FUNDAMENTAL GROUPS OF BLOW-UPS

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Abstract. Many examples of nonpositively curved closed manifolds arise as blow-ups of projective hyperplane arrangements. If the hyperplane arrangement is associated to a finite reflection group W, and the blow-up locus is W-invariant, then the resulting manifold will admit a cell decomposition whose maximal cells are all combinatorially isomorphic to a given convex polytope P. In other words, M admits a tiling with tile P. The universal covers of such examples yield tilings of $\mathbb{R}^n$ whose symmetry groups are generated by involutions but are not, in general, reflection groups. We begin a study of these "mock reflection groups", and develop a theory of tilings that includes the examples coming from blow-ups and that generalizes the corresponding theory of reflection tilings. We apply our general theory to classify the examples coming from blow-ups in the case where the tile P is either the permutohedron or the associahedron.

1. Introduction

Suppose M^n is a connected closed manifold equipped with a cubical cell structure. (In other words, M^n is homeomorphic to a regular cell complex in which each k-dimensional cell is combinatorially isomorphic to a k-dimensional cube.) It turns out that there is a rich class of examples of such manifolds satisfying the following three properties.

(1) There is a group G of symmetries of the cellulation such that the action of G on the vertex set is simply transitive and such that the stabilizer of each edge is cyclic of order 2.
(2) In the dual cell structure on M^n each top-dimensional cell is combinatorially isomorphic to some given simple convex polytope, for example, to a permutohedron or an associahedron. (Such a top-dimensional dual cell will be called a "tile".)
(3) The natural piecewise Euclidean metric on M^n (in which each combinatorial cube is isometric to a regular cube in Euclidean space) is nonpositively curved.

It follows from (2) and (3) that the universal cover $\widetilde{M}^n$ is homeomorphic to $\mathbb{R}^n$. The cubical cell structure on M^n lifts to a cellulation of $\widetilde{M}^n$ as does the dual cell structure. Although these two cell structures on M^n (the cubical one and its dual) carry exactly the same combinatorial information, they correspond to two distinct geometric pictures. Throughout this paper we shall go back and forth between these two pictures. For example, property (1), that G acts simply transitively on the vertex set of the cubical cellulation, means that G acts simply transitively on the set of n-dimensional dual cells. So, $\widetilde{M}^n$ is "tiled" by isomorphic copies of such an n-dimensional dual cell.

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Let A denote the group of all lifts of the G-action to $\widetilde{M}^n$. Fix a vertex x of the cubical structure on $\widetilde{M}^n$. By property (1) each edge containing x is flipped by a unique involution in A. Since $\widetilde{M}^n$ is connected, these involutions generate A and the 1-skeleton of $\widetilde{M}^n$ is the Cayley graph of A with respect to this set of generators. Since $\widetilde{M}^n$ is simply connected, a presentation for A can be derived by examining the 2-cells that contain x and the 2-skeleton of $\widetilde{M}^n$ is the Cayley 2-complex of this presentation. (This is explained in Section 4.7 and 5.) Furthermore, the fundamental group of M^n is naturally identified with the kernel of the epimorphism $A \to G$ induced by the projection $\widetilde{M}^n \to M^n$. One of the purposes of this paper is to initiate the study of such symmetry groups A.

This paper has two major thrusts:

- to describe a large class of examples of the above type (in Sections 1 - 4, 7 and 8), and
- to develop a general theory of tilings and their symmetry groups (in Sections 5 and 6).

We first give a rough description of the examples. The first examples are fairly standard and arise from actions of right-angled reflection groups on manifolds. (In this setting $\widetilde{M}^n$ is the manifold, and A is the reflection group.) The other examples that we discuss arise by performing an equivariant blow-up procedure to (not necessarily right-angled) reflection group actions and lifting to the universal cover. In this case, $\widetilde{M}^n$ is the universal cover, and A is the group of lifts of the reflection group action. An important guiding principle underlying this paper is that the group actions in the blow-up setting are tantalizingly similar to, but different from, reflection group actions.

Our reflection-type examples can be constructed as in [D] or [DM]. Given a simple polytope P^n that is a candidate for the fundamental tile, let W be the right-angled Coxeter group with one generator for each codimension-one face and one relation for each codimension-two face. Let $\widetilde{M}^n$ be the result of applying the reflection group construction to P^n and W, and let Γ be a torsion-free, normal subgroup of W. Then we get examples of the above type with $M^n = \widetilde{M}^n/\Gamma$, $G = W/\Gamma$, and $A = W$. Again, we note that these reflection type examples are *not* the ones of primary interest in this paper.

Our primary examples are manifolds that are constructed by blowing up certain subspaces of projective hyperplane arrangements in $\mathbb{RP}^n$. The theory of such blow-ups was developed in [DJS]. Given a hyperplane arrangement in $\mathbb{R}^{n+1}$, there is an associated $(n+1)$-dimensional convex polytope Z called a "zonotope". An equivalent formulation of the blowing-up procedure is described in [DJS]: one "blows-up" certain cells of $\partial Z/a$ (where a denotes the antipodal map). In this generality, the resulting cubical cell complex might not admit a suitable symmetry group G satisfying property (1). The condition needed is that the original zonotope Z admit a group of symmetries that is simply transitive on the vertex set of Z. The most obvious zonotopes with this property are the so-called "Coxeter cells". So, this paper is a continuation and specialization of [DJS] to the case of hyperplane arrangements associated to finite reflection groups. (N.B. a *Coxeter cell* is a zonotope corresponding to a hyperplane arrangement associated to a finite reflection group W on $\mathbb{R}^{n+1}$. A *Coxeter cell complex* is a regular cell complex in which each cell is isomorphic to a Coxeter cell. For example, since an $(n+1)$-cube is the Coxeter cell associated to $(\mathbb{Z}_2)^{n+1}$, any cubical complex is a Coxeter cell complex.)

In [DJS] we also discussed a generalization of the blow-up procedure to zonotopal cell complexes. Again, in order for property (1) to hold we need to require that the zonotopal cell complex admit a group of automorphisms that acts simply transitively on its vertex set. Examples of zonotopal cell complexes with this property are provided by Coxeter groups. Associated to any Coxeter system (W, S), there is a Coxeter cell complex $\Sigma(W, S)$ such that W acts simply transitively on its vertex set. (Here W might be infinite.) Thus, we also want to apply our blowing up procedures to the complexes $\Sigma(W, S)$.

As data for such a blowing up procedure it is necessary to specify the set of cells which are to be blown up. There are two extreme cases, the "minimal blow-up" and the "maximal blow-up". In the case of a minimal blow-up, this set of cells is the collection of all cells that cannot be decomposed as a nontrivial product. In the case of a maximal blow-up, it is the set of all cells.

Next we describe the motivating example for this paper (which was also one of the motivating examples for [DJS]). Consider the action of the symmetric group S_{n+2} as a reflection group on $\mathbb{R}^{n+1}$. The associated hyperplane arrangement is called the "braid arrangement". Let M^n denote the minimal blow-up (as in [DP] or [DJS]) of the corresponding arrangement in $\mathbb{RP}^n$. The interesting feature of M^n lies in the result of Kapranov [Ka1, Ka2] that M^n can be identified with $\overline{\mathcal{M}}_{0,n+3}(\mathbb{R})$, the real points of the Grothendieck-Knudsen moduli space of stable $(n+3)$-pointed curves of genus 0 (which, in turn, coincides with the Chow quotient $(\mathbb{RP}^1)^{n+3}//PGL(2,\mathbb{R})$). Kapranov also showed that each tile of the dual cellulation of M^n was a copy of Stasheff's polytope, the n-dimensional associahedron, K^n.

As explained in [Lee] or in Section 8, below, the set of codimension-one faces of K^n can be identified with the set of proper subintervals of $[1, n+1]$ with integer endpoints. Moreover, given two such subintervals T and T', the corresponding faces intersect if and only if either (i) the distance between T and T' (as subsets of $[1, n+1]$) is at least 2 or (ii) $T' \subset T$ (or $T \subset T'$).

In the case at hand, where $M^n = \overline{\mathcal{M}}_{0,n+3}(\mathbb{R})$, the symmetry group G is S_{n+2}. The group A has one involutory generator α_T for each proper subinterval T of $[1, n+1]$. The codimension-two faces of K^n impose additional relations of two types: (i) if the distance between T and T' is at least 2 then $(\alpha_T \alpha_{T'})^2 = 1$ and (ii) if $T' \subset T$, then $\alpha_T \alpha_{T'} \alpha_T = \alpha_{T''}$ (where T'' denotes the image of T' under the order-reversing involution of T). The epimorphism $A \to S_{n+2}$ sends α_T to the order-reversing involution in the subgroup of S_{n+2} corresponding to T. Looking at relation (ii), it is clear that if the interval T is not a single point, then α_T will not act as a reflection on $\widetilde{M}^n$. We call it a "mock reflection" and A a "mock reflection group".

Similarly, given any finite Coxeter group W, one can take the minimal blow-up of the associated projective hyperplane arrangement to obtain a manifold M^n with a cubical cell structure. When the Coxeter diagram of W is an interval, the tiles will again be associahedra.

Other examples arise by taking the maximal blow-up of an arrangement associated to a finite reflection group W. In any such example each tile is a permutohedron. (In the case where $W = (\mathbb{Z}_2)^{n+1}$ these examples occur in nature as real toric varieties associated to flag manifolds.)

In Sections 7 and 8 we prove some classification results for the universal covers of the permutohedral and associahedral tilings which arise from blow-ups. In Section 7, we show that the universal covers of all such permutohedral tilings yield the same tiling of $\mathbb{R}^n$; moreover, the various symmetry groups A that arise in this fashion are commensurable with each other (and with the right-angled reflection group associated to the permutohedron). By way of contrast, in Section 8, we show that the various associahedral tilings of $\mathbb{R}^n$ tend not to be isomorphic with each other. The reason for this dichotomy lies in the fact that the associahedron is much less symmetric than is the permutohedron. It turns out, however, that in dimensions ≤ 3 all of the symmetry groups arising from these permutohedral and associahedral tilings are quasi-isometric to each other (Theorem 8.5.6 and Theorem 8.7.1).

Section 5, the longest section of the paper, concerns the general theory of tilings. The results in this section are of a somewhat different nature than in the rest of the paper. We develop the theory in a context which is considerably more general than is indicated by the above examples. All of our previous requirements are either weakened or dropped as explained below in statements (a) through (e).

(a) The space is not required to be a manifold.
(b) The cell structure on the space need not be cubical; however, each cell is required to be a Coxeter cell. (This is to accomodate the "partial blow-ups" of [DJS] and also to include arbitrary reflection type tilings in our general theory).
(c) In view of (a), there may no longer be a well-defined dual cell structure; however, there are still "dual cones" and "tiles" (that is, cones dual to vertices).
(d) The requirement that there exist a group G that acts simply transitively on the vertex set is replaced by the requirement that the cell complex X admit a "framing" (which amounts to specifying isomorphisms between the links of any two vertices of X).
(e) The requirement of nonpositive curvature is dropped.

Thus, in Section 5 we shall largely abandon the notion of a fundamental tile (since, in view of (c), it need not be a cell). By a "tiling" we will simply mean a Coxeter cell complex X in which the links of any two vertices are isomorphic. The tiling is "symmetric" if X admits a group action which is simply transitive on its vertex set. If X is symmetric and simply connected, then one can read off a presentation for its symmetry group A (cf. Section 5.4) as before.

A key ingredient in our analysis of framed tilings and their symmetry groups is the notion of a "gluing isomorphism." This is is an isomorphism between two "codimension-one faces" of a fundamental tile. It determines how two adjacent tiles are glued together. In practice (e.g. when the symmetry group is generated by involutions), these gluing isomorphisms will always be involutions. For simplicity, let us assume this. For example, in a reflection type tiling, each gluing involution is the identity map. For blow-ups of $\Sigma(W, S)$, the gluing involutions are determined by the elements of longest length in various finite special subgroups of W. In 5.3 and 5.6, we give necessary and sufficient conditions on the sets of gluing involutions for the universal covers of two tilings to be isomorphic. (This result is then used in Sections 7 and 8 to classify certain permutohedral and associahedral tilings.) In Theorem 5.10.1, we describe necessary and sufficient conditions under which a given fundamental tile and set of gluing involutions can be realized by a symmetric tiling.

In Sections 5 and 6 we also prove, under very mild hypotheses, some general results about the symmetry group A of a symmetric and simply connected X. Since these hypotheses are satisfied in our examples, these results apply to the universal cover of a blow-up. First of all, even without the nonpositive curvature requirement, X can always be "completed" to a CAT(0)-complex $\widehat{X}$ by adding a finite number of A-orbits of cells. (This is proved in Sections 5.7 and 5.9.) Thus, each such A is a "CAT(0) group" (cf. Theorem 5.9.3). In particular, if π is any torsion-free subgroup of finite index in A, then $\widehat{X}/\pi$ is a finite $K(\pi, 1)$-complex. Secondly, A has a linear representation analogous to the canonical represention for a Coxeter group (Section 6). Whether this representation is always faithful, however, remains an interesting open question.

2. Some cell complexes associated to Coxeter groups

2.1. Coxeter systems. We first recall some standard facts about Coxeter groups; we refer the reader to [Bo], [Bro], or [H] for details.

Definition 2.1.1. Let S be a finite set. A *Coxeter matrix* on S is a symmetric $S \times S$ matrix $M = (m(s, s'))$ with entries in $\mathbb{N} \cup \{\infty\}$ such that $m(s, s') = 1$ if $s = s'$ and $m(s, s') \geq 2$ if $s \neq s'$. Given a Coxeter matrix M, the corresponding *Coxeter group* is the group W defined by the presentation

$$W = \langle S \mid (ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S\rangle.$$

The pair (W, S) is a *Coxeter system.*

A Coxeter system (W, S) determines a *length function* $l : W \to \mathbb{Z}_{\geq 0}$. Given $w \in W$, the length $l(w)$ is defined to be the minimal n such that $w = s_1 s_2 \cdots s_n$ and $s_i \in S$.

To any Coxeter system (W, S) we associate a labeled graph $\Gamma(W, S)$, called the *Coxeter diagram*, as follows. The vertex set is S and two vertices s, s' determine an edge if and only if $m(s, s') > 2$. In this case, the edge joining s and s' is labeled $m(s, s')$.

Any subset $T \subset S$ generates a subgroup $W_T \subset W$, which is itself a Coxeter group with Coxeter system (W_T, T). W_T is called a *special subgroup* of W. The Coxeter diagram $\Gamma(W_T, T)$ is the induced labeled subgraph of $\Gamma(W, S)$ with vertex set T. A subset $T \subset S$ is *spherical* if W_T is finite. Any finite special subgroup W_T has a unique element of longest length, denoted by w_T. Moreover, w_T is an involution and $w_T T w_T = T$ (Exercise 22, page 43 in [Bo]).

If W_1 and W_2 are Coxeter groups, then so is $W = W_1 \times W_2$. Let (W_1, S_1) and (W_2, S_2) be Coxeter systems for W_1 and W_2, respectively, and let $S = S_1 \times \{1\} \cup \{1\} \times S_2$. Then (W, S) is a Coxeter system for W, and the Coxeter diagram $\Gamma(W, S)$ is the disjoint union of the labeled graphs $\Gamma(W_1, S_1)$ and $\Gamma(W_2, S_2)$.

2.2. Coxeter cells. Let W be a finite Coxeter group. Then W can be represented as a group generated by orthogonal reflections on a finite dimensional Euclidean space V. The reflection hyperplanes of this representation separate V into simplicial cones, called *chambers*, and W acts transitively on the set of chambers. In fact, the representation of W can be chosen such that for any Coxeter system (W, S), the generators in S correspond to the reflections through the supporting hyperplanes of a fixed chamber C. We call this fixed chamber the *fundamental chamber*, and we call the reflections in S the *simple reflections.*

Definition 2.2.1. Let x be a point in C that is unit distance from each of the supporting hyperplanes, and let $Z = Z(W, S)$ be the convex hull of the orbit Wx. The polytope Z is called a *(normalized) Coxeter cell of type* W. The intersection of Z with C (or with any translate of C by an element of W) is called a *Coxeter block of type* W.

The group W acts isometrically on Z, and the Coxeter block $B = Z \cap C$ is a fundamental domain (Figure 1). B is combinatorially equivalent to a cube of dimension $\operatorname{Card} S$. Since W acts freely and transitively on the vertices of Z, we can identify the vertices of Z with the elements of W (once we identify x with 1). Each vertex is contained in a unique Coxeter block of type W.

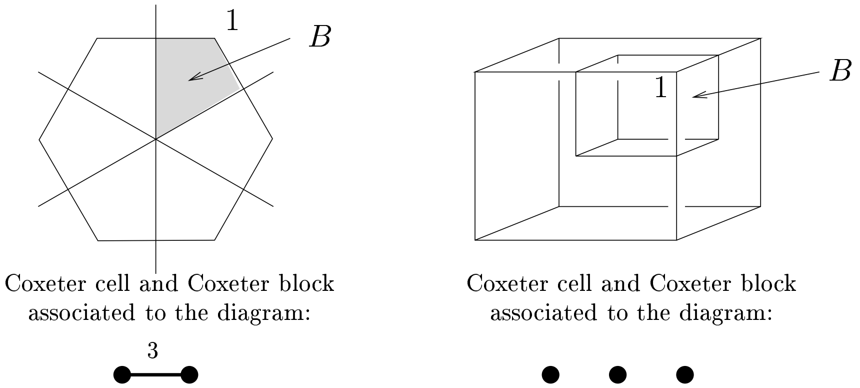


FIGURE 1.

The Coxeter block B has two types of codimension-one faces. One type is an intersection of B with a codimension-one face of C. These are the *mirrors* of B. To descibe the others, we first describe the faces of a Coxeter cell. Let W_T be a special subgroup and let Z_T be a normalized Coxeter cell of type W_T. Then the inclusion $W_T \to W$ induces an isometry from Z_T onto a face of Z (in fact, every face of Z is of the form wZ_T for some $T \subset S$ and $w \in W$). The remaining codimension-one faces of the Coxeter block B can now be identified with the Coxeter blocks associated to the codimension-one faces $Z_T \subset Z$ (i.e., where T has cardinality one less than S).

If Z_1 and Z_2 are (normalized) Coxeter cells of types W_1 and W_2, respectively, then the product $Z_1 \times Z_2$ is a (normalized) Coxeter cell of type $W_1 \times W_2$. In particular, the n-cube $[-1, 1]^n$ is a Coxeter cell; the Coxeter block containing the vertex $(1, 1, \ldots, 1)$ is $[0, 1]^n$ (Figure 1).

2.3. Coxeter cell complexes. A locally finite, regular cell complex X is a *Coxeter cell complex* if all of its cells are Coxeter cells. Since any combinatorial isomorphism between two (normalized) Coxeter cells Z and Z' is induced by an isometry, any Coxeter cell complex X has a canonical piecewise Euclidean metric, and any isomorphism of Coxeter cell complexes is induced by a unique isometry.

Let X be a Coxeter cell complex X, and let $X^{(i)}$ denote the set of i-cells in X. Let $x \in X^{(0)}$ be a vertex of X. The *link of* x, denoted $\mathrm{Lk}(x, X)$, is the (piecewise spherical) simplicial complex consisting of all points in X that are unit distance from x. The *closed star of* x, denoted $\overline{\mathrm{St}}(x, X)$, is the subcomplex of X consisting of all cells that contain x and all of their faces.

Definition 2.3.1. A Coxeter cell complex X is a *tiling* if for any two vertices x and x', there exists a combinatorial isomorphism $\overline{\mathrm{St}}(x, X) \to \overline{\mathrm{St}}(x', X)$ taking x to x'.

Let (W, S) be a (possibly infinite) Coxeter system and let $\mathcal{S}$ be the set of spherical subsets of S. (It is clear that $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex with vertex set S.) We can then define a Coxeter cell complex $\Sigma = \Sigma(W, S)$ that generalizes the Coxeter cell of a finite Coxeter group. The vertex set of Σ is W. We take a Coxeter cell of type W_T for each coset wW_T where $T \in \mathcal{S}$, and identify its vertices with the elements of wW_T. We then identify two faces of two Coxeter cells if they have the same vertex set. If Z is the cell in Σ corresponding to the coset wW_T, we define its *type* to be the subset T. (The type of a cell is well-defined since $wW_T = w'W_{T'}$ implies $T = T'$.)

It is clear that Σ is a Coxeter cell complex and that W acts via combinatorial automorphisms. Since W acts simply transitively on the vertices, Σ is a tiling. The cells of Σ that contain the vertex 1 are in bijection with the set $\mathcal{S}_{>\emptyset}$; thus, $\mathrm{Lk}(1, \Sigma)$ can be identified with $\mathcal{S}_{>\emptyset}$. More generally, we have the following:

Definition 2.3.2. Let L be a subcomplex of $\mathcal{S}_{>\emptyset}$ with the same 1-skeleton. Then the *reflection tiling* of type (W, S, L), denoted $\Sigma(W, S, L)$, is the subcomplex of $\Sigma(W, S)$ consisting of cells corresponding to the cosets wW_T where T is either the empty set or the vertex set of a simplex in L. A reflection tiling is *complete* if $L = \mathcal{S}_{>\emptyset}$.

Example 2.3.3. If L is the 0-skeleton of $\mathcal{S}_{>\emptyset}$, then $\Sigma(W, S, L)$ is the Cayley graph of (W, S). Similarly, if L is the 1-skeleton of $\mathcal{S}_{>\emptyset}$, then $\Sigma(W, S, L)$ is the Cayley 2-complex associated to the standard presentation of W.

Example 2.3.4. Suppose W is finite. Then $\Sigma(W, S)$ is the zonotope $Z(W, S)$. If L is the boundary of the simplex $\mathcal{S}_{>\emptyset}$, then $\Sigma(W, S, L) = \partial Z$.

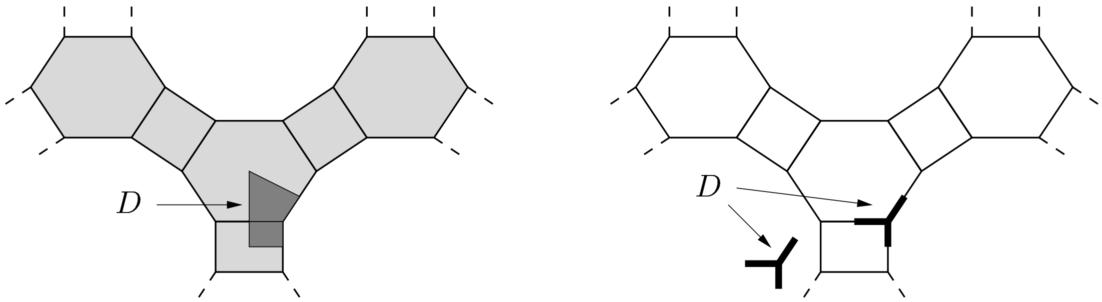
2.4. Coxeter tiles and the local geometry of Coxeter cell complexes. Let X be a Coxeter cell complex. For any vertex $x \in X^{(0)}$, the *dual cone* at x, denoted $D(x, X)$ is the union of all Coxeter blocks in X that contain the vertex x. It can be identified with a subcomplex of the barycentric subdivision of X and therefore, has a natural piecewise Euclidean metric.

Definition 2.4.1. Let Σ be the reflection tiling of type (W, S, L). Then the *Coxeter tile* of type (W, S, L), denoted $D(W, S, L)$, is the dual cone $D(1, \Sigma)$. If $L = \mathcal{S}_{>\emptyset}$, we shall denote the corresponding Coxeter tile $D(W, S)$. An *isomorphism* $D(W, S, L) \to$

$D(W', S', L')$ of Coxeter tiles is a bijection $S \to S'$ that (1) extends to a simplicial isomorphism $L \to L'$, and (2) extends to a group isomorphism $W \to W'$.

It is clear that the geometric realization of an isomorphism of Coxeter tiles is an isometry.

Example 2.4.2. Let (W, S) be the (infinite) Coxeter group with generating set $S = \{a, b, c\}$ and Coxeter diagram as in Figure 2. Then the complete reflection tiling $\Sigma(W, S)$ is an infinite 2-dimensional cell complex whose 2-cells are regular hexagons and squares. The Coxeter tile $D(W, S)$ is a union of two Coxeter blocks, one from the hexagon and one from the square. See Figure 2.



Complete reflection tiling (and Coxeter tile) associated to the (infinite) Coxeter group:

$$a \overset{3}{—} b \overset{\infty}{—} c$$

Reflection tiling (and Coxeter tile) associated to the subcomplex $L = \{\{a\}, \{b\}, \{c\}\}$

FIGURE 2.

If we take L to be the 0-skeleton of $\mathcal{S}_{>\emptyset}$ obtained by removing the spherical subsets $\{a, b\}$ and $\{a, c\}$, we obtain the second reflection tiling $\Sigma(W, S, L)$ shown in Figure 2. The associated Coxeter tile $D(W, S, L)$ is the cone on three vertices. (Note that in this example, $\Sigma(W, S, L)$ is the Cayley graph of (W, S), and $\Sigma(W, S)$ is the Cayley 2-complex.)

Example 2.4.3. Suppose W is finite. Then $\mathcal{S}_{>\emptyset}$ is a simplex and $D(W, S)$ is the Coxeter block of type (W, S), as in Definition 2.2.1. If $L = \mathcal{S}_{>\emptyset} - \{S\}$ is the boundary of the simplex $\mathcal{S}_{>\emptyset}$, then $D(W, S, L)$ is called the *fundamental simplex*. It is the intersection of ∂Z with the fundamental chamber C. The image of $\partial Z \cap C$ in projective space is a simplex. So, we think of the cell complex $\Delta = D(W, S, L)$ as being a subdivision of the simplex into Coxeter blocks.

Remark 2.4.4. The Coxeter tile $D(W, S, L)$ is a fundamental domain for the W-action on the reflection tiling $\Sigma(W, S, L)$.

If X is any Coxeter cell complex and x is a vertex of X, then the dual cone $D(x, X)$ is a Coxeter tile. To determine its type, we let $L_x = \mathrm{Lk}(x, X)$, we let V_x be the set of vertices in L_x (equivalently, the set of edges containing x), and we define a Coxeter matrix M_x on V_x as follows:

$$m_x(v,v') = \begin{cases} 1 & \text{if } v = v' \\ m & \text{if } v \text{ and } v' \text{ correspond to distinct edges of a } 2m\text{-gon in } X \\ \infty & \text{otherwise.} \end{cases}$$

Letting (W_x, V_x) be the corresponding Coxeter system, it is clear that L_x consists of spherical subsets of V_x. It follows that $D(x, X)$ can be identified with the Coxeter tile of type (W_x, V_x, L_x).

Remark 2.4.5. If X is a tiling, then there exists a Coxeter tile $D = D(W, S, L)$ and, for each $x \in X^{(0)}$, an isomorphism $\phi_x : D \to D(x, X)$. Since D may have nontrivial automorphisms, the isomorphism ϕ_x need not be unique.

3. The blow-up $\Sigma_\#$ of a reflection tiling Σ

3.1. The blow-up of a Coxeter cell. Let W be a finite Coxeter group, (W, S) be a Coxeter system, and let Z $(= Z(W, S))$ be the corresponding Coxeter cell. Then the boundary ∂Z is a Coxeter cell complex homeomorphic to S^{n-1}. It is the reflection tiling $\Sigma(W, S, L)$ where L is the boundary complex of the simplex $\mathcal{S}_{>\emptyset}$. A fundamental domain for the W-action on ∂Z is the fundamental simplex $\Delta = \partial Z \cap C$ $(= D(W, S, L))$. A *mirror* of Δ is its intersection with a codimension-one face of C. The Coxeter block $B = Z \cap C$ is homeomorphic to the cone on Δ.

The antipodal map $a : Z \to Z$ defined by $x \mapsto -x$, restricts to an isometric involution on ∂Z. This involution on ∂Z freely permutes the cells; hence the quotient $\mathbb{P}(Z) = \partial Z / a$ is a Coxeter cell complex, homeomorphic to $\mathbb{RP}^{n-1}$.

Definition 3.1.1. The *blow-up of Z at its center*, denoted $Z_\#$, is the quotient of $\partial Z \times [-1, 1]$ by the involution $\widehat{a}$ defined by $\widehat{a}(x, t) = (-x, -t)$, i.e.,

$$Z_\# = \partial Z \times_{\widehat{a}} [-1, 1].$$

The blow-up $Z_\#$ is an interval bundle over $\mathbb{P}(Z)$. Let $\pi : Z_\# \to \mathbb{P}(Z)$ be the projection. For any face $F \subset \partial Z$, $-F \cap F = \emptyset$; hence, $\pi(F)$ is embedded in $\mathbb{P}(Z)$. Moreover, $\pi^{-1}(\pi(F))$ can be identified with $F \times [-1, 1]$ (where $F \times \{1\}$ corresponds to F and $F \times \{-1\}$ to $-F$). Since F is a Coxeter cell, so is $F \times [-1, 1]$. Thus, $Z_\#$ is naturally a Coxeter cell complex. Since the antipodal map a commutes with the W-action, there is an induced W-action on $Z_\#$.

Similarly, $\pi(\Delta)$ is embedded in $\mathbb{P}(Z)$ and $\pi^{-1}(\pi(\Delta)) \cong \Delta \times [-1, 1]$. Since w_S maps Δ to $-\Delta$, the stabilizer of $\Delta \times [-1, 1]$ is the cyclic group of order two generated by w_S. Thus, $\Delta \times [0, 1]$ is a fundamental domain for W on $Z_\#$. Put

$$B_\# = \Delta \times [0, 1].$$

We think of $B_\#$ as being obtained from B by "truncating" its vertex (the cone point) and introducing a new codimension-one face corresponding to $\Delta \times \{0\}$. A *mirror* of $B_\#$ is either a codimension-one face of the form $\Delta_{\{s\}} \times [0, 1]$, where $\Delta_{\{s\}}$ is a mirror of Δ, or the codimension-one face $\Delta \times \{0\}$.

The space $Z_\#$ is tiled by the translates of $B_\#$ by elements of W. The adjacent tile to $B_\#$ across the mirror $\Delta \times \{0\}$ is $w_S B_\#$. If we identify $w_S B_\#$ with $\Delta \times [0, 1]$ via w_S, then $B_\# \cup w_S B_\#$ ($\cong \Delta \times [-1,1]$) is homeomorphic to two copies of $\Delta \times [0,1]$ glued along $\Delta \times \{0\}$. The gluing map $j_S : \Delta \to \Delta$ is given by $j_S = a \circ w_S$.

Let 1 denote the vertex of Z that is contained in the interior of C. We note that Δ is the union of all Coxeter blocks in ∂Z which contain the vertex 1 and that B is the union of all Coxeter blocks in Z that contain 1. Similarly, $B_\#$ is the union of all Coxeter blocks in $Z_\#$ that contain 1.

Remark 3.1.2. Suppose (W,S) is a finite irreducible Coxeter group. The element of longest length, w_S, is equal to the antipodal map, a, in the following cases ([Bo], Appendix I-IX, pp. 250-275): $\mathbf{A}_1$ (the cyclic group of order 2), $\mathbf{I}_2(p)$ with p even (the dihedral group of order $2p$), $\mathbf{B}_n$ (the hyperoctahedral groups), $\mathbf{D}_n$ with n even, $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{F}_4$, $\mathbf{E}_7$, and $\mathbf{E}_8$. Hence, in all these cases, j_S is the identity map.

In the remaining cases w_S is not the antipodal map, and conjugating by it induces a nontrivial diagram automorphism of $\Gamma(W,S)$. These cases are: $\mathbf{A}_n$ with $n>1$, $\mathbf{D}_n$ with n odd, $\mathbf{I}_2(p)$ with p odd, and $\mathbf{E}_6$. In each of these cases, the Coxeter diagram admits a unique non-trivial automorphism which, in fact, coincides with the diagram automorphism induced by conjugation by w_S.

If W is finite and reducible, then w_S is the product of the elements of longest length in each factor and, therefore, is the antipodal map if and only if it is antipodal in each factor.

3.2. The blow-up of a reflection tiling $\Sigma(W,S,L)$. Our goal in this section is to describe a functorial generalization of the blow-up of a Coxeter cell that (1) allows for iterated blow-ups of faces, (2) makes sense more generally for the complexes $\Sigma(W,S,L)$, and (3) preserves the W-action. Given a complex $\Sigma(W,S,L)$ and a suitable collection of cells to be blown-up, this functor will produce for every subcomplex $K \subset \Sigma$ a cell complex $K_\#$ called the "blow-up of K". Our primary interest in this paper is the topology of the blow-up $\Sigma_\# = \Sigma(W,S,L)_\#$.

Definition 3.2.1. Suppose L is a simplicial complex with vertex set V. We let $\mathcal{P}(L)$ denote the poset consisting of those subsets T of V such that either $T = \emptyset$ or T is the vertex set of a simplex in L. Thus, we can identify L with $\mathcal{P}(L)_{>\emptyset}$.

Let Σ be the reflection tiling $\Sigma(W,S,L)$, and let $\mathcal{P}$ be the poset $\mathcal{P}(L)$. Then the cells of Σ are indexed by the set $W\mathcal{P} = \{wW_T \mid T \in \mathcal{P}\}$ (see Definition 2.3.2). Any collection of cells defining one of our blow-ups $\Sigma_\#$ will satisfy the following condition: if a face Z' of a cell Z is blown-up, then either

(1) Z is blown up also, or
(2) Z is of the form $Z' \times Z''$ and the blow-up functor is applied to the two factors independently.

In addition, if the W-action on Σ is to induce an action on the blow-up $\Sigma_\#$, the collection of cells we blow-up should be W-invariant. Since the cells of Σ are indexed by the poset $W\mathcal{P}$, the blow-up collection will be indexed by a subset of the form $W\mathcal{R}$ for some subset $\mathcal{R} \subset \mathcal{P}$. With this in mind, we let $\mathcal{R}$ be any subset of $\mathcal{P}$, and define a category whose objects are all subcomplexes of Σ, but whose morphisms depend on $\mathcal{R}$.

A *subcomplex* of Σ is a Coxeter cell complex K together with an injective cellular map $K \to \Sigma$. Via this map, we will often identify K with its image in Σ. In particular, any cell Z of a subcomplex K has a well-defined type T where $T \in \mathcal{P}$ (cf. section 2.3). An $\mathcal{R}$-*morphism* $f : K_1 \to K_2$ between two subcomplexes of Σ is an injective cellular map satisfying the condition: for every cell $Z \subset K_1$, the type of Z is in $\mathcal{R}$ if and only if the type of $f(Z)$ is in $\mathcal{R}$.

There are primarily two kinds of $\mathcal{R}$-morphisms relevant to our construction. Since the W-action on Σ preserves the cells of a given type, the automorphism $w : \Sigma \to \Sigma$ is an $\mathcal{R}$-morphism for every $w \in W$. Other $\mathcal{R}$-morphisms are provided by antipodal maps on cells in Σ (though in general only some of these are $\mathcal{R}$-morphisms). To make sense of (1) above, i.e., to iterate the blow-up procedure, we need the antipodal map on Z to be a morphism in our category.

Lemma 3.2.2. *Given $T \in \mathcal{P}$, let $j_T : \mathcal{P}_{\leq T} \to \mathcal{P}_{\leq T}$ be the involution defined by $T' \mapsto w_T T' w_T^{-1}$ where w_T is the longest element in W_T. Let $Z_T \subset \Sigma$ be the cell with vertices W_T, and let $a_T : \partial Z_T \to \partial Z_T$ be the antipodal map $x \mapsto -x$. Then a_T is an $\mathcal{R}$-morphism if and only if*

$$j_T(\mathcal{R}_{\leq T}) = \mathcal{R}_{\leq T}.$$

Proof. The involution w_T is an $\mathcal{R}$-morphism and the composition $a \circ w_T$ maps the face $Z_{T'}$ onto the face $Z_{j_T T'}$. □

To make sense of (2), we recall the basic facts about product decompositions of Coxeter cells. Let Γ denote the Coxeter diagram $\Gamma(W, S)$, and for every $T \subset S$, let Γ_T denote the subdiagram $\Gamma(W_T, T)$. Two spherical sets T_1 and T_2 in $\mathcal{P}$ are *completely disjoint* if they are disjoint and no edge of $\Gamma_{T_1 \cup T_2}$ connects a vertex of Γ_{T_1} to a vertex of Γ_{T_2}. Given $T \in \mathcal{P}$, a collection $\{T_1, \dots, T_k\} \subset \mathcal{P}$ is called a *decomposition of* T if $T = T_1 \cup \dots \cup T_k$ and the T_i's are pairwise completely disjoint. If $\{T_1, \dots, T_k\}$ is a decomposition of T, then there is a corresponding direct product decomposition of groups

$$W_T = W_{T_1} \times \dots \times W_{T_k}$$

and a corresponding direct product decomposition of cells

$$Z_T = Z_{T_1} \times \dots \times Z_{T_k}.$$

If our blow-up collection $\mathcal{R}$ is to satisfy (1) and (2) above, then whenever Z_T is not blown-up and $Z_{T_1}, \dots, Z_{T_k}$ are the maximal faces of Z_T that are blown up, there should be some other subset $T_0 \subset T$ and a decomposition

$$Z_T = Z_{T_0} \times Z_{T_1} \times \dots \times Z_{T_k}.$$

We make this precise as follows.

Given $T \in \mathcal{P}$, we define $\mathcal{R}T$, the $\mathcal{R}$-*maximals in* T, to be the set of maximal elements of $\mathcal{R}_{\leq T}$, and we define T_0, the $\mathcal{R}$-*fixed part of* T, to be the complement of the union of the $\mathcal{R}$-maximals in T. To satisfy (1) and (2), $\{T_0\} \cup \mathcal{R}T$, must be a decomposition of T. Thus, if $\mathcal{R}T = \{T_1, \dots, T_k\}$, then the T_i's must be pairwise completely disjoint. Combining this requirement with the condition of Lemma 3.2.2, we make the following definition.

Definition 3.2.3. A collection $\mathcal{R} \subset \mathcal{P}$ is *admissible* if it satisfies the following two conditions:

(1) For every $T \in \mathcal{R}$, $j_T(\mathcal{R}_{\leq T}) = \mathcal{R}_{\leq T}$.
(2) For every $T \in \mathcal{P}$, the set $\{T_0\} \cup \mathcal{R}T$ is a decomposition of T.

The collection $\mathcal{R}$ is *fully admissible* if, in addition, whenever $s, t \in S$ and $2 < m(s,t) < \infty$, then $\{s,t\} \in \mathcal{R}$. The collection $\{T_0\} \cup \mathcal{R}T$ is called the *$\mathcal{R}$-decomposition of* T, and the elements $T_1, \dots, T_k$ of $\mathcal{R}T$ are the *blow-up factors of the decomposition.*

Let $\mathscr{K} = \mathscr{K}_{\mathcal{R}}(W,S,L)$ be the category whose objects are the subcomplexes of Σ and whose morphisms are $\mathcal{R}$-morphisms. Let $\mathscr{C}$ be the category whose objects are Coxeter cell complexes and whose morphisms are isomorphisms onto subcomplexes.

Proposition 3.2.4. *Let $\mathcal{R}$ be admissible. Then there exists a unique functor $\mathscr{K} \to \mathscr{C}$, denoted by $K \mapsto K_\#$ (and $f \mapsto f_\#$), satisfying the two properties:*

(1) *If $T \in \mathcal{R}$, then $(Z_T)_\#$ is the quotient of $(\partial Z_T)_\# \times [-1,1]$ by the involution $\widehat{a} : (x,t) \mapsto (a_\#(x), -t)$, where $a : \partial Z_T \to \partial Z_T$ is the antipodal map. That is,*
$$(Z_T)_\# = (\partial Z_T)_\# \times_{\widehat{a}} [-1,1].$$
(2) *If $T \in \mathcal{P}$ has $\mathcal{R}$-decomposition $\{T_0, T_1, \dots, T_k\}$, then*
$$(Z_T)_\# = Z_{T_0} \times (Z_{T_1})_\# \times \cdots \times (Z_{T_k})_\# .$$

Proof. This functor is a special case of the blow-up of "partially mined zonotopal cell complexes" as defined in [DJS]. If $\mathcal{R}$ is admissible, then the pair $(\Sigma, W\mathcal{R})$ is a partially mined zonotopal cell complex (cf., [DJS]), and for any subcomplex $K \subset \Sigma$ one obtains a partially mined subcomplex $(K, \mathcal{M})$. Any $\mathcal{R}$-morphism $f : K_1 \to K_2$ between subcomplexes of Σ correspond to a morphism $f : (K_1, \mathcal{M}_1) \to (K_2, \mathcal{M}_2)$ of partially mined zonotopal cell complexes. The functor $K \mapsto K_\#$ (and $f \mapsto f_\#$) is then precisely the blow-up functor $(K, \mathcal{M}) \mapsto K_{\#\mathcal{M}}$ (and $f \mapsto f_\#$) defined in [DJS]. (If $\mathcal{R}$ is fully admissible, then we can omit the term "partially" from the above discussion.)

Properties (1) and (2) are evident from the construction in [DJS], and uniqueness follows from the fact that the functor is defined inductively by these properties (and the W-action). □

Definition 3.2.5. Let Σ be a reflection tiling $\Sigma(W,S,L)$, and let $\mathcal{R}$ be an admissible subset of $\mathcal{P}(L)$. Then the Coxeter cell complex $\Sigma_\#$ obtained by applying the blow-up functor to Σ is called the *$\mathcal{R}$-blow-up of* Σ (or just the *blow-up of* Σ if there is no ambiguity).

Remark 3.2.6. If $\mathcal{R}$ is fully admissible, then each Coxeter cell in $\Sigma_\#$ is a cube.

Remark 3.2.7. If $T \in \mathcal{R}$, then since $(Z_T)_\#$ is an interval bundle over $(\partial Z_T)_\# / a_\#$, the fundamental group of $(Z_T)_\#$ is isomorphic to that of $(\partial Z_T)_\# / a_\#$.

Remark 3.2.8. Blow-up functors are compatible in the following sense. Suppose Σ is the complex $\Sigma(W,S,L)$, and $\mathcal{R}$ is an admissible subset of $\mathcal{P}(L)$. Let Σ' be a complex of the form $\Sigma(W', S', L')$ where $S' \subset S$, $W' \subset W$, and $L' \subset L$. Then Σ' can be naturally identified with a subcomplex of Σ. Moreover, the set $\mathcal{R}' = \mathcal{R} \cap \mathcal{P}(L')$ is an admissible subset of $\mathcal{P}(L')$, and the $\mathcal{R}'$-blow-up of Σ' coincides with the $\mathcal{R}$-blow-up of Σ' (where Σ' is viewed as a subcomplex of Σ).

Remark 3.2.9. If $\mathcal{R}$ is admissible, then so is the collection $\mathcal{R}'$, obtained by removing from $\mathcal{R}$ all elements of the form $\{s\}$ where $s \in S$. Moreover, it follows from properties (1) and (2) of the blow-up functor that for any singleton set $\{s\} \in \mathcal{P}$, the blow-up $(Z_{\{s\}})_\#$ is canonically isomorphic to $Z_{\{s\}}$ regardless of whether $\{s\}$ is in $\mathcal{R}$ or not. Thus, for any subcomplex $K \subset \Sigma$, the $\mathcal{R}$-blow-up of K and the $\mathcal{R}'$-blow-up of K are the same. For this reason, we shall assume for the remainder of the article that all admissible sets $\mathcal{R}$ contain no singleton subsets in $\mathcal{P}$.

Definition 3.2.10. $\mathcal{R}$ is the *maximal blow-up set* if it contains every spherical set in $\mathcal{P}$ of cardinality at least two.

Definition 3.2.11. $\mathcal{R}$ is the *minimal blow-up set* if it consists of the spherical sets T in $\mathcal{P}$ such that $\operatorname{Card}(T) \geq 2$ and Γ_T is connected.

3.3. The link of a vertex in $\Sigma_\#$. Let $L_\#$ denote the link of the vertex 1 in an $\mathcal{R}$-blow-up $\Sigma_\#$. In this subsection we shall decribe the poset of simplices in $L_\#$. First we introduce a subset $S_\#$ of $\mathcal{P}$ which will be used to index the vertices of the link. Let

$$S_\# = \mathcal{R} \cup \{\{s\} \mid s \in S\}.$$

Next, we want to define a poset $\mathcal{N}$ of subsets of $S_\#$. It plays the same role for $\Sigma_\#$ as does $\mathcal{P}$ for Σ.

Let $\mathcal{T}$ be a subset of $S_\#$. It is partially ordered by inclusion. The *support of* $\mathcal{T}$, denoted $\operatorname{Supp}\mathcal{T}$ is the union of all elements of $\mathcal{T}$. The $\mathcal{R}$*-fixed part* of $\mathcal{T}$, denoted $\mathcal{T}_0$ is the set of all singletons $\{s\} \in \mathcal{T}$ where $s \notin \operatorname{Supp}\mathcal{T} \cap \mathcal{R}$. (If $\mathcal{R}$ contains no singleton sets, then $\mathcal{T}_0$ consists of the elements $\{s\}$ that are maximal in $\mathcal{T}$.)

Definition 3.3.1. Given a subset $\mathcal{T} \subset S_\#$, let $T = \operatorname{Supp}\mathcal{T}$, $T_0 = \operatorname{Supp}\mathcal{T}_0$, and let $T_1, \dots, T_k$ be the maximal elements of $\mathcal{T} \cap \mathcal{R}$. We give an inductive definition of what it means for $\mathcal{T}$ to be $\mathcal{R}$-*nested*; the induction is on $\operatorname{Card}\mathcal{T}$. If $\mathcal{T} = \emptyset$, then it is $\mathcal{R}$-nested. If $\operatorname{Card}\mathcal{T} > 0$, then it is $\mathcal{R}$-nested if and only if the following two conditions hold:

(1) $T \in \mathcal{P}$ and $\{T_0, T_1, \dots, T_k\}$ is the $\mathcal{R}$-decomposition of T,
(2) $\mathcal{T}_{<T_i}$ is $\mathcal{R}$-nested (as defined by induction) for all $1 \leq i \leq k$.

It is clear that for each $T \in S_\#$, $\{T\}$ is $\mathcal{R}$-nested. The next lemma (which is easily verified) describes which two-element subsets of $S_\#$ are $\mathcal{R}$-nested.

Lemma 3.3.2. *Let T and T' be distinct elements of $S_\#$. Then $\{T, T'\}$ is $\mathcal{R}$-nested if and only if one of the following three cases holds:*

Case 1: *$T = \{s\}$ and $T' = \{s'\}$ where $s, s' \in S$, $m(s, s') < \infty$, and $\{s, s'\} \notin \mathcal{R}$.*
Case 2: *$\{T, T'\}$ is the $\mathcal{R}$-decomposition of $T \cup T'$.*
Case 3: *$T' \subset T$ (or $T \subset T'$).*

Let $\mathcal{N}$ be the set of all $\mathcal{R}$-nested subsets of $S_\#$, partially ordered by inclusion. It is not difficult to see that any subset of an $\mathcal{R}$-nested set is $\mathcal{R}$-nested. In other words, $\mathcal{N}_{>\emptyset}$ is an abstract simplicial complex with vertex set $S_\#$. A subset of $S_\#$ spans a simplex of $\mathcal{N}_{>\emptyset}$ if and only if it is $\mathcal{R}$-nested. The edges of $\mathcal{N}_{>\emptyset}$ are described explicitly in Lemma 3.3.2. It is proved in Section 3.3 of [DJS] that $\mathcal{N}_{>\emptyset}$ is a certain simplicial subdivision of $\mathcal{P}_{>\emptyset}$ and that it can be identified with the link $L_\#$ of the vertex 1 in $\Sigma_\#$.

3.4. Nonpositive curvature. Suppose that (W, S) is a Coxeter system, that L is a subcomplex of $\mathcal{S}_{>\emptyset}$, that $\Sigma = \Sigma(W, S, L)$ (as in Definition 2.3.2), and that $\mathcal{R}$ is an admissible subset of $\mathcal{P}(L)$. Let $\Sigma_\#$ denote the $\mathcal{R}$-blow-up of Σ. In this section, we want to determine when the natural piecewise Euclidean metric on $\Sigma_\#$ is nonpositively curved. Throughout this section, we shall assume, for simplicity, *that $\mathcal{R}$ is fully admissible* (cf. Definition 3.2.3), so that $\Sigma_\#$ is a cubical cell complex. Let $L_\#$ denote the link of the vertex 1 in $\Sigma_\#$, as described in the previous section.

In [G] Gromov showed that a cubical cell complex is nonpositively curved if and only if the link of each of its vertices is a flag complex. (Recall that a simplicial complex K is a *flag complex* if any finite, nonempty, collection of vertices in K that are pairwise connected by edges spans a simplex in K.) Therefore, we have the following lemma.

Lemma 3.4.1. *$\Sigma_\#$ is nonpositively curved if and only if $L_\#$ is a flag complex.*

3.5. Condition (F). Consider the following condition on a spherical set T in $\mathcal{S}_{>\emptyset}$.

(F) Suppose there exists a decomposition $\{T_1, \dots, T_k\}$ of T such that
  (1) $T_i \in S_\#$ for $1 \le i \le k$,
  (2) $T_i \cup T_j \in \mathcal{P} - \mathcal{R}$ for $1 \le i < j \le k$, and
  (3) $k \ge 3$.
  Then $T \in \mathcal{P} - \mathcal{R}$.

In [DJS], we analyzed when links of vertices in blow-ups were flag complexes. Here we are interested in the case where each such link is isomorphic to $L_\#$. The analysis in [DJS] yields the following.

Theorem 3.5.1. *$\Sigma_\#$ is nonpositively curved if and only if Condition (F) holds for each $T \in \mathcal{S}_{>\emptyset}$.*

Proof. We want to see that Condition (F) is equivalent to the condition that $L_\#$ is a flag complex. To check this we must consider collections $\{T_1, \dots, T_k\}$ where each T_i is a vertex of $L_\#$, and each pair $\{T_i, T_j\}$ spans an edge of $L_\#$ and then show that $\{T_1, \dots, T_k\}$ is $\mathcal{R}$-nested. To verify this, we only need consider the case where the T_i's are pairwise incomparable, and this is precisely the case covered by Condition (F). □

Corollary 3.5.2. *Suppose $\Sigma = \Sigma(W, S, L)$ and that $\mathcal{R}$ is the maximal blow-up set (cf., Definition 3.2.10). Then $\Sigma_\#$ is nonpositively curved.*

Proof. Condition (F) holds for any $T \in \mathcal{S}_{>\emptyset}$ (since condition (2) will never be satisfied). □

Corollary 3.5.3. *Suppose that $\Sigma = \Sigma(W, S, L)$ and that $\mathcal{R}$ is the minimal blow-up set (cf., Definition 3.2.11). Then $\Sigma_\#$ is nonpositively curved if and only if the following condition holds: given any three completely disjoint spherical sets T_1, T_2, T_3 in $\mathcal{P}(L)_{>\emptyset}$ with Γ_{T_i} connected for $i = 1, 2, 3$ and with $T_i \cup T_j \in \mathcal{P}(L)$ for $\{i, j\} \subset \{1, 2, 3\}$, then $T_1 \cup T_2 \cup T_3 \in \mathcal{P}(L)$.*

Corollary 3.5.4. *Suppose $\Sigma = \Sigma(W, S)$. Then the minimal blow-up of Σ is nonpositively curved.*

4. The dual tiling of $\Sigma_\#$

4.1. The role of $D_\#$. The quick and clean discription of the blow-up functor is the one given above in 3.2, in terms of blowing up certain collections of Coxeter cells. However, this description leaves obscure an important aspect of the construction, namely, the definition of the fundamental domain (or "fundamental tile") $D_\#$ for the W-action on $\Sigma_\#$. In fact, many of the geometric ideas about these blow-ups are best explained in terms of $D_\#$. The goal of this section is to give the definition of $D_\#$ and discuss some of its properties. The case where W is finite is particularly simple since $D_\#$ can be thought of as a convex polytope. So we shall deal with this case first.

4.2. The case where W is finite. When W is finite, D is the Coxeter block of type (W, S). It is a fundamental domain for W on Z. The *tile* $D_\#$ is obtained from D by truncating those faces that correspond to elements of $\mathcal{R}$. Thus, $D_\#$ is combinatorially equivalent to a convex polytope. The mirrors of $D_\#$ either correspond to the original mirrors of D (these are indexed by S) or to the new codimension-one faces introduced in the truncation process (these are indexed by $\mathcal{R}$).

Similarly, the simplex Δ is a fundamental domain for W on ∂Z. By Remark 3.2.8, the $\mathcal{R}$-blow-up of the ∂Z (viewed as a subcomplex of Z) coincides with the $\mathcal{R}'$-blow-up (where $\mathcal{R}' = \mathcal{R} - \{S\}$) of ∂Z (viewed as a reflection tiling in its own right). The *fundamental tile* $\Delta_\#$ is obtained by truncating the faces of Δ corresponding to elements of $\mathcal{R}'$. (Here we are thinking of Δ simply as a convex simplex, that is, we are temporarily ignoring its subdivision into Coxeter blocks.) Again, $\Delta_\#$ is a convex polytope, each codimension-one face is a mirror, and the mirrors are indexed either by the elements of S or by elements of $\mathcal{R}'$.

Example 4.2.1. The *maximal blow-up* of Z means the case where $\mathcal{R} = \mathcal{S}_{>\emptyset}$. In this case, $\Delta_\#$ is a permutohedron and $D_\# = \Delta_\# \times [0, 1]$.

Example 4.2.2. By the *minimal blow-up* of Z, we mean the case where $T \in \mathcal{R}$ if and only if the Coxeter diagram $\Gamma(W_T, T)$ is connected. If the Coxeter diagram is a straight line segment (i.e., if $\Gamma(W, S)$ is of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{I}_2(p)$, $\mathbf{H}_3$, $\mathbf{H}_4$, or $\mathbf{F}_4$), then it turns out that $\Delta_\#$ is an *associahedron* (as defined by [St]). (See Section 8.)

The next lemma gives an alternative description of $D_\#$ in terms of Coxeter blocks. Its proof follows from the discussion in 3.1.

Lemma 4.2.3. *$D_\#$ is the union of all Coxeter blocks in $Z_\#$ that contain the vertex* 1.

It follows from this lemma that $D_\#$ is a fundamental domain for the W-action on $Z_\#$ in the following sense: the W-orbit of any point x in $Z_\#$ intersects $D_\#$ in at least one point; moreover, this intersection is exactly one point if x does not belong to a mirror of any Coxeter block.

4.3. The general case. We now allow W to be infinite and consider its action on the complex $\Sigma_\#$. As before, we choose a vertex of $\Sigma_\#$ and denote it by 1. Lemma 4.2.3 suggests the following.

Definition 4.3.1. The *fundamental tile* $D_\#$ for W on $\Sigma_\#$ is the union of all Coxeter blocks in $\Sigma_\#$ that contain the vertex 1.

As before, we see that $D_\#$ is a fundamental domain for the W-action on $\Sigma_\#$. We will decribe the mirrors of $D_\#$ in the next subsection.

4.4. Mirrors. Suppose that L is a subcomplex of $\mathcal{S}_{>\emptyset}$. We begin by recalling some facts about the poset $\mathcal{P} = \mathcal{P}(L)$. We note that $\mathcal{P}_{>\emptyset}$ is an abstract simplicial complex, in other words, the vertex set of $\mathcal{P}_{>\emptyset}$ is S, and a subset $T \subset S$ spans a simplex if and only if $T \in \mathcal{P}_{>\emptyset}$. The poset $\mathcal{P}$ plays two roles in the description of Σ. First of all $\mathcal{P}_{>\emptyset}$ can be identified with the link of a vertex in the cellulation of Σ by Coxeter cells. Secondly, if the fundamental chamber D is defined to be the union of all Coxeter blocks in Σ that contain the vertex 1, then D can be identified with the geometric realization of $\mathcal{P}$. (If $\mathcal{P}$ is a poset of subsets of some set V, if $\emptyset \in \mathcal{P}$, and if $\mathcal{P}_{>\emptyset}$ is an abstract simplicial complex, then for any $T \in \mathcal{P}$, the geometric realization of $\mathcal{P}_{\leq T}$ is isomorphic to a standard simplicial subdivision of a cube of dimension $\operatorname{Card} T$. It follows that the geometric realization of $\mathcal{P}$ can be identified with the union of all such cubes.) Having made this identification, the mirror of D corresponding to $s \in S$ is then identified with the geometric realization of $\mathcal{P}_{\geq \{s\}}$.

We now apply the same construction to the blow-up $\Sigma_\#$. In this case, the link of the vertex 1 is associated to the abstract simplicial complex $\mathcal{N}_{>\emptyset}$ where $\mathcal{N}$ is the poset of $\mathcal{R}$-nested subsets of $\mathcal{P}$. It follows that there is one Coxeter block $B_{\mathcal{T}}$ containing 1 for each $\mathcal{R}$-nested set $\mathcal{T}$. (The dimension of $B_{\mathcal{T}}$ is $\operatorname{Card} \mathcal{T}$.) In other words, $D_\#$ can be identified with the geometric realization of $\mathcal{N}$.

Definition 4.4.1. For each $T \in S_\#$, the *mirror of $D_\#$ corresponding to T*, denoted $D_{\#T}$, is the geometric realization of $\mathcal{N}_{\geq \{T\}}$.

Each Coxeter block in $\Sigma_\#$ has a well-defined *type* – it is an element of $\mathcal{N}$. (Translate the Coxeter block by an element of W so that it contains the vertex 1.) Each 1-dimensional Coxeter cell also has a well-defined *type* – it is an element of $S_\#$. (In Section 5 the function that assigns to each oriented edge of $\Sigma_\#$ the type of its underlying 1-cell will be called a "framing".) In general, however, there is no consistent way to assign a "type" to the Coxeter cells in $\Sigma_\#$ of dimension ≥ 2. (See Case 3 of Figure 4.)

4.5. The involution j_T. Let $T \in S_\#$. Next we shall describe the self-homeomorphism of $D_{\#T}$ which is needed to glue together the tiles $D_{\#T}$ and $w_T D_{\#T}$. To this end we want to define an appropriate extension of the automorphism $j_T : \mathcal{R}_{\leq T} \to \mathcal{R}_{\leq T}$, defined in Lemma 3.2.2, to an automorphism of $\mathcal{N}_{\geq\{T\}}$, the closed star of the vertex T in $\mathcal{N}_{>\emptyset}$. This automorphism will also be denoted by j_T.

For any $T' \in \mathcal{P}_{>\emptyset}$, let $a_{T'}$ denote the antipodal map on the Coxeter cell $Z(W_{T'}, T')$. If $T' \in S_\#$, then it corresponds to a Coxeter 1-cell connecting the vertex 1 to $a_{T'}(1) = w_{T'}(1)$. The gluing map j_T is induced by the action of $a_T w_T$ on those vertices T' that lie in $\mathcal{N}_{\geq\{T\}}$. By Lemma 3.3.2 there are three cases to consider. First we consider Case 3, when $T' \subset T$. The face $Z(W_{T'}, T')$ of $Z(W_T, T)$ is mapped by $a_T w_T$ to the face $Z(W_{T''}, T'')$, where $T'' = w_T T' w_T$. Hence, $a_T w_T$ maps the vertex $a_{T'}(1)$ to $a_{T''}(1)$. So in this case, the correct definition of j_T is as in Lemma 3.2.2: $j_T(T') = T''$. On the other hand, if $T \subset T'$, then $a_{T'}$ commutes with a_T and with w_T. Hence $a_T w_T$ maps $a_{T'}(1)$ to $a_{T'} a_T w_T(1) = a_{T'}(1)$. Thus, in this case, we see that the appropriate definition is: $j_T(T') = T'$. A similar argument shows that in Cases 1 and 2, the appropriate definition is $j_T(T') = T'$. Thus j_T is defined on any vertex T' of $\mathcal{N}_{\geq\{T\}}$ by

$$j_T(T') = \begin{cases} w_T T' w_T & \text{if } T' \subset T \\ T' & \text{if } T' \not\subset T. \end{cases}$$

It is then not difficult to check that for any $\mathcal{R}$-nested set $\mathcal{T} \in \mathcal{N}_{\geq\{T\}}$, $j_T(\mathcal{T})$ is also $\mathcal{R}$-nested. That is to say, j_T induces an automorphism of $\mathcal{N}_{\geq\{T\}}$. The geometric realization of j_T is a self-homeomorphism of the mirror $D_{\#T}$, which we shall continue to denote by j_T.

Remark 4.5.1. In general, j_T will not extend to an automorphism of $\mathcal{N}$.

Remark 4.5.2. The subspace $D_\# \cup w_T D_\#$ of $\Sigma_\#$ is homeomorphic to two copies of $D_\#$ glued together along the mirror $D_{\#T}$ via the homeomorphism $j_T : D_{\#T} \to D_{\#T}$.

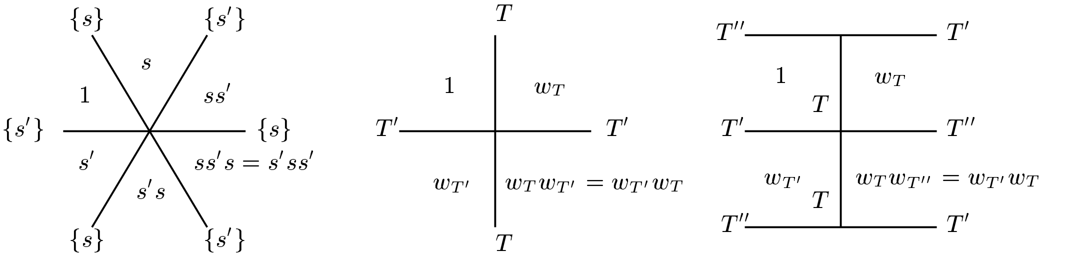
Definition 4.5.3. The mirror $D_{\#T}$ is a *reflecting mirror* if j_T is the identity map on $D_{\#T}$. If j_T is not the identity, then $D_{\#T}$ is a *mock reflecting mirror*.

4.6. Local pictures around codimension-two corners. By Lemma 3.3.2, the intersection of distinct mirrors $D_{\#T}$ and $D_{\#T'}$ is nonempty if and only if one of the three cases in the lemma holds. In Case 1 and Case 2 the picture of the tiles around $D_{\#T} \cap D_{\#T'}$ is the usual two-dimensional picture for reflection groups. In Case 3 it is slightly different. All three cases are depicted in Figure 3 below. In the figure we have labeled the tile $wD_\#$ by the corresponding element w and the mirror $wD_{\#T}$ by the corresponding element $T \in S_\#$.

There are similar pictures for the corresponding Coxeter 2-cells containing the vertex 1 as in Figure 4, below. Here we have labeled the vertices by group elements and the Coxeter 1-cells by their corresponding type.

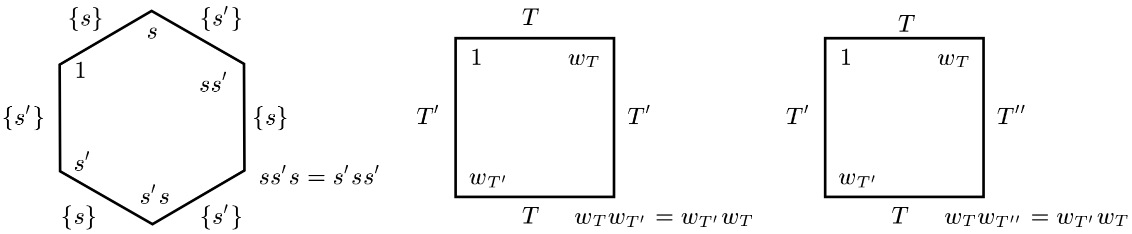
Since $D_\#$ is the dual cone in a Coxeter cell complex, it is a Coxeter tile as defined in 2.4.1. Given the local pictures around 2-dimensional cells, we can now determine the type of this dual cone (cf., Section 2.4). We define a Coxeter matrix $M_\# (= (m_\#(T, T')))$ on $S_\#$ according to the three cases of Lemma 3.3.2:

$$m_\#(T, T') = \begin{cases} 1 & \text{if } T = T' \\ m(s, s') & \text{in Case 1} \\ 2 & \text{in Cases 2 and 3} \\ \infty & \text{if } \{T, T'\} \notin \mathcal{N}. \end{cases}$$



Case 1 $(m(s, s') = 3)$ **Case 2** **Case 3** $(T' \subset T,\, T'' = j_T(T'))$

FIGURE 3.



Case 1 **Case 2** **Case 3**

FIGURE 4.

Let $(W_\#, S_\#)$ be the Coxeter system determined by $M_\#$. With respect to this Coxeter system, we have the notion of a spherical subset of $S_\#$ (namely, $\mathcal{T} \subset S_\#$ is spherical if and only if the special subgroup $(W_\#)_\mathcal{T}$ is finite). Let $\mathcal{S}(S_\#)$ denote the set of spherical subsets of $S_\#$. It is not hard to see that the simplicial complex $\mathcal{N}_{>\emptyset}$ is a subcomplex of $\mathcal{S}(S_\#)_{>\emptyset}$ (i.e., that every $\mathcal{R}$-nested set $\mathcal{T}$ is spherical). Denoting this subcomplex $L_\#$, we then have the following.

Proposition 4.6.1. *The fundamental tile $D_\#$ is isomorphic to the Coxeter tile $D(W_\#, S_\#, L_\#)$ (as defined in 2.4.1).*

4.7. The universal cover of $\Sigma_\#$ and the group A. Let $p : \widetilde{\Sigma}_\# \to \Sigma_\#$ denote the projection map of the universal cover $\widetilde{\Sigma}_\#$ onto $\Sigma_\#$. Let A denote the group of all lifts of the W-action on $\Sigma_\#$ to $\widetilde{\Sigma}_\#$. The projection map $p : \widetilde{\Sigma}_\# \to \Sigma_\#$ induces a surjective homomorphism $\phi : A \to W$, and $\pi_1(\Sigma_\#)$ is naturally identified with the kernel of ϕ. The goal of this subsection is to give a presentation for A. The method is essentially due to Poincaré.

The cellulation of $\Sigma_\#$ by Coxeter cells lifts to a cellulation of $\widetilde{\Sigma}_\#$. Thus, $\widetilde{\Sigma}_\#$ is a Coxeter cell complex. Since $D_\#$ is contractible, p maps each component of $p^{-1}(D_\#)$ homeomorphically onto $D_\#$. Hence, $\widetilde{\Sigma}_\#$ is also tiled by copies of $D_\#$. First choose a component of $p^{-1}(D_\#)$ and denote it by $\widetilde{D}_\#$. Let $\widetilde{1}$ denote the point of $\widetilde{D}_\#$ that lies above the vertex 1 in $D_\#$. ($\widetilde{D}_\#$ should be thought of as the Dirichlet domain centered at $\widetilde{1}$ for the A-action on $\widetilde{\Sigma}_\#$.) The set of elements of A that map $\widetilde{D}_\#$ to an adjacent chamber across a mirror is a set of generators for A. A set of relations can be read off by considering the local pictures around the intersections of two mirrors.

There is a dual and equivalent method to this which is easier to make mathematically precise. Consider the 2-skeleton of $\widetilde{\Sigma}_\#$ in its Coxeter cell structure. Since W acts simply transitively on the vertex set of $\Sigma_\#$, A acts simply transitively on the vertex set of $\widetilde{\Sigma}_\#$. Since we have chosen a distinguished vertex $\widetilde{1}$ in $\widetilde{\Sigma}_\#$, each vertex of $\widetilde{\Sigma}_\#$ is labeled by an element of A. Each 1-cell in $\widetilde{\Sigma}_\#$ is then labeled by its type (an element of $S_\#$). The set of edges in the star of the vertex $\widetilde{1}$ then gives a set of generators for A while the set of 2-cells in this star gives a set of relations.

We now give the details of this method. For each $T \in S_\#$, let α_T denote the unique lift of w_T that maps $\widetilde{D}_\#$ to the adjacent tile across $\widetilde{D}_{\#T}$. (Here is a more precise definition of α_T. Let x_T be the midpoint of the edge in $\Sigma_\#$ connecting the vertex 1 to $w_T(1)$. Then w_T fixes x_T. Let $\widetilde{x}_T$ denote the lift of x_T in $\widetilde{D}_\#$. Then α_T is defined to be the unique lift of w_T that fixes $\widetilde{x}_T$.) Since $(\alpha_T)^2$ also fixes $\widetilde{x}_T$ and covers the identity map on $\Sigma_\#$, $(\alpha_T)^2$ must be the identity on $\widetilde{\Sigma}_\#$, i.e., α_T is an involution.

Definition 4.7.1. The involution α_T is called a *reflection* if $j_T = \mathrm{Id}$ and a *mock reflection* otherwise.

Theorem 4.7.2. *The set $\mathcal{A} = \{\alpha_T\}_{T \in S_\#}$ is a set of generators for A. Moreover, the following relations give a presentation for A:*

(0) $(\alpha_T)^2 = 1$ *for all $T \in S_\#$.*
(1) $(\alpha_{\{s\}}\alpha_{\{s'\}})^{m(s,s')} = 1$ *whenever $m(s,s') < \infty$ and $\{s,s'\} \notin \mathcal{R}$.*
(2) $(\alpha_T\alpha_{T'})^2 = 1$ *whenever $T \cup T' \in \mathcal{S}$ and $\{T,T'\}$ is the $\mathcal{R}$-decomposition of $T \cup T'$.*
(3) $\alpha_T\alpha_{T'} = \alpha_{T''}\alpha_T$ *whenever $T' \subset T$ (where $T'' = w_T T' w_T$).*

Proof. For each $T \in S_\#$ there is a Coxeter 1-cell connecting $\widetilde{1}$ to $\alpha_T(\widetilde{1})$. Since the 1-skeleton of $\widetilde{\Sigma}_\#$ is connected, $\mathcal{A}$ is a set of generators for A.

As we have already observed, α_T is an involution, so the relations in (0) hold in A. Examining the local pictures around the intersection of two mirrors in Figure 3, we see that the relations of type (1), (2), and (3) hold in A. In terms of the 2-skeleton of $\widetilde{\Sigma}_\#$, the relations of the form (1), (2), and (3) correspond to the 2-dimensional Coxeter cells that contain the vertex $\widetilde{1}$. Hence, the Cayley 2-complex of the group defined by this presentation is a covering space of the 2-skeleton. Since the 2-skeleton is simply-connected, this covering must be trivial; hence, this presentation is a presentation for A. □

Remark 4.7.3. The homomorphism $\phi : A \to W$ is defined on the generating set $\mathcal{A}$ by $\phi(\alpha_T) = w_T$.

5. Tilings of Coxeter cell complexes

5.1. Framings. A framing is an additional structure on a Coxeter cell complex X. It is used to rigidify the situation and to provide a notion of "parallel transport" along curves. The existence of a framing implies that X is tiled (cf., Definition 2.3.1). In the converse direction, if X admits a group of automorphisms that acts simply transitively on its vertex set $X^{(0)}$, then there is an associated framing (cf., Example 5.1.6, below).

Recall that an *orientation* for an edge of X is an ordering of its two endpoints. Let $OE(X)$ denote the set of oriented edges in X. For any $e \in OE(X)$, let $i(e)$ denote its first endpoint (its "initial vertex") and $t(e)$ its second ("terminal vertex"). Also, $\overline{e}$ denotes the same edge with the opposite orientation. An *edge path* in X is a sequence $\mathbf{e} = (e_1, \dots, e_n)$ of oriented edges such that $i(e_{k+1}) = t(e_k)$, for $1 \leq k < n$. Its *initial vertex* $i(\mathbf{e})$ is defined to be $i(e_1)$ and its *terminal vertex* $t(\mathbf{e})$ is $t(e_n)$.

Definition 5.1.1. A *framing system* is a triple (V, M, L) where

(i) V is a finite set,
(ii) M $(= m(v, v'))$ is a Coxeter matrix on V, and
(iii) L is a subcomplex of $\mathcal{S}(V)_{>\emptyset}$ containing its 1-skeleton.

There is an obvious notion of an *isomorphism* between two framing systems (V, M, L) and (V', M', L') (namely, it is a bijection $\phi : V \to V'$ that induces a simplicial isomorphism $L \to L'$ and pulls back M' to M).

Associated to a framing system (V, M, L) there is the reflection tiling $\Sigma(W(M), V, L)$ defined in 2.3.2.

Example 5.1.2. Associated to any vertex $x \in X^{(0)}$, we have three sets E_x, O_x, and I_x called, respectively, the *unoriented edges*, the *outward pointing edges*, and the *inward pointing edges* at x. Their definitions are:

$$\begin{aligned} &E_x = \{\text{unoriented edges with one endpoint } x\} \\ &O_x = \{e \in OE \mid i(e) = x\} \\ &I_x = \{e \in OE \mid t(e) = x\}. \end{aligned}$$

We note that these three sets are canonically isomorphic. A Coxeter matrix M_x $(= m_x(e, e'))$ on E_x (or on O_x or I_x) is defined as follows: if $e = e'$, then $m_x(e, e') = 1$; if e and e' are distinct edges of a 2-cell in X that is a $2m$-gon, then $m_x(e, e') = m$; otherwise, $m_x(e, e') = \infty$. Let $W(M_x)$ denote the Coxeter group corresponding to M_x with fundamental set of generators E_x, and let $\mathcal{S}(E_x)$ be the poset of spherical subsets of E_x. The link, L_x, of x in X is then a subcomplex of $\mathcal{S}(E_x)$. Thus, to each $x \in X^{(0)}$ we have associated three canonically isomorphic framing systems: (E_x, M_x, L_x), (O_x, M_x, L_x), and (I_x, M_x, L_x).

Definition 5.1.3. Suppose X is a Coxeter cell complex, that (V, M, L) is a framing system and that $\nabla : OE(X) \to V$ is a function. For each $x \in X^{(0)}$, denote the restriction of ∇ to O_x and I_x by $\nabla_x : O_x \to V$ and $\overline{\nabla}_x : I_x \to V$, respectively. Then ∇ is called a *framing* if for each $x \in X^{(0)}$, both ∇_x and $\overline{\nabla}_x$ are isomorphisms of framing systems.

Remark 5.1.4. The canonical isomorphism $c_x : O_x \to I_x$ is defined by $c_x(e) = \overline{e}$. If ∇ is a framing on X, then for each $x \in X^{(0)}$, we get an involution of framing systems

$\iota_x : V \to V$ defined by the condition that the following diagram commutes

$$\begin{tikzcd} O_x \arrow[r, "c_x"] \arrow[d, "\nabla_x"] & I_x \arrow[d, "\overline{\nabla}_x"] \\ V \arrow[r, "\iota_x"] & V \end{tikzcd}$$

In all cases of interest in this paper, the involution ι_x will be the identity map for each $x \in X^{(0)}$.

Example 5.1.5. Suppose that (W, S) is a Coxeter system, that $\mathcal{R}$ is an admissible subset of $\mathcal{S}$, and that $S_{\#}$ is the subset of $\mathcal{S}$ defined in 3.3. Let $M_{\#}$ be the Coxeter matrix on $S_{\#}$ defined in 4.6, and let $L_{\#}$ be the simplicial complex $\mathcal{N}_{>\emptyset}$. Then $(S_{\#}, M_{\#}, L_{\#})$ is a framing system. If $\Sigma_{\#}$ denotes the $\mathcal{R}$-blowup of $\Sigma(W, S)$, then there is a natural framing on $\Sigma_{\#}$ with framing system $(S_{\#}, M_{\#}, L_{\#})$ which associates to each edge its type (as defined in 4.4).

Example 5.1.6. Suppose A is a group of automorphisms of a Coxeter cell complex X that acts simply transitively on $X^{(0)}$. Set $V = O_x$, $M = M_x$, and $L = L_x$. A framing $\nabla : OE(X) \to V$ is defined as follows. For any $e \in OE(X)$, let a be the unique element of A that takes $i(e)$ to x. Then $\nabla(e) = a(e)$.

Example 5.1.7. Suppose ∇ is a framing of X with framing system (V, M, L). Let (V', M', L') be another framing system and $\phi : V \to V'$ an isomorphism. Then there is an induced framing ∇^{ϕ} of X with system (V', M', L') defined by $\nabla^{\phi}(e) = \phi(\nabla(e))$.

Example 5.1.8. Suppose that $p : X \to X'$ is a covering projection of Coxeter cell complexes and that ∇' is a framing on X'. We define a framing $p^*\nabla'$ on X (with the same framing system as ∇') by the formula $(p^*\nabla')(e) = \nabla'(p(e))$.

Definition 5.1.9. Suppose that (X, ∇) and (X', ∇') are framed Coxeter cell complexes with the same framing systems. A *framed map* from (X, ∇) to (X', ∇') is a covering projection $p : X \to X'$ such that $\nabla = p^*\nabla'$.

Remark 5.1.10. Suppose p_1 and p_2 are two framed maps from (X', ∇') to (X, ∇) and that for some vertex x' of X', $p_1(x') = p_2(x')$. Since the maps preserve the framing, this implies that p_1 and p_2 also agree on any vertex adjacent to x'. (Two vertices are *adjacent* if they are connected by an edge.) Hence, if X' is connected, we must have $p_1 = p_2$. In other words, provided X' is connected, if two framed maps agree at a single vertex, then they are equal.

Henceforth, we shall assume that all Coxeter cell complexes are connected.

As a special case of Definition 5.1.9, note that any automorphism $f : X \to X$ is a covering projection. It is *framed* if $f^*\nabla = \nabla$. Let $\mathrm{Aut}(X, \nabla)$ denote the group of framed automorphisms of (X, ∇). By Remark 5.1.10, $\mathrm{Aut}(X, \nabla)$ acts freely on $X^{(0)}$.

Suppose that (X, ∇) and (X', ∇') are framed with framing systems (V, M, L) and (V', M', L'), respectively, and that $p : X \to X'$ is a covering projection (not necessarily framed). Then for each vertex $x \in X^{(0)}$ there is an *induced isomorphism* $p_x : V \to V'$

of framing systems defined by the condition that the following diagram commutes:

$$\begin{tikzcd} O_x \arrow[r, "p|O_x"] \arrow[d, "\nabla_x"] & O_{p(x)} \arrow[d, "\nabla'_{p(x)}"] \\ V \arrow[r, "p_x"] & V' \end{tikzcd}$$

Thus, after changing (V', M', L') by an isomorphism of framing systems, we may assume that $(V', M', L') = (V, M, L)$ and that at a given vertex x, p_x is the identity.

The framing ∇ is *symmetric* if $\mathrm{Aut}(X, \nabla)$ acts transitively on $X^{(0)}$. Thus, up to isomorphism of framing systems, every symmetric framing arises from the construction in Example 5.1.6.

5.2. The action of F_V on the vertex set. Suppose (X, ∇) is a framed Coxeter cell complex with framing system (V, M, L). Let F_V denote the free group on V. Given a vertex $x \in X^{(0)}$ and an element $v \in V$, let $e \in O_x$ and $e^* \in I_x$ be the oriented edges defined by $\nabla_x(e) = v$ and $\overline{\nabla}_x(e^*) = v$, respectively. Define $x \cdot v = t(e)$ and $x \cdot v^{-1} = i(e^*)$. Clearly, $(x \cdot v) \cdot v^{-1} = x = (x \cdot v^{-1}) \cdot v$; hence, these formulas define a right F_V-action on $X^{(0)}$. The action is transitive since X is connected.

If $f : (X, \nabla) \to (X', \nabla')$ is a framed map then the restriction of f to the vertex set, $f^{(0)} : X^{(0)} \to X'^{(0)}$, is obviously F_V-equivariant. Conversely, we have the following.

Lemma 5.2.1. *Suppose (X, ∇) and (X', ∇') are two framed Coxeter cell complexes with the same framing system (V, M, L). Let $\theta : X^{(0)} \to X'^{(0)}$ be an equivariant map of F_V-sets. Then there exists a unique framed map $f : (X, \nabla) \to (X', \nabla')$ with $f^{(0)} = \theta$.*

Proof. The map θ defines f on the 0-skeleton of X. If $e \in OE(X)$, then there is a unique oriented edge e' in X' from $\theta(i(e))$ to $\theta(t(e))$, namely, the edge e' with $i(e') = \theta(i(e))$ and with $\nabla'(e') = \nabla(e)$. (By F_V-equivariance, $t(e') = i(e') \cdot \nabla'(e') = \theta(i(e)) \cdot \nabla(e) = \theta(t(e))$.) We extend f to the 1-skeleton by mapping e isometrically to e'. A similar argument for higher dimensional cells shows that θ extends to a map f, which is clearly a covering projection. Uniqueness follows from Remark 5.1.10. □

Corollary 5.2.2. *Suppose that (X, ∇) and (X', ∇') are two framed Coxeter cell complexes with the same framing system. Then the correspondence $f \mapsto f^{(0)}$ is a bijection from the set of framed maps from (X, ∇) to (X', ∇') with the set of F_V-equivariant maps from $X^{(0)}$ to $X'^{(0)}$. In particular, (X, ∇) is framed isomorphic to (X', ∇') if and only if their vertex sets are isomorphic as F_V-sets.*

5.3. Gluing isomorphisms. Suppose Z is a Coxeter cell and that x is a vertex of Z. Let $O_x(Z)$ denote the set of oriented edges of Z with initial vertex x. Let W_Z denote the Coxeter group associated to Z. (W_Z is a well-defined group of symmetries of Z generated by reflections across hyperplanes.) Suppose x and x' are two vertices of Z. Then there is a unique element $w \in W_Z$ such that $wx = x'$. The element w gives us a canonical bijection, which we denote by $w_* : O_x(Z) \to O_{x'}(Z)$.

Now suppose that (X, ∇) is a framed Coxeter cell complex, that Z is a Coxeter cell in X, and that x is a vertex of Z. Set $\nabla_x(Z) = \nabla_x(O_x(Z))$. It is a spherical subset of V. If x' is another vertex of Z, then we have a canonical isomorphism $j_{(x,x',Z)} : \nabla_x(Z) \to \nabla_{x'}(Z)$ defined by $j_{(x,x',Z)} = \nabla_x \circ w_* \circ (\nabla_{x'})^{-1}$, where $w \in W_Z$ is such that $x' = wx$.

For each $v \in V$ let $\mathrm{St}(v)$ denote the star of v in the simplicial complex L. We shall now define, for each $e \in OE(X)$, an isomorphism $j_e : \mathrm{St}(\nabla(e)) \to \mathrm{St}(\nabla(\overline{e}))$. Let $x_0 = i(e)$ and $x_1 = t(e)$ be the endpoints of e. For any simplex σ in $\mathrm{St}(\nabla(e))$ let Z_σ be the corresponding Coxeter cell (i.e., Z_σ contains e, and $\nabla_{x_0}(Z_\sigma)$ is the vertex set of σ). Let σ' be the simplex in $\mathrm{St}(\nabla(\overline{e}))$ with vertex set $\nabla_{x_1}(Z_\sigma)$. Then j_e is defined to be the simplicial map that takes σ to σ' via $j_{(x_0,x_1,Z_\sigma)}$. We note that the maps j_e and $j_{\overline{e}}$ are inverses of one another; hence, j_e is an isomorphism, called the *gluing isomorphism* associated to e.

Remark 5.3.1. The definition of j_e depends only on the values of ∇ on the edges of the 2-cells that contain e.

Remark 5.3.2. Suppose that $\mathbf{e} = (e_1, \dots, e_n)$ is an edge path in a Coxeter cell Z in X with $i(\mathbf{e}) = x$ and $t(\mathbf{e}) = x'$. Then $j_{e_n} \circ \cdots \circ j_{e_1}$ maps $\nabla_x(Z)$ to $\nabla_{x'}(Z)$ and $j_{e_n} \circ \cdots \circ j_{e_1}|_{\nabla_x(Z)} = j_{(x,x',Z)}$. That is to say, the composition of gluing isomorphisms corresponding to an edge path in Z depends only on the endpoints of the path. In particular, suppose that F is a 2-dimensional face of Z and that $\mathbf{e} = (e_1, \dots, e_{2m})$ is an oriented edge path around the boundary of F. Then $j_{e_{2m}} \circ \cdots \circ j_{e_1}$ is the identity map on $\nabla_x(Z)$. This last condition can be rephrased as follows: if σ is the 1-cell in L with vertex set $\nabla_x(F)$ and if $\mathrm{St}(\sigma)$ denotes its star in L, then $j_{e_{2m}} \circ \cdots \circ j_{e_1}$ maps $\mathrm{St}(\sigma)$ to itself and $j_{e_{2m}} \circ \cdots \circ j_{e_1}|_{\mathrm{St}(\sigma)} = \mathrm{Id}_{\mathrm{St}(\sigma)}$.

Suppose $(e_1, \dots, e_n)$ is an edge path in Z and that $x_k = t(e_k)$. Then $x_k = w_k x$ for some unique element $w_k \in W_Z$. Since $\nabla_x(Z)$ is a fundamental set of generators for W_Z we get a corresponding word $(v_1, \dots, v_n)$ in $\nabla_x(Z)$ defined by $w_k = v_k \cdots v_1$. How do we express the $\nabla(e_k)$ in terms of the v_k? The answer is simple: $\nabla(e_1) = v_1$ and for $1 \le k < n$, $\nabla(e_{k+1}) = j_{e_k} \circ \cdots \circ j_{e_1}(v_{k+1})$.

Remark 5.3.3. If X is a Coxeter cell complex of reflection type (i.e., isomorphic to a reflection tiling) with the obvious symmetric framing, then each gluing automorphism j_e is the identity map on $\mathrm{St}(\nabla(e))$.

The following lemma gives a necessary condition for two framable Coxeter cell complexes to be isomorphic (not necessarily by a framed isomorphism).

Lemma 5.3.4. *Suppose that (X, ∇) and (X', ∇') are two framed Coxeter cell complexes with the same framing system (V, M, L). Let $f : X \to X'$ be a covering projection that induces the map ϕ at a given vertex x (i.e., $f_x = \phi : V \to V$). For each $e \in O_x$, let $j_e : \mathrm{St}(\nabla(e)) \to \mathrm{St}(\nabla(\overline{e}))$ and $j_{f(e)} : \mathrm{St}(\nabla'(f(e))) \to \mathrm{St}(\nabla'(\overline{f(e)}))$ be the corresponding gluing isomorphisms of X and X', respectively. Set $J_e = j_{f(e)} \circ \phi \circ (j_e)^{-1} : \mathrm{St}(\nabla(\overline{e})) \to \mathrm{St}(\nabla'(\overline{f(e)}))$. Then J_e extends to an automorphism $\widetilde{J}_e : V \to V$ of framing systems.*

Proof. Let $\overline{x}$ denote $t(e)$. Then the following diagram of isomorphisms clearly commutes:

$$\begin{tikzcd}
\mathrm{St}(\nabla(e)) \arrow[r, "f_x=\phi"] \arrow[d, "j_e"] & \mathrm{St}(\nabla'(f(e))) \arrow[d, "j_{f(e)}"] \\
\mathrm{St}(\nabla(\overline{e})) \arrow[r, "f_{\overline{x}}"] & \mathrm{St}(\nabla'(\overline{f(e)}))
\end{tikzcd}$$

Hence, $\widetilde{J}_e = f_{\overline{x}}$ is the desired extension of J_e. $\square$

5.4. Homogeneous framings.

Definition 5.4.1. A framing ∇ on X is *homogeneous* if for each $e \in OE(X)$, the gluing isomorphism $j_e : \mathrm{St}(\nabla(e)) \to \mathrm{St}(\nabla(\overline{e}))$ depends only on the value of $\nabla(e)$ (in other words, whenever $\nabla(e) = \nabla(e')$, we have $\nabla(\overline{e}) = \nabla(\overline{e}')$ and $j_e = j_{e'}$).

For homogeneous framings we shall often write $j_{\nabla(e)}$ instead of j_e.

Remark 5.4.2. If X is homogeneously framed, then the involution $\iota_x : V \to V$ defined in Remark 5.1.4 is independent of x. We shall denote this involution by $v \mapsto \overline{v}$.

Suppose that (X, ∇) is homogeneously framed. We shall now explain the consequences of homogeneity for the F_V-action on $X^{(0)}$. Fix a vertex $x \in X^{(0)}$. For each 1-simplex σ in L, let Z_σ denote the corresponding Coxeter 2-cell at x (i.e., $\nabla_x(Z_\sigma)$ is the vertex set of σ). Let $(e_1, \dots, e_{2m})$ be the edge cycle that starts at x and goes once around Z_σ, and let $\nabla(e_1) \cdots \nabla(e_{2m})$ be the corresponding word in V. Let r_σ denote the image of $\nabla(e_1) \cdots \nabla(e_{2m})$ in F_V. Finally, let A_∇ be the quotient of F_V by the normal subgroup N generated by $\{v\overline{v} \mid v \in V\} \cup \{r_\sigma \mid \sigma \text{ a 1-simplex of } L\}$. In other words, A_∇ is the group with one generator α_v for each element of V, and relations $(\alpha_v)^{-1} = \alpha_{\overline{v}}$ as well as relations corresponding to r_σ for each 1-simplex σ of L.

A different choice of vertex x' leads to a relation r'_σ that is conjugate to r_σ in F_V. Hence, the choice of x' yields the same normal subgroup N and the same quotient group A_∇. Moreover, in the F_V-action on $X^{(0)}$, the subgroup N acts trivially. Hence, we have a well-defined transitive action (from the right) of A_∇ on $X^{(0)}$.

Remark 5.4.3. Something significant has been gained. Two reduced edge paths $\mathbf{e}$ and $\mathbf{e}'$ with the same endpoints are homotopic rel endpoints if and only if one can be pushed to the other across 2-cells. Therefore, the element $a_{\mathbf{e}}$ of A_∇ corresponding to $\mathbf{e}$ depends only on the homotopy class of $\mathbf{e}$.

This remark has the following immediate consequence.

Proposition 5.4.4. *Suppose (X, ∇) is a homogeneously framed Coxeter cell complex and that A_∇ is the group defined above. Then the isotropy subgroup of the A_∇-action on $X^{(0)}$ at $x \in X^{(0)}$ is naturally identified with $\pi_1(X, x)$.*

As a consequence of Lemma 5.2.1 we get the following.

Proposition 5.4.5. *Suppose (X, ∇) and (X', ∇') are two homogeneously framed Coxeter cell complexes with the same framing system and the same set of gluing isomorphisms $\{j_v\}_{v \in V}$. Then the groups A_∇ and $A_{\nabla'}$ are canonically isomorphic. Moreover, given vertices $x \in X^{(0)}$ and $x' \in X'^{(0)}$ there is a framed map $f : (X, \nabla) \to (X', \nabla')$ taking x to x' if and only if $\pi_1(X, x)$ is a subgroup of $\pi_1(X', x')$.*

Corollary 5.4.6. *Suppose (X, ∇) is a homogeneously framed Coxeter cell complex. Let $A = A_\nabla$ and $\pi = \pi_1(X, x)$. Then the following statements are true.*

(1) $\mathrm{Aut}(X, \nabla) \cong N_A(\pi)/\pi$, *where $N_A(\pi)$ denotes the normalizer of π in A.*
(2) *∇ is symmetric if and only if π is normal in A.*
(3) *If X is simply-connected, then ∇ is symmetric and* $\mathrm{Aut}(X, \nabla) \cong A$.

Remark 5.4.7. If the framing is homogeneous and if the involution on V is trivial, then each gluing map $j_v : \mathrm{St}(v) \to \mathrm{St}(v)$ is an involution.

Suppose (X, ∇) is homogeneously framed. Let

$$\mathcal{F}_X = \{v \in V \mid j_v \text{ does not extend to an automorphism of } (V, M, L)\}.$$

Also, let $t_X = \mathrm{Card}(\mathcal{F}_X)$. The next lemma, which is a special case of Lemma 5.3.4, gives a necessary condition for two homogeneously framed Coxeter cell complexes to be isomorphic. It will be used in Section 8 to distinguish among various associahedral tilings.

Lemma 5.4.8. *Suppose that (X, ∇) and (X', ∇') are homogeneously framed with the same framing system (V, M, L). Suppose further that there is a covering projection $f : X \to X'$. Then there is an automorphism $\phi \in \mathrm{Aut}(V, M, L)$ that takes $\mathcal{F}_X$ to $\mathcal{F}_{X'}$. In particular, $t_X = t_{X'}$.*

Proof. Let ϕ be the automorphism induced by f at a given vertex x. Put $J_v = j_{\phi(v)} \circ \phi \circ (j_v)^{-1}$. By Lemma 5.3.4, J_v extends. Therefore, j_v extends if and only if $j_{\phi(v)}$ extends. □

5.5. Maximally symmetric tilings.

Definition 5.5.1. A proper action of a discrete group G on a space Y is *rigid* if for all $g \in G - \{1\}$, there does not exist a nonempty open subset U of Y that is fixed pointwise by g.

For example, if Y is a connected manifold, then it follows from Newman's Theorem (page 153 in [Bre]) that any proper effective action on Y is rigid.

Suppose (X, ∇) is a homogeneously framed Coxeter cell complex with framing system (V, M, L). Let $\mathrm{Aut}(X)$ denote the automorphism group of the cell complex X, and for x in $X^{(0)}$, let $\mathrm{Stab}(x)$ be the stabilizer subgroup of x. Assume further that the action of $\mathrm{Aut}(X)$ on X is rigid. Let $\mathrm{Aut}_0(V, M, L)$ be the subgroup of all $\phi \in \mathrm{Aut}(V, M, L)$ that are equivariant with respect to the involution $v \mapsto \overline{v}$. Then the natural homomorphism $\Phi : \mathrm{Stab}(x) \to \mathrm{Aut}_0(V, M, L)$ is injective. If, in addition, Φ is surjective we say that X is *maximally symmetric*.

Proposition 5.5.2. *Suppose X is simply connected. Then X is maximally symmetric if and only if $\phi \circ j_v \circ \phi^{-1} \circ (j_{\phi(v)})^{-1}$ extends to an automorphism of framing systems for all $v \in V$ and $\phi \in \mathrm{Aut}_0(V, M, L)$,*

Proof. If X is maximally symmetric, then the condition follows from Lemma 5.3.4. Let ∇' be the framing of X defined by $\nabla' = \nabla^{\phi^{-1}}$. Then (X, ∇') is homogeneously framed and the corresponding gluing isomorphisms j'_v are given by $j'_v = \phi^{-1} \circ j_{\phi(v)} \circ \phi$. If the condition holds, then $\{j'_v\}_{v\in V} = \{j_v\}_{v\in V}$. So, by Proposition 5.4.5, A_∇ is canonically isomorphic to $A_{\nabla'}$ (the isomorphism is induced by $v \mapsto \phi(v)$), and since X is simply connected there is a framed isomorphism $f : (X, \nabla) \to (X, \nabla')$ taking x to x. Since $\Phi(f) = \phi$, Φ is surjective. $\square$

Example 5.5.3. If $\mathrm{Aut}(V, M, L)$ is trivial, then X is maximally symmetric and $\mathrm{Aut}(X) = A_\nabla$. If $X = \Sigma(W, S, L)$ is a tiling of reflection type, then X is maximally symmetric (each j_v is trivial). In fact, in this case, the full symmetry group of X is

$$\mathrm{Aut}(X) = W \rtimes \mathrm{Aut}(V, M, L).$$

5.6. A sufficient condition for two Coxeter cell complexes to be isomorphic. Suppose that (X, ∇) is a framed Coxeter cell complex with framing system (V, M, L). Let $\Sigma = \Sigma(W, M, L)$ be the corresponding Coxeter cell complex of reflection type (where $W = W(M)$). In this section we give sufficient conditions for finding a covering projection $p : \Sigma \to X$ such that p takes the distinguished vertex $1 \in \Sigma^{(0)}$ to a given vertex $x \in X^{(0)}$ so that the induced map of framing systems $p_x : V \to V$ at the vertex 1 is the identity. The two conditions we shall give are necessary by Lemma 5.3.4 and Remark 5.3.2. The first condition is the following "Extendability Condition":

(E) For each $e \in OE(X)$, the map $j_e : \mathrm{St}(\nabla(e)) \to \mathrm{St}(\nabla(\overline{e}))$ extends to an automorphism $\widetilde{j}_e : V \to V$ of framing systems.

The second condition (H) is the condition of "no holonomy around 2-cells". It is justified by the first paragraph of Remark 5.3.2.

(H) The extended gluing isomorphisms of (E) can be chosen so that $\widetilde{j}_{\overline{e}} = (\widetilde{j}_e)^{-1}$ and so that if $(e_1, \dots, e_{2m})$ is a cycle around a 2-cell of X, then $\widetilde{j}_{e_{2m}} \circ \dots \circ \widetilde{j}_{e_1}$ is the identity map.

Remark 5.6.1. Suppose $(e_1, \dots, e_{2m})$ is a cycle around a 2-cell $F \subset X$, that $y = i(e_1) = t(e_{2m})$, and that σ is the 1-cell of L spanned by $\nabla_y(e_1)$ and $\nabla_y(\overline{e}_{2m})$. Then Remark 5.3.2 asserts that $j_{e_{2m}} \circ \dots \circ j_{e_1}|_{\mathrm{St}(\sigma)}$ is the identity map. The content of Condition (H) is that $\widetilde{j}_{e_{2m}} \circ \dots \circ \widetilde{j}_{e_1}$ is an extension of this to the identity map of V.

Remark 5.6.2. If the action of $\mathrm{Aut}(L)$ on L is rigid, then the extension $\widetilde{j}_e$ of Condition (E) is unique (if it exists); moreover, Condition (H) is then automatic.

Now suppose that (X, ∇) satisfies (E) and (H). Given a word $(v_1, \dots, v_n)$ in V we will define an edge path $\mathbf{e} = (e_1, \dots, e_n)$ beginning at $x = x_0$. We will use the notation $x_k = t(e_k)$. By definition e_1 is the unique edge at x_0 such that $\nabla_{x_0}(e_1) = v_1$. Suppose by induction that $(e_1, \dots e_k)$ has been defined. Then e_{k+1} is the unique edge at x_k satisfying $\nabla_{x_k}(e_k) = \widetilde{j}_{e_k} \circ \cdots \circ \widetilde{j}_{e_1}(v_{k+1})$. (Compare this to the formula in the second paragraph of Remark 5.3.2.)

Set $w_k = v_1 \cdots v_k$.

Lemma 5.6.3. *Suppose Conditions (E) and (H) hold. Then with notation as above, the vertex $x_n \in X^{(0)}$ depends only on the element $w = w_n \in W$ (and not on the choice of word $v_1 \cdots v_n$ representing w). Moreover, this gives a right action of W on $X^{(0)}$ defined by $x \cdot w = x_n$.*

Proof. Any two words for an element w in a Coxeter group differ by a sequence of moves of one of the following two types: (1) cancelling a subword of the form vv or (2) replacing a subword which goes half way around a 2-cell in Σ by the subword which goes half way around the 2-cell in the other direction. A subword of the form vv in $(v_1, \dots, v_n)$ corresponds to an edge subpath of the form $e\overline{e}$ and since $\widetilde{j}_e \circ \widetilde{j}_{\overline{e}} = \mathrm{Id}$, we can cancel. So the issue comes down to the following. Suppose $(v_1, \dots, v_n)$ contains a subword $\mathbf{s} = (s, t, \dots)$ which is an alternating word of length $m = m(s,t)$ in letters s and t. We want to show that if we replace this subword by the other alternating word $\mathbf{s}' = (t, s, \dots)$ of length m then the initial and final segments of the new edge path in X remain unchanged. Suppose $(e_1, \dots, e_m)$ is the portion of the edge path corresponding to $\mathbf{s}$. Let $(\overline{e}_{2m}, \dots, \overline{e}_{m+1})$ be the edge path with the same initial vertex that corresponds to $\mathbf{s}'$. Its opposite path is $(e_{m+1}, \dots, e_{2m})$ and $(e_1, \dots, e_{2m})$ is a cycle around a 2-cell in X. By Condition (H)

$$\widetilde{j}_{e_m} \circ \cdots \circ \widetilde{j}_{e_1} = \widetilde{j}_{\overline{e}_{m+1}} \circ \cdots \circ \widetilde{j}_{\overline{e}_{2m}}.$$

Hence, we can replace the segment $(e_1, \dots, e_m)$ by $(\overline{e}_{2m}, \dots, \overline{e}_{m+1})$ without affecting the definition of any succeeding edges. □

Define $p^{(0)} : \Sigma^{(0)} \to X^{(0)}$ by $p^{(0)}(1 \cdot w) = x \cdot w$. It follows as in Lemma 5.2.1 that this extends to a covering projection $p : \Sigma \to X$. We have proved the following.

Theorem 5.6.4. *Suppose that Conditions (E) and (H) hold for a framed Coxeter cell complex (X, ∇) and that Σ is the associated tiling of reflection type. Then for any vertex $x \in X^{(0)}$, there is a covering projection $p : \Sigma \to X$ taking 1 to x.*

5.7. The completed complex $\widehat{X}$. Suppose (X, ∇) is a framed Coxeter cell complex with framing system (V, M, L). Let $\widehat{L}$ denote the simplicial complex $\mathcal{S}(V)_{>\emptyset}$. Our goal in this section is to try to construct a new Coxeter cell complex $\widehat{X}$ such that

(a) X is a subcomplex of $\widehat{X}$ and they have the same 2-skeleton, and
(b) for each vertex $x \in \widehat{X}^{(0)}$ $(= X^{(0)})$ its link $\widehat{L}_x$, is isomorphic to $\widehat{L}$ and the framing gives an isomorphism $\nabla_x : \widehat{L}_x \to \widehat{L}$.

If we are successful in this construction then, since $OE(\widehat{X}) = OE(X)$, we will have an induced framing on $\widehat{X}$ with framing system $(V, M, \widehat{L})$.

There are two obvious conditions for (b) to hold: first, each gluing isomorphism must extend to an isomorphism of the appropriate stars in $\widehat{L}$ and second, there can be no holonomy around 2-cells in the star of the appropriate edge of $\widehat{L}$. For each subset T of V, let $M(T)$ denote the restriction of the Coxeter matrix to T. For each $v \in V$, let V_v denote the vertex set of $\mathrm{St}(v)$. The gluing isomorphism $j_e : \mathrm{St}(\nabla(e)) \to \mathrm{St}(\nabla(\overline{e}))$ restricts to a map of vertex sets which we also denote by $j_e : V_{\nabla(e)} \to V_{\nabla(\overline{e})}$. We next discuss a Condition (M) (for "Matrix Condition"). It has two parts (M1) and (M2). The first part (M1) is a weak version of (E) in 5.6, and the second (M2) is a weak version of (H).

(M1) For each $e \in OE(X)$ and for each pair v_1, v_2 in $V_{\nabla(e)} - \{\nabla(e)\}$, $m(j_e(v_1), j_e(v_2)) = m(v_1, v_2)$.

Condition (M1) says that j_e pulls back $M(V_{\nabla(\overline{e})})$ to $M(V_{\nabla(e)})$. Hence, if (M1) holds, then each j_e induces a simplicial isomorphism

$$\widehat{j}_e : \mathcal{S}(V)_{\geq\{\nabla(e)\}} \to \mathcal{S}(V)_{\geq\{\nabla(\overline{e})\}}$$

between the appropriate stars in $\widehat{L}$.

Note that Condition (M1) is automatic if $\{\nabla(e), v_1, v_2\}$ spans a 2-simplex in $\mathrm{St}(\nabla(e))$ (since j_e maps this 2-simplex to a 2-simplex in $\mathrm{St}(\nabla(\overline{e}))$). Thus, Condition (M1) holds automatically if L has no "missing 2-simplices".

Suppose F is a 2-cell of X, that $(e_1, \dots, e_{2m})$ is a cycle around F, and that $x = i(e_1)$. The second part of Condition (M) is the following:

(M2) For any 2-cell F of X, with notation as above,

$$\widehat{j}_{e_{2m}} \circ \cdots \circ \widehat{j}_{e_1} : \mathcal{S}_{\geq \nabla_x(F)} \to \mathcal{S}_{\geq \nabla_x(F)}$$

is the identity map.

We note that Condition (M2) is automatic if there are no missing simplices in the star of the edge σ of L which corresponds to $\nabla_x(F)$. Thus, Condition (M) holds automatically if L has no missing simplices.

Supposing that (M) holds, the idea for the construction of $\widehat{X}$ is the following. We want to fill in the "missing" cells of X that correspond to the "missing" simplicies of L. The first problem is to describe subcomplexes of X which will serve as the (partial) boundaries of the missing cells to be filled in. This is done essentially by the same method as in 5.6. Condition (M) is exactly the hypothesis needed to carry out this procedure. A second problem then arises. Our putative boundaries of missing cells might not be embedded subcomplexes of X. Instead, they might be the image of the actual boundary of a cell under a covering projection. Thus, the completion $\widehat{X}$ might turn out to be a "Coxeter orbihedron" (as defined below).

For each $T \in \mathcal{S}(V)_{>\emptyset}$ let L_T be the subcomplex of L spanned by T. By condition (iii) of Definition 5.1.1, the 1-skeleton of L_T is the complete graph on T. Let W_T be the finite Coxeter group corresponding to $M(T)$, let Z be the corresponding Coxeter cell, and let $1 \in Z^{(0)}$ be a distinguished vertex (such that $OE_1(Z) = T$). Let $Z(L_T)$ be the subcomplex of Z corresponding to L_T, i.e., $Z(L_T) = \Sigma(W_T, T, L_T)$.

Fix a vertex $x \in X^{(0)}$ and let T be any spherical subset of $\nabla_x(OE_x)$. We now propose to define a subcomplex $C(L_T)$ of X containing the vertex x, and a map $\phi : Z(L_T) \to$

$C(L_T)$ such that $\phi(1) = x$. (If T is the vertex set of a simplex σ in L, then $C(L_T)$ will be Z_σ, the corresponding Coxeter cell at x.)

We regard T as a set of fundamental generators for W_T. We proceed as in 5.6. Let $\mathbf{u} = u_1 \cdots u_n$ be a word in T. For $1 \leq k \leq n$, let w_k be the element of W_T represented by $u_k \cdots u_1$. Assuming that (M) holds, we are going to define, by induction on k, an edge path $\mathbf{e} = (e_1, \dots, e_n)$ and a sequence of subsets $T_0, \dots, T_n$ of V. We will use the notation: $v_k = \nabla(e_k)$, $x_k = t(e_k)$, and $x_0 = x$. We will also show by induction that T_k is a spherical subset of $\nabla(OE_{x_k})$. Let $T_0 = T$ and let $e_1 \in OE_{x_0}$ be the unique element satisfying $\nabla(e_1) = u_1$. Assuming that e_i and T_{i-1} have been defined for $1 \leq i \leq k$, set $T_k = \widehat{j}_{e_k}(T_{k-1}) = \widehat{j}_{e_k} \circ \cdots \circ \widehat{j}_{e_1}(T)$ and let $v_{k+1} \in T_k$ be the element defined by $v_{k+1} = \widehat{j}_{e_k} \circ \cdots \circ \widehat{j}_{e_1}(u_{k+1})$. (Here we have used induction and Condition (M) to show that $\widehat{j}_{e_{k-1}} \circ \cdots \circ \widehat{j}_{e_1}(u_{k+1})$ lies in the domain of $\widehat{j}_{e_k}$.) Let $e_{k+1} \in OE_{x_k}$ be the unique element satisfying $\nabla(e_{k+1}) = v_{k+1}$. The proof of Lemma 5.6.3 then gives the following.

Lemma 5.7.1. *Assume (M) holds for (X, ∇). With notation as above, the vertex $x_n \in X^{(0)}$, the spherical subset T_n, and the isomorphism $\widehat{j}_{e_n} \circ \cdots \circ \widehat{j}_{e_1} : T \to T_n$ depend only on the element $w = w_n \in W_T$ (and not on the choice of word $u_n \cdots u_1$ representing w). Hence, we can use the notation $w \cdot x = x_n$, $wT = T_n$, and $w_* = \widehat{j}_{e_n} \circ \cdots \circ \widehat{j}_{e_1} : T \to wT$.*

For each simplex σ of L_T and $w \in W_T$, $wW_\sigma x$ is the vertex set of a cell in X. The subcomplex $C(L_T)$ is the union of these cells. The map $\phi : Z(L_T) \to C(L_T)$ is defined on vertices by $w \mapsto w \cdot x$. The map ϕ is a covering projection and induces an isomorphism $Z(L_T)/W_{T,x} \cong C(L_T)$, where $W_{T,x}$ denotes the isotropy subgroup of W_T at x.

A *Coxeter orbicell* is the quotient of a Coxeter cell (of type W_T) by a subgroup of W_T. A *Coxeter orbihedron* is an orbihedron that is locally isomorphic to the product of a cell and some Coxeter orbicell. Thus, a Coxeter orbihedron is decomposed into Coxeter orbicells. The link of a vertex in a Coxeter orbihedron is defined as before; it is a simplicial cell complex.

The Coxeter orbihedron $\widehat{X}$ is constructed by gluing onto X a copy of $Z/W_{T,x}$ for each subcomplex of the form $C(L_T)$. In more detail, one constructs a sequence of Coxeter orbihedra $X = X_2, X_3, \dots$ where X_n is supposed to be the union of X and the n-skeleton of $\widehat{X}$, and X_{n+1} is obtained from X_n by gluing in the missing Coxeter orbicells of dimension $n+1$. We have proved the following.

Theorem 5.7.2. *Suppose (X, ∇) is a framed Coxeter cell complex and that Condition (M) holds. Then X is a subcomplex of a Coxeter orbihedron $\widehat{X}$ with the same 2-skeleton such that for each $x \in X^{(0)}$, ∇_x induces an isomorphism from $\widehat{L}_x$ to $\mathcal{S}(V)_{>\emptyset}$.*

Lemma 5.7.3. *As in Section 3, suppose $\Sigma_\#$ is the $\mathcal{R}$-blow-up of $\Sigma(W, S)$. Then the natural framing on $\Sigma_\#$ (defined in Example 5.1.5) satisfies Condition (M).*

Proof. An empty k-simplex of $L_\#$, $k \geq 2$, corresponds to a subset $\{T_1, \dots, T_{k+1}\}$ of $S_\#$ that is not $\mathcal{R}$-nested, but such that any proper subset is. In particular, $T_i \not\subset T_l$ for all $i \neq l$ (if $T_i \subset T_l$ then adding T_i to the $\mathcal{R}$-nested set $\{T_1, \dots, \widehat{T_i}, \dots, T_k\}$ would give an $\mathcal{R}$-nested set). Thus, if T is one of these vertices, say T_i, then it follows from the definition of j_T in 4.5 that $j_T(T_l) = T_l$ for $l \neq i$. In other words, j_T acts trivially on any empty k-simplex in $\mathcal{N}_{\geq \{T\}}$. This implies that Condition (M) holds. $\square$

Example 5.7.4. Suppose that $Z_\#$ is the blow-up of the n-cube at its center. Thus, $Z_\#$ is an interval bundle over $\mathbb{RP}^{n-1}$ $(= \partial Z_\#/a)$. Let $(S_\#, M_\#, L_\#)$ be the framing system for the natural framing on $Z_\#$ described in Example 5.1.5. If $S = \{s_1, \dots, s_n\}$, then $\mathcal{R} = \{S\}$ and $S_\# = \{\{s_1\}, \dots, \{s_n\}, S\}$. The link $L_\#$ is the subdivision of the $(n-1)$-simplex with vertex set $T = \{\{s_1\}, \dots \{s_n\}\}$ obtained by introducing a barycenter S. The picture for $n = 3$ is given in Figure 5.

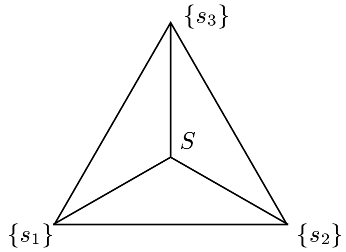


FIGURE 5.

The complex $\widehat{L} = \mathcal{S}(S_\#)_{\geq \emptyset}$ is the full n-simplex on $S_\#$. To describe the orbihedron $\widehat{Z}_\#$, we note that there are two missing simplices in $L_\#$. The first is the $(n-1)$-simplex spanned by T, and the second is the full n-simplex on $S_\#$. Clearly, $C(L_T) = \partial Z_\# = \partial([-1,1]^n)$. We fill in the missing n-cell to obtain

$$X_n = Z_\# \cup_{\partial Z_\#} [-1,1]^n$$

which is isomorphic to $\mathbb{RP}^n$ $(= \partial([-1,1]^{n+1})/a)$. The missing $(n+1)$-cell is $[-1,1]^{n+1}$, but its boundary is no longer embedded (it is the complex X_n). Thus,

$$\widehat{Z}_\# = [-1,1]^{n+1}/a.$$

Example 5.7.5. Suppose Y is the universal cover of the complex $Z_\#$ in the previous example. Thus, Y is isomorphic to $S^{n-1} \times [-1,1]$ $(= \partial([-1,1]^n) \times [-1,1])$. This time there are two missing n-cells with boundaries $S^{n-1} \times \{-1\}$ and $S^{n-1} \times \{1\}$. We fill these in to obtain $Y_n = \partial([-1,1]^{n+1})$. Filling in the missing $(n+1)$-cell we obtain $\widehat{Y} = [-1,1]^{n+1}$.

5.8. Metric flag complexes. The natural piecewise Euclidean metric on a Coxeter cell complex induces a piecewise spherical metric on the link of each vertex. Results of Gromov [G] and Moussong [M] show that the condition that a Coxeter cell complex be nonpositively curved in the sense of Aleksandrov and Gromov is equivalent to a condition on the link of each vertex. We shall now recall this condition. (For more details see [BH] or [DM].)

A spherical simplex *has size* $\geq \pi/2$ if the length of each of its edges is at least $\pi/2$. Similarly, a piecewise spherical simplicial cell complex *has size* $\geq \pi/2$ if each of its simplices does.

Example 5.8.1. The link of a vertex in a Coxeter cell of type W_T is a spherical simplex σ with vertex set T. The length of the edge connecting the vertices t_1 and t_2 is $\pi - \pi/m(t_1, t_2)$, which is $\geq \pi/2$. Hence σ has size $\geq \pi/2$. It follows that the link of any vertex in a Coxeter cell complex has size $\geq \pi/2$. Similarly, if (V, M, L) is any framing system, then the natural piecewise spherical structure on L (in which the length of an edge from v_1 to v_2 is given by $\pi - \pi/m(v_1, v_2)$) has size $\geq \pi/2$.

The definition of *flag complex* in 3.4 can be generalized as follows.

Definition 5.8.2. Suppose L is a piecewise spherical simplicial cell complex of size $\geq \pi/2$. Then L is a *metric flag complex* if it is a simplicial complex and if given any nonempty finite collection T of vertices of L such that (a) any two elements of T are connected by an edge in L and (b) it is possible to find a spherical simplex with the same set of edge lengths, then T spans a simplex in L.

Example 5.8.3. With regard to condition (b), if the length of the edge connecting any two vertices $t_1, t_2 \in T$ is given by a Coxeter matrix (i.e., if it is $\pi - \pi/m(t_1, t_2)$), then there exists a spherical simplex with this set of edge lengths if and only if the associated Coxeter group W_T is finite (see Example 6.7.3 in [DM]). It follows that if (V, M, L) is a framing system, then L is a metric flag complex if and only if $L = \widehat{L}$ (where $\widehat{L} = \mathcal{S}(V)_{>\emptyset}$).

Remark 5.8.4. If L has size $\geq \pi/2$ and if it is a flag complex, then it is a metric flag complex. Conversely, when the length of each edge is $\pi/2$, then L is a metric flag complex only if it is a flag complex.

Moussong generalized Gromov's criterion for nonpositive curvature of a cubical complex (stated in 3.4) by showing that a Coxeter cell complex is nonpositively curved if and only if the link of each vertex is a metric flag complex. (Proofs can be found in [M] or in Section 6.7 of [DM].) A corollary of this is the following.

Theorem 5.8.5. *(Moussong) Suppose X is a framed Coxeter cell complex with framing system (V, M, L). Then X is nonpositively curved if and only if $L = \widehat{L}$.*

5.9. $\widehat{X}$ is nonpositively curved. In this subsection (X, ∇) is a framed Coxeter cell complex satisfying Condition (M) of 5.7 and $\widehat{X}$ is the Coxeter orbihedron constructed in Theorem 5.7.2. It follows from Moussong's result (Thoerem 5.8.5) that $\widehat{X}$ is nonpositively curved.

Theorem 5.9.1. *If X is simply connected, then $\widehat{X}$ is actually a Coxeter cell complex.*

Some evidence for this theorem is provided by Examples 5.7.4 and 5.7.5.

Proof. Nonpositively curved orbihedra are "developable" (see page 562 in [BH]). This means that the local isotropy group at any point in the universal orbihedral cover is trivial. Thus, the universal orbihedral cover of any nonpositively curved Coxeter orbihedron is a Coxeter cell complex. The 2-skeleton of $\widehat{X}$ (as an orbicell complex) is the same as that of X. So, if X is simply connected, then so is $\widehat{X}$ and hence, $\widehat{X}$ is its own orbihedral cover. $\square$

If a complete metric space is nonpositively curved and simply connected, then it is a *CAT(0)-space* (meaning that Gromov's CAT(0)-inequalities hold for all triangles in the space). This implies, for instance, that the space is contractible. Let us say that a group G is a *CAT(0)-group* if it admits a representation as a discrete, cocompact group of isometries on a finite dimensional CAT(0)-space. In the following theorem we list some well-known properties of CAT(0)-groups. (Proofs can be found in [BH].)

Theorem 5.9.2. *Suppose G is a discrete, cocompact group of isometries of a CAT(0) polyhedron K. Then the following statements are true.*

(1) *G is finitely presented.*
(2) *The word problem and the conjugacy problem are solvable for G.*
(3) *G has only finitely many congugacy classes of finite subgroups.*
(4) *If H is any solvable subgroup of G, then H is virtually abelian.*
(5) *$H_*(G;\mathbb{Q})$ is finite dimensional and $\mathrm{cd}_{\mathbb{Q}}(G)$ is no greater than $\dim K$.*
(6) *If π is any torsion-free subgroup of G, then π acts freely on K and K/π is a $K(\pi,1)$-complex for π.*

If G is any discrete, cocompact group of automorphisms of the Coxeter cell complex X, then it extends to a cocompact group of automorphisms of $\widehat{X}$. If, in addition, X is simply connected, then $\widehat{X}$ is CAT(0) and these automorphisms are isometries. Thus, G will be a CAT(0)-group. So, as a corollary to Theorem 5.9.2 we have the following.

Theorem 5.9.3. *Suppose that X is a simply-connected, framed Coxeter cell complex satisfying Condition (M) of 5.7. Suppose further that X admits a discrete, cocompact group of automorphisms G. Then the following statements are true.*

(1) *G is finitely presented.*
(2) *The word problem and the conjugacy problem are solvable for G.*
(3) *G has only finitely many congugacy classes of finite subgroups.*
(4) *If H is any solvable subgroup of G, then H is virtually abelian.*
(5) *$H_*(G;\mathbb{Q})$ is finite dimensional and $\mathrm{cd}_{\mathbb{Q}}(G)$ is no greater than $\dim \widehat{X}$.*
(6) *If π is any torsion-free subgroup of G, then π acts freely on X. If, in addition, π has finite index in G, then we can add finitely many cells to X/π to obtain a $K(\pi,1)$-complex.*

Proof. All statements but (6) follow immediately from the previous theorem applied to the G-action on $\widehat{X}$. For (6), since the π-action is free on $\widehat{X}$, it is also free on X. The $K(\pi,1)$-complex is then $\widehat{X}/\pi$. □

From Theorem 5.9.3 and Lemma 5.7.3 we immediately get the following.

Corollary 5.9.4. *Suppose that $\widetilde{\Sigma}_{\#}$ is the universal cover of an $\mathcal{R}$-blow-up of $\Sigma(W,S)$ and that A is its symmetry group as defined in 4.7. Then A is a CAT(0)-group (and, hence, has all the properties listed in Theorem 5.9.3).*

5.10. Construction of the group and the complex from the gluing isomorphisms. Suppose we are given a Coxeter matrix M $(= m(u,v))$ on a finite set V and an involution on V denoted by $v \mapsto \overline{v}$. Call two distinct elements u and v of V *adjacent* if $m(u,v) \neq \infty$. Let L denote the 1-skeleton of $\mathcal{S}(V)_{>\emptyset}$ (i.e., L is the graph with vertex set V and with an edge connecting any two adjacent vertices). As before, V_v denotes the star of v in L (i.e., $V_v = \{u \in V \mid m(u,v) \neq \infty\}$). Also suppose that we are given as "gluing data" a set $\{j_v\}_{v\in V}$ of bijections $j_v : V_v \to V_{\overline{v}}$ such that $j_v(v) = \overline{v}$. The question we address here is: when can we find a group A and a simply connected, 2-dimensional, Coxeter cell complex X such that (a) the link of each vertex in X is L, (b) A acts simply transitively on $X^{(0)}$, and (c) $\{j_v\}_{v\in V}$ is the corresponding set of gluing isomorphisms?

There are four obvious conditions that should be imposed on the j_v. First,

(1) $j_{\overline{v}} = (j_v)^{-1}$ for all $v \in V$.

The second condition is just a rewording of Condition (M1) from 5.7:

(2) For each pair v_1, v_2 in $V_v - \{v\}$, $m(j_v(v_1), j_v(v_2)) = m(v_1, v_2)$.

Before stating the third condition we need to develop some more notation. Given an ordered pair (u,v) of adjacent elements in V, define a sequence of elements $v_0, v_1, v_2, \dots$ by the formulas:

$$v_0 = \overline{u}, \quad v_1 = v, \quad \text{and} \quad v_k = j_{v_{k-1}}(\overline{v}_{k-2}) = j_{v_{k-1}} j_{v_{k-2}}(v_{k-2}) \quad \text{for } k \geq 2 \tag{$*$}$$

(Just as in 5.7, it follows by induction that $\overline{v}_{k-2} \in V_{v_{k-1}}$ and hence, that the above formula makes sense.) We note that $v_{2l} = j_{v_{2l-1}} \cdots j_{v_1}(u)$ and $v_{2l+1} = j_{v_{2l}} \cdots j_{v_1}(v)$. Our third condition is the following:

(3) Given any ordered pair (u,v) of adjacent elements in V, let $v_0, v_1, \dots$ be the sequence defined above, and let $m = m(u,v)$. Then $v_{2m} = v_0 = \overline{u}$ and $v_{2m+1} = v_1 = v$.

We note that (3) implies that $v_{2m+k} = v_k$ for all $k \geq 0$.

These first three conditions are enough for us to be able to define the group A by the same procedure as in 5.4. For each ordered pair (u,v) of adjacent elements in V, let $r(u,v)$ be the word in V defined by

$$r(u,v) = v_1 v_2 \cdots v_{2m}.$$

Let A be the quotient of F_V (the free group on V) by the normal subgroup generated by $\{v\overline{v} \mid v \in V\} \cup \{r(u,v) \mid u \text{ and } v \text{ are adjacent}\}$. Let α_v denote the image of v in A and let $\mathcal{A} = \{\alpha_v\}_{v\in V}$ be the corresponding set of generators. Finally, for each ordered pair (u,v) of adjacent elements we define a sequence $a_1(u,v), a_2(u,v), \dots$ of elements of A by the formula:

$$a_k(u,v) = \alpha_{v_1}\alpha_{v_2}\cdots\alpha_{v_k} \tag{$**$}$$

where $v_1, v_2, \dots$ is the sequence defined by $(*)$. Our last condition is the following:

(4) (i) $\alpha_v \neq 1$, for each $v \in V$.
(ii) $\alpha_u \neq \alpha_v$, if $u \neq v$.
(iii) For each ordered pair (u,v) of adjacent elements, let $(a_k) = (a_k(u,v))$ be the above sequence and let $m = m(u,v)$. Then the elements $1, a_1, a_2, \dots, a_{2m-1}$ are distinct.

Theorem 5.10.1. *Suppose V, M, and L are as above and that $\{j_v\}_{v\in V}$ is a set of gluing isomorphisms satisfying conditions (1), (2), (3), and (4), above. Then there is a simply connected, 2-dimensional Coxeter cell complex X and a group A of automorphisms of X such that (a) the link of each vertex is L, (b) A is simply transitive on $X^{(0)}$, and (c) $\{j_v\}_{v\in V}$ is the corresponding set of gluing isomorphisms.*

Proof. The Cayley 2-complex associated to the presentation of A is such an X. (Condition (4) is needed to check that X is a Coxeter cell complex; for example, (4)(iii) implies that the 2-cell associated to $r(u,v)$ is a $2m(u,v)$-gon.) □

Remark 5.10.2. If, in addition, X satisfies Condition (M2) of 5.7, then applying Theorems 5.7.2 and 5.9.2 we see that X is actually the 2-skeleton of a CAT(0) Coxeter cell complex $\widehat{X}$ on which A acts as a group of isometries.

Example 5.10.3. Here we present an example of a finite 2-dimensional Coxeter cell complex X which violates the conclusion of the previous remark. Although X will be homogeneous and simply connected, it cannot be completed to a CAT(0) complex $\widehat{X}$. The reason is that Condition (M2) of 5.7 does not hold: there is nontrivial holonomy around each 2-cell.

Suppose that V consists of four elements $\{a,b,c,d\}$, that L is the complete graph on V (i.e., L is the 1-skeleton of a 3-simplex), and that M is the right-angled Coxeter matrix associated to L (i.e., $m(u,v)=2$ for any two distinct elements $u,v\in V$). Since the automorphism group of L is S_4 (the symmetric group on 4 letters), we can specify candidates for gluing isomorphisms by giving 4 permutations. We choose the following involutions:

$$\begin{aligned} j_a &= (b\,d)\\ j_b &= (c\,d)\\ j_c &= (a\,d)\\ j_d &= \mathrm{Id}\,. \end{aligned}$$

(Thus, j_a, j_b, and j_c are transpositions.) Conditions (1) and (2) clearly hold. Next we need to check Condition (3) for each pair (u,v) of distinct elements in V. Since reversing the order of (u,v) only reverses the order of the v_k, it suffices to check each of the 6 unordered pairs. In each case, we get a relation of length four:

$$\begin{aligned} r(a,b) &= bada, & r(a,d) &= daba\\ r(b,c) &= cbdb, & r(b,d) &= dbcb\\ r(c,d) &= dcac, & r(a,d) &= acdc. \end{aligned}$$

Notice that the relations in the same row differ by a cyclic permutation, so we really have only three independent relations.

This gives a presentation for a group

$$A = \langle a,b,c,d \mid a^2, b^2, c^2, d^2, r(a,b), r(b,c), r(c,d)\rangle.$$

It can be checked that A is isomorphic to the dihedral group of order 14, and it follows from this that condition (4) holds. As above, let X be the Cayley 2-complex of the presentation. It is simply connected and A acts simply transitively on $X^{(0)}$. Each 2-cell is a square and the link of each vertex is L. X obviously cannot be completed to a CAT(0) complex $\widehat{X}$. Indeed, the Coxeter cell corresponding to L is the 4-cube, so $\widehat{X}$ would have to be the 4-cube. But the 4-cube has 16 vertices while X has only 14.

The problem is that Condition (M2) does not hold. (It is not automatic since L is not rigid.) For example, if we compute the holonomy around a 2-cell labeled $adab$, we get $j_b \circ j_a \circ j_d \circ j_a = (c\,d)$.

6. A linear representation

6.1. The partial order. Let M be a Coxeter matrix on the set V, and let $\{j_v\}_{v\in V}$ be a set of gluing maps as defined in 5.10. In addition, we assume that the involution $v \mapsto \overline{v}$ on V is trivial, thus each j_v is an automorphism of V_v. In this section we describe conditions that allow us to define a linear representation of the resulting group A. We will show that when M, V, and $\{j_v\}$ arise from an $\mathcal{R}$-blow-up $\Sigma_\#$, these conditions are satisfied, giving a linear representation of the group A acting on the universal cover $\widetilde{\Sigma}_\#$ (as described in 4.7). This representation generalizes the standard geometric representation of a Coxeter group.

Our first assumption is that there is a partial order (denoted by $<$) on V satisfying the following condition:

(P) (i) If u and v are comparable (i.e., $u < v$ or $v < u$), then they are adjacent (i.e., $m(u, v) < \infty$).
(ii) If $3 \leq m(u, v) < \infty$, then u and v are minimal.
(iii) If $u' \leq u$, $v' \leq v$, and u and v are noncomparable and adjacent, then u' and v' are noncomparable and adjacent.
(iv) Let $\overline{M}$ denote the Coxeter matrix M with all ∞ entries replaced by 2's. Then for each v in V, the subset $V_{<v}$ $(= \{u \in V \mid u < v\})$ is spherical with respect to $\overline{M}$.

Next we consider the set of gluing automorphisms $\{j_v : V_v \to V_v\}_{v\in V}$ with $j_v(v) = v$ as in 5.10. In the presence of the partial order, we can formulate the following condition on the j_v's, which is much simpler than are the conditions in 5.10.

(C) (i) j_v is an involution.
(ii) If $v_1, v_2 \in V_v - \{v\}$, then $m(j_v(v_1), j_v(v_2)) = m(v_1, v_2)$.
(iii) j_v preserves the partial order on V_v.
(iv) If $j_v(u) \neq u$, then $u < v$.
(v) If $u < v$ and $u' = j_v(u)$, then $j_u j_v j_{u'} j_v(y) = y$ for all $y < u$.

Lemma 6.1.1. *If $\{j_v\}_{v\in V}$ satisfies Conditions (C)(i)-(iv), then Conditions (1), (2), and (3) of 5.10 hold.*

Proof. (C)(i) implies Condition (1), and (C)(ii) is the same as Condition (2). Consider Condition (3) of 5.10. If $3 \leq m(u, v) < \infty$, then by (P)(ii) u and v are minimal and hence, by (C)(iv), $j_v(u) = u$ and $j_u(v) = v$ and the sequence $v_0, v_1, v_2, v_3, \dots$ is $u, v, u, v, \dots$. The same conclusion holds if $m(u, v) = 2$ and u and v are noncomparable. Hence, Condition (3) holds in these cases. Finally, if $u < v$, then $m(u, v) = 2$ and the sequence $v_0, v_1, v_2, v_3, \dots$ begins as $u, v, j_v(u), v, \dots$ and has the same period, 4, so Condition (3) again holds. □

Remark 6.1.2. Condition (C)(v) is related to the holonomy condition (M2) of 5.7. Given any pair (u, v) with $m = m(u, v) < \infty$, the sequence $v_0 = u, v_1 = v, v_2 = j_{v_1}(v_0), \dots$ gives rise to the holonomy automorphism $j_{v_{2m}} \cdots j_{v_1} : V_{uv} \to V_{uv}$ where $V_{uv} = \{y \in V \mid \{u, v, y\} \text{ is spherical}\}$. For most pairs, this holonomy will be trivial as a consequence of the properties listed in (P) and (C)(i)-(iv). For example, if $m(u, v) \geq 3$, then u and v are minimal; thus j_u and j_v are identity maps so the holonomy automorphism $j_u j_v \cdots j_u j_v$ is trivial. However, to ensure trivial holonomy in the case $u < v$ the additional condition (C)(v) is needed.

Henceforth, we assume that (C) holds. It follows that we can define the group A as in 5.10. Condition (4) of 5.10 then becomes the following:

(4′) (i) $\alpha_v \neq 1$.
(ii) $\alpha_u \neq \alpha_v$ if $u \neq v$.
(iii) If u and v are adjacent and noncomparable, then the order of $\alpha_u \alpha_v$ is $m(u, v)$. If $u < v$ and $u' = j_v(u)$, then the elements $1, \alpha_u, \alpha_u \alpha_v, \alpha_u \alpha_v \alpha_{u'}$ are distinct.

In the next subsection, we shall define a representation for the group A and use it to show that (4′) always holds (cf., Corollary 6.2.6, below).

Lemma 6.1.3. *Let $\Sigma_\#$ denote the $\mathcal{R}$-blow-up of $\Sigma(W, S, L)$, and let $(V, M) = (S_\#, M_\#)$. Then the partial order on $S_\#$ defined by inclusion satisfies Condition (P) and the natural set of gluing involutions $\{j_T\}_{T \in S_\#}$ satisfies Condition (C).*

Proof. If T and T' are comparable, then $m_\#(T, T') = 2$, so T and T' are adjacent. Thus, (P)(i) holds. If $3 \leq m_\#(T, T') < \infty$, then T and T' are singleton subsets of S, hence they are minimal, so (P)(ii) holds. For (P)(iii), we just note that two subsets T and U are adjacent and noncomparable if and only if they are both minimal or they are completely disjoint. Thus, if T and U are noncomparable and adjacent, $T' \subset T$, and $U' \subset U$, then T' and U' must be noncomparable and adjacent.

To show that (P)(iv) holds, suppose $T \in S_\#$, and let $\overline{M}_{\#T}$ denote the restriction of $\overline{M}_\#$ to $(S_\#)_{<T}$. We have to show that $\overline{M}_{\#T}$ is the matrix for a finite Coxeter group. Let Γ_T denote the Coxeter diagram for the (finite) special subgroup W_T. Then the Coxeter diagram for $\overline{M}_{\#T}$ is obtained from Γ_T in the following two steps. First, for each subset $T' \in (S_\#)_{<T}$ of the form $T' = \{s, s'\}$, we delete the edge joining s and s'. (This corresponds to replacing the ∞-entry $m_\#(\{s\}, \{s'\})$ of $M_\#$ with a 2.) Since the resulting diagram represents a product of special subgroups of W_T, its Coxeter group is finite. Second, for each nonsingleton element $T' \in (S_\#)_{<T}$, we add a new disjoint node. (This new node is not connected to any other node since $m_\#(T', T'')$ is either 2 or ∞ for any nonminimal $T' \in S_\#$.) Since adding a disjoint node to a Coxeter diagram corresponds to adding a $\mathbb{Z}_2$ factor to its Coxeter group, the resulting diagram represents a finite Coxeter group.

Condition (C) follows directly from the definition of the j_T's given in 4.5. □

6.2. The representation. Let M be a Coxeter matrix on a set V, and let E denote the vector space $\mathbb{R}^V$ with standard basis $\{\epsilon_v\}_{v\in V}$. For each $t \in \mathbb{R}$, we define a symmetric bilinear form B_t $(= B_t(M))$ on E by

$$B_t(\epsilon_u, \epsilon_v) = \begin{cases} -\cos(\pi/m(u,v)) & \text{if } m(u,v) < \infty \\ -t & \text{if } m(u,v) = \infty. \end{cases}$$

(Note that when $t = 1$, $B_t(M)$ is the canonical bilinear form associated to the Coxeter matrix M.)

Lemma 6.2.1. *Assume that (P) holds for (V, M) and that $\{j_v\}_{v\in V}$ is a set of gluing involutions satisfying Condition (C). For each $v \in V$, let $E_v \subset E$ denote the subspace defined by*

$$E_v = \operatorname{Span}\{\epsilon_u - \epsilon_{u'} \mid u \in V_v \text{ and } u' = j_v(u)\}.$$

Then for t sufficiently large, the restriction of B_t to E_v is nondegenerate for all $v \in V$.

Proof. Let $v \in V$, and let $U_v \subset E$ be the subspace $U_v = \operatorname{Span}\{\epsilon_u \mid u \in V_{<v}\}$. We let $\overline{B}$ denote the canonical bilinear form associated to the Coxeter matrix $\overline{M}$. Then when $t = 0$, $B_t(M)$ coincides with $\overline{B}$, and the restriction $B_t|_{U_v}$ coincides with the restriction $\overline{B}|_{U_v}$. By (P)(iv), the latter is positive definite, hence $\det(\overline{B}|_{U_v}) \neq 0$. It follows that $\det(B_t|_{U_v})$ is a nonzero polynomial in t, so for t sufficiently large, $B_t|_{U_v}$ is nondegenerate. To complete the proof, we note that $E_v \subset U_v$ (by Condition (C)(iv)) and that U_v splits as an orthogonal direct sum

$$U_v = E_v \oplus \operatorname{Span}\{\epsilon_u + \epsilon_{u'} \mid u \in V_{<v} \text{ and } u' = j_v(u)\}$$

(by (C)(ii)). Thus, for t sufficiently large, $B_t|_{E_v}$ is nondegenerate. □

Let B_t be one of the bilinear forms that is nondegenerate on the subspace E_v for all $v \in V$. Let $v \in V$. The definition of the gluing involution j_v implies that ϵ_v is orthogonal to the subspace E_v. Moreover, since $B_t(\epsilon_v, \epsilon_v) = 1$ and B_t is nondegenerate on E_v, we know that B_t is nondegenerate on $\mathbb{R}\epsilon_v \oplus E_v$. Letting F_v denote the orthogonal complement of $\mathbb{R}\epsilon_v \oplus E_v$, we then have an orthogonal decomposition

$$E = \mathbb{R}\epsilon_v \oplus E_v \oplus F_v.$$

With respect to this decomposition, we define an involution $\rho_v : E \to E$ by the formula

$$\rho_v = -\operatorname{Id}|_{\mathbb{R}\epsilon_v} \oplus -\operatorname{Id}|_{E_v} \oplus \operatorname{Id}|_{F_v}.$$

It is clear that ρ_v preserves the bilinear form B_t, that $\rho_v(\epsilon_v) = -\epsilon_v$ and that $\rho_v(E_v) = E_v$ for all $v \in V$.

Lemma 6.2.2.

(1) *If v is minimal, then $E_v = \{0\}$ and ρ_v is the orthogonal reflection across the hyperplane $F_v = \epsilon_v^{\perp}$. In other words, $\rho_v(\lambda) = \lambda - 2B_t(\lambda, \epsilon_v)\epsilon_v$ for all $\lambda \in E$.*
(2) *If $u < v$, then $\rho_v(\epsilon_u) = \epsilon_{u'}$ and $\rho_v(E_u) = E_{u'}$ where $u' = j_v(u)$.*
(3) *If u and v are noncomparable and $m(u, v) = 2$, then $\rho_v(\epsilon_u) = \epsilon_u$ and $\rho_v(E_u) = E_u$.*

Proof. For (1), if v is minimal, the involution j_v is trivial. This means $E_v = \{0\}$, so ρ_v is the orthogonal reflection with the given formula. For (2), we note that both u and u' are adjacent to v, and v is nonminimal; hence, by (P)(ii), $m(u,v) = m(u',v) = 2$. It follows that the vector $\epsilon_u + \epsilon_{u'}$ is orthogonal to ϵ_v. By (C)(ii), this vector is also orthogonal to the subspace E_v. So $\epsilon_u + \epsilon_{u'}$ is in F_v and, thus, fixed by ρ_v. We then have

$$\rho_v(2\epsilon_u) = \rho_v(\epsilon_u - \epsilon_{u'}) + \rho_v(\epsilon_u + \epsilon_{u'}) = -(\epsilon_u - \epsilon_{u'}) + (\epsilon_u + \epsilon_{u'}) = 2\epsilon_{u'}.$$

Hence, $\rho_v(\epsilon_u) = \epsilon_{u'}$. To see that $\rho_v(E_u) = E_{u'}$, suppose $y < u$. By (C)(v), we have $j_u j_v j_{u'} j_v(y) = y$ or, in other words, $j_v j_u(y) = j_{u'} j_v(y)$. Applying ρ_v to $\epsilon_y - \epsilon_{j_u(y)} \in E_u$, and using the fact that $j_u(y) < u < v$, we obtain

$$\rho_v(\epsilon_y - \epsilon_{j_u(y)}) = \rho_v(\epsilon_y) - \rho_v(\epsilon_{j_u(y)}) = \epsilon_{j_v(y)} - \epsilon_{j_v j_u(y)} = \epsilon_{j_v(y)} - \epsilon_{j_{u'} j_v(y)}.$$

Since this last term is in $E_{u'}$, it follows that $\rho_v(E_u) \subset E_{u'}$. The same argument shows $\rho_v(E_{u'}) \subset E_u$; hence, $\rho_v(E_u) = E_{u'}$. The proof of (3) is similar. $\square$

Theorem 6.2.3. *The map $\alpha_v \mapsto \rho_v$ extends to a homomorphism $\rho : A \to GL(E)$. (In fact the image of ρ lies in the orthogonal subgroup $O(B_t) \subset GL(E)$.)*

Proof. A is the group with one generator α_v for each $v \in V$ and relations of the form

(a) $(\alpha_v)^2 = 1$ for all $v \in V$,
(b) $(\alpha_u\alpha_v)^m = 1$ if u and v are minimal and $m = m(u,v)$,
(c) $\alpha_u\alpha_v\alpha_{u'}\alpha_v = 1$ if $u < v$ and $u' = j_v(u)$, and
(d) $(\alpha_u\alpha_v)^2 = 1$ if u and v are noncomparable and $m(u,v) = 2$.

Since each ρ_v is an involution (preserving B_t), it suffices to show that relations (b), (c), and (d) hold under the substitution $\alpha_v \mapsto \rho_v$. In case (b), j_u and j_v are trivial; hence, the involutions ρ_u and ρ_v are the usual orthogonal reflections through the hyperplanes F_u and F_v, respectively. Since ρ_u and ρ_v fix the codimension-two subspace $F_u \cap F_v$ pointwise, it suffices to show that $\rho_u\rho_v$ has order m when restricted to $\mathrm{Span}\{\epsilon_u, \epsilon_v\}$. A simple calculation shows that B_t is positive definite on $\mathrm{Span}\{\epsilon_u, \epsilon_v\}$ and that $\rho_u\rho_v$ is a rotation through an angle of $2\pi/m$. Thus, $(\rho_u\rho_v)^m = \mathrm{Id}$.

In case (c), Lemma 6.2.2 (part 2) implies that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{R}\epsilon_u \oplus E_u) \oplus F_u \arrow[r, "\rho_v"] \arrow[d, "\rho_u = -\mathrm{Id}\oplus\mathrm{Id}"] & (\mathbb{R}\epsilon_{u'} \oplus E_{u'}) \oplus F_{u'} \arrow[d, "\rho_{u'} = -\mathrm{Id}\oplus\mathrm{Id}"] \\
(\mathbb{R}\epsilon_u \oplus E_u) \oplus F_u \arrow[r, "\rho_v"] & (\mathbb{R}\epsilon_{u'} \oplus E_{u'}) \oplus F_{u'}
\end{tikzcd} .$$

Since all of these maps are involutions, we have $\rho_u\rho_v\rho_{u'}\rho_v = \mathrm{Id}$.

In case (d), Lemma 6.2.2 (part 3) implies that we have the same commutative diagram as in case (c) but with $u' = u$. Thus, $\rho_u\rho_v\rho_u\rho_v = \mathrm{Id}$. $\square$

Remark 6.2.4. It seems likely that all of the mock reflection groups considered in this paper are linear. Indeed, the representation $\rho : A \to GL(E)$, constructed above, is probably always faithful; however, we do not have a proof of this and the linearity of A is an open question.

Example 6.2.5. Let S be the set $\{a, b, c\}$, and let (W, S) be the Coxeter system corresponding to the diagram $\mathbf{A}_3$. The corresponding Coxeter cell $Z(W, S)$ is the 3-dimensional permutohedron. The collection $\mathcal{R} = \{\{a, b\}, \{b, c\}\}$ is admissible with respect to $\mathcal{P} = \mathcal{S} - \{\{a, b, c\}\}$, and the corresponding $\mathcal{R}$-blow-up is a blow-up of ∂Z. (The Coxeter tile is a pentagon, the 2-dimensional associahedron $\Delta_\#$ as in Example 4.2.2.) The corresponding group A has a generator for each element of

$$S_\# = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}.$$

With respect to the standard basis $\epsilon_a, \epsilon_b, \epsilon_c, \epsilon_{ab}, \epsilon_{bc}$ for $E = \mathbb{R}^5$, the family of forms B_t is given by

$$[B_t] = \begin{bmatrix} 1 & -t & 0 & 0 & -t \\ -t & 1 & -t & 0 & 0 \\ 0 & -t & 1 & -t & 0 \\ 0 & 0 & -t & 1 & -t \\ -t & 0 & 0 & -t & 1 \end{bmatrix},$$

and the representation $\rho : A \to GL(\mathbb{R}^5)$ is defined by the involutions

$$\rho_a = \begin{bmatrix} -1 & 2t & 0 & 0 & 2t \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2t & -1 & 2t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2t & -1 & 2t & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho_{ab} = \begin{bmatrix} 0 & 1 & \frac{-t}{1+t} & 0 & \frac{t}{1+t} \\ 1 & 0 & \frac{t}{1+t} & 0 & \frac{-t}{1+t} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2t & -1 & 2t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho_{bc} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{t}{1+t} & 0 & 1 & \frac{-t}{1+t} & 0 \\ \frac{-t}{1+t} & 1 & 0 & \frac{t}{1+t} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2t & 0 & 0 & 2t & -1 \end{bmatrix}$$

Corollary 6.2.6. *Suppose (V, M) satisfies Condition (P), and $\{j_v\}_{v \in V}$ is a set of involutions satisfying Condition (C). Then Condition (4′) holds. In particular, the triple $(V, M, \{j_v\})$ yields a group A and an action of A on a CAT(0) Coxeter cell complex.*

Proof. (4′)(i) follows since ρ_v is a nontrivial involution for every $v \in V$. Similarly, since $u \neq v$ implies ρ_u and ρ_v have different -1-eigenspaces, we have $\rho_u \neq \rho_v$. Thus, $\alpha_u \neq \alpha_v$, so (4′)(ii) holds. Cases (b) and (d) of the proof of Theorem 6.2.3 show that if u and v are adjacent and noncomparable, then $\rho_u\rho_v$ has order $m(u, v)$. This means that the order of $\alpha_u\alpha_v$ is at least $m(u, v)$, and therefore exactly $m(u, v)$. Similarly, if $u < v$, then case (c) of Theorem 6.2.3 shows that Id, ρ_u, $\rho_u\rho_v$, and $\rho_u\rho_v\rho_{u'}$ are all distinct. Thus, $1, \alpha_u, \alpha_u\alpha_v, \alpha_u\alpha_v\alpha_{u'}$ must be distinct elements of A. By Theorem 5.10.1, A acts on a 2-dimensional Coxeter cell complex, and since condition (M2) holds (Remark 6.1.2), this complex can be completed to a CAT(0) Coxeter cell complex (Remark 5.10.2). □

7. Permutohedral tilings

Suppose that we are given a tiling of a Coxeter cell complex with fundamental tile an n-simplex. Its maximal blow-up (as defined in [DJS], Section 4.1) will then be a cubical n-manifold tiled by permutohedra. This situation can arise from a Coxeter system (W, S) of rank $n+1$ in two ways. The first is where W is finite and we take the maximal blow-up of $\partial Z(W, S)$ (the boundary of the Coxeter cell). The second way occurs when (W, S) is a "simplicial" Coxeter system (defined in subsection 7.3, below), and we take the maximal blow-up of $\Sigma(W, S)$ (the complete reflection tiling of type (W, S)). As it turns out, it follows from Theorem 5.6.4 that the universal covers of all such examples yield the same right-angled tiling of $\mathbb{R}^n$ by permutohedra. Hence, the fundamental groups of any two closed n-manifolds that arise from such constructions are commensurable. (Essentially, the same result was asserted in Section 4.2 of [DJS]; however, as we shall explain in subsection 7.5, below, there was a mistake in the proof.)

7.1. The permutohedron. There are three equivalent definitions of the n-dimensional permutohedron P. First, it can be defined as the convex polytope obtained by truncating all of the faces of an n-simplex Δ^n of codimension ≥ 2. A second definition is that it is the polytope whose boundary complex is dual to the barycentric subdivision of $\partial\Delta^n$. The third definition is that P^n is the Coxeter cell associated to the symmetric group S_{n+1} (S_{n+1} is the Coxeter group with Coxeter graph $\mathbf{A}_n$.)

Let us fix a set S of cardinality $n+1$ and suppose that the elements of S index the codimension-one faces of Δ^n. Regard S_{n+1} as the symmetric group on S. Let $V(P^n)$ denote the set of all proper nonempty subsets of S. Thus, $V(P^n)$ naturally indexes the set of codimension-one faces of P^n. Let $\mathcal{N}(P^n)$ denote the poset of all subsets of $V(P^n)$ that are chains. (If $\mathcal{R} = V(P^n)$, then $\mathcal{N}(P^n)$ is the poset of all $\mathcal{R}$-nested subsets of $V(P^n)$ as in Definition 3.3.1.) Thus, $\mathcal{N}(P^n)_{>\emptyset}$ is the barycentric subdivision of $\partial\Delta^n$. We shall also denote this simplicial complex by $L(P^n)$.

The action of S_{n+1} on S induces an action on P^n as a group of combinatorial automorphisms. A fundamental domain for this action is the associated Coxeter block B^n. As is the case for any Coxeter cell, the orbit space P^n/S_{n+1} can be identified with this Coxeter block. The permutohedron has one further symmetry: the antipodal map. In fact, it is easy to see that its full group of combinatorial symmetries, $\mathrm{Aut}(P^n)$, is just $S_{n+1} \times \mathbb{Z}_2$.

7.2. The reflection tiling. Let M' be the right-angled Coxeter matrix associated to the flag complex $L(P^n)$ and let W' be the associated Coxeter group. That is to say, W' has a generator for each codimension-one face of P^n (i.e., for each element of $V(P^n)$), and two such generators commute if and only if the corresponding faces intersect. Let Σ_{P^n} be the complete reflection tiling of type $(W', V(P_n))$. The natural framing is symmetric, and the group of frame-preserving automorphisms is $A(\Sigma_{P^n}) = W'$.

The full symmetry group of Σ_{P^n} is $G(\Sigma_{P^n}) = A(\Sigma_{P^n}) \rtimes \mathrm{Aut}(P^n)$. This has a subgroup of index two, $G_0(\Sigma_{P^n})$ defined by

$$G_0(\Sigma_{P^n}) = A(\Sigma_{P^n}) \rtimes S_{n+1}.$$

In fact, $G_0(\Sigma_{P^n})$ is a also a Coxeter group and its action on Σ_{P^n} is as a group generated by reflections. A fundamental chamber for this action is the Coxeter block B^n. Thus, $\Sigma_{P^n}/G_0(P^n) \cong P^n/S_{n+1} \cong B^n$. (The Coxeter diagram for $G_0(P^n)$ is given in Figure 7, page 536 of [DJS].)

7.3. Simplicial Coxeter systems.

Definition 7.3.1. A Coxeter system (W, S) is *simplicial* if W is infinite and each proper subset of S is spherical.

Suppose that (W, S) is simplicial and that $\mathrm{Card}(S) = n + 1$. Then the fundamental chamber for the W-action on $\Sigma(W, S)$ is an n-simplex. In 1950 in [L], Lanner showed that each such $\Sigma(W, S)$ can be identified with either hyperbolic n-space $\mathbb{H}^n$ or Euclidean n-space $\mathbb{E}^n$ so that the W-action is as a classical group of isometries generated by reflections across the faces of a hyperbolic or Euclidean n-simplex. He also listed possible Coxeter diagrams of simplicial Coxeter systems.

In the Euclidean case, there are four families in each dimension $n \geq 4$. Their Coxeter diagrams are denoted $\widetilde{\mathbf{A}}_n$, $\widetilde{\mathbf{B}}_n$ $(n \geq 2)$, $\widetilde{\mathbf{C}}_n$ $(n \geq 3)$, and $\widetilde{\mathbf{D}}_n$ $(n \geq 4)$. There are also five exceptional Euclidean simplicial Coxeter systems: $\widetilde{\mathbf{G}}_2$, $\widetilde{\mathbf{F}}_4$, $\widetilde{\mathbf{E}}_6$, $\widetilde{\mathbf{E}}_7$, $\widetilde{\mathbf{E}}_8$. In the hyperbolic case, in dimension two, there are the (p, q, r) triangle groups (where $p^{-1}+q^{-1}+r^{-1} < 1$). In dimension three, there are the nine tetrahedral hyperbolic Coxeter systems, and in dimension four there are five more hyperbolic examples. Finally, there are no hyperbolic simplicial Coxeter systems in dimensions > 4. (The Coxeter diagrams of all these groups can be found, for example, in [Bo] or [L]).

Because $\Sigma(W, S)$ is either $\mathbb{E}^n$ or $\mathbb{H}^n$, its quotient by any torsion free subgroup $\Gamma \subset W$ will be a manifold M. If $\Sigma_\#$ is any $\mathcal{R}$-blow-up, then the quotient $M_\# = \Sigma_\#/\Gamma$ is a blow-up of M.

Remark 7.3.2. The group W corresponding to the $\widetilde{\mathbf{A}}_{n+1}$ Coxeter diagram is the semidirect product of $\mathbb{Z}^n$ with the symmetric group S_{n+1}. In this case, the fundamental tile of the minimal blow-up $\Sigma_\#$ is a polytope called the “cyclohedron”. The quotient of this tiling by $\mathbb{Z}^n$ is discussed in [De] in relation to compactifications of configuration spaces.

7.4. Maximal blow-ups of simplicial tilings. Suppose (W, S) is a Coxeter system with $\mathrm{Card}(S) = n+1$ and with W finite. Let $\widetilde{Z}_\#$ be the universal cover of the maximal blow-up of the Coxeter cell $Z(W, S)$, and let X be the universal cover of $(\partial Z)_\#$. Since $Z_\#$ is an interval bundle over $(\partial Z)_\#/a_\#$ and $a_\#$ is a free involution, $\widetilde{Z}_\#$ is an interval bundle over X. It is easy to see that the natural framing on X obtained by restricting the framing on $Z_\#$ is symmetric and the group of frame-preserving symmetries is the subgroup A $(= A(X))$ of $A(W, S)$ (see 4.7) generated by all α_T where T is a proper nonempty subset of S.

Theorem 7.4.1. *Suppose (W,S) is a Coxeter system with $\mathrm{Card}(S) = n+1$ and with W finite. Let X denote the universal cover of the maximal blow-up of $\partial Z(W,S)$, and let A $(= A(X))$ be the group described above. Then*

(1) *X is isomorphic to Σ_{P^n}, and*
(2) *A is isomorphic to a subgroup of index $n!$ in $G_0(\Sigma_{P^n})$.*

Theorem 7.4.2. *Suppose (W,S) is a simplicial Coxeter system, with $\mathrm{Card}(S) = n+1$. Let $\widetilde{\Sigma}_{\#}$ denote the universal cover of the maximal blow-up of $\Sigma(W,S)$, and let A $(= A(\widetilde{\Sigma}_{\#}))$ denote its frame-preserving symmetry group from 4.7. Then*

(1) *$\widetilde{\Sigma}_{\#}$ is isomorphic to Σ_{P^n}, and*
(2) *A is isomorphic to a subgroup of index $n!$ in $G_0(\Sigma_{P^n})$.*

Both of these theorems follow from Theorem 5.6.4. We must first verify that Conditions (E) and (H) hold. For Condition (E), we must show that for each proper, nonempty subset T of S, the involution

$$j_T : \mathcal{N}(P^n)_{\geq\{T\}} \to \mathcal{N}(P^n)_{\geq\{T\}}$$

extends to an involution of $\mathcal{N}(P^n)$. The map j_T is induced by the permutation of T defined by $a_T w_T$ (we continue to denote this permutation by j_T). Extend j_T to a permutation $\widetilde{j}_T$ of S by letting $\widetilde{j}_T|_{S-T}$ be the identity permutation of $S-T$. Thus, $\widetilde{j}_T \in S_{n+1}$ and hence can naturally be regarded as an element of $\mathrm{Aut}(\mathcal{N}(P^n))$ $(= \mathrm{Aut}(P^n))$.

Condition (H) follows from Remark 5.6.2 and the fact that the link of a vertex in X (or $\widetilde{\Sigma}_{\#}$) is the boundary complex of the n-dimensional octahedron.

Example 7.4.3. As in Theorem 7.4.2 suppose that (W,S) is a simplicial Coxeter system of rank $n+1$, that $\Sigma = \Sigma(W,S)$, and that $\Sigma_{\#}$ is the maximal blow-up. Let H be a torsion-free subgroup of finite index in W. Then Σ/H and $\Sigma_{\#}/H$ are nonpositively curved, closed n-manifolds. If $\phi : A(\widetilde{\Sigma}_{\#}) \to W$ denotes the natural projection, then $\pi_1(\Sigma_{\#}/H) \cong \phi^{-1}(H)$. So, Theorem 7.4.2 implies that $\pi_1(\Sigma_{\#}/H)$ is also a subgroup of finite index in $G_0(\Sigma_{P^n})$.

7.5. Maximal blow-ups of simplicial arrangements. Suppose we are given a simplicial hyperplane arrangement in $\mathbb{R}^{n+1}$. Let Z be the corresponding zonotope, and let K be the triangulation of S^n induced by the arrangement. In Corollary 4.2.8 of [DJS], we asserted that the universal cover of the maximal blow-up $(\partial Z)_{\#}$ of ∂Z could be identified with Σ_{P^n}. Although this is correct, the proof given in [DJS] is not. We take this opportunity to correct it.

The "proof" of [DJS] had three steps.

Step 1: There is a simplicial projection (or "folding map") $p : K \to \Delta^n$ (Lemma 4.2.6 in [DJS]).
Step 2: The map p induces $p_{\#} : (\partial Z)_{\#} \to \Delta^n_{\#} = P^n$ (Corollary 4.2.7 in [DJS]).
Step 3: The map $p_{\#}$ is the projection map of an orbifold covering.

Step 2 is incorrect–the map $p_{\#}$ is not well-defined. Of course, the problem is caused by the fact that p need not be compatible with the antipodal map a: if σ^n is an n-simplex in K then, in general, $p|_{\sigma^n} \neq p \circ a|_{-\sigma^n}$. (This phenomenon causes a problem when we are considering the normal arrangement to a subspace.) However, if we divide out by S_{n+1}, the symmetry group of Δ^n, then we do get a well-defined map $(\partial Z)_\# \to \Delta^n_\#/S_{n+1}$. To see this, let Δ' be an n-simplex in the barycentric subdivision of Δ^n. Then $\Delta^n/S_{n+1} = \Delta'$, so we have a folding map $q : \Delta^n \to \Delta'$. Let $p' = q \circ p : K \to \Delta'$. Then Step 2 can be replaced by the following:

Step 2′: p' induces a map $p'_\# : (\partial Z)_\# \to \Delta_\#/S_{n+1}$.

As for the last step, we have $\Delta_\#/S_{n+1} = P^n/S_{n+1} = \Sigma_{P^n}/G_0(\Sigma_{P^n})$. Furthermore, the map $p'_\# : (\partial Z)_\# \to \Sigma_{P^n}/G_0(\Sigma_{P^n})$ is an orbifold covering. (This just means that the map is locally isomorphic to $\mathbb{R}^n \to \mathbb{R}^n/H$ where the finite group H is either a subgroup of $(\mathbb{Z}_2)^n$ or S_{n+1}.) Therefore, we have proved the following result.

Theorem 7.5.1. *Suppose that Z is an $(n+1)$-dimensional simple zonotope (i.e., it is associated to a simplicial hyperplane arrangement in $\mathbb{R}^{n+1}$) and let $(\partial Z)_\#$ denote the maximal blow-up of its boundary. Then*

(1) *The universal cover of $(\partial Z)_\#$ is isomorphic to Σ_{P^n}, and*
(2) *$\pi_1((\partial Z)_\#)$ is a subgroup of finite index in $G_0(\Sigma_{P^n})$.*

Remark 7.5.2. Note that this proof of Theorem 7.5.1 gives another proof of Theorem 7.4.2.

8. Associahedral tilings

Manifolds tiled by associahedra arise as minimal blow-ups of the boundaries of certain Coxeter cells and as the minimal blow-ups of $\Sigma(W, S)$ for certain simplicial Coxeter systems. In contrast to permutohedral tilings, the universal covers of these examples tend not to be isomorphic (although they all give tilings of $\mathbb{R}^n$). The reason is that the associahedron is less symmetric than the permutohedron. More precisely, two adjacent associahedral tiles are glued together by an involution j_T of a codimension-one face, and these gluing involutions tend not to all extend to symmetries of the full associahedron.

8.1. The associahedron. Following [Lee] we give two equivalent descriptions of a simplicial complex $\mathcal{N}_{>\emptyset}$ that is dual to the boundary complex of the n-dimensional associahedron K^n. The first description is the one given in 4.2: it shows how K^n arises as a truncation of the n-simplex. The second description is in terms of diagonals in an $(n+3)$-gon: it is more convenient for describing the group of combinatorial symmetries of K^n.

Let S be a set with $n+1$ elements and suppose that Γ is a graph with vertex set S such that Γ is homeomorphic to an interval. We might as well assume that $S = \{1, 2, \dots, n+1\}$ and that Γ is the interval $[1, n+1]$. Let V be the set of proper nonempty subsets T of S such that the full subgraph Γ_T spanned by T is connected. In other words, V can be identified with sets of consecutive integers of the form $[k, l]$ where $1 \le k \le l \le n+1$ and $l - k < n$. A *decomposition* of a subset T of S is a collection $\{T_1, \dots, T_k\}$ of disjoint subsets of T such that $T = T_1 \cup \dots \cup T_k$, each $T_i \in V$, and $\Gamma_{T_1}, \dots, \Gamma_{T_k}$ are the connected components of Γ_T. A subset $\mathcal{T}$ of V is *nested* if either $\mathcal{T} = \emptyset$ or if the maximal elements $T_1, \dots, T_k$ in $\mathcal{T}$ give a decomposition of $T_1 \cup \dots \cup T_k$. $\mathcal{N}$ is defined as the poset of all such nested subsets of V. Then $\mathcal{N}_{>\emptyset}$ is a simplicial complex of dimension $n-1$ which can be identified with a certain simplicial subdivision of $\partial\Delta^n$ (see [Lee] or [DJS]). Moreover, it is proved in [Lee] that this subdivision of $\partial\Delta^n$ can be idenitifed with the boundary complex of a convex simplicial polytope in $\mathbb{R}^n$. The dual polytope K^n is the *n-dimensional associahedron.* K^0 is a point, K^1 is an interval, and K^2 is a pentagon. We shall also denote the simplicial complex $\mathcal{N}_{>\emptyset}$ by $L(K^n)$.

The second description of this complex is more illuminating. Let P_{n+3} be a regular $(n+3)$-gon. A *diagonal* d in P_{n+3} is a line segment connecting two nonadjacent vertices of P_{n+3}. Two diagonals d and d' are *noncrossing* if their interiors are disjoint. Let V' denote the set of all diagonals of P_{n+3}. We next define a simplicial complex $\mathcal{N}'_{>\emptyset}$ with vertex set V'. A k-simplex om $\mathcal{N}'_{>\emptyset}$ is a collection $\sigma = \{d_0, \dots, d_k\}$ of pairwise noncrossing diagonals. So, a maximal simplex in $\mathcal{N}'_{>\emptyset}$ corresponds to a triangulation of P_{n+3} with no additional vertices. The dimension of such a maximal simplex is easily seen to be $n-1$.

A bijection $V \to V'$ is defined as follows. Number the vertices of P_{n+3} cyclically around the boundary by $0, 1, \dots, n+2$. Let $[k,l] \in V$. The corresponding diagonal $d_{[k,l]}$ is defined to be the diagonal connecting the vertices $k-1$ and $l+1$ of P_{n+3}. Clearly, this is a bijection. Furthermore, it is easy to see (c.f., [Lee]) that it induces an isomorphism $\mathcal{N}_{>\emptyset} \cong \mathcal{N}'_{>\emptyset}$. Henceforth, we identify V with V' and $\mathcal{N}_{>\emptyset}$ with $\mathcal{N}'_{>\emptyset}$.

Each diagonal $d \in V$ corresponds to a codimension-one face, $F(d)$, of K^n. Next, we investigate the combinatorial type of $F(d)$.

The diagonal d divides P_{n+3} into two polygons Q and Q' as indicated in Figure 6. Let $m(Q)+3$ and $m(Q')+3$ be the number of vertices of Q and Q', respectively. One

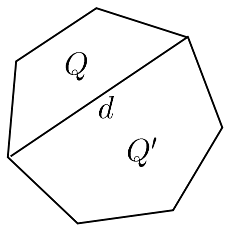


FIGURE 6.

checks easily that $m(Q)+m(Q') = n-1$. Without loss of generality, we may suppose that $m(Q) \le m(Q')$. Set $m(d) = m(Q)$.

It is completely straightforward to check that the link of d in $\mathcal{N}_{>\emptyset}$ is isomorphic to the join of the corresponding complexes of diagonals for Q and Q'. This implies the following result.

Proposition 8.1.1. *Suppose the diagonal d divides P_{n+3} into two polygons Q and Q' as above. Then*
$$F(d) \cong K^{m(Q)} \times K^{m(Q')}.$$

We note that if $m(d) = 0$, then $K^{m(Q)}$ is a point and hence $F(d)$ is an $(n-1)$-dimensional associahedron. We will need to use the following lemma in the next subsection.

Lemma 8.1.2. *K^n is not combinatorially isomorphic to a product of the form $K^i \times K^{n-i}$, with $0 < i < n$.*

Proof. Let $v(n) = \binom{n+3}{2} - (n+3)$ be the number of diagonals in P_{n+3}. A computation shows that $v(n) \geq v(i) + v(n-i)$, with equality if and only if $i = 0$ or $i = n$. In other words, for $0 < i < n$, $K^i \times K^{n-i}$ has fewer codimension-one faces than does K^n. □

8.2. Symmetries of the associahedron. Let $\mathrm{Aut}(K^n)$ $(= \mathrm{Aut}(\mathcal{N}))$ denote the group of combinatorial symmetries of K^n.

The symmetry group of P_{n+3} is D_{n+3}, the dihedral group of order $2(n+3)$. An element of D_{n+3} takes diagonals to diagonals and collections of noncrossing diagonals to collections of noncrossing diagonals. Hence, it gives an automorphism of K^n. This defines a homomorphism $\phi : D_{n+3} \to \mathrm{Aut}(K^n)$.

Lemma 8.2.1. *For $n \geq 2$, $\phi : D_{n+3} \to \mathrm{Aut}(K^n)$ is an isomorphism.*

Proof. For each $i = 0, \ldots, n+2$, we let d_i denote the diagonal with $m(d_i) = 0$ that cuts off the vertex i from P_{n+3}. We then let F_i be the corresponding codimension-one face $F(d_i)$. By Proposition 8.1.1 each F_i is isomorphic to K^{n-1} and by Lemma 8.1.2 these are the only faces isomorphic to K^{n-1}. Thus, the collection $\{F_0, \ldots, F_{n+2}\}$ is preserved by any combinatorial automorphism of K^n. In fact, the relative positions of the vertices $0, 1, \ldots, n+2$ on the circle are uniquely determined by the poset $\mathcal{N}_{>\emptyset}$. To see this, we just note that d_i and d_j are crossing diagonals (i.e., $\{d_i, d_j\} \notin \mathcal{N}_{>\emptyset}$) if and only if $j = i+1$ or $j = i-1$ (modulo $n+3$). It follows that any combinatorial automorphism of K^n must preserve the relative positions of $0, \ldots, n+2$, hence ϕ is surjective.

To see that ϕ is injective we simply note that the face F_0 is stabilized by the two-element subgroup generated by the reflection $r \in D_{n+3}$ across the line perpendicular to the diagonal d_0. The reflection r does not act trivially on K^n since it exchanges the diagonals d_1 and d_{n+2} (and, hence, the faces F_1 and F_{n+2}). On the other hand, any nontrivial element of D_{n+3} other than r moves the diagonal d_0, hence does not act trivially on K^n. It follows that ϕ is injective. □

Remark 8.2.2. For $n = 1$, $\phi : D_4 \to \mathrm{Aut}(K^1)$ is not injective. There are two types of reflections in D_4. A line of symmetry of the square P_4 can connect either two opposite vertices or the midpoints of two opposite edges: we say that the coresponding reflection is of *vertex type* or *edge type*, respectively. Clearly, ϕ takes each vertex type reflection to the identity element of $\mathrm{Aut}(K^1)$ and each edge-type reflection to the nontrivial element.

A nontrivial involution in D_{n+3} is either the antipodal map (when $n+3$ is even) or a reflection. Let us say that an involution $f : K^n \to K^n$ is of *R-type* if $f = \phi(r)$ for some reflection $r \in D_{n+3}$.

Suppose d is a diagonal of P_{n+3} subdividing it into polygons Q and Q'. Let p_d be the midpoint of d and let $L(d)$ be the line perpendicular to d at the point p_d. Thus, $L(d)$ is a line of symmetry of P_{n+3}. The corresponding reflection in D_{n+3} is denoted by $r_{L(d)}$. (See Figure 7, below.)

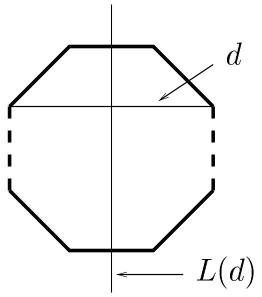


FIGURE 7.

A diagonal of P_{n+3} is a *main diagonal* if $n+3$ is even and if it connects opposite vertices of P_{n+3}.

The next lemma is geometrically clear.

Lemma 8.2.3. *Let d be a diagonal P_{n+3}.*

1. *If d is not a main diagonal, then its stabilizer in D_{n+3} is the cyclic group of order 2 generated by the reflection $r_{L(d)}$.*
2. *If d is a main diagonal, then its stabilizer in D_{n+3} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the generators can be taken to be $r_{L(d)}$ and the reflection r_d across d.*

Corollary 8.2.4. *Suppose d is a diagonal of P_{n+3}, $n \geq 2$, and that $L = L(d)$ is the corresponding line of symmetry.*

1. *If $m(d) = 0$, then $F(d) \cong K^{n-1}$ and $\phi(r_L)$ acts on $F(d)$ as a nontrivial R-type involution.*
2. *If $m(d) > 0$, then $F(d) \cong K^{m(d)} \times K^{n-m(d)-1}$ and the restriction of $\phi(r_L)$ to $F(d)$ can be written as $\phi(r_L) = r_1 \times r_2$ where $r_1 : K^{m(d)} \to K^{m(d)}$ and $r_2 : K^{n-m(d)-1} \to K^{n-m(d)-1}$ are both (nontrivial) R-type involutions.*
3. *Suppose d is a main diagonal. Then $F(d) \cong K^k \times K^k$ where $k = (n-1)/2$ and $\phi(r_d)$ acts on $F(d)$ by switching the factors. Furthermore, the restrictions of $\phi(r_L)$ and $\phi(r_d)$ are not equal.*

8.3. Gluing involutions. Manifolds tiled by associahedra arise from minimal blow-ups in cases where the Coxeter diagram is an interval. The associahedra are glued together via involutions j_T, $T \in V$, defined on the codimension-one faces of K^n. If T corresponds to a subinterval of the Coxeter diagram, then j_T is induced by an involution of the subinterval. This involution might be trivial (e.g., if the subgraph is of type $\mathbf{B}_m$) or it might be the nontrivial involution that flips the subinterval (e.g., if it is of type $\mathbf{A}_m$, $m \geq 2$). Here we are concerned with the question of when j_T extends to an automorphism of K^n. Since the identity always extends, we shall analyze the case $j_T = i_T$, where i_T is induced by the nontrivial involution of the subinterval.

As in 8.1, let S be the set of integers in $[1, n+1]$. T will denote a proper nonempty subset of S consisting of consecutive integers. By an abuse of notation, we will write $T = [k, l]$ to mean $T = \{k, \ldots, l\}$. Let i_T be the order reversing involution of T. Then i_T induces an involution of $\mathcal{N}_{\geq \{T\}}$, also denoted by i_T. (If $\{T', T\}$ is a vertex of $\mathcal{N}_{> \{T\}}$ such that $T \subset T'$ or such that the subintervals corresponding to T and T' are disjoint, then $i_T(T') = T'$.) Its geometric realization is again denoted i_T. Let $d(T)$ be the diagonal of P_{n+3} corresponding to T.

We note that $i_T : F(d(T)) \to F(d(T))$ extends to an automorphism of K^n if and only if it coincides with the action of an element of the stabilizer of $d(T)$ in D_{n+3} on this face. The main result of this subsection is the following key lemma, which determines precisely when this happens.

Lemma 8.3.1. *The involution $i_T : F(d(T)) \to F(d(T))$ lies in the stabilizer of $d(T)$ in D_{n+3} if and only if $m(d(T)) = 0$.*

Proof. Suppose $T = [k, l]$. So, $d(T)$ connects the vertices of P_{n+3} numbered $k - 1$ and $l + 1$. It divides P_{n+3} into two polygons Q_1 and Q_2 where Q_2 contains the vertices numbered $k - 1, \dots, l + 1$. Let $L = L(d(T))$ be the line of symmetry for P_{n+3} that stabilizes $d(T)$. Then L is also a line of symmetry for Q_1 and Q_2. Let r_2 denote the restriction of r_L to Q_2.

The codimension-one face $F(d(T))$ decomposes as $F(d(T)) = K^{m(Q_1)} \times K^{m(Q_2)}$. It follows from the definitions that

$$i_T = \mathrm{Id} \times \phi(r_2) : K^{m(Q_1)} \times K^{m(Q_2)} \to K^{m(Q_1)} \times K^{m(Q_2)}.$$

Comparing this with Lemma 8.2.3 and Corollary 8.2.4, we see that i_T does not belong to the stabilizer of $d(T)$ unless $m(Q_2) = 0$ (in which case i_T is the identity) or $m(Q_1) = 0$ (in which case $i_T = \phi(r_2)$). □

Remark 8.3.2. There are two ways in which it can happen that $m(d(T)) = 0$. The first is that T is a singleton. In this case i_T is the identity and $F(d(T))$ is a reflection-type face. The second way is that T is a maximal proper subinterval of $[1, n+1]$, i.e., $T = [1, n]$ or $T = [2, n+1]$. In both these cases, the involution i_T is nontrivial, but it is the restriction of a symmetry of the full associahedron.

8.4. Schläfli symbols. As before, suppose that $S = \{1, \dots, n+1\}$ and that there is a corresponding Coxeter diagram Γ with underlying graph the interval $[1, n + 1]$. The labels on the edges of Γ are then given by an n-tuple $(m_1, \dots, m_n)$ of integers, each ≥ 3, where m_i is the label on $[i, i + 1]$. Classically, this n-tuple is called the *Schläfli symbol* of Γ. For example, the Schläfli symbol for $\mathbf{A}_{n+1}$ is $(3, \dots, 3)$, while for $\mathbf{B}_{n+1}$ it is $(4, 3, \dots, 3)$.

By allowing these integers to be 2, this notation can be extended to cover certain reducible Coxeter diagrams, namely, those with underlying graph a disjoint union of subintervals $[1, n+1]$. For example, the Schläfli symbol $(2, \dots, 2)$ should be understood as representing the diagram consisting of $n+1$ vertices and no edges, i.e., $\mathbf{A}_1 \times \cdots \times \mathbf{A}_1$ ($\cong (\mathbb{Z}_2)^{n+1}$).

The Schläfli symbols that we will be interested in correspond to Coxeter systems that are either simplicial (c.f., 7.3) or spherical. Moreover, in the simplicial case, the diagram is necessarily irreducible. The Schläfli symbols corresponding to irreducible spherical or simplicial Coxeter systems are listed in Table 1, below.

TABLE 1. Schläfli symbols of irreducible spherical and simplicial Coxeter systems

dimension	spherical		Euclidean		hyperbolic	
	symbol	t	symbol	t	symbol	t
2	(3,3) (4,3) (5,3)	0 0 0	(4,4) (6,3)	0 0	(p,q) with $p^{-1}+q^{-1}+2^{-1}<1$	0
3	(3,3,3) (4,3,3) (5,3,3) (3,4,3)	3 2 3 2	(4,3,4)	1	(3,5,3) (5,3,4) (5,3,5)	3 2 3
4	(3,3,3,3) (4,3,3,3)	7 5	(4,3,3,4) (3,4,3,3)	3 4	(5,3,3,3) (5,3,3,4) (5,3,3,5)	6 4 5
$n \geq 5$	(3,3,...,3) (4,3,...,3)	a_n b_n	(4,3,...,3,4)	c_n		

$$\text{where } a_n = \frac{n(n+1)-6}{2}, \quad b_n = \frac{n(n-1)-2}{2}, \quad c_n = \frac{(n-2)(n-1)}{2}$$

8.5. Examples of symmetric associahedral tilings. As previously, we let $S = \{1, \ldots, n+1\}$ and $\mathcal{R}$ be the set of all subsets of S that correspond to proper subintervals of $[1, n+1]$ of nonzero length. In what follows we only consider spherical or simplicial Coxeter systems that can be described by Schläfli symbols as in the previous subsection.

Example 8.5.1. *(Minimal blow-ups of boundaries of Coxeter cells.)* Suppose (W, S) is spherical and irreducible with Schläfli symbol $(m_1, \ldots, m_n)$ (see Table 1) and that $Z = Z(m_1, \ldots, m_n)$ is the corresponding Coxeter cell. Then $\mathcal{R}$ is the set for the minimal blow-up $(\partial Z)_\#$ of ∂Z. Hence, $(\partial Z)_\#$ is tiled by associahedra (as is its universal cover).

Example 8.5.2. *(Nonminimal blow-ups of boundaries of Coxeter cells.)* Suppose that $(m_1, \ldots, m_n)$ is the Schläfli symbol for a finite Coxeter group W (not necessarily irreducible). For $n = 1$ or 2, the set $\mathcal{R}$ is always admissible (cf., Definition 3.2.3); however, for $n \geq 3$, it might not be. In fact, for $n \geq 3$, $\mathcal{R}$ is admissible if and only if any time a 2 occurs in $(m_1, \ldots, m_n)$, the numbers before and after it are both even. (This can be checked by using Remark 3.1.2.) For example, if $(m_1, \ldots, m_n)$ is $(2, \ldots, 2)$ or $(3, \ldots, 3, 4, 2, 4, 3, \ldots, 3)$, then $\mathcal{R}$ is admissible. For any Schläfli symbol such that W is finite and $\mathcal{R}$ is admissible, the $\mathcal{R}$-blow-up of $\partial Z(m_1, \ldots, m_n)$ will be tiled by associahedra.

Example 8.5.3. *(Minimal blow-ups of simplicial Coxeter systems.)* Let $(m_1, \ldots, m_n)$ be a Schläfli symbol of a simplicial Coxeter system (see Table 1). Then $\mathcal{R}$ is the set for the minimal blow-up of Σ $(= \Sigma(m_1, \ldots, m_n))$. Hence $\Sigma_\#$ is tiled by associahedra.

Notation 8.5.4. Given a Schläfli symbol $(m_1, \ldots, m_n)$ as above, let $X(m_1, \ldots, m_n)$ denote the universal cover of the $\mathcal{R}$-blow-up described in either Example 8.5.1, 8.5.2, or

8.5.3. Also, let $A(m_1, \ldots, m_n)$ denote the symmetry group of the natural framing on $X(m_1, \ldots, m_n)$.

Remark 8.5.5. In Example 8.5.2, when the Schläfli symbol is $(2, \ldots, 2)$, $\mathcal{R}$ is admissible. In this case, each gluing involution j_T is trivial. Hence, the universal cover $X(2, \ldots, 2)$ is the usual reflection type tiling corresponding to $(V, M, L(K^n))$.

In the next four theorems we classify some of these examples $X(m_1, \ldots, m_n)$ up to isomorphism. In fact, in the first three theorems we classify all such examples for $n \leq 4$. In the last theorem, we show that in each dimension ≥ 5, the three irreducible examples are distinct.

The basic method for showing that two such tilings are not isomorphic is to use Lemma 5.4.8. In fact, in most cases, the number t_X of mirrors (i.e., codimension-one faces) of the associahedron that have nonextendable gluing involutions is sufficient to distinguish among the examples. The number t_X is easily computable. It is the number of subsets T of $\{1, \ldots, n+1\}$ such that Γ_T is connected, $1 < \mathrm{Card}(T) < n$ and such that j_T is not the antipodal map (cf. Remark 3.1.2). (The numbers t_X are also given in Table 1.) Conversely, the basic method for showing that two such tilings are isomorphic is to use Proposition 5.4.5.

Theorem 8.5.6. (Dimension 2.) *Let (m_1, m_2) be a pair of integers ≥ 2. Then $X(m_1, m_2)$ is isomorphic to the reflection tiling $X(2, 2)$. Thus, all 2-dimensional examples are isomorphic to the tiling of the hyperbolic plane by right-angled pentagons.*

Proof. In dimension 2, all gluing involutions are extendable. □

Theorem 8.5.7. (Dimension 3.) *For $n = 3$, the universal covers of the examples in 8.5.1, 8.5.2, and 8.5.3 fall into four isomorphism classes as indicated below. (The corresponding values of $t = t_X$ are indicated in parentheses. Also, p and q denote arbitrary even integers ≥ 2.)*

(i) *($t = 0$): all three integers are even, i.e., $X(p, 2, q)$ or $X(2, p, 2)$.*
(ii) *($t = 1$): $X(4, 3, 4) \cong X(2, 4, 3)$.*
(iii) *($t = 2$): $X(4, 3, 3) \cong X(4, 3, 5) \cong X(3, 4, 3)$.*
(iv) *($t = 3$): $X(3, 3, 3) \cong X(5, 3, 3) \cong X(3, 5, 3) \cong X(5, 3, 5)$.*

Proof. The four isomorphism types can be distinguished by the indicated value of t. The existence of the indicated isomorphisms follows from Proposition 5.4.5 (possibly after changing one of the framings by an automorphism of framing systems). □

Theorem 8.5.8. (Dimension 4.) *For $n = 4$, the universal covers of the examples in 8.5.1, 8.5.2, and 8.5.3 fall into nine isomorphism classes as indicated below. (The corresponding values of $t = t_X$ are indicated in parentheses. Also, p and q denote arbitrary even integers ≥ 2.)*

(i) *($t = 0$): all four integers are even, i.e., $X(2, p, 2, q)$ ($\cong X(p, 2, 2, q)$).*

(ii) *($t = 1$):* $X(3, 4, 2, p)$.
(iii) *($t = 3$):* $X(4, 3, 3, 4) \cong X(2, 4, 3, 3)$.
(iv) *($t = 4$):* $X(3, 3, 4, 3)$.
(v) *($t = 4$):* $X(5, 3, 3, 4)$.
(vi) *($t = 5$):* $X(4, 3, 3, 3)$.
(vii) *($t = 5$):* $X(5, 3, 3, 5)$.
(viii) *($t = 6$):* $X(5, 3, 3, 3)$.
(ix) *($t = 7$):* $X(3, 3, 3, 3)$.

Proof. The only question is to distinguish the example in (iv) from (v) and to distinguish (vi) from (vii). In $X(3, 3, 4, 3)$ there are three mirrors of type $K^1 \times K^2$ with extendable gluing involutions (in fact the identity maps), namely, the mirrors corresponding to $(3, 4)$, $(4, 3)$ and (4). The mirror corresponding to (4) intersects the other two. In $X(5, 3, 3, 4)$ there are also three such mirrors corresponding to $(5, 3)$, $(3, 4)$, and (4). The mirror corresponding to $(5, 3)$ is disjoint from the other two. Hence, there is no automorphism of K^4 that takes the first set of mirrors into the second. So by Lemma 5.4.8, $X(3, 3, 4, 3) \not\cong X(5, 3, 3, 4)$. For a similar reason, $X(4, 3, 3, 3) \not\cong X(5, 3, 3, 5)$. □

In dimensions > 4, we shall only consider the examples coming from irreducible Coxeter systems.

Theorem 8.5.9. (Dimension $n > 4$.) *For $n > 4$, the universal covers of the minimal blow-ups in 8.5.1 and 8.5.3, namely $X(3, \dots, 3)$, $X(4, 3, \dots, 3)$, and $X(4, 3, \dots, 3, 4)$ are mutually non-isomorphic.*

Proof. The corresponding values of t given as a_n, b_n, and c_n in Table 1 are distinct for each $n \geq 5$. □

Remark 8.5.10. If $X(m_1, \dots, m_n) \cong X(m'_1, \dots, m'_n)$, then the corresponding groups $A(m_1, \dots, m_n)$ and $A(m'_1, \dots, m'_n)$ are commensurable.

More generally, we pose the following question.

Question 8.5.11. *When do the different associahedral tilings of Theorems 8.5.7, 8.5.8, and 8.5.9 give commensurable mock reflection groups? When do they give quasi-isometric mock reflection groups? (We answer the 3-dimensional quasi-isometry question below in 8.7.)*

8.6. Maximally symmetric associahedral tilings. Let X be one of the associahedral tilings discussed above. Recall that $d \mapsto F(d)$ defines a bijection between the set of diagonals in P_{n+3} and the set of codimension-one faces of K^n. Let $\mathscr{D}$ denote the set of diagonals, and let j_d denote the gluing involution on the face $F(d)$. Then by Proposition 5.5.2, we know that X will be maximally symmetric if and only if for all $\phi \in \mathrm{Aut}(K^n)$ $(\cong D_{n+3})$ and $d \in \mathscr{D}$, the composition $\phi \circ j_d \circ \phi^{-1} \circ (j_{\phi(d)})^{-1}$ is the restriction of an element of $\mathrm{Aut}(K^n)$. For all of the tilings in 8.5 except $X(2, 2, \dots, 2)$ and

$X(3,3,\ldots,3)$ there exists a symmetry $\phi \in \mathrm{Aut}(K^n)$ that conjugates a nonextendable gluing involution to an extendable one, hence these cannot be maximally symmetric. In the case of $X(2,2,\ldots,2)$ the tiling is of reflection type, so we already know it is maximally symmetric (Example 5.5.3). Moreover, its symmetry group is

$$\mathrm{Aut}(X) = A \rtimes D_{n+3}.$$

In the case of $X(3,3,\ldots,3)$, each gluing involution j_d $(= j_{d(T)})$ is the involution i_T described in the proof of Lemma 8.3.1. Recall the face $F(d')$ is adjacent to $F(d)$ if and only if the diagonals d' and d do not cross. Letting $\mathscr{D}_d$ denote the set of diagonals that do not cross d, we see that $j_d : F(d) \to F(d)$ induces an involution (which we also denote by j_d) on $\mathscr{D}_d$. Let e denote the edge $\{0, n+2\}$ of P_{n+3}. Then the involution $j_d : \mathscr{D}_d \to \mathscr{D}_d$ is given by

$$j_d(d') = \begin{cases} r_{L(d)}(d') & \text{if } d' \text{ and } e \text{ are on opposite sides of } d \\ d' & \text{if } d' \text{ and } e \text{ are on the same side of } d \end{cases}$$

(where $L(d)$ is as in Figure 7). Now suppose $\phi \in D_{n+3}$. If d' and $\phi(d)$ do not cross, then

$$(\phi \circ j_d \circ \phi^{-1})(d') = \begin{cases} j_{\phi(d)}(d') & \text{if } \phi(e) \text{ and } e \text{ are on the same side of } d \\ (r_{L(\phi(d))} \circ j_{\phi(d)})(d') & \text{if } \phi(e) \text{ and } e \text{ are on opposite sides of } d. \end{cases}$$

Thus, $\phi \circ j_d \circ \phi^{-1} \circ (j_{\phi(d)})^{-1}$ extends to an automorphism of K^n (it is either the restriction of Id or the restriction of $r_{L(\phi(d))}$), so by Proposition 5.5.2, $X(3,3,\ldots,3)$ is maximally symmetric.

In the remainder of this subsection, we will discuss the symmetry group of the tiling $X = X(3,3,\ldots,3)$. By Theorem 4.7.2 the subgroup A of $\mathrm{Aut}(X)$ is generated by involutions α_T where T is a proper subinterval of $[1, n+1]$. (In what follows we shall denote this generator by α_d where $d \in \mathscr{D}$ is the diagonal corresponding to T.) Since X is maximally symmetric, we also know that for any element ϕ of D_{n+3} there is a unique lift $\widetilde{\phi}$ to $\mathrm{Aut}(X)$ that stabilizes the fundamental tile K^n. We let $\rho_d : X \to X$ denote the lift of the reflection $r_{L(d)}$ in D_{n+3}. Then the group $\mathrm{Aut}(X)$ is generated by the α_d and ρ_d, $d \in \mathscr{D}$. This generating set, however, is not symmetric with respect to $\mathrm{Aut}(K^n)$. A more symmetric generating set arises from the following observation.

Lemma 8.6.1. *The involutions ρ_d and α_d commute.*

Proof. Let Q_1 and Q_2 be the two subpolygons of P with diagonal d, and assume the edge e (with vertex labels 0 and $n+2$) is contained in Q_1. Let r_1 and r_2 denote the restriction of $r_{L(d)}$ to Q_1 and Q_2, respectively. If Q_1 is an (m_1+3)-gon and Q_2 is an (m_2+3)-gon, then the face $F(d)$ is isomorphic to the product $K^{m_1} \times K^{m_2}$, and as in the proof of Lemma 8.3.1, the restriction of α_d to $F(d)$ is $\mathrm{Id} \times \phi(r_2)$. By Corollary 8.2.4, the restriction of ρ_d to $F(d)$ is $\phi(r_1) \times \phi(r_2)$. It follows that the automorphism $(\alpha_d \rho_d)^2$ takes the fundamental tile K^n to itself and fixes the face $F(d)$ pointwise. By rigidity, it must be the trivial automorphism. $\square$

Let $\mathscr{S}$ denote the set of all subpolygons of P_{n+3}, and let $\delta : \mathscr{S} \to \mathscr{D}$ be the 2-to-1 map that takes each subpolygon to its corresponding diagonal. Letting Q be an element of $\mathscr{S}$ and $d = \delta(Q)$, we define an involution $\beta_Q : X \to X$ by

$$\beta_Q = \begin{cases} \alpha_d & \text{if } e \not\subset Q \\ \rho_d \alpha_d & \text{if } e \subset Q. \end{cases}$$

It follows that if Q_1 and Q_2 are the two subpolygons sharing the diagonal d, then the two involutions β_{Q_1} and β_{Q_2} both take the fundamental tile K^n to the adjacent tile across the face $F(d)$. As in 4.7, we obtain relations among these involutions by considering local pictures around codimension-2 faces of K^n (or, dually, by considering 2-dimensional cells in the dual cellulation of X by Coxeter cells–in this case cubes). Thus, around any codimension-2 face, there are *a priori* 2^4 automorphisms of the form $\beta_{Q_1}\beta_{Q_2}\beta_{Q_3}\beta_{Q_4}$ that take K^n to itself, and since the stabilizer of K^n is D_{n+3}, there is an element ϕ $(= \phi(Q_1, Q_2, Q_3, Q_4))$ in D_{n+3} such that

$$\beta_{Q_1}\beta_{Q_2}\beta_{Q_3}\beta_{Q_4} = \widetilde{\phi}.$$

We work out these relations explicitly, below.

Suppose $b, c \in \mathscr{D}$ is a pair of noncrossing diagonals, and $I = (i_1, \dots, i_4)$ is a 4-tuple in $(\mathbb{Z}_2)^4$. Let B denote the subpolygon of P_{n+3} such that $\delta(B) = b$ and $c \not\subset B$, and let C denote the subpolygon such that $\delta(C) = c$ and $b \not\subset C$. We define a sequence of subpolygons inductively by

$$\begin{aligned} Q_0 &= B \\ Q_1 &= C \\ Q_2 &= (r_1)^{i_1}(Q_0) \\ Q_3 &= (r_2)^{i_2}(Q_1) \\ Q_4 &= (r_3)^{i_3}(Q_2) \\ Q_5 &= (r_4)^{i_4}(Q_3) \end{aligned}$$

where r_i denotes the reflection of P_{n+3} that takes the subpolygon Q_i to itself. The product $\beta_{Q_1}\beta_{Q_2}\beta_{Q_3}\beta_{Q_4}$ will take the fundamental tile of X to itself, and the resulting automorphism of K^n corresponds to the element $\phi \in D_{n+3}$ defined by $\phi(Q_0) = Q_4$ and $\phi(Q_1) = Q_5$. In other words,

$$\widetilde{\phi} = (\rho_c)^{i_1}(\rho_b)^{i_2}(\rho_c)^{i_3}(\rho_b)^{i_4}.$$

Figure 8 shows an example with $n = 5$ and $I = (1, 1, 0, 1)$. The fundamental tile K^n is labeled 1, and the tile βK^n is labeled β. (β is just one of the possible labels on the tile βK^n, since any label of the form $\beta\widetilde{\phi}$ where $\phi \in D_{n+3}$ describes the same tile). The shading indicates which side of the fixed diagonal is affected by the next gluing involution. The shaded side contains the other diagonal if and only if the corresponding element of (i_1, i_2, i_3, i_4) is 1.

Let $R_I(b, c)$ denote the word

$$R_I(b, c) = \beta_{Q_1}\beta_{Q_2}\beta_{Q_3}\beta_{Q_4}(\rho_b)^{i_4}(\rho_c)^{i_3}(\rho_b)^{i_2}(\rho_c)^{i_1}.$$

Letting $m(b, c)$ denote the order of the rotation $r_b r_c$ in D_{n+3}, we then have the following.

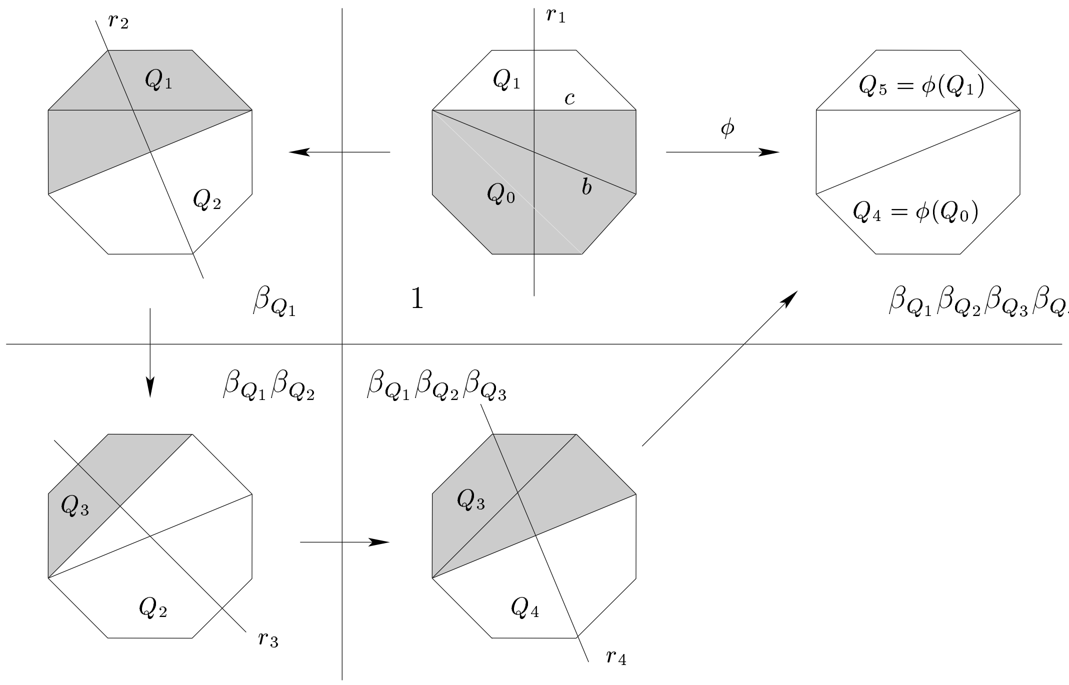


FIGURE 8.

Theorem 8.6.2. *The group* Aut(X) *has a presentation with generators* β_Q, $Q \in \mathscr{S}$, *and* ρ_d,$d \in \mathscr{D}$, *and relations:*

$(\beta_Q)^2$	*for all* $Q \in \mathscr{S}$
$(\rho_d)^2$	*for all* $d \in \mathscr{D}$
$(\beta_{Q_1}\beta_{Q_2})^2$	*whenever* $\delta(Q_1) = \delta(Q_2)$
$(\beta_Q\rho_d)^2$	*whenever* $\delta(Q) = d$
$(\rho_b\rho_c)^{m(b,c)}$	*for all* $b, c \in \mathscr{D}$
$R_I(b,c)$	*for all* $I \in (\mathbb{Z}_2)^4$ *and noncrossing diagonals* b, c

Remark 8.6.3. The generators ρ_d can be eliminated from the presentation, since $\rho_d = \beta_{Q_1}\beta_{Q_2}$ if Q_1 and Q_2 are the two subpolygons that share the diagonal d.

Let S_{n+3} be the group of permutations on the set $\{0, 1, \ldots, n+2\}$ (i.e., the set of vertex labels for P_{n+3}). Then for any $d \in \mathscr{D}$, the reflection $r_{L(d)}$ induces an involution $\overline{\rho}_d \in S_{n+3}$. Similarly, for any $Q \in \mathscr{S}$, we obtain an involution $\overline{\beta}_Q \in S_{n+3}$ as follows. Let $a_0, a_1, \ldots, a_{k+1}$ be labels on the vertices of Q ordered sequentially with a_0 and a_{k+1} being the vertices of the diagonal $\delta(Q)$. Then $\overline{\beta}_Q$ is the involution that reverses the order of the sequence $a_1, a_2, \ldots, a_k$.

Proposition 8.6.4. *There is a surjective homomorphism* $\psi : \mathrm{Aut}(X) \to S_{n+3}$ *defined by* $\rho_d \mapsto \overline{\rho}_d$, $\beta_Q \mapsto \overline{\beta}_Q$.

Proof. All of the relations in Theorem 8.6.2 hold for $\overline{\rho}_d$ and $\overline{\beta}_Q$. $\square$

Remark 8.6.5. Let M^n denote the minimal blow-up of the projectivized braid arrangement in $\mathbb{RP}^n$. (That is, $M^n = (\partial Z_\#)/a_\#$ where Z is the Coxeter cell of type $\mathbf{A}_{n+1}$.) Then M^n can be identified with the real points of the moduli space $\overline{\mathcal{M}}_{0,n+3}$ (see [Ka1, Ka2]), and the S_{n+3}-action on $\overline{\mathcal{M}}_{0,n+3}(\mathbb{R})$ respects the Coxeter cell decompositon of M^n. The homomorphism $\psi : \mathrm{Aut}(X) \to S_{n+3}$ arises when one lifts the S_{n+3}-action to the universal cover $X = \widetilde{M}^n$. In particular, $\pi_1(M^n) = \ker(\psi)$.

8.7. The 3-dimensional examples. In this subsection we discuss the question of when two 3-dimensional tilings are quasi-isometric. Nowadays any such discussion should be within the context of Thurston's Geometrization Conjecture. A closed orientable irreducible 3-manifold with infinite fundamental group has a canonical "JSJ-decomposition" into "simple pieces" and Seifert fibered pieces. Each such piece is a compact 3-manifold with boundary, and each boundary component is a torus. Thurston's Conjecture is that each simple piece is hyperbolic. By definition, a compact 3-manifold M^3 is a *hyperbolic piece* if its interior is homeomorphic to a complete hyperbolic 3-manifold of finite volume. Each boundary component then has a collared neighborhood C such that each component of the inverse image of C in $\mathbb{H}^3$ is a horoball. Identifying M^3 with the complement of a collared neighborhood of the boundary, we obtain an identification of its universal cover $\widetilde{M}^3$ and $\mathbb{H}^3$ with all these horoballs chopped off. Such an $\widetilde{M}^3$ is called a *neutered hyperbolic space*.

In the case of 3-manifolds that are tiled by associahedra or permutohedra, it turns out that (1) each piece in the JSJ-decomposition is hyperbolic, and (2) the neutered hyperbolic spaces that arise as universal covers of hyperbolic pieces are all identical. The question of whether the universal covers of two such tilings are quasi-isometric then comes down to the question of whether or not the lifts of certain gluing involutions extend to quasi-isometries of the neutered hyperbolic space. It turns out, somewhat surprisingly, that the lift of such a gluing involution extends to an an isometry of $\mathbb{H}^3$ that commensurates the lattice associated to the neutered hyperbolic space. (Hence, the gluing involution extends to a quasi-isometry of the neutered hyperbolic space.)

Theorem 8.7.1. *Suppose A_1 and A_2 are mock reflection groups associated either the 3-dimensional permutohedral tiling Σ_{P^3} (see Theorems 7.4.1 and 7.4.2) or to one of the 3-dimensional associahedral tilings of Theorem 8.5.7. Then A_1 and A_2 are quasi-isometric.*

Before proving this theorem we need to develop some notation. Let $\mathbb{R}^{3,1}$ denote Minkowski space, that is, it is a 4-dimensional real vector space with coordinates $x = (x_1, x_2, x_3, x_4)$, equipped with the indefinite bilinear form defined by

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$

Hyperbolic 3-space $\mathbb{H}^3$ can be defined as one sheet of the hyperboloid $\langle x, x \rangle = -1$, defined by $x_4 > 0$. Let $O(3,1)$ denote the isometry group of the bilinear form and let $O_+(3,1)$ be the index-two subgroup that preserves the sheets of the hyperboloid. Then $O_+(3,1)$ is the isometry group of the Riemannian manifold $\mathbb{H}^3$.

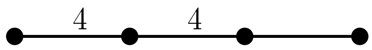
Given a spacelike vector $v \in \mathbb{R}^{3,1}$ (i.e., a vector v with $\langle v, v\rangle > 0$), define a reflection $r_v \in O_+(3,1)$ by the formula

$$r_v(x) = x - 2\frac{\langle x, v\rangle}{\langle v, v\rangle}v.$$

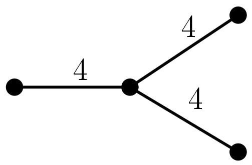
Remark 8.7.2. Let $O_+(3,1;\mathbb{Z})$ denote the subgroup of $O_+(3,1)$ that preserves the standard integer lattice $\mathbb{Z}^4 \subset \mathbb{R}^{3,1}$ (i.e., $O_+(3,1;\mathbb{Z}) = O_+(3,1) \cap GL_4(\mathbb{Z})$). If $v \in \mathbb{Z}^4$ and if $\langle v, v\rangle = 1$ or 2, then r_v preserves $\mathbb{Z}^4$, i.e., $r_v \in O_+(3,1;\mathbb{Z})$.

Suppose P is the permutohedron or the associahedron. It turns out that after collapsing each rectangular face of P to a vertex, one ends up with a polytope Q that can be realized as a right-angled convex polytope in $\mathbb{H}^3$ of finite volume. Moreover, the vertices of Q corresponding to the collapsed faces of P will be ideal vertices of the realization. This is a special case of a well-known theorem of Andreev, but we shall verify it directly. If P is a permutohedron, then Q is an octahedron. If P is an associahedron, then Q is a double pyramid on a triangular base. (In other words, Q is the suspension of a triangle.) When we write down specific realizations of these polytopes, we find an interesting surprise: the normal vectors to their faces are integral vectors v satisfying $\langle v, v\rangle = 1$ or 2. In fact, consider the following four polytopes in $\mathbb{H}^3$.

- *The fundamental* 3*-simplex* Q_0. Normal vectors to the faces are $u_1 = (1,1,1,1)$, $v_1 = (1,0,0,0)$, $w_1 = (1,-1,0,0)$ and $t_1 = (0,1,-1,0)$. Q_0 has one ideal vertex, where the faces normal to v_1, w_1, and t_1 meet. The Coxeter group generated by the reflections across its faces has Coxeter diagram



- *The pyramid* Q_1. Normal vectors to the faces are $u_1 = (1,1,1,1)$, $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, and $v_3 = (0,0,1,0)$. It is again a 3-simplex, this time with 3 ideal vertices. The corresponding Coxeter diagram is



  The *base* of the pyramid is the face normal to u_1. It is an ideal triangle. The other three faces meet at the vertex $(0,0,0,1) \in \mathbb{H}^3$.
- *The double pyramid* Q_2. The reflection r_{u_1} carries the vectors v_1, v_2, and v_3 into $v_1' = (0,-1,-1,-1)$, $v_2' = (-1,0,-1,-1)$, and $v_3' = (-1,-1,0,-1)$, respectively. The normal vectors to Q_2 are then $v_1, v_2, v_3, v_1', v_2', v_3'$.
- *The regular ideal octahedron* Q_3. The normal vectors to the faces are the eight vectors $(\pm1,\pm1,\pm1,1)$.

Observations 8.7.3.

(1) $Q_0 \subset Q_1$, $Q_1 \subset Q_2$, and $Q_1 \subset Q_3$.

(2) The Coxeter group generated by reflections across the faces of Q_0 is $O_+(3,1;\mathbb{Z})$.
(3) The symmetry group of the associahedron (i.e., the dihedral group D_6 of order 12) acts on Q_2, and Q_0 is a fundamental domain.
(4) The symmetry group of the permutohedron (i.e., $S_4 \times \mathbb{Z}_2$) acts on the octahedron Q_3 as its full symmetry group. Again Q_0 is a fundamental domain. (Q_0 is a simplex in the barycentric subdivision of Q_3).

We are now in a position to prove Theorem 8.7.1. Let X_1 and X_2 be associahedral tilings corresponding to groups A_1 and A_2. Consider a nonextendable gluing involution i defined on a face of the 3-dimensional associahedron. By Lemma 8.3.1, the face is rectangular. By the proof of Lemma 8.3.1, i is a reflection of the rectangular face about a line of symmetry connecting the midpoints of two opposite edges. Any such rectangular face corresponds to an ideal vertex of the double pyramid Q_2. For the sake of definiteness, let us fix this vertex to be the one where the faces normal to v_1 and v_2 intersect the base triangle (normal to u_1). The reflection r_w defined by the vector $w = (0,2,1,1)$ then has the desired effect – its restriction to the corresponding rectangular face in the horosphere is i. Since $\langle w, w\rangle = 4$, r_w is not represented by an integral matrix, rather its entries are rational numbers with denominators at most 2. It follows that r_w commensurates $O_+(3,1;\mathbb{Z})$. (In fact, conjugation by r_w maps the congruence subgroup, consisting of all matrices that are congruent to the identity mod 2, into itself.)

Let Ω denote the neutered hyperbolic space for $O_+(3,1;\mathbb{Z})$. The isometry r_w does not quite map Ω into itself (r_w does map the set of lifts of all cusps into itself, but it might not preserve their horoball neighborhoods). However, it can be modified to a homeomorphism preserving Ω that extends the gluing involution i and that is a bounded distance from r_w. The hyperbolic pieces of X_i, $i = 1,2$, give a partition into copies of Ω, and these copies are glued together via quasi-isometries. Hence, X_1 is quasi-isometric to X_2. Similarly, X_1 and X_2 are both quasi-isometric to the simply-connected, symmetric permutohedral tiling. This completes the proof of Theorem 8.7.1.

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