March 29

Eigenvectors, Eigenvalues, (2x2) determinants.

<u>Def'n</u> If $F : V \to V$ is a linear transformation a nonzero vector v is <u>eigenvector</u>

$$F(v) = cv$$

where c is scalar

$c =$ an eigenvalue.

A a square matrix

$$x \to Ax \qquad \mathbb{R}^n \to \mathbb{R}^n$$

v is eigenvector for A if $Av = cv$

$$A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 \\ 6 & -2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)(-2-\lambda) + 12$$

$$= \lambda^2 - 3\lambda - 10 + 12$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 2)(\lambda - 1)$$

<u>Eigenvector for $\lambda = 1$</u> (Null space)

$$\begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\downarrow$ row reduce

$$\begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$x_2 = 1 \qquad x_1 = \frac{1}{2} \qquad \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

Eigenvector for $\lambda = 2$

$$\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix} \qquad x_1 - \tfrac{2}{3} x_2 = 0$$

$x_2 = 1 \quad x_1 = 2/3$

$$\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \leftarrow \text{eigenvector for } 2$$

Stupid example $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Ae_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 e_1$

eigenvalue eigenvector

$Ae_2 = e_2$

eigenvalue

~~(scribble)~~ λ = eigenvalue

eigenvector

$$A v = \lambda v = \lambda (I v)$$

$$(A - \lambda I)\, v = \vec{0}$$

$$v \in \mathcal{N}(A - \lambda I)$$

Point: is eigenvalue λ is such that $A - \lambda I$ has a null-space ($\neq 0$)

General Method

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$A - \lambda I$$

$$= \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc$$

① quadratic poly in λ.

Suppose λ_0 & λ_1 are solutions

② For λ_0 & λ_1 Find Null space $A - \lambda_0 I$.

March 28

16. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix}$

Eigen values for $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

3 x 3 determinants
cofactor expansion

March 30

If A is a square matrix ($n \times n$), then $\det A$ is a real number.

What to know about determinants

- $\boxed{\det A = 0 \text{ iff } A \text{ is singular}}$ (i.e. $\mathcal{N}(A) \neq \{0\}$ or columns of A are lin dep)

- $\boxed{\det(AB) = \det A \det B}$

- If A is upper triangular

$$A = \begin{pmatrix} d_1 & & * \\ & d_2 & \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \quad \text{then}$$

$$\det A = d_1 \cdots d_n$$

Consequences of $\det(AB) = \det A \det B$

$$\det(AA^{-1}) = \det(I) = 1$$

$$\det(A)\det(A^{-1}) \quad \text{so}$$

$$\boxed{\det A^{-1} = \frac{1}{\det A}}.$$

If S is nonsingular

Then $\det(S^{-1}AS) = \det(S^{-1})\det A \det(S)$

$$= \det A$$

Effect of row and column operations on det.

April 2

- Characteristic polynomial
$$p(\lambda) = \det(A - \lambda I).$$

- Roots of $p(\lambda)$ are the eigenvalues.

- If λ is an eigenvalue then $A - \lambda I$ is singular

So Null space $(A - \lambda I) \neq \{0\}$

Null space is <u>eigenspace</u> for λ.

Procedure for finding eigenspaces (or eigenvectors)

1. Calculate characteristic poly.

2. Factor it (possibly using complex numbers)

3. For each root (= eigenvalue)

find corresponding eigenspace)

Fund. Thm of Algebra.

Using complex numbers any polynomial can be factored completely

So $p(t) = a(t-\lambda_1)\cdots(t-\lambda_n)$

where λ_i's are the roots

Fact det A is product of eigenvalues of A

Pf $p(t) = \det(A - tI)$

$(\lambda_1 - t) \cdots (\lambda_n - t)$

$= (-1)^n (t-\lambda_1) \cdots (\cancel{t} - \lambda_n)$

$p(0) = \det A = (-1)^n (-1)^n \lambda_1 \cdots \lambda_n$

Repeated roots. The algebraic multiplicity of an eigenvalue is number of times it is repeated.

Geometric Multiplicity is dimension of the eigenspace

(This can be strictly less than the algebraic multiplicity)

Ex
$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$p(\lambda) = -(\lambda-2)^3$$

$$A-2I \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = 0$

x_1 is indep

dim(Null Space) = 1

Eigenspace has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 0 \\ & 0 & 2 \end{bmatrix}$$

2 is eigen

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

dim(Eigenspa

= 2

Basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

April 4

Thm Suppose $u_1, \ldots u_m$ are eigenvectors with distinct eigenvalues $\lambda_1, \ldots \lambda_m$.

Then $u_1, \ldots u_m$ are linearly independent.

Pf induction on m

$$c_1 u_1 + \cdot + c_m u_m = 0 \qquad (*)$$

Apply A

$$c_1 \lambda_1 u_1 + c_2 \lambda_2 u_2 + c_n \lambda_m u_m = 0$$

multiply $*$ by λ_1 & subtract

$$c_1(\lambda_1 - \lambda_m) u_1 + \quad + c_{m-1}(\lambda_{m-1} - \lambda_n) u_{m-1} =$$

Homework Problem

$$\begin{bmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{bmatrix}$$

$$p(t) = -(t-1)^3$$

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characteristic polynomials

$$p(t) = \det(A - tI)$$

eigenvalues are roots

λ is root (eigenvalue

$$(A - \lambda I)v = 0$$

null space of $A - \lambda I$

E_λ = "eigenspace ~~of~~ for λ"

v is in E_λ

$A v = \lambda v.$

<u>Ex</u> $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$p(t) = \det \begin{bmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{bmatrix}$

$= t^2 - (2\cos\theta)t + \cos^2\theta + \sin^2\theta$

$= t^2 - 2(\cos\theta)t + 1$

$t = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \cos\theta \pm i \sin\theta$

Complex eigenvectors:

$\lambda = \cos\theta + i\sin\theta$

$$\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}$$

null-space in $\mathbb{C}^2$ $\begin{bmatrix} 1 \\ i \end{bmatrix}$

~~If algebraic~~

alg. multiplicity of λ

= # times λ is repeated

geom mult of λ = $\dim E_\lambda$

alg mult $\geq$ geom mult

A is "defective" if
alg mult > geom mult.

Ex $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$p(t) = \det \begin{bmatrix} 2-t & 1 \\ 1 & 2-t \end{bmatrix}$$

$$= t^2 - 4t + 4 - 1$$

$$= t^2 - 4t + 3$$

$$\frac{\cancel{16}\,4 \pm \sqrt{16-12}}{2}$$

$$= (t-1)(t-3)$$

$t = 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$x_1 = -x_2$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\lambda = 3$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $x_1 = x_2$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

orthogonal

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

April 6

A matrix D is <u>diagonal</u>

$\iff$ $De_i = \lambda_i e_i$

In other words, D is diagonal

$\iff$ the eigenvectors are the standard basis vectors & eigenvalues = diagonal entries

Suppose A has eigenvectors $S_1, \ldots, S_n$ with eigenvalues $\lambda_1, \ldots \lambda_n$. $S = [S_1, \ldots S_n]$

$$S^{-1} A S = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = D$$

A is <u>diagonalizable</u> iff it has a basis of eigenvectors

Give you 3×3 matrix A diagonalize

<u>Step 1</u>: Find eigenvalues say $\lambda_1, \lambda_2, \lambda_3$

Step 2 For λ_i Find eigenvector, S_i

$$S^{-1} A S = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$