Quiz on Friday (probably on definitions)

Definition of a vector space

(Section 5.2 p 361)

Def A vector space V is a set (of vectors) together with 2 operations (addition and scalar multiplication):

- For any $\vec{v}, \vec{w} \in V$, there is vector $\vec{v}+\vec{w} \in V$
- For any $\vec{v} \in V$ and real number $c \in \mathbb{R}$, there is a vector $c\vec{v}$ satisfying conditions:

(a1) $v + w = w + v$

(a2) $(u + v) + w = u + (v+w)$

(a3) There a $\vec{0} \in V$, $\vec{0} + \vec{v} = \vec{v}$

(a4) For any $\vec{v}$, There is $-\vec{v}$ s.t.
$v + (-v) = \vec{0}$

(m1) $a(b\vec{v}) = (ab)\vec{v}$

(m2) $a(v+w) = av + aw$

(m3) $(a+b)v = av + bv$

(m4) If $1 \in \mathbb{R}$, $1v = v$

Main Example: (Sections 3.1, 3.2)

$$\mathbb{R}^n = \left\{ \vec{v} \;\middle|\; \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},\ v_i \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

Define

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \\ v_n + w_n \end{bmatrix}$$

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

Def'n A nonempty subset $W \subset V$ is a <u>subspace</u> if it is a vector space. This amounts to saying that it is closed under addition & scalar mult:

- if $v, w \in W$, then $v+w \in W$
- if $v \in W$ & $c \in \mathbb{R}$, then $cv \in W$.

Examples:

1) $\left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} v_2 = 2v_1 \\ v_3 = 3v_1 \end{matrix} \right\}$

2) $\left\{ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; av_1 + bv_2 + cv_3 = 0 \right.$

or $= \left\{ \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; ax + by + cz = 0 \right.$

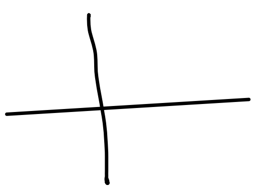
$\left[z \right]$

Another description $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$= \left\{ \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \vec{n} \cdot \vec{v} = 0 \right.$$

Non-examples:

3) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x = 0 \text{ or } y = 0 \right\}$



4) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; z = 1 \right\}$

Why not?

<u>Span</u> of a subset (3.3 or 5.3)

$Q = \{\vec{v}_1, \dots \vec{v}_k\}$ vectors in V (or $\mathbb{R}^n$)

$Sp(Q)$ $= \text{Span } Q$ is set ~~of~~ all linear combinations of form

$\vec{y} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ ← linear combo

<u>Remark</u> Q does not have to be a finite set of vectors

<u>Thm</u> $Sp(Q)$ is a subspace

<u>Pf</u> Closed under addition & mult

$y = c_1\vec{v}_1 + \dots c_k\vec{v}_k$

$y' = d_1 v_1 + \quad d_k \dot{v}_k$

$y + y' = (c_1+d_1)\bar{v}_1 + \dots (c_k+d_k)\vec{v}_k$

$\square$

<u>Examples</u>: 1) The span of one vector $\vec{v}$ (a line)

2) Span of two vectors

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- Linear combination of vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^m$, something of form $\vec{y} = c_1\vec{v}_1 + \dots c_n\vec{v}_n$
  c_i scalar

- Span of set of vectors
  $Q = \{\vec{v}_1, \dots, \vec{v}_n\}$

  $Sp(Q) = \{$all possible linear combin of vectors in $Q\}$

This a subspace: closed under addition & mult.

Solving systems of linear equations (Chapter 1)

$$x_1 + 2x_2 = 5$$
$$3x_1 + 4x_2 = 7$$

Subtract 3 (1st) from 2nd

$$x_1 + 2x_2 = 5$$
$$0 - 2x_2 = -8$$
$$x_2 = 4$$

$x_2 = 4, \; x_1 = -3$

Matrix form

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 7 \end{array}\right)$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \qquad \vec{b} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

In general, for m equations in n unknowns

$$a_{11}x_1 + \quad + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + \quad + a_{mn}x_n = b_n$$

augmented matrix

$$A = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots a_{2n} & \\ & & & \\ a_{m1} & & a_{mn} & b_n \end{array}\right]$$

A is $m \times n$ matrix. m rows n columns

Row operations

- Add (or subtract) multiple

of one row to another)

- Multiply a row by nonzero scalar

- Interchange two rows

Row echelon form

- rows of zeros are at bottom
- In a nonzero row, first entry is 1
- In $(i+1)^{st}$ row 1^{st} non-zero entry is in column to right of 1^{st} non zero entry in i^{th} row

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Reduced row echelon

entries above any leading 1 are all $= 0$.

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m equations in n unknowns

Basic Question: How many solutions?
Possible Answers: $0, 1, \infty$

Vector Form of system of equations

$$A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

$m \times n$ matrix
m rows
n columns

$$\vec{x} = \begin{bmatrix} x_1 \\ \\ \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

Multiplying vector by matrix not yet defined.

$$\vec{A}_i = i^{th} \text{ column vector} = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} \in \mathbb{R}^m$$

Solving

$$A\vec{x} = \vec{b} \quad \text{means}$$

Finding numbers $x_1, \dots x_n$ ~...

$$x_1\vec{A}_1 + \dots x_n\vec{A}_n = \vec{b} \quad ?$$

i.e answering question

Q. Is $\vec{b}$ a linear combination of $\{\vec{A}_1, \dots \vec{A}_n\}$?

Q. Is $\vec{b}$ in span of $\{\vec{A}_1, \dots \vec{A}_n\}$?

Augmented matrix $[A|\vec{b}]$ means make last column vector $\vec{b}$
it is $m \times (n+1)$ matrix

Important Special Case: $\vec{b} = \vec{0}$
called Homogeneous system

$$A\vec{x} = \vec{0}$$

This system is always consistent
i.e., always has a least one solution; namely
i.e all $x_i = 0$.

$$x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Method for determing # of solutions
. Put $[A|\vec{b}]$ in echelon form

or reduced echelon form

- If # equations = # unknowns we expect unique solution

$$\left[\begin{array}{cccc|c} 1 & & * & & * \\ 0 & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ 0 & \cdots & & 0 \; 1 & * \end{array}\right]$$

echelon form

$$x_1 + * x_2 + \cdots * x_n = *$$

$$\vdots$$

$$x_{n-1} + * x_n = *$$

$$x_n = *$$

Reduced echelon form is even easier

Expected

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 \; - & 0 & * \\ & & \ddots & & \\ 0 & \cdots & & 1 & * \end{array}\right]$$

$x_1 = *$

$x_n = *$

Inconsistent system: A row of 0's except in last column

$$\left[\begin{array}{ccccc} & & & & \\ 0 & \cdots & \cdots & 0 & * \end{array}\right]$$

Ex 1 (p 29) $(C|d) =$

5 equations 6 unknowns

$$\begin{array}{cccccc|c} & & & & & & rd \\ 1 & 2 & 0 & 3 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$x_1 + 2x_2 \qquad + 3x_4 \quad + \quad 4x_6 = 1$$

$$x_3 + 2x_4 \qquad + 3x_6 = 2$$

leading 1 $\qquad x_5 + x_6 = 2$

x_1, x_3, x_5 are called dependent variables, x_2 x_4 x_6 are independent variables

r = # of nonzero rows in $[C|d]$

Actually we should say we have r equations in n unknowns instead of m equations in n unknowns