Problem given vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

when is $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$ in the span of 2 given vectors?

when can $\vec{b}$ be written as linear combination of given vectors. Find x_1, x_2

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solve 3 equations in 2 unknowns

$$\begin{bmatrix} x_1 + x_2 \\ 2x_1 + -1\,x_2 \\ 3x_1 + 4\,x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & b_1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & b_2 \\ 3 & 4 & b_3 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & -3 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 3b_1 \end{bmatrix} \quad -\frac{1}{3} R_2$$

$$\rightarrow \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & -\frac{b_2}{3} + \frac{2}{3} b_1 \\ 0 & 1 & b_3 - 3b_1 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & -\frac{b_2}{3} + \frac{2}{3} b_1 \\ 0 & 0 & b_3 - 3b_1 + \frac{b_2}{3} - \frac{2}{3} b_1 \end{bmatrix}$$

Ans

$$b_3 + \frac{1}{3} b_2 - \frac{11}{3} b_1 = 0$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$-\frac{11}{3}x + \frac{1}{3}y + z = 0$$

$$-11x + y + 3z = 0$$

<u>Problem</u> Show $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

are linearly independent

Show

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

only solution $x_1 = 0$, $x_2 = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

Show this unique solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_2 = 0$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is only solution

So $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ are linearly independent

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$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ ← Augmented matrix

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2$$

$$-2x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Mult 2 matrices A, B

Only defined when
\# columns of A = \# rows of B

$A = m \times n$
$B = n \times p$

$AB = m \times p$ matrix

$A = 2 \times 3$ $\quad B = 3 \times 1$

2×1

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1(-1) + 2(-2) + 3(-3) \\ 4(-1) + 5(-2) + 6(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -14 \\ -32 \end{bmatrix}$$

1st Midterm - Feb 14

Matrices.

$M(m,n) = \{\text{all } m \times n \text{ matrices}\}$

Vector space properties

Addition and scalar mult.

$\mathbb{O}$ = 0-matrix

Usual Properties

- $A+B = B+A$
- $(A+B)+C = A+(B+C)$
- $A + \mathbb{O} = A \quad (A + (-A) = \mathbb{O})$

Matrix mult.

$A = m \times n \qquad B = n \times p$

$AB = m \times p$

$C = (c_{ij})$ then

$c_{ij} = (i^{th} \text{ row of } A) \cdot (j^{th} \text{ column of } B)$

- Multiplying matrix by a ~~vector~~ column vector

Gives a function $F: \mathbb{R}^n \to \mathbb{R}^m$

$F(\vec{x}) = A\vec{x}.$

- $A\vec{e}_i = A_i =$ i^{th} column vector

Properties of Matrix Mult.

- Not Commutative
- It is Associative

Easy properties

$$A(B+C) = AB + AC$$

Similarly in other direction

$c \in \mathbb{R}$

$$(cA)(B) = c(AB)$$

or $(A)(cB) = c(AB)$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{\ }$ $\begin{bmatrix} 0 \end{bmatrix}$

$$e_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{positio}$$

$$A\vec{e}_i = A_i = i^{th} \text{ column vector of } A$$

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Matrix Mult is associative

Show $(AB)C = A(BC)$

Enough to show it for $C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_j$

$(AB)e_j = j^{th}$ column vector of AB

component in row i = $(i^{th} \text{ row } A)(j^{th} \text{ colum of } B)$

is ij entry of (AB) $= \sum a_{ik} b_{kj}$

$$A(Be_j) = A(\underset{j^{th} \text{ column of } B}{B_j}) = b_{1j}A_1 + \quad + b_{nj}A_n$$

i^{th} component $= b_{..} a_{i1} + \cdots + b . a.$

$$ij \quad n1 \qquad nj \quad in$$

$$= a_{i1} b_{1j} + \qquad + a_{in} b_{nj}$$

$$= \sum_k a_{ik} b_{kj}$$

- Definitions linear combination, span, linear dependent, lin independent

- Determine if $\vec{v}_1, \dots \vec{v}_n$ are lin. dep. or ind. Write $\vec{v}_i$ as column vectors $A_1, \dots A_n$ of matrix A

Does $A\vec{x} = \vec{0}$ have unique solution or not?

- Row reduce A

$r =$ # non zero rows
$=$ #. dep variables.
$n-r =$ independent variable

lin dep means one of vectors can be written as linear combination of others

Problem 1 $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

Row reduce $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{bmatrix}$

Prob 2 $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

Prob 3 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 3 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

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Non singular Matrix (= Invertible matrix)

linear independent vectors
(say column vectors of matrix A)

$A\vec{x} = \vec{0}$ has unique solution

<u>Q 1</u> When does $A\vec{x} = \vec{0}$ have unique solution?

<u>Q 2</u> When does $A\vec{x} = \vec{b}$ have a solution for every $\vec{b}$

$A = n \times m$ r = # of nonzero rows in row reduction

<u>Ans 1</u> $r = n$ (no independent variables)

Ans 2 $r = m$

Both Questions can have positive answer when n=m (square matrix)

Def $A = n \times n$. A is nonsingular if its column vectors are linear indep, i.e., if $A\vec{x} = \vec{0}$ has unique solution

Thm A is nonsingular then $A\vec{x} = \vec{b}$ has unique solution for every $\vec{b}$.

Homogeneous & Inhomogeneous systems

Suppose $\vec{v}$ is a particular solution to $Ax=b$

$$\{\vec{y} \mid A\vec{y} = \vec{b}\} = \{\vec{x} + \vec{v} \mid Ax = 0\}$$

Feb 2

- $\{\vec{v}_1, \dots \vec{v}_k\}$ are "linearly dependent or independent"
- $\{\vec{v}_1, \dots, \vec{v}_k\}$ "span" a vector space V.
- Def'n $\{\vec{v}_1, \dots \vec{v}_k\}$ is a basis for V if
  - $\{\vec{v}_1, \dots \vec{v}_k\}$ is lin. ind.
  - $Sp(\{\vec{v}_1, \dots \vec{v}_k\}) = V$
- Interpretation $\{\vec{v}_1, \dots \vec{v}_k\}$ is a "basis" iff each vector $\vec{b}$ can be written uniquely as a linear combination

$$\vec{b} = t_1 \vec{v}_1 + \dots + t_k \vec{v}_k$$

Now singular matrices

A is $n \times n$

A is <u>nonsingular</u> if its column vectors $\{A_1, \dots A_n\}$ are lin. independent. in $\mathbb{R}^n$

<u>Thm</u> If A is nonsingular then its column vectors span $\mathbb{R}^n$. In other words, A is nonsingular iff its column vectors form a basis for $\mathbb{R}^n$.

- <u>Invertible</u> <u>Matrices</u>

- I_n = $n \times n$ Identity matrix

If D is $m \times n$ Then

$$DI_n = D \quad \& \quad I_m D = D$$

$n \times n$ <u>Def'n</u>

- A is invertible $\Leftrightarrow$ there is $n \times n$ matrix B s.t. $AB = I_n$ (or if there is C

C, s.t. $CA = I_n$)

$B = A^{-1} = C$

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what are the column vectors of B?

<u>Thm</u> A is nonsingular $\iff$ A is invertible

<u>Pf</u> A nonsingular. Column vectors of B are unique solutions to $AB_i = e_i$

These give $AB = I_n$

Since I_n is nonsingular B is nonsingular. So B has inverse C, i.e, $BC = I$

$C = (AB)C = A(BC) = A$

$\therefore \; BA = I_n$

Find the Inverse

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 7 \end{pmatrix}$$