A is nonsingular $n \times n$ means column vectors of A are a basis for $\mathbb{R}^n$.

Nonsingular $\Longleftrightarrow$ invertible, i.e There is a matrix A^{-1} s.t

$$AA^{-1} = I_n$$

Fact: A^{-1} is also nonsingular &

$$A^{-1}A = I$$

So $(A^{-1})^{-1} = A$

Fact: If A, B are invertible Then (AB) is invertible & $(AB)^{-1} = B^{-1}A^{-1}$

Ex Inverse of diagonal matrix

Basis = linearly independent, spanning set.

= minimal spanning set

Ex standard basis $\{e_1, \ldots e_n\}$ for $\mathbb{R}^n$

- Any element of $\mathbb{R}^n$ can be written uniquely as linearly combination of basis elements.

- Fact Any two bases for a subspace W of $\mathbb{R}^n$ have same # of elements

This # is called the dimension of W = $\dim W$

Inverse of 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Calculate A^{-1}

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

General Formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $\Delta = ad - bc$ is determinant.

Fact $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nonsingular $\Longleftrightarrow \Delta \neq 0$

Facts about

Bases for $\mathbb{R}^n$

- Def'n A basis for $\mathbb{R}^n$ is a linearly independent spanning set.

- Fact: The number of vectors in any basis for $\mathbb{R}^n$ is n.

  Pf.

- Fact: If $\{v_1, \ldots v_n\}$ is a basis for $\mathbb{R}^n$. Then any vector $b = \begin{bmatrix} b_1 \\ b_n \end{bmatrix}$ in $\mathbb{R}^n$ can be written uniquely as a linear combination of $v_1, \ldots, v_n$,

  i.e $b = x_1 \vec{v}_1 + \ldots x_n \vec{v}_n$ and the x_i are unique.

How to find the x_i?

- Standard basis for $\mathbb{R}^n$

$\{e_1, \ldots e_n\}$ where $e_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Then $\vec{b} = b_1 \vec{e}_1 + \quad + b_n \vec{e}_n$

- Fact. A basis is $\overbrace{\text{as a}}^{\text{the same thing}}$ minimal spanning set.

Minimal means that if you discard one of the vectors you no longer have a spanning set

Pf of Fact

not minimal $\Rightarrow$ lin dep

Suppose $\{v_1, \ldots v_m\}$ not minimal & we could throw away say v_m

lin dep $\Longrightarrow$ not minimal

$x_1 v_1 + \quad + x_m v_m = 0$ Not all $x_i = 0$

~~pf~~ then one x_i, say $x_m \neq 0$

In other words, if $v_1, \ldots v_m$ is a spanning set we can discard some of these vectors to get a spanning set

Basis for a subspace W of $\mathbb{R}^n$ (or for that matter for any vector space W) Same definition

<u>Fact</u> ANY 2 bases for W have the same number of elements. This number is called <u>dimension</u> of W, $\dim W$

Relationship with nonsingular matrices

Def $A = n \times n$ matrix is nonsingular iff its column vectors form a basis

Problem
Given a vector $\vec{b}$ find the unique scalars $x_1, \dots x_n$ s.t $x_1 A_1 + \dots + x_n A_n = \vec{b}$

i.e $A\vec{x} = \vec{b}$

Ans $\vec{x} = A^{-1} b$

Problem Find $x_1, \dots x_n$ s.t

$x_1 A_1 + \dots x_n A_n = e_i$

Ans $\vec{x} = (A^{-1} e_i)$

i.e $x =$ i^{th} column vector of A_i

Feb 9

FIRST EXAM – Monday Feb 12
HW is also due that day

Comments about Exam
- 7 problems

<u>Some topics</u>

- From Ch 2: Dot products, length, orthogonal projection

- Definitions (from Ch 1 & 3)
  linear combination of vectors
  linear independence of set of vectors
  Span of a set of vectors
  Subspace
  Basis for a subspace or vector space
  Dimension of a subspace
  Vector Space properties of $\mathbb{R}^n$
- Matrices
  - Row echelon form & solving linear systems
  - Matrix multiplication

- Nonsingular $n \times n$ matrices and Invertible matrices
- Calculating the inverse of a matrix
- Determing when a set of column vectors is lin. independent

Null space and Range of an $(m \times n)$ matrix A

$$x \longrightarrow Ax$$

is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$

In other words, to each column vector $\vec{x} \in \mathbb{R}^n$ we associate a vector Ax in $\mathbb{R}^m$

There are 2 subspaces associated to this situation

$$\mathcal{N}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

$$\mathcal{R}(A) = \{ y \in \mathbb{R}^m \mid Ax = y \text{ has a solution} \}$$

Image (A) = range of A

Claim $\mathcal{N}(A)$ is subspace

$\tilde{x}, \vec{y} \in \mathcal{N}(A)$ then

$x+y \in \mathcal{N}(A)$

$Ax = 0$ & $Ay = 0$

$A(x+y) = Ax + Ay = 0 + 0$

So $x+y \in \mathcal{N}(A)$

$A(cx) = cA(x) = 0$

$\mathcal{R}(A) = Sp\{A_1, \ldots A_n\}$

= span of column vectors.

Prob What are dimensions of $\mathcal{N}(A)$ & $\mathcal{R}(A)$

Put A in reduced row echelon

$\dim \mathcal{R}A = r. = \#$ of nonzero rows

= # of leading 1's

= rank of A

$\dim N(A) = n - r$

= # columns $- r$