BUILDINGS ARE $CAT(0)$

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Introduction. Given a finitely generated Coxeter group W, I described, in [D, Section 14], a certain contractible simplicial complex, here denoted $|\mathbf{W}|$, on which W acts properly with compact quotient. After writing [D], I realized that there was a similarly defined, contractible simplicial complex associated to any building C. This complex is here denoted $|C|$ and called the "geometric realization" of C. The definition is such that the geometric realization of each apartment is isomorphic to $|\mathbf{W}|$. (N. B. Our terminology does not agree with standard usage. For example, if W is finite, then the usual Coxeter complex of W is homeomorphic to a sphere, while our $|\mathbf{W}|$ is homeomorphic to the cone on this sphere.)

There is a natural piecewise Euclidean metric on $|\mathbf{W}|$ (described in §9) so that W acts as a group of isometries. Following an idea of Gromov ([G, pp. 131-132]), Gabor Moussong proved in his Ph.D. thesis [M] that with this metric $|\mathbf{W}|$ is "$CAT(0)$" (in the sense of [G]). This is equivalent to saying that it is simply connected and "nonpositively curved". Moussong's result implies, via a standard argument, the following theorem.

Theorem. *The (correctly defined) geometric realization of any building is $CAT(0)$.*

Although this theorem was known to Moussong, it is not included in [M].

The theorem implies, for example, that the Bruhat Tits Fixed Point Theorem can be applied to any building. (See Corollary 11.9.)

One of the purposes of this paper is to provide the "correct definition" of the geometric realization $|C|$ and to give the "standard argument" for deducing the above theorem from Moussong's result.

Another purpose is to explain and expand the results of [HM] and [Mei] on graph products of groups. In fact, I was inspired to write this paper after listening to J. Harlander's lecture at the Durham Conference on the results of [HM]. I realized that graph products of groups provide nice examples of buildings where the associated W is a right-angled Coxeter group.

Part I of this paper concerns the combinatorial theory of buildings. In §1, §2, §3 and §6 we recall some relevant definitions and results from Ronan's book [R]. In §4 we state a theorem of Tits [T] and use it in §5 to prove that the chamber system associated to a graph product of groups is a building (Theorem 5.1).

Part II concerns the geometric and topological properties of buildings. In §7, we recall some basic definitions concerning $CAT(0)$ spaces from [G], [Bri], [Bro] and [CD1]. The definitions of $|\mathbf{W}|$ and $|C|$ are given in §9 and §10, respectively. In §11 we prove the main theorem and deduce some consequences. In particular, in Corollary 11.7, we apply the theorem in the case of a graph products of groups.

I. The combinatorial theory of buildings

§1. Chamber systems. A *chamber system over a set I* is a set C together with a family of partitions of C indexed by I. The elements of C are *chambers*. Two chambers are *i-adjacent* if they belong to the same subset in the partition corresponding to i.

Example 1.1. Let G be a group, B a subgroup and $(P_i)_{i\in I}$ a family of subgroups containing B. Define a chamber system $C = C(G, B, (P_i)_{i\in I})$ as follows: $C = G/B$ and the chambers gB and $g'B$ are *i-adjacent* if they have the same image in G/P_i.

Let C be a chamber system over I. Let I^* denote the free monoid on I. (An element of I^* is a word $\mathbf{i} = i_1 \cdots i_k$, where each $i_j \in I$.) A *gallery* in C is a finite sequence of chambers $(c_0, c_1, \cdots, c_k)$ such that c_{j-1} is adjacent but not equal to $c_j, 1 \le j \le k$. The gallery has *type* $\mathbf{i} = i_1 \cdots i_k$ if c_{j-1} is i_j-adjacent to c_j. If each i_j belongs to a given subset J of I, then it is a *J-gallery*. A chamber system is *connected* (or *J-connected*) if any two chambers can be joined by a gallery (or a J-gallery). The J-connected components of a chamber system C are called its *J-residues*. The *rank* of a chamber system over I is the cardinality of I. A *morphism* $\varphi : C \to C'$ of chamber systems over the same set I is a map which preserves i-adjacency.

Suppose that G is a group of automorphisms of C which acts transitively on C. Fix a chamber $c \in C$ and for each $i \in I$ choose an $\{i\}$-residue containing c. Let B denote the stabilizer of c and P_i the stabilizer of the $\{i\}$-residue. Then, clearly, C is isomorphic to the chamber system $C(G, B, (P_i)_{i\in I})$ of Example 1.1.

Suppose $C_1, \cdots, C_k$ are chamber systems over $I_1, \cdots, I_k$. Their *direct product*, $C_1 \times \cdots \times C_k$, is a chamber system over the disjoint union $I_1 \coprod \cdots \coprod I_k$. Its elements are k-tuples $(c_1, \cdots, c_k)$ where $c_t \in C_t$. For $i \in I_t$, $(c_1, \cdots, c_k)$ is *i-adjacent* to $(c'_1, \cdots, c'_k)$ if $c_j = c'_j$ for $j \neq t$ and c_t and c'_t are i-adjacent.

§**2. Coxeter groups.** A *Coxeter matrix* M over a set I is a symmetric matrix $(m_{ij}), (i,j) \in I \times I$, with entries in $\mathbb{N} \cup \{\infty\}$ such that for all $i, j \in I, m_{ii} = 1$ and $m_{ij} \ge 2$ for $i \neq j$. If J is a subset of I, then M_J denotes the restriction of M to J, i.e., it is the matrix formed by the entries of M which are indexed by $J \times J$.

For each $i \in I$ introduce a symbol s_i and let $S = \{s_i | i \in I\}$. The *Coxeter group determined by* M is the group W (or $W(M)$) given by the presentation, $W = \langle S | (s_i s_j)^{m_{ij}} = 1$, for all $(i,j) \in I \times I$ with $m_{ij} \neq \infty \rangle$. The natural map $S \to W$ is an injection ([B, p. 92]) and henceforth, we identify S with its image in W. For any subset J of I, denote by W_J the subgroup of W generated by $\{s_j | j \in J\}$. The case $J = \emptyset$ corresponds to the trivial subgroup of W.

The Coxeter matrix M (and the group W) are *right-angled* if each off-diagonal entry of M is 2 or ∞.

The matrix M is *spherical* if W is finite. A subset J of I is *spherical* if $W_J\big(\cong W(M_J)\big)$ is finite. Associated to M we have the poset $\mathcal{S}^f$ (or $\mathcal{S}^f(M)$) of all spherical subsets of I.

Suppose the $\mathbf{i} = i_1 \cdots i_k$ is an element in the free monoid I^*. Its *value* $s(\mathbf{i})$ is the element of W defined by $s(\mathbf{i}) = s_{i_1} \cdots s_{i_k}$. Two words $\mathbf{i}$ and $\mathbf{i}'$ are *equivalent* (with respect of M) if $s(\mathbf{i}) = s(\mathbf{i}')$. The word $\mathbf{i}$ is *reduced* (with respect to M) if the word length of $s(\mathbf{i})$ is k. (Alternative definitions of these concepts in terms of "homotopies of words" can be found in [R, p. 17].)

§**3. Buildings.** Let I, I^*, M and W be as in §2.

A *building of type* M is a chamber system C over I such that

(1) for each $i \in I$, each subset of the partition corresponding to i contains at least two chambers, and

(2) there exists a "W-valued distance function" $\delta : C \times C \to W$ such that if $\mathbf{i}$ is a reduced word in I^* (with respect to M), then chambers c and c' can be joined by a gallery of type $\mathbf{i}$ if and only if $\delta(c, c') = s(\mathbf{i})$

Example 3.1. Let $\mathbf{W}$ be the chamber system $C(W, \{1\}, (W_{\{i\}})_{i\in I})$ where the notation is as in Example 1.1. In other words, the set of chambers is W and two chambers w and w' are i-adjacent if and only if $w' = w$ or $w' = ws_i$. There is a

W-valued distance function $\delta : W \times W \to W$ defined by $\delta(w, w') = w^{-1}w'$. Thus, $\mathbf{W}$ is a building, called the *abstract Coxeter complex* of W.

Example 3.2. Suppose that C is a building of rank 1. Then W is the cyclic group of order 2. There is only one possibility for $\delta : C \times C \to W$; it must map the diagonal of $C \times C$ to the identity element and the complement of the diagonal to the nontrivial element. Thus, any two chambers are adjacent.

Example 3.3. Suppose that $M_1, \cdots, M_k$ are Coxeter matrices over $I_1, \cdots, I_k$, respectively. Let I denote the disjoint union $I_1 \coprod \cdots \coprod I_k$ and define a Coxeter matrix M over I by setting m_{ij} equal to the corresponding entry of M_t whenever i, j belong to the same component I_t of I and $m_{ij} = 2$ when they belong to different components. Then $W = W_1 \times \cdots \times W_k$ where $W = W(M)$ and $W_t = W(M_t)$. Suppose that $C_1, \cdots, C_k$ are buildings over $I_1, \cdots, I_k$. As in §1, their direct product $C = C_1 \times \cdots \times C_k$ is a chamber system over I. Moreover, the direct product of the W_t-valued distance functions gives a W-valued distance function on C. Hence, C is a building of type M.

§**4. A theorem of Tits.** I, M, W and $\mathcal{S}^f$ are as in §2. Suppose that G is a chamber-transitive group of automorphisms on a building C of type M. Choose a chamber $c \in C$ and for each $J \in \mathcal{S}^f$ a J-residue containing c. Let P_J denote the stabilizer of this J-residue and B $(= P_\emptyset)$ the stabilizer of c. Set $P_i = P_{\{i\}}$, so that $C \cong C(G, B, (P_i)_{i \in I})$. We note that P_J is just the subgroup generated by $(P_j)_{j \in J}$.

Thus, the chamber-transitive automorphism group G gives the following data: for each $J \in \mathcal{S}^f$, there is a group P_J, and whenever $J, J' \in \mathcal{S}^f$ with $J \leq J'$, there is an injective homomorphism $\varphi_{JJ'} : P_J \to P_{J'}$ such that $\varphi_{JJ} = id$ and $\varphi_{JJ''} = \varphi_{J'J''} \circ \varphi_{JJ'}$ when $J \leq J' \leq J''$. (This is the data for a "complex of groups" in the sense of [H] or more precisely a "simple complex of groups " in the sense of [CD2].)

For a natural number m, let $\mathcal{S}^f_m$ denote the subposet of $\mathcal{S}^f$ consisting of all J with $\operatorname{Card}(J) \leq m$.

Proposition 4.1. (Tits [T, Proposition 2]). *Suppose that $C(G, B, (P_i)_{i \in I})$ is a building of type M. Then G is the direct limit of the system of groups $\{P_J | J \in \mathcal{S}^f_2\}$.*

The "direct limit" of a system of groups is defined in [S, p. 1]; the term "amalgamated sum" is used in [T] for this concept.

There is a converse result to Proposition 4.1. Suppose we are given a simple complex of groups (P_J) over $\mathcal{S}^f_3$. (In other words, a system of groups as in the paragraph preceding Proposition 4.1, where the data is only specified for those J with $\operatorname{Card}(J) \leq 3$.) For any subset K of I, let G_K denote the direct limit of the system of groups $\{P_J | J \in \mathcal{S}^f_2 \cap K\}$. Set $B = P_\emptyset, P_i = P_{\{i\}}$ and $G = G_I$.

Theorem 4.2. (Tits [T, Theorem 1]). *With notation as above, suppose that for all $J \in \mathcal{S}^f_3 - \{\emptyset\}, C(P_J, B, (P_j)_{j \in J})$ is a building of type M_J. Then $C(G, B, (P_i)_{i \in I})$ is a building of type M.*

§**5. Graph products.** Let Γ be a simplicial graph with vertex set I. Define a Coxeter matrix $M(= M(\Gamma))$ by setting

$$m_{ij} = \begin{cases} 1; & \text{if } i = j \\ 2; & \text{if } \{i, j\} \text{ spans an edge of } \Gamma \ . \\ \infty; & \text{otherwise} \end{cases}$$

Let W be the associated right-angled Coxeter group. Note that a subset J of I is spherical if and only if any two elements of J span and edge; furthermore, if this is the case, then W_J is the direct product of $|J|$ copies of the cyclic group of order two.

Suppose we are given a family of groups $(P_i)_{i\in I}$. For each $J \in \mathcal{S}^f$, let P_J denote the direct product

$$P_J = \prod_{j\in J} P_j$$

($P_\emptyset = \{1\}$). If $J \le J' \in \mathcal{S}^f$, then $\varphi_{JJ'} : P_J \to P_{J'}$ is the natural inclusion. The direct limit G of $\{P_J | J \in \mathcal{S}_2^f\}$ is called the *graph product* of the $(P_i)_{i\in I}$ (with respect to Γ).

Alternatively, G could have been defined as the quotient of the free product of the $(P_i)_{i\in I}$ by the normal subgroup generated by all commutators of the form $[g_i, g_j]$, where $g_i \in P_i, g_j \in P_j$ and $m_{ij} = 2$.

Notice that for all $J \in \mathcal{S}^f$, $J \neq \emptyset, C(P_J, \{1\}, (P_j)_{j\in J})$ is a building of type M_J (it is a direct product of the rank one buildings $C(P_j, \{1\}, P_j)$ as in Example 3.3). Hence, Tits' Theorem 4.2 implies the following.

Theorem 5.1. *Let G be a graph product of groups $(P_i)_{i\in I}$, as above. Then $C(G, \{1\}, (P_i)_{i\in I})$ is a building of type M.*

Actually, it is not difficult to give a direct proof of this, without invoking Tits' Theorem, by producing the W-valued distance function $\delta : G \times G \to W$. It is clear that any g in G can be written in the form $g = g_{i_1} \cdots g_{i_k}$, where $g_{i_t} \in P_{i_t} - \{1\}$ and where $\mathbf{i} = i_1 \cdots i_k$ is a reduced word in I^* (with respect to M). Moreover, if $g' = g_{i'_1} \cdots g_{i'_k}$ is another such representation, then $g = g'$ if and only if we can get from one representation to the other by a sequence of replacements of the form, $g_i g_j \to g_j g_i$ where $m_{ij} = 2$. In particular, the words $\mathbf{i} = i_1 \cdots i_k$ and $\mathbf{i}' = i'_1 \cdots i'_k$ must have the same image in W. If $g = g_{i_1} \cdots g_{i_k}$, then define $\delta(1, g) = s(\mathbf{i})$ and then extend this to $G \times G$ by $\delta(g, g') = \delta(1, g^{-1}g')$. This δ has the desired properties.

§**6. Apartments and retractions.** Suppose that C is a building of type M. Let $\mathbf{W}$ be the abstract Coxeter complex of W (defined in Example 3.1). A W-isometry of $\mathbf{W}$ into C is a map $\alpha : W \to C$ which preserves W-distances, i.e.,

$$\delta_C(\alpha(w), \alpha(w')) = \delta_{\mathbf{W}}(w, w')$$

for all $w, w' \in W$. (Here δ_C is the W-valued distance function on C and $\delta_{\mathbf{W}}(w, w') = w^{-1}w'$.)

An *apartment* in C is an isometric image $\alpha(W)$ of $\mathbf{W}$ in C.

The set of W-isometries of $\mathbf{W}$ with itself is bijective with W. Indeed given $w \in W$, there is a unique isometry $\alpha_w : W \to W$ sending 1 to w. It is defined by $\alpha_w(w') = ww'$. It follows that an isometry $\alpha : W \to C$ is uniquely determined by its image $A = \alpha(W)$ together with a chamber $c = \alpha(1)$.

Fix an apartment A in C and a chamber c in A. Define a map

$$\rho_{c,A} : C \to A,$$

called the *retraction of C onto A with center c*, as follows. Let $A = \alpha(W)$ with $\alpha(1) = c$. Set $\rho_{c,A}(c') = \alpha(\delta(c, c'))$.

The following lemma is an easy exercise which we leave for the reader.

Lemma 6.1. *The map $\rho_{c,A} : C \to A$ is a morphism of chamber systems.*

In other words, if $\delta(c', c'') = s_i$, then either $\rho(c') = \rho(c'')$ or $\delta\big(\rho(c'), \rho(c'')\big) = s_i$, where $\rho = \rho_{c,A}$.

Corollary 6.2. *Let $\rho = \rho_{c,A}$. If $\delta(c', c'') \in W_J$, then $\delta(\rho(c'), \rho(c'')) \in W_J$.*

Lemma 6.3. *Let $\rho = \rho_{c,A} : C \to A$. If A' is another apartment containing c, then $\rho|_{A'} : A' \to A$ is a W-isometry (i.e., $\rho|_{A'}$ is an isomorphism).*

Proof. Let $\alpha : W \to A$ and $\beta : W \to A'$ be W-isometries such that $\alpha(1) = c = \beta(1)$. If $c' \in A'$, then $\rho(c') = \alpha(\delta(c, c'))$ and $\beta(\delta(c, c')) = c'$. Therefore, $\rho(c') = \alpha \circ \beta^{-1}(c')$, i.e., $\rho|_{A'} = \alpha \circ \beta^{-1}$.

II. Geometric properties of buildings

§**7.** $CAT(0)$ **spaces.** A metric space (X, d) is a *geodesic space* if any two points x and y can be joined by a path of length $d(x, y)$; such a path, parametrized by arc length, is a *geodesic*. A *triangle* in a geodesic space X is a configuration of three points and three geodesic segments connecting them. For each real number ϵ, let M^2_ϵ denote the plane of curvature ϵ. (If $\epsilon > 0, M^2_\epsilon = S^2_\epsilon$, the sphere of radius $1/\sqrt{\epsilon}$; if $\epsilon = 0, M^2_0 = \mathbb{E}^2$, the Euclidean plane; if $\epsilon < 0, M^2_\epsilon = \mathbb{H}^2_\epsilon$, the hyperbolic plane of curvature ϵ.) Let T be a triangle in X and T^* a "comparison triangle" in M^2_ϵ. This means that the edge lengths of T^* are the same as those of T. (If $\epsilon > 0$, assume that the perimeter of T is $\leq 2\pi/\sqrt{\epsilon}$.) Denote the canonical isometry $T \to T^*$ by $p \to p^*$. Suppose x, y, z are the vertices of T and that, for $t \in [0, 1], p_t$ is the point on the edge from x to y of distance $td(x, y)$ from x. The *$CAT(\epsilon)$-inequality* is the inequality,

$$d(z, p_t) \leq d^*(z^*, p^*_t),$$

where d^* denotes distance in M^2_ϵ. The space X is $CAT(\epsilon)$ if this inequality holds for all triangles T (of perimeter $\leq 2\pi/\sqrt{\epsilon}$ when $\epsilon > 0$) and for all $t \in [0, 1]$.

If x, y, z are points in $\mathbb{E}^2$, then a simple argument ([Bro, p. 153]) shows

$$d^2(z, p_t) = (1 - t)d^2(z, x) + td^2(z, y) - t(1 - t)d^2(x, y)$$

Here d denotes Euclidean distance and $d^2(x, y) = d(x, y)^2$.

We return to the situation where T is a triangle in X with vertices x, y, z. Since the edge lengths of T are the same as those in the Euclidean comparison triangle, the equation in the previous paragraph immediately yields the following well-known lemma.

Lemma 7.1. *With notation as above, the $CAT(0)$-inequality for the triangle T in X is equivalent to*

$$d^2(z, p_t) \leq (1 - t)d^2(z, x) + td^2(z, y) - t(1 - t)d^2(x, y)$$

A *piecewise Euclidean polyhedron* X is a space formed by gluing together convex cells in Euclidean space via isometries of their faces. Each cell in X can then be identified with a Euclidean cell, well-defined up to isometry. It follows that arc-length makes sense in X. Thus, X has a natural "path metric": $d(x, y)$ is the infimum of the lengths of all piecewise linear paths joining x to y. One says that X

has *finitely many shapes of cells* if there are only a finite number of isometry types of cells in the given cell structure on X. If X has finitely many shapes of cells, then the path metric gives it the structure of a complete geodesic space ([Bri]). We are interested in the question of when such piecewise Euclidean polyhedra are $CAT(0)$.

It also makes sense to speak of geodesically convex cells in S^n. This leads to the analogous notion of a piecewise spherical polyhedron. Piecewise spherical polyhedra play a distinguished role in this theory since they arise naturally as “links” of cells in piecewise constant curvature polyhedra. A basic result is that if X is a piecewise constant curvature polyhedron (of curvature ϵ), then X satisfies $CAT(\epsilon)$ locally if and only if the link of each cell is $CAT(1)$. (See [Bri] or Ballman's article in [GH, Ch. 10].)

§**8. The geometric realization of a poset.** Let P be a poset. Its *derived complex*, denoted by P', is the poset of finite chains in P (partially ordered by inclusion). It is an abstract simplicial complex: the vertex set is P, the simplices are the elements of P'. The geometric realization of this simplicial complex is denoted by $geom(P)$ and called the *geometric realization* of P.

There are two different decompositions of $geom(P)$ into closed subspaces. Both decompositions are indexed by P. Given $p \in P$, let $geom(P)_{\leq p}$ (respectively, $geom(P)_{\geq p}$) denote the union of all simplices with maximal vertex p (respectively, minimal vertex p). The subcomplex $geom(P)_{\leq p}$ is a *face*, while $geom(P)_{\geq p}$ is a *coface*. Thus, the poset of faces of $geom(P)$ (partially ordered by inclusion) is naturally identified with P, while the poset of cofaces is identified with P^{op} (where P^{op} denotes the same set as P but with the order relations reversed).

§**9. The geometric realization of the Coxeter complex.** Let I, M, W and $\mathcal{S}^f$ be as in §2 and $\mathbf{W}$ be the abstract Coxeter complex of Example 3.1. Consider the poset

$$W\mathcal{S}^f = \coprod_{J \in \mathcal{S}^f} W/W_J,$$

where the partial ordering is inclusion of cosets. We remark that $W\mathcal{S}^f$ can be identified with the poset of all residues in $\mathbf{W}$ of spherical type (where a residue of type $\emptyset$ is interpreted to be a chamber).

The *geometric realization of* $\mathbf{W}$, denoted $|\mathbf{W}|$, is defined to be the geometric realization of the poset $W\mathcal{S}^f$, i.e.,

$$|\mathbf{W}| = geom(W\mathcal{S}^f).$$

Also, we will use the notation,

$$|1| = geom(\mathcal{S}^f).$$

Subscripts will be used to denote the cofaces of $|\mathbf{W}|$ and $|1|$. Thus, if $J \in \mathcal{S}^f$ and $wW_J \in W\mathcal{S}^f$, then

$$\begin{aligned} |1|_J &= geom(\mathcal{S}^f)_{\geq J} \\ |\mathbf{W}|_{wW_J} &= geom(W\mathcal{S}^f)_{\geq wW_J}. \end{aligned}$$

The maximal cofaces (when $J = \emptyset$) correspond to chambers in $\mathbf{W}$. If $w \in W$ is such a chamber, then the corresponding maximal coface is also called a *chamber* and will be denoted simply by $|w|$ (instead of $|\mathbf{W}|_{wW_\emptyset}$). Similarly, we will write $|w|_J$ instead of $|\mathbf{W}|_{wW_J}$.

Our next goal is to show that the faces of $|\mathbf{W}|$ are cells.

First suppose that W is finite. Then it has a representation as an orthogonal reflection group on $\mathbb{R}^n, n = \text{Card}\,(I)$. The reflection hyperplanes divide $\mathbb{R}^n$ into simplicial cones each of which is a fundamental domain for the W-action. Two of these simplicial cones (which are images of each other under the antipodal map) are bounded by the hyperplanes corresponding to the $s_i, i \in I$. Choose one and call it the *fundamental simplicial cone.* (See [B, Ch. V §4].) For each $\lambda \in (0, \infty)^I$ there is a unique point p_λ in the interior of the fundamental simplicial cone so that the distance from p_λ to the hyperplane fixed by s_i is λ_i. The convex hull of the W-orbit of p_λ is a convex cell in $\mathbb{R}^n$ called a *Coxeter cell of type* W and denoted by $P_W(\lambda)$. The intersection of $P_W(\lambda)$ with a fundamental simplicial cone is also a convex cell in $\mathbb{R}^n$ called a *Coxeter block* and denoted by $B_W(\lambda)$.

For example, if W is a direct product of cyclic groups of order two, then $P_W(\lambda)$ is the Cartesian product of intervals $([-\lambda_i, \lambda_i])_{i\in I}$, while $B_W(\lambda)$ is the Cartesian product of $([0, \lambda_i])_{i\in I}$.

The next lemma is an easy exercise. A proof can be found in [CD3, Lemma 2.1.3].

Lemma 9.1. *Suppose W is finite. The vertex set of $P_W(\lambda)$ is Wp_λ (the W-orbit of p_λ). Let $\theta : W \to Wp_\lambda$ denote the bijection $w \to wp_\lambda$. Given a subset V of W, $\theta(V)$ is the vertex set of a face of $P_W(\lambda)$ if and only if $V = wW_J$ for some $wW_J \in W\mathcal{S}^f$. Thus, θ induces an isomorphism from $W\mathcal{S}^f$ to the poset of faces of $P_W(\lambda)$.*

The map θ of Lemma 9.1 induces a simplicial isomorphism from $|\mathbf{W}|$ to the barycentric subdivision of $P_W(\lambda)$ taking faces to faces. Moreover, the restriction of this identification to $|1|$ yields an identification of $|1|$ with $B_W(\lambda)$. Thus, when W is finite each chamber of $|\mathbf{W}|$ is identified with a Coxeter block.

We also note that the face of $P_W(\lambda)$ corresponding to wW_J is isometric to $P_{W_J}(\lambda_J)$ where λ_J denotes the image of λ under the projection $(0, \infty)^I \to (0, \infty)^J$.

We return to the general situation where W may be infinite. As before, choose $\lambda \in (0, \infty)^I$. By Lemma 9.1 we can identify the face of $|\mathbf{W}|$ corresponding to wW_J with the Coxeter cell $P_{W_J}(\lambda_J)$. Moreover, this identification is well-defined up to an element of W_J (which acts by isometries on $P_{W_J}(\lambda_J)$). Hence, we have given $|\mathbf{W}|$ the structure of a piecewise Euclidean cell complex in which each cell is isometric to a Coxeter cell.

We make a few observations.

(1) The natural W-action on $|\mathbf{W}|$ is by isometries.

(2) Any chamber $|w|$ is a fundamental domain for the W-action.

(3) The projection $W\mathcal{S}^f \to \mathcal{S}^f$ induces a projection $|\mathbf{W}| \to |1|$ which is constant on W-orbits. The induced map $|\mathbf{W}|/W \to |1|$ is a homeomorphism

(4) If I is finite (a mild assumption), then $|1|$ is a finite complex and $|\mathbf{W}|$ has only finitely many shapes of cells.

We henceforth assume that I is finite.

Theorem 9.2. (Moussong [M]). *For any finitely generated Coxeter group W, the space $|\mathbf{W}|$, with the piecewise Euclidean structure defined above, is a complete $CAT(0)$ geodesic space.*

Remark 9.3. The choice of $\lambda \in (0,\infty)^I$ plays no role in Moussong's Theorem (or anywhere else). Henceforth, we normalize the situation by setting $\lambda_i = 1$, for all $i \in I$.

Remark 9.4. Suppose W is right-angled. Then (with the above normalization) each Coxeter cell is a regular Euclidean cube of edge length 2. In this situation $|\mathbf{W}|$ is a cubical complex and Theorem 9.2 was proved by Gromov, [G, p. 122], by a relatively easy argument.

§**10. The geometric realization of a building.** Let C be a building of type M. The quickest way to define the geometric realization of C is as follows. Let $\mathcal{C}$ denote the poset of all J-residues in C with $J \in \mathcal{S}^f$. Then $|C|$ is defined to be the geometric realization of $\mathcal{C}$. For each $c \in C$ let $|c|$ denote the maximal coface $geom(\mathcal{C})_{\geq c}$. The map Type: $\mathcal{C} \to \mathcal{S}^f$ which associates to each residue its type induces a map of geometric realizations $|C| \to |1|$. Moreover, the restriction of this map to each chamber $|c|$ is a homeomorphism. Since we showed in §9 how to put a piecewise Euclidean structure on $|1|$ (a union of Coxeter blocks), this defines a piecewise Euclidean structure on $|C|$.

We shall now describe an alternate approach to this definition which is probably more illuminating. Let X be a space and $(X_i)_{i\in I}$ a family of closed subspaces. For each $x \in X$, set $J(x) = \{j \in I | x \in X_j\}$. Define an equivalence relation $\sim$ on $C \times X$ by $(c,x) \sim (c',x')$ if and only if $x = x'$ and $\delta(c,c') \in W_{J(x)}$. The X-*realization* of C, denoted by $X(C)$, is defined to be the quotient space $(C \times X)/\sim$. (Here C has the discrete topology.)

If c is a chamber in C, then $X(c)$ denotes the image of $c \times X$ in $X(C)$. If A is an apartment, then

$$\begin{aligned} X(A) &= \bigcup_{c\in A} X(c) \\ &\cong (A \times X)/\sim \end{aligned}$$

Let $\rho = \rho_{c,A} : C \to A$ be the retraction onto A with center c. By Corollary 6.2, the map $\rho \times id : C \times X \to A \times X$ is compatible with the equivalence relation $\sim$. Hence, there is an induced map $\overline{\rho} : X(C) \to X(A)$. The map $\overline{\rho}$ is a retraction in the usual topological sense.

We shall apply this construction in the following two special cases.

Case 1. $X = |1|$, the geometric realization of $\mathcal{S}^f$, and for each $i \in I, |1|_i = |1|_{\{i\}}$, the coface corresponding to $\{i\}$.

Case 2. W is finite (so that it acts as an orthogonal reflection group on $\mathbb{R}^n, n = \text{Card}\,(I)$) and $X = \Delta$, the spherical $(n-1)$-simplex which is the intersection the fundamental simplicial cone with S^{n-1}. The subspace Δ_i is the codimension-one face of Δ which is the intersection of the hyperplane corresponding to s_i and Δ.

In Case 1, we will use the notation $|C|$ for $|1|(C)$ and call it the *geometric realization of* C. It is a simple matter to check that this agrees with the definition in the initial paragraph of this section. Similarly, $|c| = |1|(c)$ and $|A| = |1|(A)$.

In Case 2 (where C is a building of spherical type), $\Delta(C)$ will be called the *spherical realization of* C.

As explained in §9, $|1|$ has a natural piecewise Euclidean cell structure: the cells are Coxeter blocks. Thus,

$$|1| = \bigcup_{J \in \mathcal{S}^f} B_{W_J}.$$

Here the Coxeter block B_{W_J} is identified with the face of $|1|$ corresponding to J, i.e., with $geom(\mathcal{S}^f)_{\leq J}$. This induces a piecewise Euclidean cell structure (and a resulting path metric) on $|C|$. Thus, if A is an apartment of C, then $|A|$ is isometric to $|\mathbf{W}|$.

Similarly, if C is of spherical type, then $\Delta(C)$ inherits a piecewise spherical simplicial structure from the spherical simplex Δ. If A is an apartment, then $\Delta(A)$ is isometric to the round sphere S^{n-1}. These spherical realizations of buildings of spherical type arise naturally as links of certain cells in $|C|$ for a general building C.

Example 10.1. (A continuation of Example 3.2). Suppose that C is a building of rank one. Then $|1| = [0,1]$. Hence, $|C| = (C \times [0,1])/\sim$ where $(c,0) \sim (c',0)$ for all $c, c' \in C$, i.e., $|C|$ is the cone on C. Similarly , $\Delta(C) = C$ (with the discrete topology).

Example 10.2. (A continuation of Example 3.3). Suppose that $C = C_1 \times \cdots \times C_k$ is a direct product of buildings. We use the notation of Example 3.3 Let $|1_t|$ denote the geometric realization of $\mathcal{S}^f(M_t), 1 \leq t \leq k$, and let $|1|$ denote the geometric realization of $\mathcal{S}^f(M)$. Then $|1| = |1_1| \times \cdots \times |1_k|$, where the Cartesian product has the product piecewise Euclidean structure and the product metric. It follows that

$$|C| = |C_1| \times \cdots \times |C_k|.$$

Similarly, if C is of spherical type, then $\Delta(C)$ is the "orthogonal join" (defined in [CD1, p. 1001]) of the $\Delta(C_i)$.

In particular suppose that each C_i is a rank one building. Then W is the direct product of k copies of the cyclic group of order two; $|C|$ is a Cartesian product of k cones; each chamber in $|C|$ is isometric to the k-cube $[0,1]^k$ and each apartment is isometric to $[-1,1]^k$. Similarly, $\Delta(C)$ is the orthogonal join of the C_i; each chamber is an "all right" spherical $(k-1)$-simplex and each apartment is isometric to S^{k-1}, triangulated as the boundary of a k-dimensional octahedron.

Remark 10.3. The usual definition of the geometric realization of a building C, say in [Bro] or [R], is as $X(C)$ where X is a simplex of dimension $\mathrm{Card}\,(I) - 1$ and $(X_i)_{i \in I}$ is the set of codimension-one faces. If C is spherical, then $X(C) = \Delta(C)$, and if C is irreducible and of affine type, then $X(C) = |C|$. But, if, for example, C is the direct product of two buildings of affine type, then $X(C)$ is not a Cartesian product (it is a join). In [R, p. 184], Ronan comments to the effect that in the general case, the geometric realization of a building should be defined as above.

Remark 10.4. Another possibility is to take X and $(X_i)_{i \in I}$ to be a "Bestvina complex" as in [Bes] or [HM]. This means that X is a CW-complex, each X_i is a subcomplex and that if, for J a subset of I, we set

$$X_J = \bigcap_{i \in J} X_j$$

(and $X_\emptyset = X$), then X_J is nonempty and acyclic if and only if $J \in \mathcal{S}^f$. Furthermore, X is required to have the smallest possible dimension among all such complexes with this property. The argument in [D] then shows that $X(C)$ is acyclic (and contractible if X is contractible). In the case where C admits a chamber-transitive automorphism group $G, X(C)$ can be used to determine information about the cohomological dimension (or virtual cohomological dimension) of G.

§11. The main theorem and some of its consequences.

Theorem 11.1. *The geometric realization of any building is a complete, $CAT(0)$ geodesic space.*

In the case where the building C is irreducible and of affine type, this result is well-known. A proof can be found in [Bro, Ch. VI §3]. Our proof follows the argument given there.

Suppose $\rho = \rho_{c,A} : C \to A$ is the retraction onto A with center c. Its geometric realization $\overline{\rho} : |C| \to |A|$ (defined in §10) takes each chamber of $|C|$ isometrically onto a chamber of $|A|$. Hence, $\overline{\rho}$ maps a geodesic segment in $|C|$ to a piecewise geodesic segment in $|A|$ of the same length. From this observation we conclude the following.

Lemma 11.2. *The retraction $\overline{\rho} : |C| \to |A|$ is distance decreasing, i.e., for all $x, y \in |C|$,*

$$d_{|A|}(\overline{\rho}(x), \overline{\rho}(y)) \leq d_{|C|}(x, y),$$

where $d_{|A|}$ and $d_{|C|}$ denote distance in $|A|$ and $|C|$, respectively. In particular, if $x, y \in |A|$, then $d_{|A|}(x, y) = d_{|C|}(x, y)$.

Lemma 11.3. *There is a unique geodesic between any two points in $|C|$.*

Proof. One of the basic facts about buildings is any two chambers are contained in a common apartment ([R, Theorem 3.11, p. 34]). This implies that, given $x, y \in |C|$, there is an apartment A such that $x, y \in |A|$. (Choose chambers $c, c' \in C$ so that $x \in |c|, y \in |c'|$ and an apartment A containing c and c'; then $x, y \in |A|$.) A basic fact about $CAT(0)$ spaces is that any two points are connected by a unique geodesic segment. Since $|A|$ is $CAT(0)$ (Moussong's Theorem), there is a unique geodesic segment in $|A|$ from x to y. Let $\gamma : [0, d] \to |A|$ be a parametrization of this segment by arc length, where $d = d(x, y)$. By the last sentence of the previous lemma, γ is also a geodesic in $|C|$. Let $\gamma' : [0, d] \to |C|$ be another geodesic from x to y. If $\overline{\rho} : |C| \to |A|$ is the geometric realization of any retraction onto A, then $\overline{\rho} \circ \gamma'$ is a geodesic in $|A|$ from x to y (since it is a piecewise geodesic of length d). Hence, $\overline{\rho} \circ \gamma' = \gamma$. Let $t_0 = \sup\{t | \gamma|_{[0,t]} = \gamma'|_{[0,t]}\}$. Suppose that $t_0 < d$. Then, for small positive values of $\epsilon, \gamma(t_0 + \epsilon)$ lies in the relative interior of some coface $|c|_J$, with $c \in A$, while $\gamma'(t_0 + \epsilon)$ lies in the relative interior of a different coface $|c'|_J$, where $|c'|_J \not\subset |A|$. Set $\rho = \rho_{c,A}$. Since $|c|_J \neq |c'|_J, \delta(c, c') \notin W_J$. Hence, $\overline{\rho}(\gamma'(t_0 + \epsilon)) \neq \gamma(t_0 + \epsilon)$, a contradiction. Therefore, $t_0 = d$ and $\gamma = \gamma'$.

Lemma 11.4. *Suppose $\rho = \rho_{c,A}$. If $x \in |c|$, then $d(x, \overline{\rho}(y)) = d(x, y)$ for all $y \in |C|$.*

Proof. Choose an apartment A' so that $|A'|$ contains both x and y. By the previous lemma, the image of the geodesic γ from x to y is contained in $|A'|$. By Lemma

6.3, $\rho|_{A'} : A' \to A$ is a W-isometry. It follows that $\overline{\rho}|_{|A'|} : |A'| \to |A|$ is an isometry. Hence, $\overline{\rho} \circ \gamma$ is actually a geodesic (of the same length as γ).

Proof of Theorem 11.1. Suppose $x, y, z \in |C|$. For $t \in [0,1]$, let p_t be the point on the geodesic segment from x to y such that $d(x, p_t) = td(x,y)$. By Lemma 7.1, to prove that $|C|$ is $CAT(0)$ we must show that

$$d^2(z, p_t) \leq (1-t)d^2(z,x) + td(z,y) - t(1-t)d^2(x,y).$$

Choose an apartment A so that $x, y \in |A|$. Since the geodesic segment from x to y lies in $|A|, p_t \in |A|$. Hence, we can choose a chamber c in A so that $p_t \in |c|$. Let $\rho = \rho_{c,A}$. By Lemma 11.4, $d(z, p_t) = d(\overline{\rho}(z), p_t)$. Hence,

$$\begin{aligned} d^2(z, p_t) &= d^2(\overline{\rho}(z), p_t) \\ &\leq (1-t)d^2(\overline{\rho}(z), x) + td^2(\overline{\rho}(z), y) - t(1-t)d^2(x,y) \\ &\leq (1-t)d^2(z,x) + td^2(z,y) - t(1-t)d^2(x,y). \end{aligned}$$

The first inequality holds since $\overline{\rho}(z), x, y$ all lie in $|A|$ and since $|A|$ is $CAT(0)$. The second inequality follows from Lemma 11.2. Therefore, $|C|$ is $CAT(0)$.

Since $CAT(0)$ spaces are contractible (via geodesic contraction), we have the following corollary.

Corollary 11.5. $|C|$ *is contractible.*

As we mentioned previously (in Remark 10.4) this can also be proved as in [D].

Suppose, for the moment, that C is spherical. Then $|C|$ has a distinguished vertex v, namely the coface $|C|_{W_I} (= |c|_I$ for any $c \in C)$. The link of v in C is $\Delta(C)$. Since the link of a vertex in a $CAT(0)$ space is $CAT(1)$ ([G, p. 120]), this gives the following corollary.

Corollary 11.6. *Suppose C is spherical, then $\Delta(C)$ is $CAT(1)$.*

Of course, one could also give a direct argument for this by proving the analogs of Lemmas 11.2 and 11.3 for $\Delta(C)$ (in the case of 11.3 one shows the uniqueness of geodesics only in the case where the endpoints are of distance less than π).

Corollary 11.7. (Meier) *With notation as in §5, let G be a graph product of groups $(P_i)_{i \in I}$ and let C be the building $C(G, \{1\}, (P_i)_{i\in I})$ (cf. Theorem 5.1). Then $|C|$ is $CAT(0)$.*

Remark 11.8. Since the Coxeter block associated to a direct product of cyclic groups of order two is a Euclidean cube, $|C|$ is a cubical complex. In the proof of Theorem B in [Mei], in the case of graph products, the complex $|C|$ is defined (without mentioning buildings) and proved to be $CAT(0)$. There is a simple proof of this by induction on $\operatorname{Card}(I)$. One first notes that for any subset J of I, $|C_J|$ is a totally geodesic subcomplex of $|C|$, where $C_J = C(G_J, \{I\}, (P_j)_{j\in J})$. If I is spherical, then $|C|$ is a direct product of cones on the P_i and hence, is $CAT(0)$. If I is not spherical, then there exist $i, j \in I$ with $m_{ij} = \infty$. Then G is the amalgamated product of $G_{I-\{i\}}$ and $G_{I-\{j\}}$ along $G_{I-\{i,j\}}$ and $|C|$ is a union of components each of which is a translate of $|C_{I-\{i\}}|$ or $|C_{I-\{j\}}|$. Furthermore, the intersection of two such components is a translate of the totally geodesic subcomplex $|C_{I-\{i,j\}}|$.

The result follows from a gluing lemma of Gromov ([G, p. 124] and [GH, p. 192]) and induction.

The Bruhat-Tits Fixed Point Theorem (see [Bro, p. 157]) states that if a group of isometries of a complete, $CAT(0)$ space has a bounded orbit, then it has a fixed point. Applying this result to the case of a building with a chamber-transitive automorphism group, we get the following.

Corollary 11.9. *With notation as in §4, suppose that $C = C(G, B, (P_i)_{i\in I})$ is a building and that H is a subgroup of G which has a bounded orbit in $|C|$. Then H is conjugate to a subgroup of P_J, for some $J \in \mathcal{S}^f$.*

Remark 11.10. First suppose W is finite. Then it can be represented as a reflection group on hyperbolic n-space. Given $\lambda \in (0, \infty)^I$, one defines a "hyperbolic Coxeter cell" $P^h_W(\lambda)$ and a "hyperbolic Coxeter block" $B^h_W(\lambda)$, exactly as in §9. Returning to the situation where W and C are arbitrary, given $\lambda \in (0, \infty)^I$, we get a piecewise hyperbolic structure on $|C|$. Let us denote it $|C|^h(\lambda)$. If, for the link of each cell in the $CAT(0)$ structure on $|\mathbf{W}|$, the length of the shortest closed geodesic is strictly greater than 2π, then for sufficiently small values of $\lambda, |\mathbf{W}|^h(\lambda)$ will be $CAT(-1)$ (and hence, W will be word hyperbolic). If this is the case, then the proof of Theorem 11.1 shows that $|C|^h(\lambda)$ is also $CAT(-1)$. In [M] Moussong determined exactly when this holds. His condition is that for each subset J of I neither of the following occur:

a) W_J is of affine type, with $\operatorname{Card}(J) \geq 3$,

b) W_J is a direct product $W_{J_1} \times W_{J_2}$ where both J_1 and J_2 are infinite.

Thus, if neither condition holds, then $|C|$ can be given a $CAT(-1)$ structure.

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