Null space and RANGE of $m \times n$ matrix A

$$\left.\begin{array}{l} \cdot\ \mathcal{N}(A) \subset \mathbb{R}^n \\ \cdot\ R(A) \subset \mathbb{R}^m \end{array}\right\} \text{subspaces}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{bmatrix}$$ Find $\mathcal{N}(A)$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 6 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\swarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dim \mathcal{N}(A) = \overset{n-r}{4 - 3} = 1$$

x_3 is independent variable

$x_4 = 0$

$x_2 = -3x_3$ set $x_3 = 1$

$x_1 = 2x_3$

Basis for $\mathcal{N}(A) = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Range of A, dim = 3

$x_1 = 2$, $x_2 = -3$, $x_3 = 1$, $x_4 = 0$

$$2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \vec{0}$$

So I can eliminate any column vector except the 3rd to get a basis

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Suppose Row reduction had been

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = 2$

$n - r = 4 - 2 = 2$

x_3 & x_4 are ind. variabl.

$$x_1 = 2x_3$$
$$x_2 = -3x_3$$

(a) Set $x_3 = 1$ $x_4 = 0$

Basis $\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

(b) Set $x_3 = 0$, $x_4 = 1$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $R(A)$

$$2A_1 - 3A_2 + A_3 = 0$$
$$A_4 = 0$$

Row rank

$\overline{R(A)}$ = column space

Can also look at span

of row vectors

Ex. 2 p 180

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rk = 2

nullity = 4 - 2

Start with

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 4 \\ 1 & 2 & 4 & -1 \end{bmatrix}$$

Basis $\mathcal{N}(A)$

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$x_3 = 1 \quad x_4 = 0$

$x_1 + 2x_3 = 0 \quad x_1 = -2$

$x_2 = -1$

$x_3 = 0 \quad x_4 = 1$

$\mathcal{R}(A)$ $\quad x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4$

$x_3 = 1, x_4 = 0$ eliminate A_3 & A_4

Find Basis for Range of   Feb 16

Subspace of $\mathbb{R}^4$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 3 & 5 & 4 \\ 1 & 4 & -1 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 4 \\ 1 & 2 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}$$

Ans $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}$

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Linear Transformation from

$\mathbb{R}^n$ to $\mathbb{R}^m$. $F : \mathbb{R}^n \to \mathbb{R}^m$

<u>Def</u> $F(v + v') = F(v) + F(v')$

$$F(tv) = t\,F(v)$$

<u>Thm</u> F is determined by what it does on a basis $\{v_1, \ldots, v_n\}$

<u>Pf</u> Any $v = c_1 v_1 + \ldots c_n v_n$

$$F(v) = c_1 F(v_1) + c_n F(v_n)$$

Say $w = \{w_1, \ldots w_m\}$ basis for $\mathbb{R}^m$.

$$F(\vec{v}_j) = \sum a_{ij} \vec{w}_i$$

$$F(e_j) = \sum a_{ij} e_i$$

$$F(x_1 e_1 + \ldots x_n e_n) = x_1 F(e_1) + \ldots x_n F(e_n)$$
$$= x_1 A_1 + \quad + x_n A_n$$
$$= A \vec{x}$$

Feb 19

Ch 5: Vector space, subspace, basis, dimension

__Def'n__ Linear transformation between 2 vector spaces (Notation: $F: V \longrightarrow W$).

__Thm__ A linear transformation is determined by what it does on a basis

__Pf__ Suppose $\vec{v}_1, \ldots \vec{v}_n$ basis for V. Any vector $\vec{v}$ can be written uniquely as

$$\vec{v} = x_1 v_1 + \cdots x_n v_n$$

$$F(\vec{v}) = x_1 F(v_1) + \quad x_n F(v_n)$$

Write $F(\vec{v}_j)$ $j = 1, \ldots n$

as a linear combination

of basis vectors $w_1, \ldots w_m$

$$F(v_j) = \sum a_{ij} w_i$$

a_{ij} is a matrix

<u>Key point</u> ~~F is deter.~~

Pick basis $\vec{v}_1 \ldots \vec{v}_n$

for V. Then F is

determined by $F(\vec{v}_i)$

<u>Pf</u> ANY $\vec{v} \in V$ can

written uniquely as

lin combi of basis

$$\vec{v} = x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n$$

$$F(\vec{v}) = F(x_1 v_1) + \dots + F(x_n v_n)$$
$$= x_1 F(v_1) + \qquad x_n F(v_n)$$

Let $\vec{w}_1, \dots \vec{w}_m$ be basis for $\mathbb{R}^n$

$F(\vec{v}_j) =$ linear comb of w_i

$$= \sum a_{ij} w_i$$

$m \times n$ matrix

$V = \mathbb{R}^n$

Standard basis $e_1, \dots e_n$

$$e_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$W = \mathbb{R}^m$. Standard basi

$\hat{e}_1, \ldots \hat{e}_m$

$F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$F(e_i) = A_i$ = column vector in $\mathbb{R}^m$

$A = [A_1, \ldots A_n]$

write A_j as a sum of $\hat{e}_1, \ldots \hat{e}_m$

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = a_{1j}\hat{e}_1 + \cdots a_{mj}\hat{e}_m$$

F is matrix by

$F(\vec{v}) = A\vec{v}$

$$Ae_j = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A_j$$

Feb 21

- Definition of vector space V (5.2, page 361)
- Example: $P_2 = \{$polynomials of degree $\leq 2\}$
- Basis for P_2, $\dim P_2 = 3$
- $\mathcal{B} = \{v_1, \ldots v_n\}$ a basis for V.

Any $\vec{v} \in V$ can be written uniquely as

$$\vec{v} = x_1\vec{v}_1 + \ldots x_n\vec{v}_n$$

- Coordinates $[\vec{V}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- Linear Transformation between two vector spaces.
  Def (5.7: p 403)
- If $\dim V = n$, then there is linear transformation (isomorphism)

$F: V \longrightarrow \mathbb{R}^n$ sending

$\mathcal{B}$ to standard basis

$F(v_i) = e_i$ means

$$F(\vec{v}) = [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Matrix of a linear transfm

$T: V \longrightarrow W$ $\quad \mathcal{B} =$ basis for V

$\quad C =$ basis for W

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "F_{\mathcal{B}}"] & W \arrow[d, "F_C"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^m \end{tikzcd}$$

Matrix mult by matrix A

$$A[v]_{\mathcal{B}} = [T\vec{v}]_C$$

<u>Def</u> vector space V

$V =$ a set

Add two of them

or scalar multiply

1) $v + w = w + v$

2) $u + (v+w) = (u+v) + w$

3) There is vector 0
s.t $v + 0 = v$

4) There is vector $-v$
s.t $v + -v = 0$

1) $a(v+w) = av + aw$

2) $a(b\vec{v}) = (ab)\vec{v}$

3) $1 \cdot v = v$

4) $(a+b)v = av + bv$

$P_2 = \{$ polynomials of degree $\leq 2 \}$

$p(x)$ is a poly.

$p(x) = a_0 + a_1 x + a_2 x^2$

$q(x) = b_0 + b_1 x + b_2 x^2$

$(p+q)(x) = p(x) + q(x)$

$=$

$a_0+b_0 + (a_1+b_1)x +$
$(a_2+b_2)x^2$

could hav

$p(x) = 1 + 2x + 3x^2$

$3\,p(x) = 3(1+2x+3x^2)$

Basis $= \mathcal{B}$

$\mathcal{B} = \{1, x, x^2\}$

linear ~~dep~~ indep

$a_0 + a_1x + a_2x^2 = 0$

$x=0 \qquad a_0=0$

$x=1 \qquad a_0+a_1+a_2=0$

$x=-1 \qquad a_0-a_1+a_2=0$

$\# \mathcal{B} = 3$

$\dim P_2 = 3$

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a different basis

$\{1, (x-1), (x-1)^2\}$

$u = x-1$

$u+1 = x$

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$= a_0 + a_1(u+1) + a_2(u+1)^2$$

$$= a_0 + a_1 u + a_1 + a_2 u^2 + 2a_2 u + a_2$$

$$= (a_0 + a_1 + a_2) + (a_1 + 2a_2)u + a_2 u^2$$

$$= (a_0 + a_1 + a_2) + (a_1 + 2a_2)(x-1) + a_2 (x-1)^2.$$

$\mathcal{P}_m = \{$ poly's of deg $\leq m \}$

$\dim \mathcal{P}_m = m+1.$

Ex $C([0,1])$

$= \{$ ~~f~~ all continuous functions from $[0,1]$ to $\mathbb{R} \}$

$f(x)$

Linear Trans $T: V \to W$

Def ~~B~~ T :

$T : \mathcal{P}_2 \longrightarrow \mathbb{R}^1$

$T(p) = p(1) = a_0 + a_1 + a_2$

$T(p+q) = T(p) + T(q)$

Feb 27

- Basis for a vector space V
- dimension = # of elements in basis
- Coordinates
- Linear Transformations
- Thm linear Transformation $T : V \longrightarrow W$ is determined by what it does on a basis.

- Ex $A = m \times n$ matrix
- Null space, Range,
- Invertible linear transformation (= "isomorphism")

- Thm $\dim V = n \Rightarrow V \cong \mathbb{R}^n$

- $\mathcal{B}$ = Basis for V
  $\mathcal{E}$ = standard basis for $\mathbb{R}^n$

$T : V \longrightarrow W$ ($\mathbb{R}^n$ and $\mathbb{R}^n$,

T is linear transform from v to w

(i.e. $T(v + v') = T(v) + T(v')$
$T(cv) = cT(v)$

Thm T is completely determined by what does on basis ~~for~~ B for V

$\mathcal{B} = \{ \vec{v}_1, \dots \vec{v}_n \}$

$T(v_i) = \vec{w}_i$ in W

$\vec{v} = t_1 \vec{v}_1 + \dots + t_n \vec{v}_n$

$\left([\vec{v}]_{\mathcal{B}} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \right)$

Then

$T(\vec{v}) = t_1 T(\vec{v}_1) + \dots t_n T(\vec{v}_n)$

$= t_1 \vec{w}_1 + \dots t_n \vec{w}_n$