March 5

- Vector space & related concepts (subspace, basis, dimension, etc)

- Linear Transformation between $T: U \to V$ from vector space U to another on V

  Related ideas

  - T is one-to-one, onto, invertible

- Linear Transformation is determined by its effect on a basis $B = \{u_1, \dots u_n\}$

- Fact if $\dim U = n$, then

There is invertible linear transformation $T: U \to \mathbb{R}^n$ defined by $T(u_i) = e_i$ U is isomorphic $\mathbb{R}^n$

$U \cong \mathbb{R}^n$

(T is invertible because $T^{-1}: \mathbb{R}^n \to U$ defined by $T^{-1}(e_i) = u_i$

- nullity + rank = $\dim U$

- Coordinates: B = basis $T: U \to \mathbb{R}^n$

$$[u]_B = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \quad \text{where}$$

$$u = t_1 u_1 + \dots + t_n u_n$$

- Matrix representation of

$$S : U \longrightarrow V$$

basis B Basis C

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[d, "\cong"'] & V \arrow[d, "\cong"] \\ \mathbb{R}^n \arrow[r, "Q"] & \mathbb{R}^m \end{tikzcd}$$

$$Q\,[u]_B = [S(u)]_C$$

i.e. $[S(u_i)]_C = Q_i$ = column

$$S(u_i) = t_1 v_1 + \dots t_m v_m$$

$$Q_i = \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}$$

Change of Basis Matrix

= transition matrix

Example 3 (p. 422)

$S : \mathcal{P}_2 \to \mathcal{P}_3$

$S(f) = x^2 f'' - 2f' + xf$

$C = \{1, x, x^2\}$ Basis for $\mathcal{P}_2$

$D = \{1, x, x^2, x^3\}$

S is rep by 4×3 matrix Q

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$= a_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -4 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2a_1 \\ a_0 - 4a_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 + 2a_2 \\ a_2 \end{bmatrix}$$

March 7

Orthogonal Projection

u vector in $\mathbb{R}^n$

$$\mathrm{proj}_u(v) = \frac{v \cdot u}{|u|^2} u$$

$\shortparallel$

$T(v)$ linear transformation

What is matrix representation of T?

Say $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

V, W vector spaces

$B = \{v_1, \dots v_n\}$ basis ~~for~~ V

$C = \{w_1, \dots w_m\}$ basis for W

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "F_B"'] & W \arrow[d, "F_C"] \\ \mathbb{R}^n \arrow[r, dashed, "Q"] & \mathbb{R}^m \end{tikzcd}$$

- i^{th} column of Q is

  $[T(v_i)]_C$

Change of basis, transition matrix $V = W$

$T =$ identity transformation

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ \mathbb{R}^n \arrow[rr, "Q"] & & \mathbb{R}^n \end{tikzcd}$$

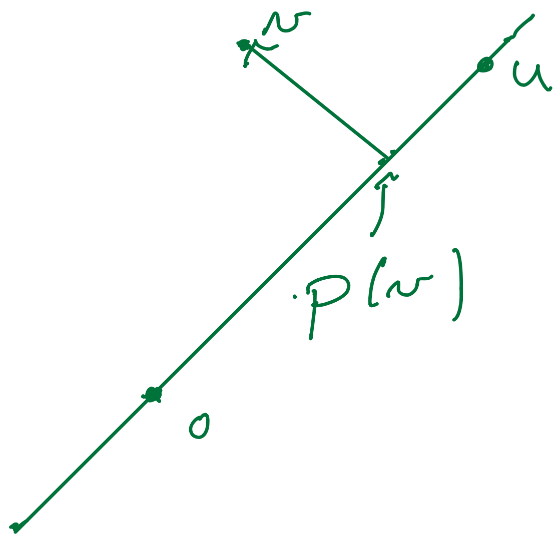
$$[v]_B \to [v]_C$$

March 19

Orthogonal Projection onto a subspace of $\mathbb{R}^n$

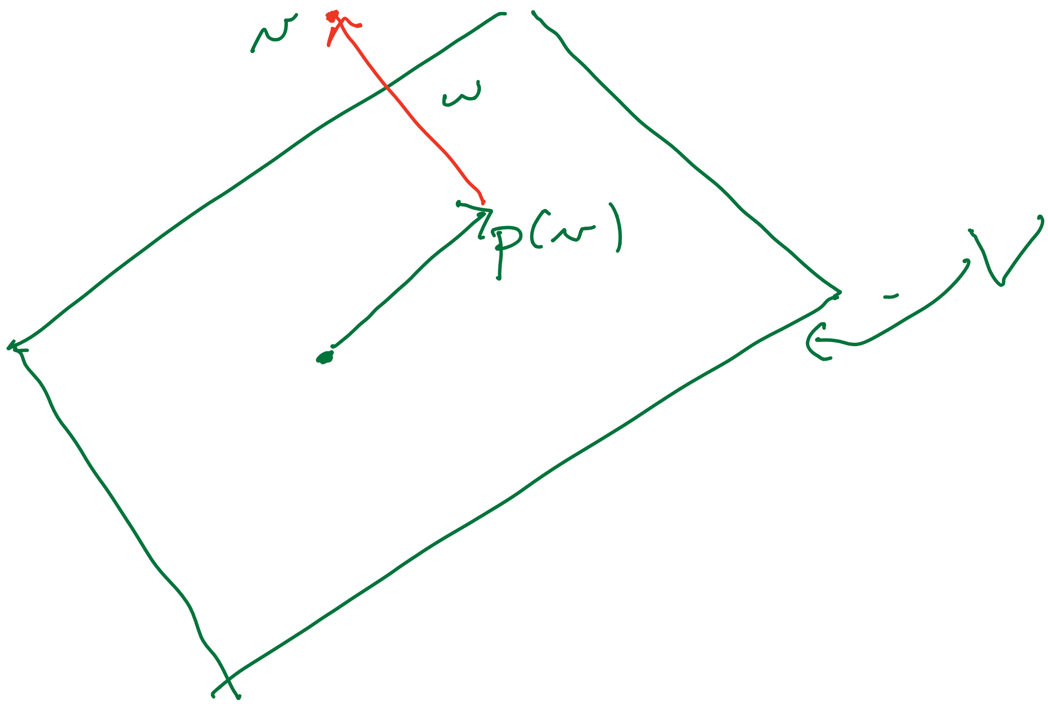
$V \subset \mathbb{R}^n$. Define $p : \mathbb{R}^n \to V$

$p(v) =$ closest pt in V to v



Projection on a line

$$p(v) = \frac{(v \cdot u)}{|u|^2} u$$



Orthogonal Basis and

Orthonormal Basis

$\{u_1, \dots, u_n\}$ is orthogonal

basis if $u_i \cdot u_j = 0$ $i \neq j$

and $u_i \cdot u_i > 0$ (i.e $u_i \neq 0$)

Any such subset is a basis

orthonormal basis means,

in addition, $u_i \cdot u_i = 1$

<u>Thm</u> If $\{u_1, \dots u_n\}$ is orthonormal basis for $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, Then

$$v = (v \cdot u_1)\vec{u}_1 + \dots + (v \cdot u_n)\vec{u}_n.$$

Remark. The standard basis is orthonormal

Proof of Theorem. Since $\{u_1, .. u_n\}$ is basis

$$v = c_1 u_1 + \dots + c_n u_n$$

Then

$$c_1 = (v \cdot u_1)$$

$$\vdots$$

$$c_n = (v \cdot u_n)$$

why? $v \cdot u_i = c_i$ $\square$

$L = \{ t v_1$

$n = 2$

p = projection onto line spanned by $\vec{v_1}$

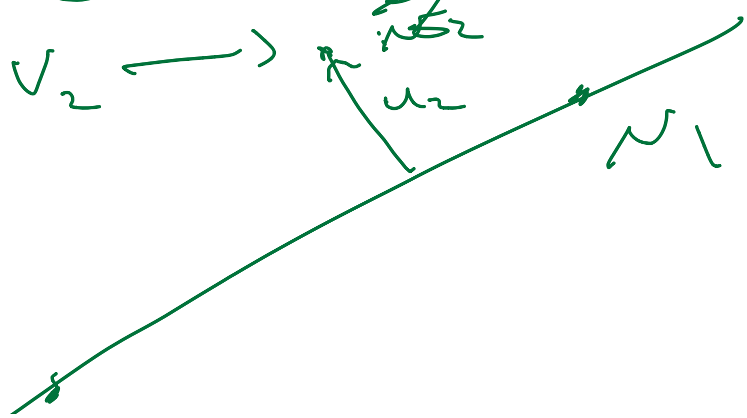
$p : \mathbb{R}^2 \to L$

Given $\vec{v_1}$ and another vector $\vec{v_2}$

Convert into orthogonal basis $\{u_1, u_2\}$

$u_1 = v_1$

u_2 is $\perp$ to u_1



write v_2 as sum of vector in same direction as v_1 & one perpendicular to v_1

$$u_1 = \frac{v_2 \cdot u_1}{|u_1|^2} v_1$$

$$= p(v_2)$$

$$u_2 = v_2 - u_1$$

<u>Conclusion</u> to convert ~~$u_1, \dots$~~ $\{v_1, v_2\}$ to orthogonal basis

" $p(v_2)$

$$u_1 = v_1$$

$$u_2 = u_1 - \text{proj}_{v_1}(v_2)$$

Check $u_1 \cdot u_2 = 0$?

$$\underbrace{\left[v_2 - \left(\frac{u_1 \cdot v_2}{u_1 \cdot u_1}\right) u_1 \right]}_{u_2} \cdot u_1 = 0$$

$$\cancel{= u_1 \cdot u_2 - \left(\frac{u_1 \cdot v_2}{u_1 \cdot u_1}\right) u_1 \cdot u_2}$$

$$u_1 \cdot v_2 - \left(\frac{u_1 \cdot v_2}{(u_1 \cdot u_1)}\right)(u_1 \cdot u_1)$$

Given basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$.

$$u_1 = v_1$$

Pick u_2 s.t. $u_2 \perp u_1$

$$\operatorname{Span}\{u_1, u_2\} = \operatorname{Span}\{v_1, v_2\}$$

$$u_2 = v_2 - p_{u_1}(v_2)$$

$$u_3 = v_3 - P_{Sp(u_1, u_2)}(v_3)$$

$$= v_3 - \left(\left(\frac{v_3 \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{v_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2 \right)$$