).

Also $\pi_0\Gamma(\mathcal{N}) = [X,N]$, which acts on $[B,X^{\circledast}; S\setminus N, N]_{\tau^{\circledast}}$ via the right translation actions of N on $S\setminus N$ and N. Thus we have the following corollary.

2.2 Corollary. *If the bundle* ρ *of principal orbits is trivial, then the correspondence of* 2.1 *is with*

$$[B, X^{\circledast}; S\setminus N, N]_{\tau^{\circledast}} / [X,N].$$

The main case of interest is that for which X is contractible. In this case ρ will be trivial, so that 2.2 applies, and $[X,N] = \pi_0(N)$, the group of arc components of N.

Remark 4. Although we have kept the singular isotropy group K fixed throughout this section, there is no difficulty in generalizing the Theorem to the case in which K (and β) is allowed to vary over the components of B. Thus this result, and the absolute case, can be regarded as classification theorems for general G-manifolds over the *top two strata* of their orbit spaces; see [1;IV.6.3]. This remark surely indicates the general importance of these theorems. It is remarkable that the classification, *given* a smooth structure on the orbit space, shows no difference between the differentiable and topological cases on the top two strata. It is only with the third strata level that differentiability begins to make itself felt. (In these remarks we regard a singular stratum which occurs as in [1;IV.6.3(ii)] to be at the third level or below. We decline to make the meaning of "level" precise here.)

We shall conclude this section with a discussion of the naturality properties of the classification theorem. For simplicity we shall limit this discussion to the case in which the bundle ρ of principal orbits is <u>trivial</u>, but the general case can be treated easily in the same way.

Suppose we are given another triple $(\widetilde{X},\widetilde{X}^{\bullet},\widetilde{B})$ and a map

$$k: \widetilde{X} \longrightarrow X$$

taking $\widetilde{X}^{\bullet}$ (at least) to $X^{\bullet}$, $\widetilde{B}$ to B, and with $k^{-1}B = \widetilde{B}$. We assume, moreover, that k is transverse regular on B; that is, it takes some collar of $\widetilde{B}$ in $\widetilde{X}$ by a collar map to a collar of B in X.

Assume we are also given a proper G-manifold $\widetilde{Y}^{\bullet}$ over $\widetilde{X}^{\bullet}$ and a smooth equivariant map $\widetilde{Y}^{\bullet} \longrightarrow Y^{\bullet}$ such that the diagram

$$\begin{array}{ccc} \widetilde{Y}^{\bullet} & \longrightarrow & Y^{\bullet} \\ \downarrow \mu & & \downarrow \mu \\ \widetilde{X}^{\bullet} & \xrightarrow{k} & X^{\bullet} \end{array}$$

commutes. Then there is clearly a topological equivalence of $\widetilde{Y}^{\bullet}$ with the pull-back

$$k^{*}Y^{\bullet} = \left\{(x,y) \in \widetilde{X}^{\bullet} \times Y^{\bullet} \;\middle|\; k(x) = \mu(y)\right\}.$$

This pull-back is easily seen to be a <u>smooth</u> submanifold of $\widetilde{X}^{\bullet} \times Y^{\bullet}$. (Compare section 5 where more complicated pull-backs are discussed in more detail.) Since $\widetilde{Y}^{\bullet}$ is topologically equivalent over $\widetilde{X}^{\bullet}$ to $k^{*}Y^{\bullet}$, the classification theorem in the absolute case [1;VI.6.2] implies that $\widetilde{Y}^{\bullet}$ is <u>smoothly</u> equivalent over $\widetilde{X}^{\bullet}$ to $k^{*}Y^{\bullet}$. Without loss of generality, then, we may as well assume that $\widetilde{Y}^{\bullet} = k^{*}Y^{\bullet}$.

If (M,φ_M) is a proper G-manifold over X extending $Y^{\circledast}$ then we have the pull-back

$$\begin{array}{ccc} k^*M & \longrightarrow & M \\ \downarrow & & \downarrow \mu_M \\ \tilde{X} & \xrightarrow{\ k\ } & X \end{array}$$

(where, again, k^*M is seen, as in section 5, to be a smooth submanifold of $\tilde{X} \times M$). The part of k^*M over $\tilde{X}^{\circledast}$ is just the pull-back $k^*(M^{\circledast})$, and the given equivalence $\varphi_M: M^{\circledast} \longrightarrow Y^{\circledast}$ over $X^{\circledast}$ induces the equivalence $k^*\varphi_M: k^*M^{\circledast} \longrightarrow k^*Y^{\circledast} = \tilde{Y}^{\circledast}$.

Now, via the absolute classification theorem, $Y^{\circledast}$ is associated with a map $\tau^{\circledast}: B^{\circledast} \longrightarrow S\backslash N$ (assuming ρ is trivialized). It is then clear that the pull-back $k^*Y^{\circledast}$ over $\tilde{X}^{\circledast}$ is associated with the composition $\tau^{\circledast} \circ k: \tilde{B}^{\circledast} \longrightarrow B^{\circledast} \longrightarrow S\backslash N$. (This is, of course, a special case of the following statement.)

2.3 <u>Theorem</u>. <u>In the above situation, suppose that the principal orbit bundle</u> ρ <u>of</u> M <u>is trivial and suppose that</u> (M,φ_M) <u>is associated, via</u> 2.2, <u>to the pair</u> (τ,σ) <u>where</u> $\tau: B \longrightarrow S\backslash N$, $\sigma : X^{\circledast} \longrightarrow N$ <u>are such that</u> $\tau = \tau^{\circledast}\sigma$ <u>on</u> $B^{\circledast}$. <u>Then the pull-back</u> $(k^*M, k^*\varphi_M)$ <u>is associated to</u> $(\tau \circ k, \sigma \circ k)$.

The proof is straightforward and will be omitted.

3. Preliminaries on regular actions.

In order to combine the cases of regular orthogonal, unitary and symplectic actions, we introduce the following notation. Let $d = 1, 2$ or 4. In these three cases, let G_n^d stand, respectively, for O_n, U_n or Sp_n; $n \geqslant 2$. Of particular importance are the groups G_2^d. Let

$$C_2^d = \text{center } G_2^d \quad \text{and} \quad \Gamma_{d+1} = G_2^d / C_2^d.$$

Note that $\Gamma_2 \approx O_2$, $\Gamma_3 \approx SO_3$ and $\Gamma_5 \approx SO_5$ for $d=1,2,4$ respectively; see the appendix.

By a regular G_n^d-manifold $(n \geqslant 2)$ we shall mean a smooth manifold M^{2d+k} with a smooth G_n^d-action such that:

1. The orbit types are G_n^d/G_{n-2}^d, G_n^d/G_{n-1}^d and fixed points.
2. The representation of G_n^d about a fixed point is twice the standard representation plus a trivial k-dimensional real representation.
3. The slice representation of G_{n-1}^d on the normal space to the orbit of a point with isotropy group G_{n-1}^d is one standard representation plus a trivial (k+d+1)-dimensional real representation.
4. The bundle of principal orbits is trivial.

We remark that, in fact, condition 3 follows from 1 and 2. The condition 4 is not really needed to carry out a classification, but will be necessary in order to frame the results in their final geometric form. It is also useful for notational convenience. We are mainly interested in the case in which the orbit space is contractible (since this is true for the most interesting examples), and this implies 4. An assumption that the orbit space is contractible would not, however, simplify the discussion appreciably beyond the simplification already resulting from condition 4.

Consider the fixed point set M^G (where $G = G_n^d$) of a regular G_n^d-manifold M. This has a euclidean invariant tubular neighborhood for which the fiber can be considered as the euclidean space of $n \times 2$-matrices (real, complex, or quaternionic, according as $d = 1,2$ or 4) with the fiber representation being the left multiplication by matrices in G_n^d. The structure group of this bundle is the centralizer of G_n^d in O_{2dn} and some straightforward matrix manipulations show that this centralizer is just G_2^d acting by <u>right</u> multiplication on the $n \times 2$-matrices. Note that, since $G_2^d \approx N(G_{n-2}^d)/G_{n-2}^d$ in G_n^d, G_2^d is also the structure group of the bundle of principal orbits. It is not hard to see, in fact, that the normal bundle (tubular neighborhood) of M^G is <u>associated</u> with a restriction of the principal orbit bundle to a certain copy of M^G in M/G. (A short proof of this in the orthogonal case $d=1$ is given in [1;VI.7.1] and it generalizes directly to the other cases $d=2,4$. An extended proof can also be found in [5].) Since the principal orbit bundle is assumed to be trivial, we conclude that the normal G_n^d-bundle of M^G in M is also trivial. Thus M^G has an invariant tubular neighborhood of the form

$$\theta_M: R^{dn} \times R^{dn} \times M^G \longrightarrow M$$

where G_n^d acts on $R^{2dn} = R^{dn} \times R^{dn}$ via twice the standard representation. Given one such tubular neighborhood, all others which are <u>orthogonally G_n^d-bundle equivalent</u> to it are obtained by operating on it, via the right action of G_2^d on $R^{dn} \times R^{dn}$, by a smooth map $M^G \longrightarrow G_2^d$.

In the appendix we show that the orbit space of G_n^d on $R^{dn} \times R^{dn}$ can be identified with $R^+ \times R^{d+1}$ with a smooth orbit map

$$\pi: R^{dn} \times R^{dn} \longrightarrow R^+ \times R^{d+1}$$

which is equivariant with respect to the given right G_2^d-action on $R^{dn} \times R^{dn}$ and the right G_2^d-action on $R^+ \times R^{d+1}$ given by a specific representation

$$\lambda: G_2^d \longrightarrow O_{d+1}.$$

Moreover, this representation has kernel $C_2^d = \text{center}\, G_2^d$ and image O_2 for $d=1$, SO_3 for $d=2$ and SO_5 for $d=4$. We shall identify the image with $\Gamma_{d+1} = G_2^d/C_2^d$ via this representation.

The explicit homeomorphism

$$\pi^*: \frac{R^{dn} \times R^{dn}}{G_n^d} \xrightarrow{\approx} R^+ \times R^{d+1}$$

given in the appendix can be seen to be a diffeomorphism on the complement of the origin (the fixed point), where the left hand side has the functional structure induced by the orbit map from C^∞ on $R^{dn} \times R^{dn}$. (At the origin, π^* is not a diffeomorphism, and the left hand side probably is really a solid cone based on D^{d+1}.)

The composition

$$\theta_{M^*} = \theta_M^*(\pi^* \times 1)^{-1}: (R^+ \times R^{d+1}) \times M^G \longrightarrow \frac{R^{dn} \times R^{dn}}{G_n^d} \times M^G \longrightarrow M^*$$

is a topological tubular neighborhood of M^G in $M^* = M/G$ and we can define a differentiable structure on its image by demanding that it be a diffeomorphism. Since this structure coincides, on the complement of M^G, with the functional structure on M^* induced from M, these may be amalgamated to give a smooth structure on M^* which makes M^* into a smooth $(k+d+2)$-manifold with boundary. The boundary consists of the singular orbits and the fixed point set M^G is a k-dimensional (codimension $d+1$) smooth submanifold of ∂M^*. (See [1;VI.5] for background on the induced functional structure.)

This structure on M^* definitely depends on the choice of tubular neighborhood of M^G in M. It is also, unfortunately, not natural; that is, an equivariant diffeomorphism $M \longrightarrow M$ need not induce a diffeomorphism $M^* \longrightarrow M^*$. However, the structure is well defined by M up to diffeomorphism, and in fact up to concordance. These facts are the reason that we will have to control the tubular neighborhoods of M^G in our discussion, although they play no essential role in the classification theorems we shall prove. Note that changing the tubular neighborhood via the operation by a smooth map $M^G \longrightarrow G_2^d$ does not affect the differentiable structure on M^*.

Note that since the structure group Γ_{d+1} is connected for $d=2,4$, the normal bundle of M^G in M^* has a <u>canonical</u> (independent of the choice of the tube θ_M of M^G in M) fiber orientation in the unitary and symplectic cases.

We shall let D^{2dn} denote the unit disk in $R^{2dn} = R^{dn} \times R^{dn}$. Note that G_n^d operates on D^{2dn} on the left and G_2^d on the right.

The image $\pi(D^{2dn}) \subset R^+ \times R^{d+1}$ is, according to the appendix, the solid truncated paraboloid

$$D_+^{d+2} = \{(x,y) \in R^+ \times R^{d+1} \mid 0 \leq x \leq 1 - \|y\|^2\}.$$

(Of course, one could change π by a diffeomorphism so that this is taken into the unit half-disk, but the explicit map π of the appendix is sometimes useful.) We put

$$D^{d+1} = \{(x,y) \in R^+ \times R^{d+1} \mid 0 \leq x = 1 - \|y\|^2\}$$

and

$$S^d = \partial D^{d+1} = \{(0,y) \in R^+ \times R^{d+1} \mid \|y\|^2 = 1\}.$$

This notation will be fixed throughout this paper. Thus note that, in particular, D^{d+1} will <u>never</u> be used for $\{(0,y) \in R^+ \times R^{d+1} \mid \|y\|^2 \leq 1\}$. See figure 3.

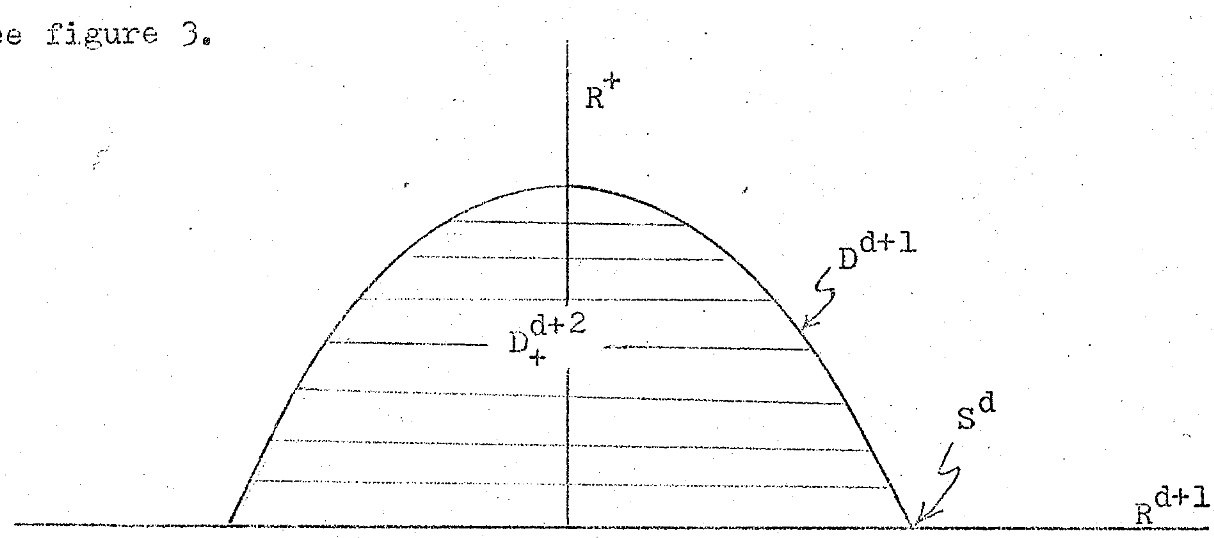

Figure 3.

4. Classification of regular actions over a base manifold.

Suppose we are given a smooth $(k+d+2)$-manifold W with boundary ∂W. Suppose we are given a smooth submanifold $\Sigma^k \subset \partial W$ of codimension $d+1$, and that we are also given a (trivial) tubular neighborhood

$$\theta_W : D_+^{d+2} \times \Sigma^k \longrightarrow W$$

of Σ^k in W. This data

$$\mathcal{W} = (W^{k+d+2}, \Sigma^k, \theta_W)$$

will be kept fixed throughout this section.

By a G_n^d-manifold over $\mathcal{W}$ we shall mean a triple

$$\mathcal{M} = (M^{k+2dn}, \theta_M, \mu_M)$$

where M is a regular G_n^d-manifold, $\theta_M : D^{2dn} \times M^G \longrightarrow M$ is a closed invariant smooth tubular neighborhood of M^G in M, and $\mu_M : M \longrightarrow W$ is an invariant map which induces a diffeomorphism $\mu_M^* : M^* \xrightarrow{\approx} W$ (where θ_M is used to define the smooth structure on M^* as in section 3), such that the following diagram commutes:

$$\begin{array}{ccccc}
D_+^{d+2} \times M^G & \xrightarrow[\approx]{1 \times \mu_M | M^G} & D_+^{d+2} \times \Sigma^k & \xrightarrow[\approx]{\overline{\lambda}} & D_+^{d+2} \times \Sigma^k \\
\downarrow \theta_{M^*} & & & & \downarrow \theta_W \\
M^* & & \xrightarrow[\approx]{\mu_M^*} & & W
\end{array}$$

where $\lambda = \lambda(\mathcal{M})$ is a smooth map $\lambda : \Sigma^k \longrightarrow \Gamma_{d+1}$ and $\overline{\lambda}$ is defined by

$$\overline{\lambda}(x, y) = (x \cdot \lambda(y), y),$$

where $\Gamma_{d+1} \subset O_{d+1}$ is regarded as acting on the right of $D_+^{d+2} \subset R^+ \times R^{d+1}$ in the canonical manner. Note that $\lambda(\mathcal{M})$ is uniquely determined by the other data.

Two such G_n^d-manifolds

$$\mathcal{M} = (M, \theta_M, \mu_M) \quad \text{and} \quad \mathcal{N} = (N, \theta_N, \mu_N)$$

over $\mathcal{W}$ are said to be <u>equivalent over</u> $\mathcal{W}$ if there is an equivariant diffeomorphism

$$\varphi : M \xrightarrow{\approx} N$$

and a smooth map

$$\psi : \Sigma^k \longrightarrow G_2^d$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
D^{2dn} \times \Sigma^k & & \xrightarrow{\quad \bar{\psi} \quad} & & D^{2dn} \times \Sigma^k \\
{\scriptstyle 1 \times \mu_M | M^G} \uparrow \approx & & & & \approx \uparrow {\scriptstyle 1 \times \mu_N | N^G} \\
D^{2dn} \times M^G & & W & & D^{2dn} \times N^G \\
{\scriptstyle \theta_M} \downarrow & \nearrow^{\mu_M} & & \nwarrow^{\mu_N} & \downarrow {\scriptstyle \theta_N} \\
M & & \xrightarrow[\approx]{\quad \varphi \quad} & & N
\end{array}$$

Note then that

$$\lambda(\mathcal{M}) = \psi \lambda(\mathcal{N})$$

(which follows easily from the diagrams on orbit spaces induced by the above diagrams), where the juxtaposition $\psi\lambda$ denotes the projection $G_n^d \longrightarrow \Gamma_{d+1}$ and the product in Γ_{d+1}.

(Note that if $\mathcal{M} = (M, \theta_M, \mu_M)$ and $\mathcal{M}' = (M, \theta'_M, \mu_M)$ are both G^d_n-manifolds over $\mathcal{W}$, then $\lambda(\mathcal{M})^{-1}\lambda(\mathcal{M}') : \Sigma^k \longrightarrow \Gamma_{d+1}$ lifts to a map $\Sigma^k \longrightarrow G^d_2$ and they are equivalent over $\mathcal{W}$.)

In this section, we shall classify these equivalence classes.

Now let $\mathcal{F}(\mathcal{W})$ denote the set of framed cobordism classes (relative to Σ^k) of normally framed manifolds $V^{k+1} \subset \partial W$ with $\Sigma^k = \partial V^{k+1}$ such that, in cases d=2,4 only, the framing of V induces the same fiber orientation of the normal bundle of Σ in W as does the tube θ_W. We shall define below a right operation of the group $[W, G^d_2]$ on $\mathcal{F}(\mathcal{W})$. (In the most interesting case in which W is <u>contractible</u>, this operation is trivial for d=2,4 since G^d_2 is then connected; and, for d=1, it is operation by $[W, O_2] \approx Z_2$ which simply multiplies the framing of V by -1, simultaneously on all components of V.)

Then the main theorem of this section can be stated as follows. Recall that, by assumption, the bundles of principal orbits, of the G^d_2-manifolds we consider, are trival.

4.1 <u>Theorem.</u> <u>There is a one-one correspondence between the set of equivalence classes over</u> $\mathcal{W}$ <u>of</u> G^d_n<u>-manifolds over</u> $\mathcal{W}$ <u>and the set</u>

$$\mathcal{F}(\mathcal{W})/[W, G^d_2]$$

<u>of orbits of the operation of</u> $[W, G^d_2]$ <u>on the set</u> $\mathcal{F}(\mathcal{W})$ <u>of framed cobordism classes of (normally) framed submanifolds of</u> ∂W <u>cobounding</u> Σ <u>(and consistent with the fiber orientation of the normal bundle of</u> Σ <u>in</u> W <u>given by</u> θ_W, <u>for</u> d=2,4).

<u>Proof.</u> We shall regard the given tube

$$\theta_W \colon D_+^{d+2} \times \Sigma^k \longrightarrow W^{k+d+2}$$

as an inclusion. Then let

$$X = W - \mathrm{int}(D_+^{d+2} \times \Sigma^k)$$
$$B = X \cap \partial W$$
$$X^{\circledast} = D^{d+1} \times \Sigma^k$$
$$B^{\circledast} = X^{\circledast} \cap B = S^d \times \Sigma^k.$$

(Recall the notation convention introduced at the end of section 3.) Now it is well known (see [12] for example) that the Thom-Pontriagin construction gives a one-one correspondence of $\mathcal{F}(\mathcal{W})$ with the set

$$[B, \Sigma;\ S^d, \Gamma_{d+1}]$$

of homotopy classes of pairs (τ, σ) where

$$\tau \colon B \longrightarrow S^d$$
$$\sigma \colon \Sigma \longrightarrow \Gamma_{d+1}$$

and

$$\tau(x,y) = x \cdot \sigma(y) \quad \text{for} \quad (x,y) \in S^d \times \Sigma^k = B^{\circledast}.$$

(Here, as always, we regard $\Gamma_{d+1} \subset O_{d+1}$ acting on the right of S^d in the standard fashion.)

Using the fact that Σ is a deformation retract of $X^{\circledast} = D^{d+1} \times \Sigma$, and using the homotopy extension theorem, we see that there is a canonical one-one correspondence between $[B, \Sigma;\ S^d, \Gamma_{d+1}]$ and

$$[B, X^{\circledast};\ S^d, \Gamma_{d+1}]_p ,$$

which stands for the homotopy classes of pairs (τ, σ) where

$$\tau \colon B \longrightarrow S^d$$
$$\sigma \colon X^{\circledast} \longrightarrow \Gamma_{d+1}$$

and

$$\tau = p\sigma \quad \text{on} \quad B^{\circledast} = X^{\circledast} \cap B = S^d \times \Sigma^k$$

where $p: S^d \times \Sigma^k \longrightarrow S^d$ is the projection. (As in section 2, we use the juxtaposition $(p\sigma)$ here to denote the right operation of the set of maps $X^{\circledast} \longrightarrow \Gamma_{d+1}$ on the set of maps $B^{\circledast} \longrightarrow S^d$; etc.)

Now there is, as in section 2, the right operation

$$[B, X^{\circledast};\ S^d, \Gamma_{d+1}]_p \times [X, G_2^d] \longrightarrow [B, X^{\circledast};\ S^d, \Gamma_{d+1}]_p$$

given by $[\tau,\sigma][\omega] \longmapsto [\tau\omega, \sigma\omega]$; via the homomorphism $G_2^d \longrightarrow \Gamma_{d+1}$.

Since X is a deformation retract of W we can use $[X, G_2^d]$ and $[W, G_2^d]$ interchangeably. Through the above one-one correspondences, we finally have defined the required right action of $[W, G_2^d]$ on $\mathcal{F}(\mathcal{W})$. Unfortunately, we do not know how to describe this action in a geometrical way, but it is clear that the case in which W is contractible works as stated above 4.1.

We have reduced the proof of 4.1 to setting up a one-one correspondence between

$$[B, X^{\circledast};\ S^d, \Gamma_{d+1}]_p / [X, G_2^d]$$

and the set of equivalence classes over $\mathcal{W}$ of G_n^d-manifolds over $\mathcal{W}$. To do this we shall relate both of them to a third object, which we now describe.

Let us choose maps

$$\eta_i: X^{\circledast} \longrightarrow \Gamma_{d+1}$$

which are a complete set of representatives for the left cosets of the image of $[X^{\circledast}, G_2^d]$ in $[X^{\circledast}, \Gamma_{d+1}]$ We may assume that η_i

factors through the projection $X^{\circledast} = D^{d+1} \times \Sigma^k \longrightarrow \Sigma^k$ and shall use η_i to also denote the induced map $\Sigma^k \longrightarrow \Gamma_{d+1}$. Then we have the maps

$$p\eta_i : B^{\circledast} \longrightarrow S^d$$

given by $(p\eta_i)(b) = p(b) \cdot \eta_i(b)$ where $p: B^{\circledast} = S^d \times \Sigma^k \longrightarrow S^d$ is the projection, as above.

Consider the disjoint union

$$\Delta = \bigcup_i [B, X^{\circledast}; S^d, G_2^d]_{p\eta_i}$$

and put

$$\Delta' = [B, X^{\circledast}; S^d, \Gamma_{d+1}]_{p}.$$

Right order?

We map $k: \Delta \longrightarrow \Delta'$ by putting $k[\tau,\sigma]_i = [\tau, \eta_i\sigma]$, where the subscript i denotes an element of the i-th part of Δ.

Suppose $[\tau', \sigma'] \in \Delta'$. Then $\sigma' \simeq \eta_i \sigma$ for some $\sigma: X^{\circledast} \longrightarrow G_2^d$ and a unique index i. On $B^{\circledast}$ we have $\tau' = p\sigma' \simeq p\eta_i\sigma$ and this homotopy extends to a homotopy of τ' to $\tau: B \longrightarrow S^d$ with $\tau = p\eta_i\sigma$ on $B^{\circledast}$. Thus $k[\tau,\sigma]_i = [\tau, \eta_i\sigma] = [\tau', \sigma']$, which shows that k is onto.

Now $[X, G_2^d]$ acts on the right of both Δ and Δ' and k is clearly equivariant.

Recalling that $C_2^d = \text{center } G_2^d$ acts trivially on S^d, we have the left action of $[X^{\circledast}, C_2^d]$ on Δ defined by

$$[\gamma][\tau,\sigma]_i = [\tau, \gamma\sigma]_i.$$

Let $[X^{\circledast}, C_2^d]$ act trivially on Δ'. Then k is clearly equivariant for this action also.

By the choice of the η_i it is clear that k cannot take elements of Δ belonging to different values of i to the same element of Δ'.

Thus suppose that $k[\tau,\sigma]_i = k[\tau',\sigma']_i$; that is,

$$[\tau,\eta_i\sigma] = [\tau',\eta_i\sigma'].$$

Without loss of generality we may suppose that σ and σ' are both compositions of maps $\Sigma \longrightarrow G_2^d$ with the projection $X^{\circledast} = D^{d+1} \times \Sigma \longrightarrow \Sigma$. By the covering homotopy theorem we see that we may also assume that $\tau' = \tau$. But on $B^{\circledast} = S^d \times \Sigma$, we have

$$p\eta_i\sigma = \tau = \tau' = p\eta_i\sigma'$$

and hence

$$\sigma'\sigma^{-1}: B^{\circledast} \longrightarrow C_2^d.$$

Since σ,σ' are constant on the fiber D^{d+1} of $X^{\circledast} = D^{d+1} \times \Sigma$, we conclude that $\sigma'\sigma^{-1}: X^{\circledast} \longrightarrow C_2^d$ and hence

$$[\sigma'\sigma^{-1}][\tau,\sigma]_i = [\tau,\sigma']_i = [\tau',\sigma']_i.$$

This shows that two elements of Δ taken by k to the same element of Δ' are in the same orbit of the action by $[X^{\circledast},C_2^d]$.

It follows that k induces a one-one correspondence

$$[X^{\circledast},C_2^d]\backslash\Delta/[X,G_2^d] \xrightarrow{\approx} \Delta'/[X,G_2^d] .$$

It now remains to exhibit a one-one correspondence between

$$[X^{\circledast},C_2^d]\backslash\Delta/[X,G_2^d]$$

and the set of equivalence classes over $\mathcal{W}$ of G_n^d-manifolds over $\mathcal{W}$.

Thus let

$$[\tau,\sigma]_i \in [B,X^{\circledast};\ S^d, G_2^d]_{p\eta_i} \subset \Delta.$$

Consider the composition

$$R^{dn}\times R^{dn}\times \Sigma^k \xrightarrow{\pi} R^+\times R^{d+1}\times\Sigma^k \xrightarrow{\bar{\eta}_i^{-1}} R^+\times R^{d+1}\times\Sigma^k.$$

Restricting this to the unit sphere bundle gives the map

$$Y^{\circledast} = S^{2dn-1}\times\Sigma^k \xrightarrow{\pi} D^{d+1}\times\Sigma^k \xrightarrow{\bar{\eta}_i^{-1}} D^{d+1}\times\Sigma^k$$

which defines a proper G_n^d-manifold $Y^{\circledast}$ over $X^{\circledast} = D^{d+1}\times\Sigma^k$; see section 2. This $Y^{\circledast}$ will remain fixed throughout this section. It follows from 2.3 and the appendix that this $Y^{\circledast}$ over $X^{\circledast}$ corresponds to

$$[\tau^{\circledast}]^* \in [B^{\circledast},\ S^d]/[X^{\circledast}, N]$$

by the absolute version of the classification theorem 2.2, where

$$\tau^{\circledast} = p\eta_i:\ B^{\circledast} = S^d\times\Sigma^k \longrightarrow S^d.$$

(Note that $p\eta_i$ can also be interpreted as the composition

$$S^d\times\Sigma^k \xrightarrow{\bar{\eta}_i} S^d\times\Sigma^k \xrightarrow{p} S^d.)$$

Thus the class

$$[\tau,\sigma]_i \in [B,X^{\circledast};\ S^d, G_2^d]_{\tau^{\circledast}}$$

gives rise, via the classification theorem 2.2, to a proper G_n^d-manifold (T,φ_T) over X and extending $Y^{\circledast}$. Thus we can form

$$M = T \cup_{\varphi_T} (D^{2dn}\times\Sigma^k)$$

and we have the projection $\mu_M: M \longrightarrow W$ given by $\mu_T: T \longrightarrow X$ on T and $\bar{\eta}_i^{-1}\circ\pi$ on $D^{2dn}\times\Sigma^k \longrightarrow D_+^{d+2}\ \Sigma^k$. (Recall that

$\theta_W: D_+^{d+2} \times \Sigma \longrightarrow W$ is regarded as inclusion.) Let $\theta_M: D^{2dn} \times \Sigma \longrightarrow M$ be the inclusion. Then $\mathcal{M} = (M, \theta_M, \mu_M)$ is a G_n^d-manifold over $\mathcal{W}$ and it is easily checked that $\lambda(\mathcal{M}) = \eta_i^{-1}: \Sigma^k \longrightarrow \Gamma_{d+1}$.

Clearly, changing (T, φ_T) by an equivalence over X changes $\mathcal{M}$ only by an equivalence over $\mathcal{W}$. Thus $\mathcal{M}$ depends only on the orbit of $[\tau,\sigma]_i$ under right operation by $[X, G_2^d]$. Let $\gamma: \Sigma \longrightarrow C_2^d$ be a smooth map, and let

$$\bar{\gamma}: D^{2dn} \times \Sigma^k \longrightarrow D^{2dn} \times \Sigma^k$$

be $\bar{\gamma}(x,y) = (x \cdot \gamma(y), y)$ which is an equivalence commuting with the orbit map. Then the identity on T and $\bar{\gamma}$ on $D^{2dn} \times \Sigma^k$ give an equivalence

$$T \bigcup_{\varphi_T} (D^{2dn} \times \Sigma^k) \xrightarrow{\approx} T \bigcup_{\varphi_T \circ \bar{\gamma}} (D^{2dn} \times \Sigma^k)$$

over $\mathcal{W}$. However, the proper G_n^d-manifold $(T, \varphi_T \circ \bar{\gamma})$ over X and extending γ^* corresponds to

$$[\tau, \gamma^{-1}\sigma]_i = [\gamma^{-1}][\tau,\sigma]_i \in [B, X^*; S^d, G_2^d]_{p\eta_i}$$

according to Remark 3 of section 2. This shows that $\mathcal{M}$ depends only on the orbit

$$^*[\tau,\sigma]_i^* \in [X^*, C_2^d] \backslash \Delta / [X, G_2^d].$$

Now suppose that $[\tau', \sigma']_j \in \Delta$ gives rise in the same way to $\mathcal{M}' = (M', \theta_{M'}, \mu_{M'})$ with

$$M' = T' \bigcup_{\varphi_{T'}} (D^{2dn} \times \Sigma^k),$$

etc. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are equivalent over $\mathcal{W}$ via $\varphi: M \longrightarrow M'$ and $\psi: \Sigma^k \longrightarrow G_2^d$. As noted at the beginning of this section, this implies that $\lambda(\mathcal{M}) = \psi\lambda(\mathcal{M}')$ and hence that $\eta_i^{-1} = \psi\eta_j^{-1}$; that is, $\eta_i = \eta_j\psi^{-1}$. But the choice of the η_i then implies that $i = j$ and the equation $\eta_i = \eta_i\psi^{-1}$ implies that $\psi\colon \Sigma^k \longrightarrow C_2^d = \text{center } G_2^d$. The diagram

$$\begin{array}{ccc} D^{2dn} \times \Sigma^k & \xrightarrow{\bar{\psi}} & D^{2dn} \times \Sigma^k \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & M' \end{array}$$

then shows that $(T, \varphi_T \circ \bar{\psi})$ is equivalent over X to $(T', \varphi_{T'})$ and hence that

$$[\psi^{-1}][\tau,\sigma]_i^* = [\tau, \psi^{-1}\sigma]_i^* = [\tau', \sigma']_i^* ,$$

whence

$${}^*[\tau,\sigma]_i^* = {}^*[\tau',\sigma']_i^* .$$

Finally, let $\mathcal{M} = (M, \theta_M, \mu_M)$ be an arbitrary G_n^d-manifold over $\mathcal{W}$. It remains to show that $\mathcal{M}$ can be constructed, up to equivalence over $\mathcal{W}$, as above.

The map $\lambda = \lambda(\mathcal{M})\colon \Sigma^k \longrightarrow \Gamma_{d+1}$ is homotopic via $\lambda_t\colon \Sigma \longrightarrow \Gamma_{d+1}$ to $\lambda_1 = \psi\eta_i^{-1}$ for a unique i and some map $\psi\colon \Sigma^k \longrightarrow G_2^d$ (by the choice of the η_i). Then the homotopy $\lambda\lambda_t^{-1}$ can be lifted to a homotopy $\psi_t\colon \Sigma^k \longrightarrow G_2^d$ starting at the identity. Thus ψ_1 covers $\lambda\lambda_1^{-1} = \lambda\eta_i\psi^{-1}$ and hence $\lambda(\mathcal{M}) = (\psi_1\psi)\,\eta_i^{-1}$. As remarked at the beginning of this sectinn, this implies that (up to equivalence over $\mathcal{W}$) we may <u>assume</u> that $\lambda(\mathcal{M}) = \eta_i^{-1}$.

Then let T be that part of M over X and $T^\bullet = \partial T$, that over $X^\bullet = D^{d+1} \times \Sigma^k$. Define $\varphi_T\colon T^\bullet \longrightarrow \mathcal{Y}^\bullet = S^{2dn-1} \times \Sigma^k$ to be the composition

$$T^{\circledast} \xrightarrow{\theta_M^{-1}} S^{2dn-1} \times_M G \xrightarrow{1 \times \mu_M|_M G} S^{2dn-1} \times \Sigma^k = Y^{\circledast}.$$

Then (T, φ_T) is a proper G_n^d-manifold over X extending $Y^{\circledast}$.
The map

$$T \cup_{\varphi_T} (D^{2dn} \times \Sigma^k) \xrightarrow{\ \varphi\ } M$$

which is the inclusion on T and is the composition

$$D^{2dn} \times \Sigma^k \xrightarrow{(1 \times \mu_M|_M G)^{-1}} D^{2dn} \times_M G \xrightarrow{\theta_M} M$$

on $D^{2dn} \times \Sigma^k$, together with the trivial map $\psi: \Sigma^k \longrightarrow G_2^d$, then gives an equivalence over $\mathcal{W}$, as desired. (The reader may check that φ commutes with the maps to W.)

Let us now note the following addendum to Theorem 4.1. Let $\mathcal{M} = (M, \theta_M, \mu_M)$ be a G_n^d-manifold over $\mathcal{W}$. Let $f: (W, \Sigma) \longrightarrow (W, \Sigma)$ be a diffeomorphism that is an orthogonal bundle isomorphism on the tubular neighborhood θ_M (and preserving fiber orientation for $d=2,4$); that is, there is a smooth map $\lambda_f: \Sigma^k \longrightarrow \Gamma_{d+1}$ such that the diagram

$$\begin{array}{ccccc}
D_+^{d+1} \times \Sigma^k & \xrightarrow{1 \times f|\Sigma} & D_+^{d+1} \times \Sigma^k & \xrightarrow{\bar{\lambda}_f} & D_+^{d+1} \times \Sigma^k \\
\downarrow \theta_W & & & & \downarrow \theta_W \\
W & & \xrightarrow{\quad f \quad} & & W
\end{array}$$

commutes. Then $\mathcal{M}_f = (M, \theta_M, f \circ \mu_M)$ is a G_n^d-manifold over $\mathcal{W}$. (Note that $\lambda(\mathcal{M}_f) = (\lambda(\mathcal{M}) \circ f^{-1}) \lambda_f$.)

4.2 <u>Proposition.</u> *If $\mathcal{M} = (M, \theta_M, \mu_M)$ corresponds to a framed manifold V_M^{k+1} cobounding Σ^k via 4.1, then $\mathcal{M}_f = (M, \theta_M, f \circ \mu_M)$ corresponds to $f(V_M)$.*

<u>Proof.</u> This fact follows from the naturality of all of our constructions and from 2.3. The details are straightforward and will be omitted. It is most easily viewed as a pull-back

$$\begin{array}{ccc} M & \xrightarrow{\ 1\ } & M \\ {\scriptstyle \mu_M}\downarrow & & \\ W & & \downarrow {\scriptstyle \mu_M} \\ {\scriptstyle f}\downarrow & & \\ W & \xrightarrow{\ f^{-1}\ } & W \end{array}$$

and the result generalizes to appropriate pull-backs in general; see section 5.

We now turn to the special case in which W^{k+d+2} is acyclic and Σ^k is a homology sphere. The following result also holds in the orthogonal case d=1, but will be excluded since it is a very special case of stronger known results which will be treated separately in 4.4 and 4.5.

4.3 <u>Corollary.</u> *Consider the unitary and symplectic cases $d = 2, 4$. Assume that W^{k+d+2} is acyclic and that Σ^k is a homology sphere which bounds a framed manifold in ∂W. Then the equivalence classes over $\mathcal{W}$ of G_n^d-manifolds over $\mathcal{W}$ are in one-one correspondence with*

$$\pi_{k+1}(G_{d+1}, SO_{d+1}),$$

where G_{d+1} is the space of maps $S^d \longrightarrow S^d$ of degree one; see [6,12].

<u>Proof</u>. Note that $\Gamma_{d+1} = SO_{d+1}$ for $d = 2,4$. Let $R_+^{k+d+2} \subset W^{k+d+2}$ be the interior of a disk neighborhood of a point in Σ^k with $R_+^{k+d+2} \cap \Sigma^k = R^k$ (standardly embedded in R_+^{k+d+2}). We may assume that this intersects the given tubular neighborhood of Σ^k nicely:

$$R_+^{k+d+2} \cap (D_+^{d+2} \times \Sigma^k) = D_+^{d+2} \times R^k.$$

Suppose that the given framed manifold V^{k+1} cobounding Σ^k is represented by

$$\bar{\tau}: B \longrightarrow S^d, \qquad \bar{\sigma}: \Sigma \longrightarrow SO_{d+1}$$

(with $\bar{\tau} = p\bar{\sigma}$ on $B^{\bullet} = S^d \times \Sigma^k$). We may assume that V^{k+1} intersects $R^{k+d+1} = R_+^{k+d+2} \cap \partial W$ in a linear half space with standard framing. (That is, on $R^{k+d+1} \cap B = R^{k+d+1} - (D^{d+1} \times R^k)$, $\bar{\tau}$ is projection to the equator S^d.)

Let $\tau: B \longrightarrow S^d$, $\sigma: \Sigma \longrightarrow SO_{d+1}$ represent some other class $[\tau,\sigma] \in [B,\Sigma; S^d, SO_{d+1}]_p$. Since $\Sigma^k - R^k$ is acyclic, obstruction theory implies that σ is homotopic to a map which equals $\bar{\sigma}$ on $\Sigma^k - R^k$. Suppose that (τ_0,σ_0) and (τ_1,σ_1) are such maps with $[\tau_0,\sigma_0] = [\tau_1,\sigma_1]$ via the homotopy (τ_t,σ_t). Then $\sigma_t^{-1}\sigma_0 | \Sigma^k - R^k$ is a homotopy between the constant maps to $e \in SO_{d+1}$. Again, obstruction theory implies that this homotopy (and hence (τ_t,σ_t)) can be changed so that $\{\sigma_t^{-1}\sigma_0\}: (\Sigma^k - R^k) \times I \longrightarrow SO_{d+1}$ is given by $(x,t) \longmapsto g_t \in SO_{d+1}$ (independent of x), with $g_0 = e = g_1$. Then $(\tau_t', \sigma_t') = (\tau_t g_t, \sigma_t g_t)$ is a homotopy between (τ_0,σ_0) and (τ_1,σ_1) such that $\sigma_t' = \bar{\sigma}$ on $\Sigma^k - R^k$.

Thus we may as well consider only maps (τ,σ), and homotopies between them, such that $\sigma = \bar{\sigma}$ on $\Sigma^k - R^k$, and hence $\tau = \bar{\tau}$ on $S^d \times (\Sigma^k - R^k) \subset S^d \times \Sigma^k = B^{\bullet}$.

Now the inclusions

$$S^d \longrightarrow S^d \times (\Sigma^k - R^k) \longrightarrow B - R^{k+d+1}$$

induce isomorphisms in cohomology by Alexander duality. (Note that $B - R^{k+d+1} \longrightarrow B$ is a homotopy equivalence.) Thus $(B - R^{k+d+1}, S^d \times (\Sigma^k - R^k))$ has trivial cohomology, and it follows from obstruction theory and the homotopy extension theorem that we may restrict our attention to maps (τ,σ), and homotopies of them, such that

$$\tau = \bar{\tau} \text{ on } B - R^{k+d+1}$$
$$\sigma = \bar{\sigma} \text{ on } \Sigma^k - R^k.$$

Since, by choice, $\bar{\tau}$ and $\bar{\sigma}$ are standard on the boundary of the coordinate neighborhood R^{k+d+1} in W, it follows that the homotopy classes of such pairs are in one-one correspondence with the same thing for the standard sphere $S^k \subset S^{k+d+1}$. That is, we have a one-one correspondence with

$$[S^d \times D^{k+1}, S^k; S^d, SO_{d+1}]_p = \pi_{k+1}(G_{d+1}, SO_{d+1}),$$

as claimed.

Remark. Note that the proof of 4.3 shows that given one G_n^d-manifold over $\mathcal{W}$ (as in 4.3) corresponding to a framed manifold $V^{k+1} \subset \partial W$, then all other G_n^d-manifolds over $\mathcal{W}$ correspond to framed manifolds which coincide with V^{k+1} outside a given coordinate neighborhood in ∂W, in which V^{k+1} is a linear subspace with standard framing.

For completeness, we shall end this section with a discussion of the implications of the main Theorem 4.1 for the orthogonal case $d=1$. Because of the very fortuitous fact that, in the orthogonal case, the attachment of a tubular neighborhood of the fixed point set to its complement is essentially unique, these results can be derived from the absolute classification theorem for two orbit types, and hence these results have been known for some time; see [1,10]. However, it takes no effort to deduce them from 4.1.

4.4 Corollary [10]. The set of equivalence classes over $\mathcal{W} = (W^{k+3}, \Sigma^k, \theta_W)$ of O_n-manifolds over $\mathcal{W}$ is in one-one correspondence with

$$\frac{H^1(\partial W - \Sigma;\ Z)_\Sigma}{2j^*H^1(W;\ Z)} \Big/ Z_2$$

where $H^1(\partial W - \Sigma;\ Z)_\Sigma$ denotes the subset of $H^1(\partial W - \Sigma;\ Z)$ consisting of elements whose restriction to each unit circle S^1 in a normal plane to Σ in ∂W is a generator (± 1) of $H^1(S^1;\ Z)$; j^* is induced by inclusion $\partial W - \Sigma \subset W$; and Z_2 acts via the automorphism -1 of the coefficients Z.

Proof. Suppose that $\tau: B \longrightarrow S^1$ is a map whose restriction to $S^1 \times \{x\} \longrightarrow S^1$ has degree ± 1 for each point $x \in \Sigma^k$. Let Σ_0 be a component of Σ^k and consider the restriction $\tau: S^1 \times \Sigma_0 \longrightarrow S^1$. If $\iota \in H^1(S^1;\ Z)$ is the canonical generator, then in

$$H^1(S^1 \times \Sigma_0;\ Z) \approx H^1(S^1;\ Z) \oplus H^1(\Sigma_0;\ Z)$$

we have

$$\tau^*(\iota) = (\pm\iota, \alpha).$$

For a map $\sigma_0 : \Sigma_0 \longrightarrow O_2$ consider the map $p\sigma_0 : S^1 \times \Sigma_0 \longrightarrow S^1$ given by $(x,y) \longmapsto x \cdot \sigma_0(y)$. It is a routine matter to compute what this does to $H^1(S^1 \times \Sigma_0; Z) = H^1(S^1; Z) \oplus H^1(\Sigma_0; Z)$ and one sees from this that there is a unique $[\sigma_0] \in [\Sigma_0, O_2]$ such that $(p\sigma_0)^*(\iota) = (\pm \iota, \alpha) = \tau^*(\iota)$. Thus there is a map $\sigma : \Sigma \longrightarrow O_2$, unique up to homotopy, with $p\sigma \simeq \tau$ on $B^0 = S^1 \times \Sigma$. Up to homotopy we may suppose that $p\sigma = \tau$. Moreover, when τ can be written in the form $p\sigma$ on $S^1 \times \Sigma$, it is clear that the equation $\tau = p\sigma$ determines σ uniquely. This shows that the map $[\tau, \sigma] \longmapsto [\tau]$ of

$$[B, \Sigma;\ S^1, O_2]_p \longrightarrow [B, S^1]$$

is a one-one correspondence to the subset of those elements $[\tau] \in [B, S^1] = H^1(B; Z)$ having degree ± 1 on each $S^1 \times \{x\}$ for $x \in \Sigma$. Recalling that $G_2^1 \longrightarrow \Gamma_2$ is a double covering $O_2 \longrightarrow O_2$, it is routine to check that the operation by $[W, G_2^1]$ on $[B, \Sigma;\ S^1, O_2]_p$ corresponds to addition by elements of $2j^*H^1(W; Z)$ and possible change of sign.

4.5 Corollary. If, in 4.4, we have $H^1(\partial W; Z) = 0 = H^2(\partial W; Z)$, then there is a one-one correspondence between the equivalence classes over $\mathcal{W}$ of O_n-manifolds over $\mathcal{W}$ and the set of choices of orientations of Σ^k modulo simultaneous change of orientation on all components of Σ^k. Given an orientation of Σ^k, the corresponding O_n-manifold is that one associated (via 4.1) to a framed manifold $V^{k+1} \subset \partial W$ cobounding Σ^k which is consistent with the given orientation of Σ^k.

Proof. Note that ∂W, and hence Σ, is orientable since $H^1(\partial W; Z_2) = 0$. Via the exact sequence

$$0 = H^1(\partial W) \longrightarrow H^1(\partial W - (R^2 \times \Sigma^k)) \xrightarrow{\partial^*} H^2(\partial W, \partial W - (R^2 \times \Sigma^k)) \longrightarrow H^2(\partial W) = 0$$

and the excision and Thom isomorphisms

$$H^2(\partial W, \partial W - (R^2 \times \Sigma^k)) \approx H^2((D^2, S^1) \times \Sigma^k) \approx H^0(\Sigma^k; Z)$$

we see that $H^1(\partial W - (R^2 \times \Sigma^k; Z)_\Sigma$ corresponds to the set of elements of $H^0(\Sigma^k; Z)$ which are ± 1 on each component of Σ^k. The operation by $2j^*H^1(W;Z)$ is trivial since $H^1(\partial W;Z) = 0$, and Z_2 operates by simultaneous change of sign. The actual signs ± 1 depend on the choice of the original tubular neighborhood $\theta_W: R^2 \times \Sigma^k \longrightarrow \partial W$. For a given $[\tau,\sigma] \in [B,\Sigma; S^1, O_2]_p$ it is clear that the sign for a component Σ_0 of Σ is $+1$ or -1 corresponding to whether $\sigma: \Sigma^k \longrightarrow O_2$ takes Σ_0 into SO_2 or not. Choose an orientation for ∂W, and orient Σ_0 by demanding that $\theta_W: R^2 \times \Sigma_0^k \longrightarrow \partial W$ preserve or reverse orientation according as the above sign is $+1$ or -1. This orientation of Σ^k is then clearly consistent with that of a framed manifold cobounding Σ^k corresponding to $[\tau,\sigma]$, and the result follows.

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5. The pull-back construction.

In this section we shall give a useful explicit construction of the G_n^d-manifold over $\mathcal{W} = (W^{k+d+2}, \Sigma^k, \theta_W)$ corresponding to a given framed manifold cobounding Σ^k; that is, to a given pair of maps τ, σ representing an element of $[B, \Sigma; S^d, \Gamma_{d+1}]_p$. We shall apply this construction to identify the restriction of an O_{2n}-action to a U_n-action and the restriction of a U_{2n}-action to an Sp_n-action in terms of the correspondence with framed manifolds. As in section 4, we shall regard the given tube $\theta_W: D_+^{d+2} \times \Sigma^k \to W$ as an inclusion.

Suppose we are given

$$\tau: B \longrightarrow S^d$$

$$\sigma: \Sigma \longrightarrow \Gamma_{d+1}$$

with

$$\tau(x,y) = x \cdot \sigma(y) \quad \text{for} \quad (x,y) \in S^d \times \Sigma^k = B^{\circ}.$$

We shall first extend τ to a smooth map $\bar{\tau}: W \longrightarrow D_+^{d+2} \subset R_+^{d+2}$ in a particular way. To do this, define

$$\bar{\tau}: D_+^{d+2} \times \Sigma^k \longrightarrow D_+^{d+2}$$

to be $\bar{\tau}(x,y) = \text{stretch}(x) \cdot \sigma(y)$ where

$$\text{stretch}: D_+^{d+2} \longrightarrow D_+^{d+2}.$$

is as defined in the appendix. (It is smooth, Γ_{d+1}-equivariant, the identity near 0, and collapses a collar of D^{d+1} to D^{d+1}.) Then we extend this to a collar $[0,\epsilon] \times B$ of the pair $(B, B^{\circ}) = (B, S^d \times \Sigma^k)$ in $(X, X^{\circ}) = (X, D^{d+1} \times \Sigma^k)$ by means of a collar map:

$$\bar{\tau} = 1 \times \tau: [0,\epsilon] \times B \longrightarrow [0,\epsilon] \times S^d \subset D^{d+1} \subset D_+^{d+2}.$$

(See the end of section 3.) Then we can clearly extend this to the remainder of X (i.e., the remainder of W) taking this remainder into the complement of the collar $[0,\varepsilon] \times S^d$ in D^{d+1}. (This can be done in an explicit manner, or by application of the Tietze extension theorem and smoothing.) This completes the description of the map $\bar{\tau}: W \longrightarrow D_+^{d+2}$ extending $\tau: B \longrightarrow S^d$.

Recall the map (constructed explicitly in the appendix) $\pi: R^{2dn} \longrightarrow R^+ \times R^{d+1}$ which is regarded as the orbit map of twice the standard representation of G_n^d on R^{2dn}. We define

$$M = \left\{(w,x) \in W \times R^{2dn} \;\middle|\; \bar{\tau}(w) = \pi(x)\right\};$$

that is, M is defined by the pull-back diagram

$$\begin{array}{ccc} M & \longrightarrow & R^{2dn} \\ \mu_M \big\downarrow & & \big\downarrow \pi \\ W & \xrightarrow{\ \bar{\tau}\ } & R^+ \times R^{d+1}. \end{array}$$

The representation of G_n^d on R^{2dn} induces a G_n^d-action on M. Since the image of $\bar{\tau}$ is in D_+^{d+2} we may also consider this as the pull-back diagram

$$\begin{array}{ccc} M & \longrightarrow & D^{2dn} \\ \mu_M \big\downarrow & & \big\downarrow \pi \\ W & \xrightarrow{\ \bar{\tau}\ } & D_+^{d+2} \end{array}$$

We have the map

$$\theta_M: D^{2dn} \times \Sigma^k \longrightarrow M$$

given by

$$\theta_M(x,y) = ((\pi(x)\sigma(y)^{-1}, y),\ \text{stretch}(x)) \in (D_+^{d+2} \times \Sigma^k) \times R^{2dn} \subset W \times R^{2dn},$$

where "stretch" is defined in the appendix and satisfies $\pi(\text{stretch}(x)) = \text{stretch}(\pi(x))$.

5.1 Theorem. *With the functional structure on* M *induced from that of* $W \times R^{2dn}$, (M, θ_M, μ_M) *is a smooth* G^d_n*-manifold over* $\mathcal{W}$ *and it corresponds, via* 4.1, *to the given pair* (τ, σ). (*That is, it corresponds to the framed manifold* $V^{k+1} = \bar{\tau}^{-1}(R^+ z)$ *where* $z \in S^d$ *is a regular value of* $\tau: B \longrightarrow S^d$ *and the ray* $R^+ z$ *is given the standard normal framing in* $R^+ \times R^{d+1}$.)

Proof. We must show that M is a smooth submanifold of $W \times R^{2dn}$. (Note that M will be tangent to the boundary $\partial W \times R^{2dn}$ of $W \times R^{2dn}$ where they meet. If questions about the boundary worry the reader, he should note that $\bar{\tau}$ extends to an exterior collar of W to an exterior collar of $R^+ \times R^{d+1}$ in $R \times R^{d+1} = R^{d+2}$.) To show this, consider the smooth map

$$\varphi: W \times R^{2dn} \longrightarrow R^{d+2}$$

given by

$$\varphi(w,x) = \pi(x) - \bar{\tau}(w).$$

We claim that the origin $0 \in R^{d+2}$ is a regular value of φ. The verification of this breaks up into the following three cases. Let $\varphi(w,x) = 0$; so that $\pi(x) = \bar{\tau}(w)$.

(i) If $\bar{\tau}(w) = 0$, then $w \in \Sigma^k$ and, by fixing x and varying w we see that the differential of φ at (w,x) is onto since that of $\bar{\tau}$ at w is onto.

(ii) If $\bar{\tau}(w) \notin \{0\} \times R^{d+1}$ then $\pi(\overset{x}{w}) \notin \{0\} \times R^{d+1}$ and hence the differential of π at $\overset{x}{w}$ is onto, and thus so is that of φ.

(iii) If $0 \neq \bar{\tau}(w) \in \{0\} \times R^{d+1}$ then, by fixing x and varying w, we see that the image of the differential of φ contains a non-zero normal vector to $\{0\} \times R^{d+1}$ in $R \times R^{d+1}$ and, by fixing w and varying x, we see (since $\pi(w) \neq 0$) that it also contains the tangent space to $\{0\} \times R^{d+1}$.

Note that this also shows that M has a trivial normal bundle in $W \times R^{2dn}$.

Since the composition

$$D^{2dn} \times \Sigma^k \xrightarrow{\theta_M} M \longrightarrow W \times R^{2dn}$$

is clearly an embedding, θ_M is an embedding. It is then routine to verify that (M, θ_M, μ_M) is a G_n^d-manifold over $\mathcal{W}$. The fact that it corresponds to (τ, σ) is a straightforward, if long, check through the proof of the classification theorem 4.1 using 2.3, and will be omitted. (It is simply a consequence of the naturality of the invariants in the classification theorem.) It is clear that the corresponding framed manifold V^{k+1} is as described, once one makes the easy check that $\bar{\tau}$ is transverse regular on $R^+ z$.

We shall now study the restriction of U_{2n}-actions to Sp_n and of O_{2n}-actions to U_n.

Consider the maps

$$R^+ \times R^5 \xrightarrow{\omega_{2,4}} R^+ \times R^3 \xrightarrow{\omega_{1,2}} R^+ \times R^2$$

defined by

$$\omega_{1,2}(x, a, b, c) = (x + c^2, a, b)$$

$$\omega_{2,4}(x, a, b, c, d, e) = (x + d^2 + e^2, a, b, c).$$

The diagrams

$$\begin{array}{ccc} & R^{8n} & \\ {}^{\pi_4}\swarrow & & \searrow^{\pi_2} \\ R^+ \times R^5 & \xrightarrow{\omega_{2,4}} & R^+ \times R^3 \end{array} \qquad\qquad \begin{array}{ccc} & R^{4n} & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ R^+ \times R^3 & \xrightarrow{\omega_{1,2}} & R^+ \times R^2 \end{array}$$

then commute (see the appendix) where π_d denotes the orbit map for twice the standard representation of G_n^d or G_{2n}^d. Note that these maps restrict to

$$D_+^6 \xrightarrow{\omega_{2,4}} D_+^4 \xrightarrow{\omega_{1,2}} D_+^3 .$$

Also let $\omega_1 \colon R^+ \times R \longrightarrow R^+$ and $\omega_2 \colon R^+ \times R^2 \longrightarrow R^+$ be defined by

$$\omega_1(x,c) = x + c^2, \qquad \omega_2(x,d,e) = x + d^2 + e^2.$$

Then the diagrams

$$\begin{array}{ccc} R^+ \times R^3 & \longrightarrow & R^+ \times R \\ \downarrow{\scriptstyle \omega_{1,2}} & & \downarrow{\scriptstyle \omega_1} \\ R^+ \times R^2 & \longrightarrow & R^+ \end{array} \qquad\qquad \begin{array}{ccc} R^+ \times R^5 & \longrightarrow & R^+ \times R^2 \\ \downarrow{\scriptstyle \omega_{2,4}} & & \downarrow{\scriptstyle \omega_2} \\ R^+ \times R^3 & \longrightarrow & R^+ \end{array}$$

are pull-back diagrams where the horizontal maps are projections. (Explicitly, the maps on top are $(x,a,b,c) \longmapsto (x,c)$ and $(x,a,b,c,d,e) \longmapsto (x,d,e)$.)

Now let $\mathcal{W} = (W^{k+3}, \Sigma^k, \theta_W)$, $\sigma \colon \Sigma^k \longrightarrow O_2$, and $\bar{\tau} \colon W^{k+3} \longrightarrow R^+ \times R^2$ be as above, and consider the diagram

$$\begin{array}{ccccc} M^{4n+k} & \longrightarrow & R^{4n} & & \\ \downarrow{\scriptstyle \mu'_M} & & \downarrow{\scriptstyle \pi_2} & & \\ TW & \xrightarrow{\bar{\tau}'} & R^+ \times R^3 & \longrightarrow & R^+ \times R \\ \downarrow & & \downarrow{\scriptstyle \omega_{1,2}} & & \downarrow{\scriptstyle \omega_1} \\ W & \xrightarrow{\bar{\tau}} & R^+ \times R^2 & \longrightarrow & R^+ \end{array}$$

in which M, TW, and $\bar{\tau}'$ are defined so that the squares are pull-backs. Let $\varphi: W \longrightarrow R^+$ denote the composition along the bottom, whose image is $I = [0,1]$. (We note, but will not use, that if more care is taken in the definition of $\bar{\tau}$ in the interior of W, one can do it in a canonical way for which φ is then independent of the particular τ we start with, and thus depends only on $\mathcal{N}$.) Note that φ maps a collar of ∂W in W by a collar map to a collar of $\{0\}$ in $[0,1]$ and maps the rest of W away from $\{0\}$.

The diagram

$$\begin{array}{ccc} TW & \longrightarrow & R^+ \times R \\ \downarrow & & \downarrow{\scriptstyle \omega_1} \\ W & \xrightarrow{\varphi} & R^+ \end{array}$$

is a pull-back diagram (and can be taken as the definition of the "thickening" TW of W). Thus

$$TW = \{(w,x,c) \in W \times R^+ \times R \mid \varphi(w) = x + c^2, \quad x \geq 0\}$$

$$\approx \{(w,c) \in W \times R \mid c^2 \leq \varphi(w)\}$$

and it is clear from this that TW is diffeomorphic to $I \times W$ with the corners straightened. We may regard W as a <u>submanifold</u> of TW via the inclusion $W \longrightarrow TW$ taking $w \longmapsto (w,0) \in W \times R$; that is, setting $c = 0$ above.

Define

$$\theta_{TW}: D^4_+ \times \Sigma^k \longrightarrow TW$$

by

$$\theta_{TW}(x,y) = ((\omega_{1,2}(x),y),\ \text{stretch}(x)\cdot \sigma'(y)) \in W \times D^4_+$$ } This is dir

where

$$\sigma'(y) = \begin{bmatrix} \sigma(y) & 0 \\ 0 & \pm 1 \end{bmatrix} \in SO_3$$

(with sign ± 1 according as $\sigma(y) \in SO_2$ or $\sigma(y) \in O_2 - SO_2$). Regarding θ_{TW} as an inclusion, we then see that $\tau' = p\sigma'$ on $S^2 \times \Sigma^k = D^3 \times \Sigma^k \cap \partial(TW)$ where τ' is the restriction of $\bar{\tau}'$ to $\partial(TW) - \text{int}(D^4_+ \times \Sigma^k)$. Put

$$T\mathcal{W} = (TW, \Sigma^k, \theta_{TW}).$$

Now the outer square of the diagram

$$\begin{array}{ccc} M & \longrightarrow & R^{4n} \\ \downarrow & & \downarrow \pi_2 \\ TW & \xrightarrow{\bar{\tau}'} & R^+ \times R^3 \\ \downarrow & & \downarrow \omega_{1,2} \\ W & \xrightarrow{\bar{\tau}} & R^+ \times R^2 \end{array}$$

is a pull-back and thus M is the O_{2n}-manifold over $\mathcal{W}$ associated with the framed manifold $\bar{\tau}^{-1}(R^+ z)$ where z is a regular value of τ.

Since π_2 is the orbit map for the action of the subgroup $U_n \subset O_{2n}$, we see easily that $\mathcal{M}' = (M, \theta_M, \mu_M')$, with θ_M as defined above via W (or using the analogous construction via TW), is a U_n-manifold over $T\mathcal{W}$ and it clearly corresponds to the pair (τ', σ'). Note that the framed manifold associated with (τ', σ') is simply the framed manifold $V^{k+1} \subset \partial W \subset \partial(TW)$ with the given framing extended by the positive normal field of ∂W in $\partial(TW)$. This framed manifold in $\partial(TW)$ is called the "suspension" of $V^{k+1} \subset \partial W$ in $\partial(TW)$. Thus we have shown the following fact.

5.2 Theorem. *If $\mathcal{M} = (M^{4n+k}, \theta_M, \mu_M)$ is an O_{2n}-manifold over $\mathcal{W} = (W^{k+3}, \Sigma^k, \theta_W)$ associated to the framed manifold V^{k+1} in ∂W cobounding Σ^k, then the restriction of the action to U_n gives a U_n-manifold $\mathcal{M}' = (M, \theta_M, \mu_M')$ over the thickening $T\mathcal{W} = ((TW)^{k+4}, \Sigma^k, \theta_{TW})$ which is associated to the suspension of V^{k+1}.*

Similarly, in the U_{2n} case, consider $\mathcal{W} = (W^{k+4}, \Sigma^k, \theta_W)$, $\sigma: \Sigma^k \longrightarrow SO_3$ and $\bar{\tau}: W^{k+4} \longrightarrow R^+ \times R^3$ given as above and consider the diagram

$$\begin{array}{ccccc}
M^{8n+k} & \longrightarrow & R^{8n} & & \\
\downarrow{\scriptstyle \mu_M'} & & \downarrow{\scriptstyle \pi_4} & & \\
T^2W & \xrightarrow{\bar{\tau}'} & R^+ \times R^5 & \longrightarrow & R^+ \times R^2 \\
\downarrow & & \downarrow{\scriptstyle \omega_{2,4}} & & \downarrow{\scriptstyle \omega_2} \\
W & \xrightarrow{\bar{\tau}} & R^+ \times R^3 & \longrightarrow & R^+ .
\end{array}$$

As above, the "double thickening" T^2W can be defined by the pull-back diagram

$$\begin{array}{ccc} T^2W & \longrightarrow & R^+ \times R^2 \\ \downarrow & & \downarrow \omega_2 \\ W & \xrightarrow{\varphi} & R^+ \end{array}$$

and hence

$$T^2W = \left\{(w,x,d,e) \in W \times R^+ \times R^2 \;\middle|\; \varphi(w) = w + d^2 + e^2, \quad x \geqslant 0\right\}$$

$$\approx \left\{(w,d,e) \in W \times R^2 \;\middle|\; d^2 + e^2 \leqslant \varphi(w)\right\}$$

which is diffeomorphic to $W \times D^2$ with corners straightened. Again $W \subset T^2W$ as the set where $d = e = 0$ and the d, e coordinates define a normal frame of ∂W in $\partial(T^2W)$. This normal frame added to a framed submanifold V^{k+1} of ∂W give V^{k+1} as a framed submanifold of $\partial(T^2W)$ called the "double suspension" of V^{k+1}. Just as in the O_{2n} case, we have the following theorem.

5.3 <u>Theorem</u>. *If $\mathcal{M} = (M^{8n+k}, \theta_M, \mu_M)$ is a U_{2n}-manifold over $\mathcal{W} = (W^{k+4}, \Sigma^k, \theta_W)$ associated to the framed manifold V^{k+1} in ∂W cobounding Σ^k, then the restriction of the action to Sp_n gives an Sp_n-manifold $\mathcal{M}' = (M, \theta_M, \mu'_M)$ over the double thickening $T^2\mathcal{W} = ((T^2W)^{k+6}, \Sigma^k, \theta_{T^2W})$ which is associated to the double suspension of V^{k+1}.*

<u>Remark</u>. One would expect that the pull-back type of construction used here will be very useful in studying other questions and other types of actions. Indeed it has provided an elegant setting for the proof of the results announced in [3]; see [2].

We shall conclude this section by applying Theorem 5.1 to some questions of embeddability. Note that any regular G_n^d-manifold can be given the structure of a G_n^d-manifold over some $\mathcal{W} = (W, \Sigma, \theta_W)$ where, of course, $(W^{k+d+2}, \Sigma^k) \approx (M^*, M^G)$.

If W^{k+d+2} embeds in R^m then M embeds in $R^m \times R^{2dn}$ by the pull-back construction. This proves the following result.

5.4 Theorem. *Let M^{2dn+k} be a regular G_n^d-manifold and assume that the orbit space M^* embeds smoothly in R^m. Then M embeds smoothly and equivariantly in $R^m \times R^{2dn}$ where G_n^d acts trivially on R^m and by twice the standard representation on $R^{2dn} = R^{dn} \times R^{dn}$.*

5.5 Corollary. *Let M be a regular G_n^d-manifold such that M^* is a disk $M^* \approx D^{k+d+2}$. Then M embeds in the representation space $R^{k+d+2} \times R^{2dn}$.*

If $d = 2, 4$, M^G is a homology sphere and M^* is a disk D^{k+d+2}, we shall prove in section 7 that the action extends to a regular O_{dn}-action with orbit space D^{k+3}. Thus, in this case, 5.5 implies that M^{2dn+k} embeds in $R^{k+3} \times R^{2dn}$. Since a contractible manifold (of dimension greater than 4) always embeds with codimension one in euclidean space, these results constitute a substantial improvement and simplification of the embeddability results in [9].

5.6 Proposition. *Let M^{2dn+k} be a regular G_n^d-manifold. Then M is a π-manifold if and only if M^* is a π-manifold.*

<u>Proof.</u> By the proof of 5.1, M is the inverse image of a regular value of a map $M^* \times R^{2dn} \longrightarrow R^{d+2}$ and hence M embeds in $M^* \times R^{2dn}$ with trivial normal bundle. Thus M is a π-manifold if M^* is one. Conversely, note that since $M^* \approx M^* - (\text{collar})$, and since, by assumption, M has trivial principal orbit bundle, there is a codimension zero embedding $M^* \times R^{2dn-(d+2)} \longrightarrow M$. Thus M^* is a π-manifold if M is one.

5.7 <u>Theorem.</u> <u>Let</u> M <u>be a regular</u> G_n^d<u>-manifold such that the orbit space</u> M^* <u>is a</u> π<u>-manifold.</u> <u>Then</u> M <u>is the boundary of a</u> π<u>-manifold</u> $\overline{M}$ <u>with a regular</u> G_n^d<u>-action extending that of</u> M.

<u>Proof.</u> If M is a G_n^d-manifold over $\mathcal{W} = (W^{k+d+2}, \Sigma^k, \theta_W)$ and corresponds to a framed manifold $V^{k+1} \subset \partial W$ then consider the thickening TW. We may also thicken V^{k+1} to $TV \subset \partial(TW)$ (with a canonical extension of the framing). (Note that $\Sigma^{k+1} = \partial(TV)$ is the double of V^{k+1}.) Now there is a regular G_n^d-manifold $\overline{M}'$ over TW corresponding to TV. The inclusion $W \subset TW$ divides TW, and hence $\overline{M}'$ into two pieces. If $\overline{M}$ is the closure of one of these pieces then $\partial \overline{M}$ is that part of $\overline{M}'$ over $W \subset TW$ and it is clearly equivalent to M. Now TW is a π-manifold since W is one and thus $\overline{M}$ is a π-manifold by 5.6.

Note that the fact that M bounds a regular action does not depend on the assumption that M^* is a π-manifold.

6. Classification of regular actions.

In this section we shall consider the classification of regular G_n^d-manifolds up to equivariant diffeomorphism, not necessarily respecting the base manifold (or tubular neighborhoods of fixed point sets).

Let

$$\mathcal{W} = (W^{k+d+2}, \Sigma^k, \theta_W)$$

be as in section 4. In the unitary and symplectic cases $d = 2,4$, the given tubular neighborhood θ_W induces a fiber orientation o_Σ of the normal bundle to Σ^k in ∂W. As noted in section 3, if M^{k+2dn} is a regular G_n^d-manifold, $d = 2,4$, then there is also an induced canonical fiber orientation o_M of the normal bundle of M^G in ∂M^* (defined via any choice of an invariant tubular neighborhood $R^{dn} \times R^{dn} \times M^G \longrightarrow M$ and the induced tubular neighborhood of M^G in M^*). In the orthogonal case $d = 1$, we regard the data o_M and o_Σ as deleted from the statements of all results in this section.

6.1 Theorem. *Suppose that M^{k+2dn} is a regular G_n^d-manifold and suppose we are given a diffeomorphism*

$$h: (M^*, M^G, o_M) \xrightarrow{\approx} (W, \Sigma, o_\Sigma).$$

Given an invariant tubular neighborhood $\theta_M: D^{2dn} \times M^G \longrightarrow M$, through which the smooth structure of M^* is defined, there exists a map $\mu_M: M \longrightarrow W$ such that $\mathcal{M} = (M, \theta_M, \mu_M)$ is a G_n^d-manifold over $\mathcal{W}$ and such that the diffeomorphism $\mu_M^*: M^* \longrightarrow W$ is smoothly isotopic, relative to M^G, to h.

Proof. The invariant tubular neighborhood θ_M induces $\theta_{M^*}: D_+^{d+2} \times M^G \longrightarrow M^*$. The composition

$$D_+^{d+2} \times \Sigma^k \xrightarrow{1 \times h^{-1}|\Sigma} D_+^{d+2} \times M^G \xrightarrow{\theta_{M^*}} M^* \xrightarrow{h} W$$

defines a tubular neighborhood of Σ^k in W which may not coincide with θ_W but which gives the correct fiber orientation. By the uniqueness theorem for closed tubular neighborhoods, we know that this tubular neighborhood is ambient isotopic (relative to Σ^k) to a tubular neighborhood which is orthogonally bundle equivalent to θ_W; see [1;VI.2.6]. Moreover, this preserves fiber orientation when $d = 2,4$. Thus, h is isotopic to a diffeomorphism k such that there is a smooth map $\lambda: \Sigma^k \longrightarrow \Gamma_{d+1}$ for which the diagram

$$\begin{array}{ccccc} D_+^{d+2} \times M^G & \xrightarrow{1 \times h|M^G} & D_+^{d+2} \times \Sigma^k & \xrightarrow{\bar{\lambda}} & D_+^{d+2} \times \Sigma^k \\ \downarrow{\scriptstyle \theta_{M^*}} & & & & \downarrow{\scriptstyle \theta_W} \\ M^* & & \xrightarrow{\quad k \quad} & & W \end{array}$$

commutes. Then $\mathcal{M} = (M, \theta_M, k \circ \pi)$, where $\pi: M \longrightarrow M^*$ is the orbit map, is a G_n^d-manifold over $\mathcal{W}$, as required.

6.2 Theorem. Let $\mathcal{M} = (M, \theta_M, \mu_M)$ and $\mathcal{N} = (N, \theta_N, \mu_N)$ be G_n^d-manifolds over $\mathcal{W} = (W, \Sigma, \theta_W)$ and let V_M^{k+1} and V_N^{k+1} be framed manifolds in ∂W cobounding Σ^k to which $\mathcal{M}$ and $\mathcal{N}$ correspond via 4.1. Then M and N are equivariantly diffeomorphic if and only if there is a diffeomorphism

$f: (W^{k+d+2}, \Sigma^k, o_\Sigma) \longrightarrow (W^{k+d+2}, \Sigma^k, o_\Sigma)$ <u>such that</u> $[f(V_M^{k+1})]^* = [V_N^{k+1}]^*$ <u>where</u> $[V]$ <u>denotes the frame cobordism class of</u> V <u>and</u> $[V]^*$ <u>its orbit under the right action by</u> $[W, G_2^d]$.

<u>Proof</u>. If $\varphi': M \longrightarrow N$ is an equivariant diffeomorphism then, by the uniqueness theorem for invariant tubular neighborhoods, [1;VI.2.6] and [1;VI.3.1], φ' is equivariantly isotopic to a diffeomorphism $\varphi: M \longrightarrow N$ which is an equivariant orthogonal bundle equivalence on the tubular neighborhoods θ_M and θ_N. Thus φ induces a diffeomorphism $\varphi^*: M^* \longrightarrow N^*$ and there is a smooth map $\psi: \Sigma^k \longrightarrow G_2^d$ such that the following diagram commutes (where f is defined by commutativity).

$$\begin{array}{ccccccccc}
D^{2dn} \times \Sigma^k & \xleftarrow{1 \times \mu_M | M^G} & D^{2dn} \times M^G & \xrightarrow{\theta_M} & M & \xrightarrow{\pi_M} & M^* & \xrightarrow{\mu_M^*} & W \\
\downarrow \bar{\psi} & & & & \varphi \downarrow \approx & & \varphi^* \downarrow \approx & & f \downarrow \approx \\
D^{2dn} \times \Sigma^k & \xleftarrow{1 \times \mu_N | N^G} & D^{2dn} \times N^G & \xrightarrow{\theta_N} & N & \xrightarrow{\pi_N} & N^* & \xrightarrow{\mu_N^*} & W
\end{array}$$

Thus $\mathcal{M}_f = (M, \theta_M, f \circ \mu_M)$ is a G_n^d-manifold over $\mathcal{W}$ which corresponds to the framed manifold $f(V_M)$ by 4.2 (or by section 5). The above diagram shows that $\mathcal{M}_f$ is equivalent over $\mathcal{W}$ to $\mathcal{N}$ and it follows from 4.1 that $[f(V_M)]^* = [V_N]^*$.

Conversely, suppose that $f: (W, \Sigma) \longrightarrow (W, \Sigma)$ is a diffeomorphism such that $[f(V_M)]^* = [V_N]^*$. By changing f by an isotopy, we may suppose that f is an orthogonal bundle equivalence on the given tubular neighborhood θ_W to itself. Then the G_n^d-manifold $\mathcal{M}_f = (M, \theta_M, f \circ \mu_M)$ over $\mathcal{W}$ corresponds to the framed manifold $f(V_M)$ by 4.2. Since $[f(V_M)]^* = [V_N]^*$, $\mathcal{M}_f$ and $\mathcal{N}$ are equivalent over $\mathcal{W}$. In particular, M and N are equivariantly diffeomorphic.

6.3 Corollary. *The set of equivariant diffeomorphism classes of regular G_n^d-manifolds M^{k+2dn} such that (M^*, M^G, o_M) is diffeomorphic to $(W^{k+d+2}, \Sigma^k, o_\Sigma)$ is in one-one correspondence with*

$$\frac{\mathcal{F}(\partial W, \Sigma, o_\Sigma)/[W, G_2^d]}{\mathrm{Diff}(W, \Sigma, o_\Sigma)}$$

where $\mathcal{F}(\partial W, \Sigma, o_\Sigma)$ is the set of framed cobordism classes of framed submanifolds of ∂W cobounding Σ and (when $d=2,4$) consistent with the given fiber orientation o_Σ of the normal bundle of Σ in ∂W.

6.4 Corollary. *The set of equivariant diffeomorphism classes of regular G_n^d-manifolds M with $(M^*, M^G, o_M) \approx (D^{k+d+2}, \Sigma^k, o_\Sigma)$ is in one-one correspondence with*

$$\frac{\mathcal{F}(S^{k+d+1}, \Sigma^k, o_\Sigma)/Z_2}{\mathrm{Diff}(S^{k+d+1}, \Sigma^k)}$$

where Z_2 acts by multiplying the frame by -1 (trivial for $d=2,4$).

Proof. If V_1 and V_2 are framed manifolds cobounding Σ and if $f: S^{k+d+1} \longrightarrow S^{k+d+1}$ is a diffeomorphism carrying V_1 into V_2, it suffices to extend f to D^{k+d+2}. This may not be possible. However, it is well known that there is a diffeomorphism h of S^{k+d+1} which is the identity outside a disk D^{k+d+1}, not touching V_2, and such that hf is isotopic to an orthogonal map. Thus hf extends to D^{k+d+2} and takes V_1 to V_2. Hence 6.4 follows from 6.3.

We note also that 4.4 and 4.5 have corresponding formulations of this type. In fact, this is the version in which 4.4 occurs in [10].

7. Classification of unitary and symplectic knot manifolds

By a G_n^d-knot manifold we shall mean an oriented regular G_n^d-manifold M^{2dn+k} such that (M^*, M^G) is diffeomorphic to (D^{k+d+2}, Σ^k) where Σ^k is a homotopy k-sphere in S^{k+d+1}. If Σ^k is required only to be a homology sphere, then M will be called a homology G_n^d-knot manifold.

When $d = 2$ or $d = 4$, "oriented" simply means that M is oriented in the usual sense. Since the normal bundle of M^G in M has a canonical fiber orientation, it follows that M^G inherits an orientation from that of M. Also, the normal bundle of M^G in M^* has the canonical fiber orientation o_M so that M^* inherits an orientation. (Note that the connection between the orientation of M and that of M^* can also be specified by choosing an orientation of the principal orbit. This is canonical since the group G_2^d of self-equivalences of G_n^d/G_{n-2}^d is connected for $d = 2, 4$.) Thus, specification of an orientation for any one of the three manifolds M, M^G, M^* determines that of the others. We regard D^{k+d+2} as having its canonical orientation and can demand the given diffeomorphism $(M^*, M^G) \approx (D^{k+d+2}, \Sigma^k)$ to preserve orientation. Thus Σ^k inherits an orientation and, as before, its normal bundle has an induced fiber orientation (denoted by o_Σ in section 6).

When $d = 1$ and n is even, "oriented" means that orientations are specified for both M and M^*. Again, the normal bundle of M^G in M (but not in M^*) has a canonical fiber orientation and hence M^G (and consequently Σ^k) inherits an orientation The choice of orientation for both M and M^*

can also be thought of as an orientation of M^G and a specification of a reduction of the structure group O_2 of the equivariant normal bundle of M^G in M to the subgroup SO_2. It is done so as to make the operation of equivariant connected sum well defined.

The case $d = 1$ and n <u>odd</u> will not concern us (primarily) in this section but we remark that, in this case, "oriented" means the choice of an orientation of M and one of M^G. The orientation of M induces one of M^* since the group O_2 of self equivalences of the principal orbit O_n/O_{n-2} preserves orientation for n odd.

Our previous classification theorems all have obvious analogs for the oriented case and we shall make use of this without further comment.

With these conventions, the equivariant connected sum operation (for given n, d and k) is well defined by [1;VI.3.2].

Two G_n^d-knot manifolds M_0 and M_1 will be said to be <u>equivariantly</u> h-<u>cobordant</u> if there is an oriented regular G_n^d-manifold M with boundary $\partial M = M_1 - M_0$ such that (M^*, M^G) is an h-cobordism (of pairs) between (M_0^*, M_0^G) and (M_1^*, M_1^G).

7.1 <u>Theorem</u>. <u>For</u> $d = 2,4$ <u>and</u> $k \geqslant 5$, <u>equivariantly h-cobordant</u> G_n^d-<u>knot manifolds with</u> k-<u>dimensional fixed point sets are equivariantly diffeomorphic (preserving orientation)</u>.

<u>Proof</u>. Since M_0^* is contractible, M_0^G is simply connected of dimension $k \geqslant 5$, and the codimension of M_0^G in ∂M_0^* is $d+1 \geqslant 3$, the relative h-cobordism theorem implies that (M^*, M^G) is diffeomorphic to $(D^{k+d+2}, \Sigma^k) \times I$ for some knot $\Sigma^k \subset S^{k+d+1}$

Then M corresponds to some framed manifold $V^{k+2} \subset S^{k+d+1} \times I$ cobounding $\Sigma^k \times I$ and which is "constant" near $S^{k+d+1} \times \partial I$. (This follows easily from the "closed" case of 4.1 by simply doubling M and using the interpretation in terms of homotopy classes $[\tau, \sigma]$ before passing to V^{k+2}.) Then

$$V_0^{k+1} = V^{k+2} \cap (S^{k+d+1} \times \{0\}) \quad \text{and} \quad V_1^{k+1} = V^{k+2} \cap (S^{k+d+1} \times \{1\})$$

are framed manifolds to which M_0 and M_1 correspond. Since V^{k+2} is a cobordism between them, M_0 and M_1 are equivalent by 4.1.

We remark that 7.1 also holds for $k=1$. See the remark following 7.2.

Let $\Theta^k(G_n^d)$ denote the set of equivariant h-cobordism classes of G_n^d-knot manifolds with k-dimensional fixed sets. By 7.1, $\Theta^k(G_n^d)$ can be interpreted as the set of equivariant oriented equivalence classes of G_n^d-knot manifolds when $d = 2, 4$ and $k \geq 5$. Standard remarks show that $\Theta^k(G_n^d)$ is an abelian group with respect to connected sum when $k \geq 1$.

Recall the standard notation

$$P_{k+1} = \begin{cases} Z & \text{for } k \equiv 3 \pmod 4 \\ Z_2 & \text{for } k \equiv 1 \pmod 4 \\ 0 & \text{for } k \text{ even.} \end{cases}$$

7.2 __Theorem.__ __There is a homomorphism__ $\varphi_d: \Theta^k(G_n^d) \longrightarrow P_{k+1}$. __For__ $k \neq 3$, $\varphi_1: \Theta^k(O_n) \longrightarrow P_{k+1}$ __is onto__, $\varphi_2: \Theta^k(U_n) \longrightarrow P_{k+1}$ __is an isomorphism, and__ $\varphi_4: \Theta^k(Sp_n) \longrightarrow P_{k+1}$ __is an isomorphism.__

Proof. Let M be a G_n^d-knot manifold and suppose that it corresponds to the knot (D^{k+d+2}, Σ^k) and the framed manifold $V^{k+1} \subset S^{k+d+1}$ cobounding Σ^k. We define $\varphi_d(M)$ to be zero for k even, $\frac{1}{8}\,\mathrm{index}(V^{k+1})$ for $k \equiv 3 \pmod 4$, and the Arf invariant of the framed manifold V^{k+1} for $k \equiv 1 \pmod 4$; see [12]. The proof of 7.1 shows that the framed cobordism class (not ambient and with an h-cobordism on the boundary) of V^{k+1} depends only on the equivariant h-cobordism class of M, and hence that φ_d is well defined. (For $d=2,4$ and $k \geqslant 5$ this also follows directly from 7.1 and 6.3.) Using, for example, the construction in section 5, we see that a connected sum of framed manifolds $(D^{k+d+2}, S^{k+d+1}, V^{k+1}, \Sigma^k)$ gives rise to the connected sum of the corresponding G_n^d-manifolds, and it follows that φ_d is a homomorphism. For $k \neq 3$ there is a framed, almost closed, submanifold V^{k+1} of S^{k+2} (and hence of S^{k+3} and S^{k+5}) with any given Arf invariant for $k \equiv 1 \pmod 4$, or any given multiple of 8 as index for $k \equiv 3 \pmod 4$. Thus φ_d is onto for $k \neq 3$, by 4.1.

Suppose now that $d = 2,4$ and $k \neq 3$ and let M be a G_n^d-knot manifold with $\varphi_d(M) = 0$. Then M corresponds to (D^{k+d+2}, V^{k+1}) where V^{k+1} is a framed submanifold of S^{k+d+1} cobounding a homotopy sphere Σ^k. Now $\varphi_d(M) = 0$ which, by its definition, is the only obstruction to making V^{k+1} ambiently frame cobordant (relative to Σ^k) to a contractible manifold since $k \neq 3$; see [12; 4.7]. Thus we may assume that V^{k+1} is contractible. (If $k \neq 2,3,4$ then, of course, V^{k+1} is a disk.)

Consider the half disk $D_+^{(k+1)+d+3}$ as half the thickening of D^{k+d+3} and consider a (half) thickening $(TV)^{k+2}$ of V^{k+1} in $S^{(k+1)+d+3} \cap D_+^{(k+1)+d+3}$. Then there is a G_n^d-manifold $\overline{M}$ over $D_+^{(k+1)+d+3}$ with boundary M and with fixed set the "boundary" of $(TV)^{k+2}$, which is diffeomorphic to V^{k+1} and hence is contractible; compare 5.7. By deleting an invariant open disk neighborhood of a fixed point in the interior of $\overline{M}$ we then obtain an h-cobordism between M and the linear action. Thus φ_d is a monomorphism.

<u>Remark</u>. Suppose $k = 1$ and $d = 2,4$. It can be computed that $\pi_2(G_3,SO_3) \approx Z_2 \approx \pi_2(G_5,SO_5)$ and hence there are <u>at most</u> two inequivalent G_n^d-knot manifolds for $k = 1$ by 4.3 and the fact that S^1 does not knot in S^4 or S^6. But $\Theta^1(U_n) \approx Z_2 \approx \Theta^1(Sp_n)$ by 7.2, and it follows that 7.1 also holds true for $k = 1$. Since $\pi_3(G_5,SO_5) = 0$ and S^2 does not knot in S^7, this also holds for $k=2$ and $d=4$.

Let $\Theta^{k+d+1,k}$ be the group of h-cobordism classes of oriented k-knots (homotopy spheres) in S^{k+d+1} (oriented). We have the homomorphism

$$\Theta^k(G_n^d) \longrightarrow \Theta^{k+d+1,k}$$

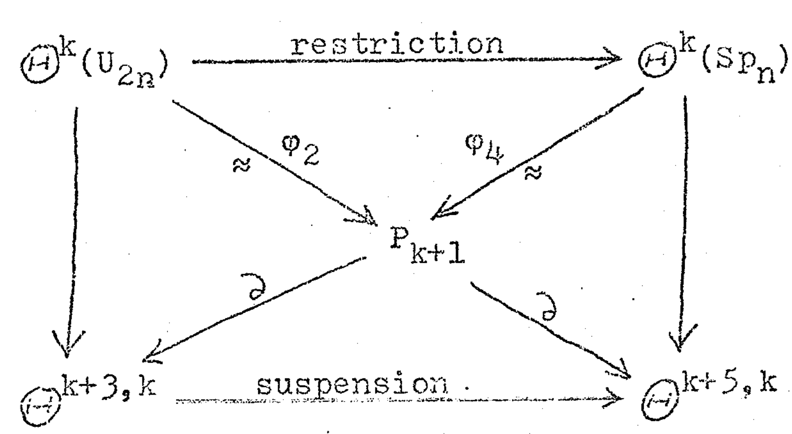
given by assignment of the orbit knot.

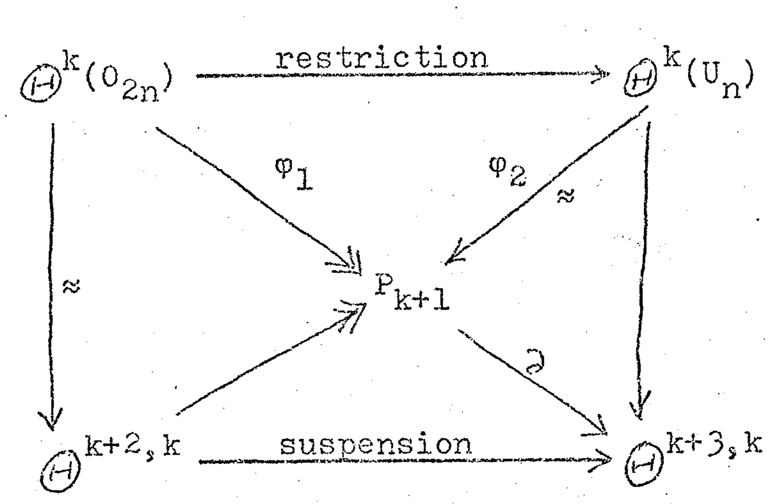
There are the homomorphisms $\partial : P_{k+1} \to \Theta^{k+d+1,k}$ for $d = 2,4$ and $k \neq 3$ defined in [12]. They are given by embedding an appropriate framed almost closed (k+1)-manifold in S^{k+d+1} and passing to its boundary $\Sigma^k \subset S^{k+d+1}$. There is also the epimorphism

$$\Theta^{k+2,k} \longrightarrow P_{k+1}$$

given by assigning the appropriate index or Arf invariant of the knot.

It then follows from 5.2, 5.3 and 4.5 that, for $k \neq 3$, the following diagrams commute:





7.3 Theorem. Let $k \neq 3$. If $\partial : P_{k+1} \longrightarrow \Theta^{k+3,k}$ is a monomorphism (which is the case for $k \neq 1$) then a U_n-knot manifold is completely determined, up to equivariant h-cobordism, by its orbit knot $\Sigma^k \subset S^{k+3}$. If $\partial : P_{k+1} \longrightarrow \Theta^{k+5,k}$ is a monomorphism (which is the case for $k > 5$ and $k \neq 2^i - 3$) then an Sp_n-knot manifold is completely determined by its orbit knot $\Sigma^k \subset S^{k+5}$, up to equivariant h-cobordism (and hence up to equivariant diffeomorphism for $k \neq 2,3,4$).

Proof. This is immediate from the diagrams above. The parenthetical remarks follow immediately from the results of Levine [12] and Browder [4], except for the case of U_n-actions with $k = 2^i - 3$. The fact that $\partial : P_{4r+2} \longrightarrow \Theta^{4r+4,4r+1}$ is a monomorphism for $r \geq 1$ (even when $r = 2^j - 1$) will be shown in section 8.

7.4 Theorem. Suppose that the knot dimension $k \neq 3$. Then each homology Sp_n-knot manifold is the restriction to $Sp_n \subset U_{2n}$ of a homology U_{2n}-knot manifold. Also each homology U_n-knot manifold is the restriction to $U_n \subset O_{2n}$ of a homology O_{2n}-knot manifold. In particular, the knot $\Sigma^k \subset S^{k+5}$ (resp. $\Sigma^k \subset S^{k+3}$) desuspends to S^{k+3} (resp. S^{k+2}). If this desuspension is specified then the extension of the Sp_n-action to U_{2n} (resp. U_n to O_{2n}) is unique up to equivariant h-cobordism.

Proof. If the knot Σ^k is a homotopy sphere then this is immediate from the diagrams preceding 7.3. (In fact, in this case, the first diagram shows that the desuspension of $\Sigma^k \subset S^{k+5}$ to $\Sigma^k \subset S^{k+3}$ is unique.) Thus assume that $k > 3$.

First let us show that if Σ^k bounds a framed manifold V^{k+1} in S^{k+5} (resp. S^{k+3}) then, up to frame cobordism, $V^{k+1} \subset S^{k+5}$ is the suspension of a framed manifold in S^{k+2}. The extendability part of the theorem follows immediately from this, 5.2 and 5.3. Let V_1^{k+1} be a framed manifold in S^{k+2} having the same index or Arf invariant as does V^{k+1} and cobounding a homotopy k-sphere Σ_1^k. (We shall use the same notation V_1^{k+1} for the suspension.) Then V^{k+1} is frame cobordant in S^{k+5} (resp. S^{k+3}) to $V \# (-V_1) \# V_1$ and $V \# (-V_1)$ is frame cobordant to a contractible manifold U^{k+1}. It then suffices to show that U^{k+1} compresses into S^{k+2}. Using one of the normal frames, U^{k+1} can be thickened to $TU \approx U \times I \approx D^{k+2}$ (since $k+2 \geq 6$). But D^{k+2}, and hence U^{k+1}, is ambient isotopic to a subspace of $S^{k+2} \subset S^{k+5}$ (resp. $S^{k+2} \subset S^{k+3}$), by the uniqueness theorem of tubular neighborhoods.

For the uniqueness statement suppose, for example, that M_1 and M_2 are U_{2n}-manifolds over (D^{k+4}, Σ^k) and that they correspond to the framed manifolds V_1^{k+1}, V_2^{k+1} in S^{k+3}. According to 4.3, and the remark below it, we can assume that $V_2 = V_1 \# V$ where V^{k+1} is a framed manifold cobounding the standard $S^k \subset S^{k+3}$. Then $M_2 \approx M_1 \# M$ where M is the U_{2n}-manifold corresponding to V^{k+1}. Assuming that M_1 and M_2 are equivalent as Sp_n-manifolds, we conclude that V_1 and V_2 have the same index or Arf invariant, since these are frame cobordism and diffeomorphism invariants. But then V has zero index or Arf invariant and thus is frame cobordant in S^{k+3} to a disk. (This holds when $k = 4$ since $\partial V \approx S^k$.) Thus M is the linear U_{2n}-manifold and hence $M_1 \approx M_2$ as U_{2n}-manifolds, as claimed.

We now turn to the determination of the diffeomorphism type of the total spaces of U_n and Sp_n-knot manifolds. These are homotopy spheres which bound π-manifolds by 5.7, 7.3, and the known result (see [1;V.11.2]) in the O_{2n}-case. (A direct homology calculation is also easy to give; see [5] or the proof of [1;V.11.2].)

Let $b: P_{k+1} \longrightarrow \Theta^{4n+k}$ be the usual homomorphism assigning the boundary of an almost closed, framed $4n+(k+1)$-manifold. Let $\Theta^k(G_n^d) \longrightarrow \Theta^{2dn+k}$ be the assignment of the total space of an action; where n is to be even when $d = 1$. Then we claim that the following diagrams ($k \neq 3$) commute:

$$\begin{array}{ccc} \Theta^k(O_{2n}) & \twoheadrightarrow & P_{k+1} \\ & \searrow & \swarrow b \\ & \Theta^{4n+k} & \end{array} \qquad \begin{array}{ccc} \Theta^k(U_n) & \xrightarrow{\approx} & P_{k+1} \\ & \searrow & \swarrow b \\ & \Theta^{4n+k} & \end{array} \qquad \begin{array}{ccc} \Theta^k(Sp_n) & \xrightarrow{\approx} & P_{k+1} \\ & \searrow & \swarrow b \\ & \Theta^{8n+k} & \end{array}$$

In fact these follow immediately from the diagrams preceding 7.3 and the commutativity in the O_{2n}-case. The latter case follows from [2,3].

In fact one can give explicit examples of all U_n and Sp_n-knot manifolds as follows. Let $k = 2r+1$ and consider the Brieskorn manifold $W_{p,q}^{2dn+k} = W_{p,q}^{2(dn+r)+1}$ which is the intersection of the unit sphere in C^{dn+r+2} with the variety

$$u^p + v^q + z_1^2 + \dots + z_{dn+r}^2 = 0.$$

As an O_{dn+r}-manifold, this is well known to correspond to the torus link (knot, when p and q are relatively prime) of type (p,q); see [1;V.10] for example. Also, this knot bounds a Seifert surface V^2 in S^3 of Arf-Robertello invariant 1 when $p=3$ and $q=2$, and of signature $\pm 8k$ when $p=6k-1$ and $q=3$. By the results of [2,3] it follows that the restriction $(W^{2dn+k}_{p,q}, O_{dn})$ of this O_{dn+r}-action to the subgroup O_{dn} satisfies

$$\varphi_1(W^{2dn+k}_{3,2}, O_{dn}) = 1 \in P_{k+1} = Z_2 \quad \text{for} \quad k \equiv 1 \pmod 4$$

$$\varphi_1(W^{2dn+k}_{5,3}, O_{dn}) = \pm 1 \in P_{k+1} = Z \quad \text{for} \quad k \equiv 3 \pmod 4.$$

Taking $d = 2,4$ and restricting to $U_n \subset O_{2n}$ or $Sp_n \subset O_{4n}$ we conclude from the diagrams preceding 7.3 that $(W^{4n+k}_{3,2}, U_n)$ and $(W^{8n+k}_{3,2}, Sp_n)$ respectively generate $\Theta^k(U_n) \approx Z_2$ and $\Theta^k(Sp_n) \approx Z_2$ for $k \equiv 1 \pmod 4$, and $(W^{4n+k}_{5,3}, U_n)$ and $(W^{8n+k}_{5,3}, Sp_n)$ respectively generate $\Theta^k(U_n) \approx Z$ and $\Theta^k(Sp_n) \approx Z$ for $3 \neq k \equiv 3 \pmod 4$.

8. Cutting out the knot.

Previous to the present paper, the only method available to study regular actions was the method of cutting out a tubular neighborhood of the fixed point set, studying the pieces, and gluing them back together. For O_n-actions that method works well, since the gluing is seen to be essentially unique. For U_n and Sp_n-actions, however, the gluing is badly nonunique, and, for this reason, that method runs into serious homotopy theoretic complications. The Sp_n case is particularly difficult to understand from that point of view. However, for U_n-knot manifolds, the method can be pushed to yield an interestingly different perspective on our results in the unitary case. In this section, we shall indicate this method for U_n-knot manifolds. Since our main aim in studying this method concerns only the standard knot (and because of the remark below 4.3 which essentially reduces the general case to that of the standard knot) we will confine our attention to this case, in order to simplify the discussion.

Thus suppose we are given a U_n-manifold over

$$\mathcal{W} = (D^{k+4}, S^k, \theta)$$

where $\theta: D^4_+ \times S^k \longrightarrow D^{k+4}$ is a standard tubular neighborhood of the standardly embedded S^k in D^{k+4}, regarded as inclusion. The notation X, B, $X^{\circledast} = D^3 \times S^k$ and $B^{\circledast} = S^2 \times S^k$ is as in section 4. For $x \in S^k$, and hence $(D^3 \times \{x\}, S^2 \times \{x\}) \subset (X^{\circledast}, B^{\circledast})$, the inclusion $(D^3 \times \{x\}, S^2 \times \{x\}) \longrightarrow (Y, B)$ is a homotopy equivalence, and it follows that for any U_n-manifold $\mathcal{M} = (M^{4n+k}, \theta_M, \mu_M)$ over $\mathcal{W}$, the part of $\mathcal{M}$ over X is equivalent over X to the same thing for the <u>linear</u> action. This also holds for the part over the tube $D^4_+ \times S^k$ since $\pi_k(U_2) \longrightarrow \pi_k(SO_3)$ is onto.

Thus any such U_n-manifold M can be constructed by cutting apart the linear action along $S^{4n-1} \times S^k$ (above $X^{\circledast} = D^3 \times S^k$) and pasting back via an equivariant diffeomorphism of $S^{4n-1} \times S^k$ over the identity on $X^{\circledast} = D^3 \times S^k$.

Now the group

$$\pi_0 \mathrm{Diffeo}^{U(n)}_{D^3 \times S^k}(S^{4n-1} \times S^k)$$

of isotopy classes of such self equivalences over the identity are classified in [1;V.7.1] and [1;VI.6.4]. This classification is described as follows.

Let $\mathcal{N} = \mathcal{M}ap^{U(n)}(\rho, \rho)$ which is a trivial bundle (since ρ is trivial) over $X^{\circledast} = D^3 \times S^k$ with fiber $N = U_2$. Let $\mathcal{S} = S\text{-}\mathcal{M}ap^{U(n)}(\xi, \xi)$, which is a certain subbundle of $\mathcal{N}|S^2 \times S^k$ with fiber $S = U_1 \times U_1$ and structure group S acting by conjugation. (The meaning of this notation is given in [1;V.7] and is not needed here.) The bundle $\mathcal{S}$ does <u>not</u> have the trivialization induced from that of $\mathcal{N}$, but it <u>is</u> trivial since S is abelian. The fibers S of $\mathcal{S}$ lie in the fiber N of $\mathcal{N}$ as (varying) conjugates of the standard $U_1 \times U_1 \subset U_2$; see [1;V.7.2]. Then there is a canonical isomorphism

$$\pi_0 \Gamma(\mathcal{N}, \mathcal{S}) \approx \pi_0 \mathrm{Diffeo}^{U(n)}_{D^3 \times S^k}(S^{4n-1} \times S^k)$$

where $\pi_0\Gamma(\mathcal{N},\mathcal{S})$ is the group of homotopy classes of sections of $\mathcal{N}$ over $D^3 \times S^k$ with values in $\mathcal{S}$ over $S^2 \times S^k$.

A similar fact holds over the complement X of the tubular neighborhood. That is, $\mathcal{N}$ and $\mathcal{S}$ extend to (trivial) bundles $\mathcal{N}'$ and $\mathcal{S}'$ over X and $B \approx S^2 \times D^{k+1}$ respectively, and the isotopy classes of smooth self equivalences over X correspond to $\pi_0\Gamma(\mathcal{N}',\mathcal{S}')$. This acts on the right of $\pi_0\Gamma(\mathcal{N},\mathcal{S})$ in the obvious way. The orthogonal bundle equivalences of $D^{4n} \times S^k$ over the identity of $D^4_+ \times S^k$ are given by maps $S^k \longrightarrow C = \text{center}(U_2)$ and there is an induced left action of $[S^k,C]$ on $\pi_0\Gamma(\mathcal{N},\mathcal{S})$. Then it is clear that the collection

$$[S^k,C]\backslash \pi_0\Gamma(\mathcal{N},\mathcal{S})/\pi_0\Gamma(\mathcal{N}',\mathcal{S}')$$

of double cosets, classifies U_n-actions over $\mathcal{W}$. We are mainly concerned with the case $k>1$, for which $[S^k,C] = 0$.

The restriction of sections to $S^2 \times S^k$ gives the pull-back diagram

$$\begin{array}{ccc} \Gamma(\mathcal{N},\mathcal{S}) & \longrightarrow & \Gamma(\mathcal{N}) = U_2^{D^3 \times S^k} \\ \downarrow & & \downarrow \text{restriction} \\ \Gamma(\mathcal{S}) & \xrightarrow{\text{inclusion}} & \Gamma(\mathcal{N}|S^2 \times S^k) = U_2^{S^2 \times S^k}. \end{array}$$

(Similarly for $\mathcal{N}'$, $\mathcal{S}'$.) Since the right hand map is a fibration with fiber $\Gamma(\mathcal{N}, e) = (U_2,e)^{(D^3,S^2)\times S^k}$, so is the left hand map.

Thus we have the diagram

$$\begin{array}{ccccc} (U_2,e)^{(X,S^2\times D^{k+1})} & \longrightarrow & \Gamma(\mathcal{N}',\mathcal{S}') & \longrightarrow & \Gamma(\mathcal{S}') \approx (U_1\times U_1)^{S^2\times D^{k+1}} \\ \downarrow & & \downarrow & & \downarrow \\ (U_2,e)^{(D^3,S^2)\times S^k} & \longrightarrow & \Gamma(\mathcal{N},\mathcal{S}) & \longrightarrow & \Gamma(\mathcal{S}) \approx (U_1\times U_1)^{S^2\times S^k} \end{array}$$

in which the rows are fibrations and the vertical maps are restrictions.

Suppose now that $k > 1$. Then the evaluation $(U_1\times U_1)^{S^2\times S^k} \longrightarrow U_1\times U_1$, at a point, is a homotopy equivalence. Thus $\pi_0\Gamma(\mathcal{S})$ and $\pi_0\Gamma(\mathcal{S}')$ are trivial and $\pi_1\Gamma(\mathcal{S}') \longrightarrow \pi_1\Gamma(\mathcal{S}) \approx Z\times Z$ is an isomorphism. Thus the induced diagram of homotopy sequences of these fibrations for $k > 1$ has the form

$$\begin{array}{ccccccc} \pi_1\Gamma(\mathcal{S}') & \longrightarrow & [X,S^2\times D^{k+1};U_2,e] & \longrightarrow & \pi_0\Gamma(\mathcal{N}',\mathcal{S}') & \longrightarrow & 1 \\ \downarrow\approx & & \downarrow & & \downarrow & & \\ \pi_1\Gamma(\mathcal{S}) & \longrightarrow & [(D^3,S^2)\times S^k;U_2,e] & \longrightarrow & \pi_0\Gamma(\mathcal{N},\mathcal{S}) & \longrightarrow & 1. \end{array}$$

Since X is contractible, $X/(S^2\times D^{k+1}) \simeq S^3$. Also $(D^3\times S^k)/(S^2\times S^k) \simeq S^3\vee S^{k+3}$. It is clear that the induced map $S^3\vee S^{k+3} \longrightarrow S^3$ has degree one on S^3. It is easily shown that any such map can be changed by a homotopy equivalence of $S^3\vee S^{k+3}$ commuting with projection to S^{k+3} such that it becomes projection to S^3. Thus the above diagram has the form

$$\begin{array}{ccccccc} Z\times Z & \longrightarrow & \pi_3(U_2) & \longrightarrow & \pi_0\Gamma(\mathcal{N}',\mathcal{S}') & \longrightarrow & 1 \\ \downarrow\approx & & \downarrow(1,0) & & \downarrow & & \\ Z\times Z & \longrightarrow & \pi_3(U_2)\times\pi_{k+3}(U_2) & \longrightarrow & \pi_0\Gamma(\mathcal{N},\mathcal{S}) & \longrightarrow & 1. \end{array}$$

(for $k > 1$) and hence the induced map

$$\pi_{k+3}(U_2) \longrightarrow \pi_0\Gamma(\mathcal{N},\mathcal{S})/\pi_0\Gamma(\mathcal{N}',\mathcal{S}')$$

is an isomorphism.

Similarly, in the case $k = 1$ it can be shown that the same map $\pi_4(U_2) \longrightarrow \pi_0\Gamma(\mathcal{N},\mathcal{S})$ induces an isomorphism

$$\pi_4(U_2) \xrightarrow{\approx} \pi_1(C)\backslash\pi_0\Gamma(\mathcal{N},\mathcal{S})/\pi_0\Gamma(\mathcal{N}',\mathcal{S}').$$

The proof of this is more difficult and needs some special arguments. Since we shall not need the case $k = 1$, we shall not discuss it further.

It is clear that the geometric meaning of this is as follows: Let D^{k+3} be an oriented disk in the <u>interior</u> of D^{k+4}. Let $f: (D^{k+3},S^{k+2}) \longrightarrow (U_2,e)$ be a smooth map representing an element $\alpha \in \pi_{k+3}(U)$. Define a U_n-manifold over $\mathcal{W} = (D^{k+4},S^k,\theta)$ by cutting the space S^{4n+k} of the linear action apart over the oriented disk D^{k+3} and pasting it back together by the equivariant diffeomorphism given by right translation by the map f. (Here we assume the bundle of principal orbits to be trivialized as $D^{k+3}\times(U_n/U_{n-2})$ over D^{k+3}.) Denote this U_n-manifold over $\mathcal{W}$ by M_α^{4n+k}.

Then the above considerations prove the following theorem.

8.1 <u>Theorem</u>. <u>Each U_n-manifold over $\mathcal{W} = (D^{k+4},S^k,\theta)$ is equivalent over $\mathcal{W}$ to M_α^{4n+k} for a unique</u> $\alpha \in \pi_{k+3}(U_2)$.

<u>Remark</u>. Since $\pi_{k+3}(U_2) \approx \pi_{k+3}(S^3) \approx \pi_{k+3}(S^2) \approx \pi_{k+1}(F_2) \approx \pi_{k+1}(G_3,SO_3)$, this result is consistent with 4.3 and gives another interpretation of that classification. We do not know how to get this interpretation directly out of 4.3. The case

of Sp_n-actions, from this point of view, is quite unclear, but it is easily seen that there is no strict analog of 8.1 for it.

8.2 Corollary. *The homomorphism*

$$\partial : P_{4r+2} \longrightarrow \Theta^{4r+4, 4r+1}$$

of Levine [12], is a monomorphism for all $r \geq 1$.

Proof. This is trivially true when the Kervaire sphere is exotic; that is, for $r+1$ not a power of two. (It is more trivially false for $r = 0$.) Suppose that $\partial : P_{4r+2} \longrightarrow \Theta^{4r+4, 4r+1}$ is trivial. Then $r+1$ is a power of two. Consider the action of $U_2 \subset O_4$ on the last four coordinates of the Brieskorn manifold $M = W_3^{4r+9}$:

$$z_0^3 + z_1^2 + z_2^2 + \dots + z_{2r+5}^2 = 0 ; \quad \|z\| = 1$$

which is the Kervaire $(4r+9)$-sphere. This is a U_2-knot manifold over $(D^{4r+5}, \Sigma^{4r+1})$ where Σ^{4r+1} is the Kervaire sphere. The hypothesis implies that $(S^{4r+4}, \Sigma^{4r+1})$ is the standard knot. By 8.1 we can construct this action from the linear action by changing the linear action in the neighborhood of some principal orbit. Consider the fixed set M^{U_1} which touches no principal orbits. It follows that M^{U_1} is diffeomorphic to the fixed set of U_1 in the linear action and hence it is a standard sphere. But $M^{U_1} = W_3^{4r+5}$, the Kervaire sphere, and this is not standard since $r+2$ is not a power of 2 (since $r+1$ is a power of 2 and $r \neq 0$). This contradiction finishes the proof.

9. Appendix.

In this appendix we shall give some information concerning the orbit map of twice the standard representation of G_n^d on $R^{dn} \times R^{dn}$; $n \geq 2$. Much of this material can hardly be described as anything other than classical, but we feel that too many readers will not be sufficiently familiar with it in the present context to omit it.

Let F denote the reals R, complexes C, or quaternions Q, according as $d = 1, 2$, or 4. For column vectors $u, v \in F^n$ $(\approx R^{dn})$ let $\langle u, v\rangle = \sum \bar{u}_i v_i$ denote the inner product.

Let $(u,v) \in F^n \times F^n$. Then $\|u\|^2$, $\|v\|^2$ and $\langle u, v\rangle$ are invariants of twice the standard representation of G_n^d on $F^n \times F^n$ and are easily seen to be a complete set of invariants as is well known.

Let us regard $F^n \times F^n$ as the space of $n \times 2$ matrices over F. For any element $(u,v) \in F^n \times F^n$ we then have the 2×2 Hermitian matrix

$$(u,v)^*(u,v) = \begin{bmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle v,u\rangle & \langle v,v\rangle \end{bmatrix} = \begin{bmatrix} \|u\|^2 & \langle u,v\rangle \\ \overline{\langle u,v\rangle} & \|v\|^2 \end{bmatrix}$$

which forms a complete set of invariants.

Let H_2^d denote the space of 2×2 Hermitian matrices over F. Then $g \in G_2^d$ acts on $M \in H_2^d$ by $M \longmapsto g^{-1}Mg = g^*Mg$, which defines a right action. The map $(u,v) \longmapsto (u,v)^*(u,v)$ of

$$F^n \times F^n \longrightarrow H_2^d$$

then gives a G_2^d-equivariant map where G_2^d acts by right multiplication on $F^n \times F^n$, and this map can be regarded as the orbit map of the left G_n^d-action on $F^n \times F^n$, once its image is determined. It is not hard to see, in fact, that (since $n \geq 2$) its image consists of those elements satisfying the Schwarz inequality $|\langle u, v\rangle|^2 \leq \|u\|^2 \|v\|^2$ and the inequalities $\|u\|^2 \geq 0$, $\|v\|^2 \geq 0$; that is, the image consists of the <u>positive semidefinite</u> Hermitian matrices.

We will need to know that the G_2^d-action on H_2^d preserves the trace; that is, $tr(g^{-1}Mg) = tr(M)$ for Hermitian M and $g \in G_2^d$. This is clear in the real and unitary cases. For the symplectic case, recall the monomorphism

$$M = C + jD \longrightarrow \varphi(M) = \begin{bmatrix} C & \bar{D} \\ -D & \bar{C} \end{bmatrix}$$

of quaternionic 2×2 matrices into the complex 4×4 matrices. It is evident that $Tr(\varphi(M)) = 2 \, Re \, Tr(M)$ and hence $Re \, Tr(M)$ is invariant under conjugation by elements of the quaternionic 2×2 general linear group. Since $Tr(M)$ is real for Hermitian M, the contention follows.

Now if $M = (m_{i,j})$, then $Tr(M^*M) = \sum |m_{i,j}|^2$. Since $Tr(M^*M)$ is preserved under conjugation by G_2^d, it follows that the G_2^d-action on H_2^d preserves $\sum |m_{i,j}|^2$ which is the sum of squares of the coefficients of M considered as a real $2d \times 2d$ matrix. Thus, with this norm on the <u>real</u> vector space H_2^d, the G_2^d-action is <u>orthogonal</u>. Since the trace is preserved, the subspace of H_2^d consisting of matrices of trace zero is preserved. Its orthogonal complement is the space of scalar matrices rI for real r and this is fixed under G_2^d.

Let SH_2^d denote the space of Hermitian 2×2 matrices of trace zero. Suppose that

$$N = \begin{bmatrix} a & w \\ \bar{w} & -a \end{bmatrix}$$

with $w = b + ci + dj + ek$ (and with d, e omitted in the complex case and c, d, e omitted in the real case). The remarks above show that the map

$$\psi: SH_2^d \longrightarrow R^{d+1}$$

$$\psi(N) = (a, b, c, d, e)$$

is an isomorphism of real vector spaces (halving the norm square) and carries the right G_2^d-action (conjugation) on SH_2^d to an orthogonal action of G_2^d on R^{d+1}. That is, there is a homomorphism $\lambda: G_2^d \longrightarrow O_{d+1}$ such that

$$\psi(g^{-1}Ng) = \psi(N) \cdot \lambda(g).$$

The kernel of λ is that of the G_2^d-action on H_2^d and this is clearly the center C_2^d of G_2^d. Thus λ induces a monomorphism

$$\Gamma_{d+1} = G_2^d / C_2^d \longrightarrow O_{d+1},$$

which we shall regard as an inclusion. Consideration of connectivity and dimension of Γ_{d+1} then shows that $\Gamma_2 = O_2$, $\Gamma_3 = SO_3$ and $\Gamma_5 = SO_5$.

For $M = (u, v)^*(u, v)$ we have

$$2M - Tr(M) \cdot I = \begin{bmatrix} \|u\|^2 - \|v\|^2 & 2<u, v> \\ 2\overline{<u, v>} & \|v\|^2 - \|u\|^2 \end{bmatrix}$$

which is in SH_2^d. Thus it follows that the map

$$\bar{\pi}_d : R^{dn} \times R^{dn} = F^n \times F^n \longrightarrow R \times (R \times F) = R \times R^{d+1}$$

$$(u,v) \longmapsto (\|u\|^2 + \|v\|^2, \|u\|^2 - \|v\|^2, 2\langle u,v\rangle)$$

can be identified with the orbit map of the G_n^d-action and is <u>equivariant</u> with respect to the right G_2^d-action on $R^{dn} \times R^{dn}$ and the right G_2^d-action (via $\lambda : G_2^d \longrightarrow O_{d+1}$) on $R \times R^{d+1}$ (trivial on the first coordinate). The image of this map consists of those points $(y,a,b,c,d,e) \in R \times R^{d+1}$ such that $y \geq (a^2 + b^2 + c^2 + d^2 + e^2)^{1/2}$.

It follows from a theorem of Glaeser (Annals of Math., 77 (1963), 193-209) that $\bar{\pi}_d$ is a <u>diffeomorphism</u>, where the orbit space is given the functional structure induced by the orbit map. (I am indebted to E. Bierstone for pointing this out to me.) If somewhat more were known about this map (equivariant lifting of isotopies) and if we used this structure on the orbit space throughout, then a stronger version of the classification theorem 4.1 would follow immediately; compare [1;VI.6]. (On the other hand, smooth manifolds with boundary are better understood than manifolds with this type of singularity on the boundary.)

The image of $\bar{\pi}_d$ consists of a positive solid cone. Since we wish to have the structure of a manifold with boundary on the orbit space, we alter the map $\bar{\pi}_d$ by using the homeomorphism

$$(y,a,b,c,d,e) \longmapsto (x,a,b,c,d,e)$$

where $x = y^2 - (a^2 + b^2 + c^2 + d^2 + e^2)$, which converts the image of $\bar{\pi}_d$ into the half space $R^+ \times R^{d+1}$. Notice that this is a <u>diffeomorphism</u> outside the origin (the only part of the cone with $y = 0$) and is O_{d+1}-equivariant.

Thus, finally, we may regard the map

$$\pi_d: R^{dn} \times R^{dn} = F^n \times F^n \longrightarrow R^+ \times (R \times F) = R^+ \times R^{d+1}$$

$$(u,v) \longmapsto (x, \|u\|^2 - \|v\|^2, 2\langle u,v\rangle)$$

as the orbit map for the G_n^d-action, where

$$x = (\|u\|^2 + \|v\|^2)^2 - (\|u\|^2 - \|v\|^2)^2 - 4|\langle u,v\rangle|^2$$
$$= 4(\|u\|^2\|v\|^2 - |\langle u,v\rangle|^2).$$

Clearly, this is also equivariant with respect to the right G_n^d-actions.

Note that under $\bar{\pi}_d$ the unit disk in R^{2dn} goes to the set of points in the cone with $y^2 \leq 1$. Thus under π_d it goes to the set of points $(x,w) \in R^+ \times R^{d+1}$ with $0 \leq x \leq 1 - \|w\|^2$.

Now let us study the restriction of twice the standard representation of U_{2n} to Sp_n and that of O_{2n} to U_n. A quaternionic column vector $u \in Q^n$ can be considered as a complex vector

$$u_C = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $u = u_1 + ju_2$. For $u,v \in Q^n$ it is immediate that $\langle u_C, v_C\rangle$ is the complex part of $\langle u,v\rangle$. Thus the diagram of orbit maps

$$\begin{array}{ccc} Q^n & \xrightarrow{\approx} & C^{2n} \\ \downarrow{\scriptstyle \bar{\pi}_4} & & \downarrow{\scriptstyle \bar{\pi}_2} \\ R \times R^5 & \xrightarrow{\text{proj.}} & R \times R^3 \end{array}$$

commutes, where $\text{proj}(y,a,b,c,d,e) = (y,a,b,c)$. Similarly, we have the commutative diagram

$$\begin{array}{ccc} C^n & \xrightarrow{\approx} & R^{2n} \\ \downarrow{\bar{\pi}_2} & & \downarrow{\bar{\pi}_1} \\ R \times R^3 & \xrightarrow{\text{proj.}} & R \times R^2 \end{array}$$

This is o.k

of orbit maps, where $\text{proj}(y,a,b,c) = (y,a,b)$. Modification, as above, of the $\bar{\pi}_d$ to the orbit maps π_d clearly yield the maps $\omega_{2,4}$ and $\omega_{1,2}$, on the $R^+ \times R^{d+1}$, defined in section 5.

We shall now define the maps denoted by "stretch" in section 5. First let

$$f: R^+ \longrightarrow R^+$$

be a smooth nondecreasing for $x \leq 1$ map such that, for some $\epsilon > 0$,

$$f(x) = \begin{cases} x & \text{for } x < \epsilon \\ 1 & \text{for } x > 1-\epsilon. \end{cases}$$

This is to be fixed once and for all. Then simply define

$$\text{stretch}: F^n \times F^n \longrightarrow F^n \times F^n$$

by

$$\text{stretch}(u,v) = \frac{f((\|u\|^2 + \|v\|^2)^{1/2})}{(\|u\|^2 + \|v\|^2)^{1/2}} (u,v).$$

Note that its image is the unit disk (and, in section 5, we restrict the domain to the unit disk). This map is the identity near the origin and is clearly equivariant with respect to the left G_n^d-action <u>and</u> the right G_2^d-action. Passing to orbit spaces, via the orbit map π_d, this induces a map

$$\text{stretch}: R^+ \times R^{d+1} \longrightarrow R^+ \times R^{d+1}$$

which is equivariant with respect to the Γ'_{d+1}-action. It is not hard to see that this map is smooth. (Indeed this follows from the fact that it is the identity near the origin and that, outside the origin, π_d induces a <u>diffeomorphism</u> $F^n \times F^n/G_n^d \longrightarrow R^+ \times R^{d+1}$ where the orbit space is given the functional structure induced by the orbit map.)

We will now conclude this appendix by showing where the linear action of G_n^d on the unit sphere S^{2dn-1} in $F^n \times F^n$ fits into the classification theorem [1;V.6.2] (the absolute version of 2.1 and 2.2). This action, and the given map $\pi_d\colon S^{2dn-1} \longrightarrow D^{d+1} = \{(x,y) \in R^+ \times R^{d+1} \mid 0 \leq x = 1 - \|y\|^2\}$, corresponds to an element

$$[\tau]^* \in [\partial D^{d+1}, S\backslash N]/\pi_0(N)$$

where $N = N(G_{n-2}^d)/G_{n-2}^d \approx G_2^d$ and $S = (N(G_{n-2}^d) \cap N(G_{n-1}^d))/G_{n-2}^d \approx G_1^d \times G_1^d \subset G_2^d$. The space $S\backslash N$ has a canonical orientation (for $d = 2,4$) as the (left) projective line $FP^1 \approx S^d$. The disk D^{d+1} has a canonical orientation from the coordinate $y \in R^{d+1}$. (We may as well ignore the coordinate x in D^{d+1} in the following remarks since it plays no role for the action on the sphere. Thus D^{d+1} is regarded as the unit disk in R^{d+1}.) Thus ∂D^{d+1} inherits an orientation. (The boundary of an oriented manifold M is oriented by demanding that the front extension by the outward normal vector of a compatible frame for ∂M should be a compatible frame for M.) Then the invariant to be associated to the linear G_n^d-manifold (S^{2dn-1}, π_d) is the <u>degree</u> of $\tau\colon \partial D^{d+1} \longrightarrow S\backslash N = FP^1 \approx S^d$. (For $d=1$, it is well defined only up to sign and is well known to be ± 1. Thus we shall concentrate on the cases $d = 2,4$.)

This invariant is known to be ± 1, as shown in [5]. (This can also be shown analogously to the proof of the case $d = 1$ given in [1;V.6.3]; viz: actions corresponding to invariants <u>other</u> than ± 1 have total spaces which are not homology spheres since they are cyclic branched coverings with non-homology spheres as branch sets.) For $d = 2, 4$ the sign is unambiguous, and must be determined if one wishes to make one's orientation conventions explicit. (It is false, for $d=2,4$, that there exists, as stated in [5], a self equivalence of S^{2dn-1} inducing an orientation reversing diffeomorphism on the orbit space.) We shall show, by what we believe to be the definitive method, that the invariant for (S^{2dn-1}, π_d) is $+1$. (It follows that if $\varphi: D^{d+1} \longrightarrow D^{d+1}$ is an orientation reversing diffeomorphism, then the invariant for $(S^{2dn-1}, \varphi\circ\pi_d)$ is -1.)

For the sake of simplicity we shall give the proof in the case of symplectic actions; $d = 4$. The unitary case, $d=2$, is easier. We take $K \subset Sp_n$ to be that copy of Sp_{n-1} fixing the <u>first</u> coordinate axis, and $H \subset K \subset Sp_n$ that copy of Sp_{n-2} fixing the first two axes. By $Sp_2 \subset Sp_n$ we mean the subgroup <u>acting</u> on the first two coordinates and we identify this with $N = N_H/H$ in the canonical way. Then $S = Sp_1 \times Sp_1 \subset Sp_2$ and $S\backslash N = (Sp_1 \times Sp_1)\backslash Sp_2$.

By checking the proof of the classification theorem, one sees immediately that if one constructs an Sp_2-equivariant map $\Phi: Sp_2 \times D^5 \longrightarrow (S^{4n-1})^H$ such that the diagram

$$\begin{array}{ccc} Sp_2 \times D^5 & \xrightarrow{\Phi} & (S^{4n-1})^H \\ & \underset{\text{proj.}}{\searrow} \quad \swarrow_{\pi} & \\ & D^5 & \end{array}$$

commutes, then the invariant $[\tau] \in [S^4, S\backslash N]$ is represented by the map $\tau: S^4 \longrightarrow S\backslash N$ such that $\Phi(\tau(y), y) \subset (S^{4n-1})^K$; that is, if $\tau(y) = Sg$ for $g \in N$, then $\Phi(g, y)$ is fixed by K. We shall give a canonical construction of such a map Φ and check this condition.

Recall the epimorphism $\lambda: Sp_2 \longrightarrow SO_5$ constructed earlier in this appendix, and note that $S = Sp_1 \times Sp_1$ is the subgroup such that $\lambda(S)$ is the copy of SO_4 in SO_5 fixing the first coordinate axis. As before, it is convenient for notational purposes to regard R^5 as $R \times Q$. Then any given point in $D^5 \subset R \times Q$ can be written in the form $(r, 0) \cdot \lambda(g)$ for $0 \leqslant r \leqslant 1$ in R and $g \in Sp_2$. Of course, only the coset Sg of g in $S\backslash N$ is well determined. Then we define Φ as that map taking $(h, (r,0) \cdot \lambda(g)) \in Sp_2 \times D^5$ to the $n \times 2$ quaternionic matrix

$$\Phi(h, (r,0)\cdot\lambda(g)) = (u,v) = \begin{bmatrix} hg^{-1} \begin{bmatrix} (\frac{1+r}{2})^{1/2} & 0 \\ 0 & (\frac{1-r}{2})^{1/2} \end{bmatrix} g \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \end{bmatrix}$$

which is clearly in $(S^{4n-1})^H$. Then

$$(u,v)^*(u,v) = \begin{bmatrix} \frac{r+1}{2} & 0 \\ 0 & \frac{1-r}{2} \end{bmatrix} g .$$

Subtracting (trace • I) from twice this and taking the first row we then see that

$$\pi(u,v) = (r,0)\cdot\lambda(g)$$

as desired.

Now $(r,0)\cdot\lambda(g) \in S^4$ exactly when $r = 1$, and then

$$\Phi(h, (1,0)\cdot\lambda(g)) = \begin{bmatrix} hg^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} g \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \end{bmatrix}$$

which is fixed by $K = \{1\} \times Sp_{n-1}$ exactly when $h \in Sg$. Thus

$$\tau((1,0)\cdot\lambda(g)) = Sg \in S\backslash N.$$

In other words, τ is just the inverse of the canonical equivariant diffeomorphism

$$S\backslash N \longrightarrow S^d$$

$$Sg \longmapsto (1,0)\cdot\lambda(g)$$

taking the coset space $S\backslash N$ to the orbit of the point $(1,0)$ in the <u>transitive</u> right N-space S^d. As indicated, we choose the orientation of $S\backslash N$ given by the similar diffeomorphism

$$S\backslash N \longrightarrow QP^1$$

$$Sg \longmapsto (1:0)g$$

where QP^1 is the <u>left</u> quaternionic projective line. Now the orientation of S^d from that of $D^{d+1} \subset R \times Q$ via front extension by the outward normal is clearly the same as the orientation given via the stereographic projection $S^d - \{(-1,0)\} \longrightarrow Q$ from $(-1,0)$.

Now if

$$g = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in Sp_2$$

then

$$g^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g = \begin{bmatrix} \bar{p} & \bar{r} \\ \bar{q} & \bar{s} \end{bmatrix}\begin{bmatrix} p & q \\ -r & -s \end{bmatrix} = \begin{bmatrix} |p|^2 - |r|^2 & \bar{p}q - \bar{r}s \\ * & * \end{bmatrix}$$

which implies that

$$(1,0)\cdot\lambda(g) = (|p|^2 - |r|^2,\ \bar{p}q - \bar{r}s) = (2|p|^2 - 1,\ 2\bar{p}q) \in R \times Q$$

which has stereographic projection

$$\frac{\bar{p}q}{|p|^2} = p^{-1}q \in Q$$

(assuming $p \neq 0$ as we may). On the other hand, in QP^1 we have

$$(1:0)g = (p:q) = (1:p^{-1}q)$$

and hence $\tau: S\setminus N \longrightarrow S^d$ preserves orientation, as claimed. Because of this fact, we often substitute S^d for $S\setminus N$ in this paper, by implicit use of this diffeomorphism.

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