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DIFFERENTIABLE MANIFOLDS[1]

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Introduction

The main purpose of this paper is to provide tools of a purely analytic character for a general study of the topology of differentiable manifolds, and maps of them into other manifolds. A differentiable manifold is generally defined in one of two ways; as a point set with neighborhoods homeomorphic with Euclidean space E_n, coördinates in overlapping neighborhoods being related by a differentiable transformation,[2] or as a subset of E_n, defined near each point by expressing some of the coördinates in terms of the others by differentiable functions.[3]

The first fundamental theorem is that the first definition is no more general than the second; any differentiable manifold may be imbedded in Euclidean space. In fact, it may be made into an analytic manifold in some E_n. As a corollary, it may be given an analytic Riemannian metric. The second fundamental theorem (when combined with the first) deals with the smoothing out of a manifold. Let f be a map of any character (continuous or differentiable, without an inverse) of a differentiable manifold M of dimension m into another, N, of dimension n. (Either manifold might be an open subset of Euclidean space.) Then if $n \geqq 2m$, we may alter f as little as we please, forming a regular map F. (A map is *regular* if, near each point, it is differentiable and has a differentiable inverse.) Moreover, if $n \geqq 2m + 1$, F may be made (1-1). We show in Theorem 6 that if $n \geqq 2m + 2$, then any two regular maps f_0, f_1 of M into E_n are equivalent, in the following sense. $f_0(M)$ may be deformed into $f_1(M)$ by maps $f_t (0 \leqq t \leqq 1)$ so that the path crossed by the manifold is the regular map of an $(m + 1)$-dimensional manifold. Moreover, if $n \geqq 2m + 3$, and $f_0(M)$ and $f_1(M)$ are non-singular, so is the $(m + 1)$-manifold.

A fundamental unsolved problem is the following: *Can any analytic manifold be mapped in an analytic manner into Euclidean space?*[4]

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[1] Presented to the Am. Math Soc. Sept. 1935. An outline of the paper will be found in Proc. Nat. Ac. of Sci., vol. 21 (1935), pp. 462–463.

[2] Differentiable manifolds have been studied for instance by O. Veblen and J. H. C. Whitehead, *The foundations of differential geometry*, Cambridge Tracts, 1932. An example of a differentiable (in fact, analytic) manifold is the manifold of k-planes through a point in n-space. See §24.

[3] Manifolds in E_n which are defined by the vanishing of a set of differentiable functions are of a special character; see H. Whitney, *The imbedding of manifolds* $\cdots$, in the October 1936 issue of these Annals.

[4] This seems quite probable. It is proved for some special analytic manifolds in §§23–24.

Theorem 1 shows only that there is a differentiable map (with all derivatives), such that the resulting point set forms an analytic manifold.

Many portions of the proofs are based on the Weierstrass approximation theorem, if the manifolds are closed; if they are open, this theorem must be replaced by a corresponding theorem on functions defined in open sets. This and other theorems which will be useful may be found in a previous paper.[5] In proving both fundamental theorems, the following method is used continually. Let f be a differentiable map of M into E_n, and let U be a small portion of M. We consider a class S of maps f' of U into E_n which approximate to f in U; S forms a part of a Euclidean space. The maps f' we do not wish are characterized by subsets of S whose dimensions may be learned,—we use here the notion of "k-extent" of a set similar to a definition of Carathéodory.[6] We find a desirable map f' in U, and do the same in other neighborhoods until we have found F in the whole manifold M.

The arrangement of the paper is indicated by the sentences introductory to each part.

Contents

I. Definitions and Preliminary Results

In this part we collect definitions and facts which will be used constantly in what follows. In one section, §4, we assume a knowledge of the imbedding theorem and of Lemma 23.

1. **Manifolds of class C^r.** By an *m-dimensional manifold M of class C^r*, or, a *C^r-m-manifold* (r finite or $r = \infty$),[7] we shall mean a system composed of a set of points, which we shall also call M, and certain maps, as follows: Let $Q = Q_m$ be the interior of the unit $(m-1)$-sphere in Euclidean space E_m.[8] Let $\theta_1, \theta_2, \cdots$ be a finite or denumerable number of (1-1) maps of Q into M. Define the sets of points

$$U_i = \theta_i(Q), \qquad U_{ij} = U_{ji} = U_i \cdot U_j, \qquad Q_{ij} = \theta_i^{-1}(U_{ij}), \tag{1.1}$$

[5] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Am. Math. Soc., vol. 36 (1934), pp. 63–89. We refer to this paper as AE.

[6] C. Carathéodory, *Über das lineare Mass von Punktmengen*, Gött. Nachr., 1914, p. 426.

[7] We always suppose r is finite and > 0 unless otherwise stated. However, the results of §§1–4 all hold for $r = \infty$.

[8] That is, the space of all ordered sets of n real numbers.

and the (1-1) maps

$$h_{ij}(x) = \theta_j^{-1}(\theta_i(x)) \text{ in } Q_{ij}. \tag{1.2}$$

We make four assumptions:

(α) The maps θ_i cover M: For each p in M there is an i and an x in Q such that $\theta_i(x) = p$.

(β) The Q_{ij} are open.

(γ)[9] There are no i, j, and sequence of points $\{x^k\}$ such that

$$x^k \text{ is in } Q_{ij}, \qquad x^k \to x \text{ in } Q - Q_{ij}, \qquad h_{ij}(x^k) \to x' \text{ in } Q - Q_{ji}.$$

(δ) $h_{ij}(x)$ (if defined) is of class C^r (see §2), and if $r > 0$, it has a non-vanishing Jacobian.

If, further, the h_{ij} are analytic, we say M is *analytic*. A 0-manifold consists of a finite or denumerable number of isolated points. A k-manifold ($k < 0$) contains no points.

We call the θ_i the *maps defining* M, and the U_i, *neighborhoods* in M. Note that M *need not be connected*.

We define *limit points* in M as follows: $p_k \to p$ if and only if there are an i, an s, a point x of Q and a sequence $\{x^k\}$ of points of Q such that

$$\theta_i(x^k) = p_{s+k}, \qquad \theta_i(x) = p, \qquad x^k \to x. \tag{1.3}$$

We prove two facts. *If $p_k \to p$ and p is in U_j, then for some s, p_{s+k} is in U_j and $\theta_j^{-1}(p_{s+k}) \to \theta_j^{-1}(p)$.* For, say (1.3) holds. Then p is in U_{ij}, and hence $x = \theta_i^{-1}(p)$ is in Q_{ij}. As Q_{ij} is open and $x^k \to x$, there is an s such that x^{s+k} is in Q_{ij}. Hence $h_{ij}(x^{s+k}) = \theta_j^{-1}(p_{s+k})$ is in Q_{ji}, and as h_{ij} is continuous, $\theta_j^{-1}(p_{s+k}) \to \theta_j^{-1}(p)$. *If $p_k \to p$ and $p_k \to p'$, then $p = p'$.* For suppose $p \neq p'$, (1.3) holds, and similar relations hold with i, p, x^k, x replaced by j, p', x'^k, x'. (We may evidently take $s' = s$.) Then $h_{ij}(x^k) = x'^k$. $\cdot$ x is not in Q_{ij}; for if it were, then $h_{ij}(x) = x'$ as h_{ij} is continuous, and

$$p' = \theta_j(x') = \theta_j(h_{ij}(x)) = \theta_i(x) = p.$$

Similarly x' is not in Q_{ji}. But this contradicts (γ).

These two facts show that the obvious criteria for p_k not $\to p$ hold: If p is in U_i and there are no $\{x^k\}$, x such that (1.3) hold, then p_k does not $\to p$; if $\theta_i(x^k) = p_k$, $x^k \to x$ in Q, and $\theta_i(x) \neq p$, then p_k does not $\to p$. We can now define *open, closed, compact* sets, etc. as usual. A manifold M is closed or open according as the set of points M is compact or not.

Note that *any manifold of class C^r is of class C^s for $s < r$; any open subset of a manifold is a manifold* (with suitably chosen maps); *a finite or denumerable number of manifolds together form a manifold. E_n is a manifold with the obvious maps.*

Given two sets of maps in M, we say they define *C^r-equivalent manifolds* if not only each set separately, but also the two sets together, satisfy the above condi-

[9] (β) and (γ) correspond to (C^1) and (C^2) in Veblen and Whitehead, loc. cit.

tions. We may speak of a point being the identical point on both manifolds, of a function being identical on them, etc. Two manifolds are *C^r-homeomorphic* if there is a (1-1) correspondence between their points such that, on identifying corresponding points, the two sets of maps define C^r-equivalent manifolds. In other words there is a (proper) (1-1) regular C^r-map of either one into the other (see §2). It is often convenient not to distinguish between two C^r-equivalent manifolds. We may then say for instance "choose maps defining M such that $\cdots$." The number r should then be definitely associated with the manifold; for a C^r-manifold may be C^r-equivalent to a C^s-manifold with $s > r$ (see Theorem 1).

2. **Functions defined in manifolds.** We shall use the words "function" and "map" interchangeably. Let R be an open set in E_m, and let f be a real-valued function defined in R. It is *of class C^r* if it has continuous partial derivatives through the r^{th} order. If R is any subset of E_m,[10] we say f is *of class C^r in the subset R of E_m* if its definition can be extended through an open set containing R so that it is of class C^r there (see AE). It is sufficient that the extension be possible separately about each point of R. If the values of f are points of E_n, we say it is of class C^r (or of class C^r in a subset) if each of its coördinates is.

Suppose f is a function defined in a subset R of a C^r-manifold M, with values in a C^s-manifold N. Let θ_i, χ_j; U_i, V_j; Q_m, Q_n be the maps etc. defining M and N respectively.[11] Take any p_0 in R, and say p_0 is in U_i, $q_0 = f(p_0)$ is in V_j. The function $f_i(x) = f(\theta_i(x))$ is defined in the set $R_i = \theta_i^{-1}(R \cdot U_i)$. Suppose that, for some neighborhood U of $x^0 = \theta_i^{-1}(p_0)$ in Q_m, x in $R_i \cdot U$ implies $f_i(x)$ in V_j; set $f_{ij}(x) = \chi_j^{-1}(f_i(x))$, and suppose that f_{ij}, defined in $R_i \cdot U$ and with values in Q_n, is of class C^t ($t \leqq r, s$). If this is true of each p_0 and each corresponding i, j, we say f is *of class C^t in R*, or in the subset R of M, if R is not open. (If the condition is satisfied at p_0 for one pair (i, j), it is satisfied at p_0 for each such pair (i, j), on account of (δ).) If M and N are analytic, f may be analytic.

Suppose M is of class C^r, $r \geqq 1$. Let x be a point of Q, and let $C_1, \cdots, C_\mu$ be differentiable curves ending at x, whose tangents at x form a set of independent vectors. If x is in Q_{ij}, then parts of these curves, and the vectors, transform under h_{ij} into other such curves and vectors in Q; by (δ), the new vectors are independent. The corresponding curves in M we shall say define a set of *independent directions* at $\theta_i(x)$. If f is a C^1-map of M into N, these curves, and hence "directions," go into curves and directions in N. We define: f is a *regular*[12] map of M into N if it is of class C^1, and any set of independent directions at a point in M goes into such a set in N. If f is defined in a subset of M, we say it is regular if its definition can be extended through an open subset of M so that it is regular there.

[10] This case does not occur in the fundamental theorems. In this connection, see also H. Whitney, *Differentiable functions* $\cdots$, Trans. Am. Math. Soc., vol. 40 (1936).

[11] We shall always use these symbols in this manner.

[12] In Veblen and Whitehead, loc. cit., it is also required that the map be (1-1).

The map f of M into N is *completely regular* if it is regular, and has the following property: at most two points of M go into any single point of N; if $f(p_1) = f(p_2)$, $p_1 \neq p_2$, then a set of m independent directions at p_1 together with such a set at p_2 go into a set of $2m$ independent directions at $f(p_1)$ in N. This is of course only possible (if f is not (1-1)) if $n \geqq 2m$.

Given the map f of M into N, we define the *limit set* Lf as follows: A point q of N is in Lf if there exist sequences $\{p_k\}$ in M and $\{q_k\}$ in N such that $q_k \to q$, $f(p_k) = q_k$, and the sequence $\{p_k\}$ has no limiting point in M. The map f is *proper* if $f(M)$ does not intersect its limit set: $f(M)\cdot Lf = 0$. If M and N are C^r-m- and C^r-n-manifolds, and f is a (1-1) regular proper C^r-map of M into N, we shall say f *C^r-imbeds* M in N. $f(M)$ is then a C^r-m-manifold in N (see §3).

3. **Manifolds in manifolds.** If f is a regular C^r-map of M into N, we shall call the combination (M, f) a *local C^r-manifold* in N. Each point p of M is then in a neighborhood U in which f is (1-1). In general, the nature of M is determined by the nature of the point set $f(M)$; but this is not necessarily the case. We shall commonly speak of $f(M)$ as a local manifold in N, keeping in mind that M and f must both be given. (But see below.) The *limit set* $Lf(M)$ of $f(M)$ is the limit set Lf. The local manifold is *proper* if f is; has at most *regular singularities* if f is completely regular; is *non-singular* if f is (1-1). (M, f), or the resulting point set $f(M)$, is a *C^r-manifold in N* if it is non-singular and proper. If $f(M)$ is a local manifold in N, and f is of class C^r, we shall say M is *C^r-homeomorphic* with $f(M)$.

We shall show now that if (M, f) is a C^r-manifold in N, then, using the point set $f(M)$ alone, we may determine a C^r-manifold M', which is necessarily C^r-homeomorphic with M. This justifies calling the point set $f(M)$ a manifold in N. Also, setting $N = E_n$, we justify our original definition of a manifold. (See also Theorem 1.) Moreover, as M and $f(M)$ are C^r-homeomorphic, there is in general no harm in identifying them. This justifies the phrase "the manifold M in N."

LEMMA 1. *Let (M, f) be a C^r-m-manifold in the C^r-n-manifold N. Then the subset $f(M)$ of N has the following property. Any q_0 in $f(M)$ is in a neighborhood U in $f(M)$*[13] *such that U is in some V_j, and the points $\chi_j^{-1}(U)$ are given by*

$$y_{m+k} = \psi_k(y_1, \cdots, y_m) \qquad (k = 1, \cdots, n - m) \tag{3.1}$$

(*if $y_1, \cdots, y_m$ are suitably chosen rectangular coördinates in E_n), where $(y_1, \cdots, y_m)$ runs through an open set in the $(y_1, \cdots, y_m)$-plane. Moreover, if M' is any subset of N with the above property, then maps in M' determined by* (3.1) *make M' a C^r-m-manifold; if, further, $M' = f(M)$, then M is C^r-homeomorphic with M'.*[14]

Given M, f and q_0, say $q_0 = f(p_0)$; we may take a neighborhood U^* of p_0 in M such that U^* is in some U_i and $U = f(U^*)$ is in some V_j (as f is continuous).

[13] That is, q_0 is in U which is in $f(M)$, and U is open in $f(M)$, i.e., no q in U is the limit (in N) of a sequence of points of $f(M) - U$.

[14] It is easily seen that the lemma holds if f is completely regular and proper.

Let $(x_1, \cdots, x_m)$ be rectangular coördinates in E_m. $f_{ij} = \chi_j^{-1} f \theta_i$ maps the open subset $\theta_i^{-1}(U^*)$ of Q_m into Q_n. As f is regular, the matrix $\| \partial y_k / \partial x_l \|$ of partial derivatives of f_{ij} is of rank m at $\theta_i^{-1}(p_0)$; we may suppose that the first determinant is $\neq 0$. Then, taking U^* small enough, (3.1) holds. To show that U is a neighborhood of q_0 in $f(M)$, take any q in U, and suppose there is a sequence $\{q_k\}$ in $f(M) - U$, $q_k \to q$. Say $q = f(p)$, $q_k = f(p_k)$. Suppose there were a subsequence $\{p_{\nu_k}\}$ of $\{p_k\}$ such that $p_{\nu_k} \to p'$ in M. Then $f(p_{\nu_k}) = q_{\nu_k} \to f(p')$, hence $f(p') = q$, and $p' = p$, as f is (1-1). As $p_{\nu_k} \to p$ in U^*, there is an s such that $p_{\nu_{s+k}}$ is in U^* (see §1). But then $q_{\nu_{s+k}}$ is in U, a contradiction. Therefore $\{p_k\}$ has no limit in M. But then f is not proper, again a contradiction.

Next suppose that M' is a subset of N defined by equations (3.1). In each such equation, the domain of definition $R = (y_1, \cdots, y_m)$ is open; we may cover R by spheres, and map Q_m into each sphere and hence into M', defining maps in M'. Using the fact that the h_{ij} in N are of class C^r with non-vanishing Jacobian, it is easily shown that the same is true for the maps in M'. Suppose further that $M' = f(M)$. The equations $y = f_{ij}(x)$ previously considered, when solved as before, give a further set of equations (3.1) and thus another set of maps defining M'; but this set is clearly C^r-equivalent to the other, and thus defines a C^r-homeomorphism between M and M'.

LEMMA 2. *Let M be a C^r-m-manifold in N, and N, a C^r-n-manifold in the C^r-n'-manifold N'. Then M is a C^r-m-manifold in N'. We may replace "manifold" by "local manifold."*

Say f maps M into N and g maps N into N'; then $f' = gf$ maps M into N'. As f and g are regular and of class C^r, so is f'. If f and g are (1-1), so is f'; the same is easily seen to be true with "(1-1)" replaced by "proper."

4. **Functions defined in submanifolds of a manifold.** Let M be a submanifold of N, and let f be defined in one of the manifolds; we propose to study the relation of f to the other manifold. The values of f may be points of another manifold. The results of this section will be used only occasionally.

LEMMA 3. *Let M be a local C^r-m-manifold in the C^r-n-manifold N, let R be an open subset of N, and let f be of class C^r in R. Then f is of class C^r in the subset $R \cdot M$ of M.*

By this we mean, if g is the map of M into N, that $f' = fg$ is of class C^r in that open subset R' of M for which $g(R')$ is in R. Take q_0 in $R \cdot M$, and say $q_0 = g(p_0)$, p_0 in U_i, q_0 in V_j. The hypothesis is that $f^*(y) = f\chi_j(y)$ is of class C^r in a neighborhood of $y^0 = \chi_j^{-1}(q_0)$ in Q_n. As θ_i, g, and χ_j^{-1} are of class C^r, so is

$$f'(\theta_i(x)) = f^* \chi_j^{-1} g \theta_i(x)$$

in a neighborhood of $x^0 = \theta_i^{-1}(p_0)$, as required.

A converse of this lemma is

LEMMA 4. *Let M be a C^r-m-manifold in the C^r-n-manifold N, let R' be an open subset of M, and let f be of class C^r in R'. Then its definition may be* *extended throughout an open subset R of N containing R' so that it is of class C^r there.*

As R' is a C^r-m-manifold in N, we may suppose without loss of generality that $R' = M$. By Theorem 1, we may C^r-imbed N in E_ν ($\nu = 2n + 1$); then, by Lemma 2, M becomes a C^r-m-manifold in E_ν. Define f in $R(M)$ by setting $f(p) = f(H(p))$ (Lemma 23). f is now of class C^r in $R(M)$. For, let U be a neighborhood of a point p_0 in M, and let S be the product of $\theta_i^{-1}(U)$ and $E_{\nu-m}$: $z = (x, y)$, x in $\theta_i^{-1}(U)$, y in $E_{\nu-m}$, z in S. S may be considered as a subset of E_n. Setting $f'(z) = f(\theta_i(x))$, f' is obviously of class C^r in S. Let ϕ be a congruent map of $E_{\nu-m}$ into $P(p_0)$, and for any p in U, let $T_p = T_{p,P(p)}$ be the map of $P(p_0)$ into $P(p)$ of §19. Set

$$q = \psi(x, y) = T_{\theta_i(x)}\phi(y).$$

This is a C^r-map of S into E_ν. If $S' = (x, y)$ for $\| y \| <$ some α, and $\phi(O) = p_0$, ψ maps S' into part of $R(M)$. Moreover,

$$x = \theta_i^{-1}(H(q)), \qquad y = \phi^{-1}T_{H(q)}^{-1}\, q \text{ in } \psi(S');$$

hence ψ has an inverse of class C^r in $\psi(S')$. But in $\psi(S')$, by the definitions of f and f', $f(q) = f'(\psi^{-1}(q))$, which shows that f is of class C^r in $R(M)$. Set $R = R(M)\cdot N$; by Lemma 3, f is of class C^r in R.

We remark that the lemmas hold if we replace everywhere "of class C^r" by "analytic." The lemmas show that if M and N are as given, and f is defined in M, then *"f is of class C^r in M" is the same as "f is of class C^r in the subset M of N."*

5. **Admissible sets of maps in a manifold.** Let M be a C^r-m-manifold with maps θ_i. If (a) each θ_i is of class C^r in $\bar{Q}$,[14a] and (b) any compact subset of M has points in common with but a finite number of the U_i, we say the maps form an *admissible set.* If, further, Q' is the sphere concentric with Q and of half the radius, $U_i' = \theta_i(Q')$, and the U_i' cover M, we say the maps form a *completely admissible set.* Any manifold may be defined by a completely admissible set of arbitrarily small maps:

Lemma 5. *Let M be a C^r-m-manifold, and let $R_1, R_2, \cdots$ be a set of open sets covering M. Then M is C^r-equivalent to a manifold with completely admissible maps θ_i such that each $\bar{U}_i$ is in some R_j.*

Let θ_i^* be the given maps in M. Let $Q_1, Q_2, \cdots$ be the spheres of rational center and rational radius such that each $\bar{Q}_k$ is in Q. Q may be mapped into Q_k by a linear transformation ϕ_k. Set $\theta_{ki}(x) = \theta_i^*(\phi_k(x))$; then the θ_{ki} are defined in $\bar{Q}$, and obviously define a manifold C^r-equivalent to M. Call these maps χ_i, and the corresponding neighborhoods, V_i.

Set $W_i' = V_1 + \cdots + V_i$; then $\bar{W}_i'$ is compact and closed. Each point of $\bar{W}_i'$ is in some W_j', and hence $\bar{W}_i'$ is in the sum of a finite number of the W_j' and is thus in some W_k'. Hence we may pick out a finite or infinite subsequence $W_1, W_2, \cdots$ of the W_i' such that $\bar{W}_i$ is in W_{i+1} and the W_i cover M. Each

[14a] $\bar{Q} = Q$ plus limit points.

point of $\overline{W_i - W_{i-1}}$ is in a $V_j' = \chi_j(Q')$ such that $\bar{V}_j$ lies in some R_k and has no points in W_{i-2}; a finite number of these V_j' may be chosen such that they cover $\overline{W_i - W_{i-1}}$. Choose such neighborhoods for each i; the corresponding maps, arranged in a sequence $\theta_1, \theta_2, \cdots$, obviously have the required properties.

6. **Approximations to functions defined in manifolds.** Let M and N be C^r-m- and C^r-n-manifolds with admissible maps θ_i, χ_j, and let $f(p)$ be a C^r-map of M into N. Take any p_0 in M, and say p_0 is in U_i, $f(p_0)$ is in V_j. Then $f_{ij}(x) = \chi_j^{-1} f\theta_i(x)$ is a C^r-map of part of Q_m into Q_n, with derivatives

$$D_k f_{ij}(x) = \frac{\partial^{k_1 + \cdots + k_m}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f_{ij}(x) \qquad (\sigma_k \leqq r),$$

where $\sigma_k = k_1 + \cdots + k_m$. Each $D_k f_{ij}(x)$ is a vector function defined in part of Q_m. Let $\eta(p)$ be a positive continuous function in M, and let $F(p)$ be another C^r-map of M into N. We shall say *F approximates to f in M through the s^{th} order with an error $< \eta$*, or, *F approximates (f, M, s, η)*, if the following is true: For any point p in any U_i there is a j such that $f(p)$ and $F(p)$ are in V_j and

$$\| D_k F_{ij}(p) - D_k f_{ij}(p) \| < \eta(p) \qquad (\sigma_k \leqq s).$$

If $M = E_m$ and $N = E_n$, this reduces to the ordinary definition. For $s = 0$, this inequality is independent of the maps defining M.

LEMMA 6. *Given two sets of admissible maps θ_i, θ_i^* in M and two sets χ_j, χ_j^* in N, and given $f(p)$ and $\eta(p)$ as above, there is a positive continuous function $\zeta(p)$ in M such that if F approximates (f, M, s, ζ) in terms of the θ_i^* and χ_j^*, then F approximates (f, M, s, η) in terms of the θ_i and χ_j.*

Let $f_{ij}, f_{ij}^*, F_{ij}, F_{ij}^*$ be the corresponding maps of Q_m into Q_n, and set $u_{ik}(x) = \theta_k^{*-1}(\theta_i(x))$, $v_{jl}(y) = \chi_l^{*-1}(\chi_j(y))$ where defined. Now given i, j, there are numbers k, l such that near a given point

$$f_{kl}^* = \chi_l^{*-1} f\theta_k^* = v_{jl}\chi_j^{-1} f\theta_i u_{ik}^{-1} = v_{jl} f_{ij} u_{ik}^{-1};$$

hence the derivatives of f_{kl}^* of order $\leqq s$ are polynomials in those of v_{jl}, f_{ij}, and u_{ik}^{-1}, of order $\leqq s$. It is sufficient to show that all such derivatives are bounded in the neighborhood of any point p_0 of M. p_0 is in a finite number of U_i and U_k^*, and $f(p_0)$ is in a finite number of V_j and V_l^*. Each θ_i etc. is defined in a bounded closed set, and hence its derivatives are bounded; therefore the same is true of the derivatives of the above functions.

We remark that *if f is regular, completely regular, proper, or* (1-1) *regular and proper, then the same is true of any function which approximates to f through the first order closely enough.*

II. THE THEOREMS

We collect here the principal results of the paper (apart from Lemma 23). Using the first two theorems and the results of §9, the remaining theorems may be proved with little difficulty; hence we give these proofs practically in full in this part.

7. Two types of properties of maps. The purpose of this section is to explain (e) and (f) of Theorem 2. See also §9. Let M and N be C^r-m- and C^r-n-manifolds, and let f be a fixed C^r-map of M into N ($r \geqq 0$). Let $\eta(p)$ be a positive continuous function in M. We shall say a property Ω of maps of M into N is an *(f, r, η)-property* if the following is true:

(a) Ω is defined for all maps f' which approximate (f, M, r, η).

(b) There is a compact open subset W' of M such that whether any f' has the property Ω depends only on the values of f' in $\overline{W}'$.

(c) If f' approximates (f, M, r, η) and has the property Ω, then for some continuous $\eta'(p)$, $0 < \eta'(p) < \eta(p)$, any map f'' which approximates (f', M, r, η') has the property Ω.

(d) If f' approximates (f, M, r, η), then for an arbitrary continuous $\eta'(p)$, $0 < \eta'(p) < \eta(p)$, there is a map f'' which approximates (f', M, r, η') and has the property Ω.[15]

Before defining the second type of property, we shall consider certain functionals. We suppose now that M and N are analytic manifolds in E_μ and E_ν respectively.

Let $\Delta(p, q)$ be a continuous function of the pair of points p, q of M such that $\Delta(p, p) = 0$ and $\Delta(p, q) > 0$ for $p \neq q$. Let $\zeta(p)$ be a positive continuous function in M. To each C^r-map f of M into E_ν such that

$$\| D_k f(q) - D_k f(p) \| \leqq \Delta(p, q) \qquad (\sigma_k \leqq r), \tag{7.1}$$

(see §6),[16] let there correspond a map $\mathfrak{L}f$ of M into E_ν. We shall say $\mathfrak{L}$ is an *analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional* if $\mathfrak{L}f$ is analytic, and approximates (f, M, r, ζ), and, if f_1, f_2 and $f_1 + f_2$ satisfy (7.1), then[17]

$$\mathfrak{L}(f_1 + f_2) = \mathfrak{L}f_1 + \mathfrak{L}f_2 .$$

The existence of such a functional is given by Lemma 27. Note that if $\mathfrak{L}$ is an analytic linear $(M, E_\nu, r, \Delta', \zeta')$-functional, then it is an analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional for $\Delta \leqq \Delta'$, $\zeta \geqq \zeta'$.

We shall say a property Ω is an *$[f, r, \eta, \Delta, \zeta]$-property* if (a), (b) and (c) hold, and also:

(d′) There is a compact open subset W of M containing $\overline{W}'$, and there are C^r-maps $G_1, \cdots, G_h$ of M into E_ν such that $G_i(p) = O$ for p in $M - W$, with the following property. If $\mathfrak{L}$ is any analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional and if f' approximates (f, M, r, η) and satisfies (7.1), then there is an arbitrarily

[15] (c) and (d) may be phrased as follows: If $\mathfrak{S}$ is the space of maps of M into N which approximate (f, M, r, η), using a very strong topology, then those maps which have the property Ω form an everywhere dense open subset of $\mathfrak{S}$.

[16] $D_k f(p)$ shall mean some $D_k f_i(x)$, where $p = \theta_i(x)$. If p and q are both in some U_i, we shall use the same i in defining $D_k f(p)$ and $D_k f(q)$.

[17] We add points, etc., by considering them as vectors from the origin O. If $p = \Sigma \alpha_i p_i$ with $\Sigma \alpha_i = 1$, then p is independent of the choice of O.

small $\alpha = (\alpha_1, \cdots, \alpha_h)$ such that

$$f_\alpha(p) = H\mathfrak{L}[f'(p) + \sum \alpha_i G_i(p)]$$

has the property Ω. (We suppose η and ζ are so small that f_α is in $R(N)$ for $|\alpha_i| <$ some $\bar{\alpha} > 0$, and thus H is defined; see Lemma 23.)

Note that any $[f, r, \eta, \Delta, \zeta]$-property is an $[f, r, \eta, \Delta', \zeta']$-property for $\Delta' \geqq \Delta$, $\zeta' \leqq \zeta$; also, for large enough Δ, it is an (f, r, η)-property.

8. **The fundamental theorems.** We state here the two theorems on which most of the other results of the paper are based.

THEOREM 1. *Any C^r-m-manifold ($r \geqq 1$ finite or infinite) is C^r-homeomorphic with an analytic manifold in Euclidean space E_{2m+1}.*

See also Theorem 3 and footnote [32].

THEOREM 2. *Let M and N be analytic m- and n-manifolds in Euclidean spaces E_μ and E_ν respectively. Let f be a C^r-map of M into N ($r \geqq 0$ finite). Let η be a positive continuous function in M. Then there is a C^r-map F of M into N with the following properties:*

(a) *F approximates (f, M, r, η).*

(b) *If $n \geqq 2m$, then F is completely regular.*

(c) *If $n \geqq 2m + 1$, then F is* (1-1).

(d) *F is analytic.*

(e) *Let $\Omega_1, \Omega_2, \cdots$ be (f, r, η)-properties, let $W'_1, W'_2, \cdots$ be the corresponding subsets of M* (§7, (b)), *and suppose any compact subset of M has points in common with at most a finite number of the W'_i. Then F has the properties $\Omega_1, \Omega_2, \cdots$.*

(f) *For some functions $\Delta(p, q)$, $\zeta(p)$ as in* §7, *let $\Omega'_1, \Omega'_2, \cdots$* be $[f, r, \eta, \Delta, \zeta]$-*properties. Then F has these properties.*

In place of (d), (e) *and* (f) *we may have if we choose* (e′): (e) *holds without the finiteness restriction.*

If we are satisfied with a function F of class C^r, we would naturally use (e′) in place of (d), (e) and (f). Note that if f is proper in M, then we can insure that F be proper in M by taking η small enough; then, if $n \geqq 2m + 1$, (b) and (c) show that F is a homeomorphism, and thus $F(M)$ is a C^r- (or analytic) manifold in N. We might generalize the theorem by making $F = f$ at certain isolated points of M, or, if r is replaced by ∞, letting $F(p)$ approximate to $f(p)$ together with higher and higher partial derivatives as p approaches the limit set LM. (Compare AE, Theorem III.) We could also consider manifolds of different classes in different subsets; an example is given in Theorem 5.

9. **Consequences of** (e) **and** (f) **of Theorem 2.** We may give the function F various properties, either because these are of one of the two types, or because they are the logical sums of a denumerable number of such properties. We give some examples below; for the proofs that they are of the required nature, see §§34–35.

(A) and (B) are (b) and (c) of the theorem.

(C) If K is a subset of N which is the sum of at most a denumerable number of sets of zero $(n - m)$-extent (see §17), then $F(M)$ does not intersect K.

(D) If $f(M)$ and N' are local C^1-m- and C^1-n'-manifolds in N, then if $m + n' < n$, the local manifold $F(M)$ does not intersect N', while if $m + n' \geqq n$, $N^* = F(M) \cdot N'$ is a local $(m + n' - n)$-manifold in N. At each point p of N^*, there are n independent directions in N, each being parallel to $F(M)$ or to N'. If $f(M)$ and N' are non-singular, or proper, so is N^*.

10. **A further imbedding theorem; Riemannian manifolds.** We may replace E_{2m+1} by E_{2m} in Theorem 1 as follows:

THEOREM 3. *Any C^r-m-manifold M ($r \geqq 1$ finite or infinite) is C^r-homeomorphic with a proper analytic local manifold with at most regular singularities in E_{2m}.*

To prove this, let M' be a C^r-homeomorphic analytic manifold in E_{2m+1}. Let $\rho(p)(p$ in $M')$ be the smaller of (a) 1, (b) the reciprocal of the distance from p to a fixed point of E_{2m+1}, (c) the distance from p to the limit set LM' if $LM' \neq 0$. Let q_0 be a fixed point distinct from the origin O in E_{2m}, and set $f(p) = \rho(p)q_0$ in M'. This is a continuous map of M' into E_{2m} such that $f(p) \neq O$, and either $Lf(M')$ is void or $Lf(M') = O$; hence f is proper. Let F be the analytic completely regular proper map given by Theorem 2 with $N = E_{2m}$; $F(M')$ is the required local manifold.

THEOREM 4. *Any C^r-manifold M ($r \geqq 1$ finite or infinite) may be given an analytic Riemannian metric, the coefficients of the fundamental quadratic form being of class C^r in terms of the original maps defining M.*

This follows from Theorem 1 or 3 on using the ds^2 of E_{2m+1} or E_{2m}.

11. **An extension theorem.** Suppose a closed subset of a manifold M is mapped into a manifold N; can the map be extended over the rest of M so as to be differentiable? Or, M might be replaced by a manifold M with boundary B, and the closed subset, by B. An answer is given by the following theorem.

THEOREM 5. *Let A be a separable metric space, let B be a closed subset of A, and let $M = A - B$ be a C^r-m-manifold ($r \geqq 1$ finite or infinite).*[18] *Let N be a C^s-n-manifold ($s \geqq r$), and let f be a continuous map of B into N. Suppose f can be extended so as to be continuous throughout M.*[19] *Then there is a map F of A into N with the following properties:*

(a) *F is continuous in A and of class C^r in M; $F = f$ in B.*

(b) *If $n \geqq 2m$, F is completely regular in M.*

(c) *If $n \geqq 2m + 1$, F is (1-1) in M.*

Suppose, in addition, that A is a $C^{r'}$-m-manifold,[20] *$1 \leqq r' \leqq r$; f is of class C^t in the subset B of A (see §2), $1 \leqq t \leqq r'$. Then we have also*

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[18] We suppose that continuity in M agrees with continuity in A.

[19] This is always possible if $N = E_n$. See for instance Kuratowski, *Topologie* I, p. 211, or Alexandroff-Hopf, *Topologie* I, p. 76.

[20] We suppose that the maps defining M are a subset of those defining A.

(d) *F is of class C^t in A.*

(e) *If $n \geqq 2m$ and f is regular [completely regular] in B, then F is regular [completely regular] in A.*

(f) *If $n \geqq 2m + 1$ and f is regular and (1-1) in B, then F is regular and (1-1) in A.*

We might also apply (e) and (f) of Theorem 2. If the extension of f over M is proper in M [in A], we can make F proper in M [in A]. If N is analytic, we may make $f(M)$ analytic, etc., as in Theorem 2.

By Theorem 1, there are analytic manifolds M' and N' in E_μ and E_ν, C^r- and C^s-homeomorphic with M and N respectively. f[20a] gives a map f' of M' into N'. Choose $\eta(p)$ positive and continuous in M so that $\eta(p) \to 0$ as $p \to B$. Applying Theorem 2 with its r replaced by 0, we replace the extension f' over M' by a function F'; the resulting map F of M into N is of class C^r and has the properties (a), (b) and (c) (setting $F = f$ in B).

Now suppose that A is a $C^{r'}$-manifold, $1 \leqq r' \leqq r$. Let A'_1 be a $C^{r'}$-homeomorphic analytic manifold in E_μ; then M is $C^{r'}$-homeomorphic with the corresponding subset M'_1 of A'_1. Let M'' be an analytic manifold in E'_μ, C^r-homeomorphic with M. The map g of M'' into M'_1 thus defined is of class $C^{r'}$. We may choose $\eta(p)$ positive and continuous in M so that $\eta(p) \to 0$ as $p \to B$, and so that, considering $\eta(p)$ in M'', if g'' approximates (g, M'', r', η), then g'' is a homeomorphism and $g''(M'')$ does not intersect $B' = A'_1 - M'_1$. Let g'' be such a function which is analytic (Theorem 2); the resulting map g' of M into $g'(M)$ is a C^r-homeomorphism. Letting g', in B, be the map already given, g' is (1-1) in A. From Lemma 10 below it is seen that g' is of class $C^{r'}$ in A. It is regular, and taking $\eta(p)$ small enough insures that it is proper; hence *g' is a $C^{r'}$-homeomorphism in A and a C^r-homeomorphism in M.* Let $A' = g'(A)$, $M' = g'(M)$. M' (but not A') is analytic. Let N' and f' be as before.

f' is continuous in A'; considering the values of f' in B' alone, it is of class C^t in the subset B' of E_μ (Lemma 4). Suppose that we have proved Lemma 7; then there is a function F'' of class C^t in $R(A')$ (see Lemma 23) which equals f' together with derivatives of order $\leqq t$ in B', and such that $F''(M')$ is in $R(N')$.[21] Then $F''' = HF''$ (see Lemma 23) is of class C^t in A', and maps A' into N'. Define $\eta(p)$ in M' as before, and let F' be the approximation to F''' in M' given by Theorem 2. Set $F' = F''$ in B'; then F' is of class C^t in A', as is easily seen from Lemma 10. The resulting map F of A into N is of class C^t in A and of class C^r in M, and has the properties (a) through (d). The regularity condition in (e) is satisfied automatically; we obtain complete regularity if f is completely regular in B by applying the method of proof of (A) of §9 in using Theorem 2. Suppose $n \geqq 2m + 1$ and f is (1-1) in B. Then $F(B)$ is the sum of a denumerable number of sets of zero $(m + 1)$-extent, by Lemmas 13, 14 and 15, and hence we may make $F(M)$ avoid $F(B)$ by applying (C) of §9.

There remains to prove

[20a] I.e. the extension of f over M.

[21] If $N = E_\nu$, we could obviously avoid Lemma 7.

LEMMA 7. *Let A^* be a subset of the open set R in E_μ, $\bar{A}^* \cdot R = A^* \cdot R$, and let B^* be a subset of A^*, $\bar{B}^* \cdot A^* = B^* \cdot A^*$. Let $\zeta(p)$ be a positive continuous function in A^*. Let f' be a continuous map of A^* into E_ν, and let f', considered in B^* alone, be of class C^t in B^*. Then there is a C^t-map F of A^* into E_ν which equals f' in B^*, together with partial derivatives of order $\leqq t$, and approximates $(f', A^* - B^*, 0, \zeta)$.*

We may suppose that $E_\nu = E_1$, the space of real numbers. By a direct application of the method of proof of AE Lemma 3,[22] we find a function f which is continuous in R, is of class C^t in a neighborhood R' of B^* (R' in R), $= f'$ in B^*, and approximates $(f', A^* - B^*, 0, \zeta)$. Let R'' be a neighborhood of B^* such that $S = \bar{R}'' \cdot A^*$ is in R'. Set

$$A_s = E_\mu - R \qquad (-1 \leqq s \leqq t - 1),$$

$$A = A' = A_s = A_0 + B^* \qquad (s \geqq t),$$

$$B_s = A^* - S \qquad (0 \leqq s \leqq t - 1),$$

$$B = B_s = A^* - B^* \qquad (s \geqq t).$$

The conditions of AE Theorem III are seen to be satisfied; the function F given by the theorem (if $\epsilon(x)$ is small enough) has the required properties.

12. **A deformation theorem.** We first introduce some definitions. Let M and N be C^r-m- and C^r-n-manifolds, let I be the closed interval $(0, 1)$, and let I' be an open interval containing I. Let $M' = M \times I'$ be the product of M and I'; this is a C^r-$(m + 1)$-manifold with easily defined maps. Let f be a regular C^r-map of M' into N. For each p in M and t in I', set $\phi_t(p) = f(p \times t)$. Each ϕ_t is a regular C^r-map of M into N. Set $M_0 = \phi_0(M)$, $M_1 = \phi_1(M)$; these are local C^r-manifolds in N. If, given M_0 and M_1, there exist such M, I' and f, we shall say the set of maps ϕ_t, t in I, forms a *regular C^r-deformation* of M_0 into M_1. A regular deformation such that M' has at most regular singularities, we shall call *completely regular*; one in which each ϕ_t is (1-1), we call *topological*; one in which f is (1-1), we call *completely topological*. If there is such a map f which is merely continuous, it defines a *deformation* of M_0 into M_1.

THEOREM 6. *Let M_0 and M_1 be C^r-homeomorphic local C^r-m-manifolds in the C^r-n-manifold N ($r \geqq 1$ finite or infinite, $n \geqq 2m + 2$). Suppose there exists a*

[22] Let f_1 be a continuous extension of f' throughout R. Take a subdivision of $R - B^*$ as in AE §8, and define an extension f_2 of class C^t of f', considered in B^* alone, throughout R. (In using AE §9, we take for x^ν a point of B^* whose distance from y^ν is less than twice the distance from y^ν to B^*). Define the ϕ_k in $R - B^*$ as in AE. Given a neighborhood R^* of B^*, R^* in R, let ϕ_{i_k} be those functions which are $\neq 0$ somewhere in $R - R^*$, and set

$$f = f_2 + \sum \phi_{i_k}(f_1 - f_2).$$

As $\sum \phi_{i_k} = 1$ in $R - R^*$, $f = f_1$ in $R - R^*$; also $f = f_2$ in a neighborhood of B^*. For R^* small enough, f evidently approximates to f' as required.

deformation of M_0 *into* M_1 *in* N.[23] *Then there is a regular* C^r*-deformation of* M_0 *into* M_1. *If* M_0 *and* M_1 *are completely regular, so is the deformation. If* M_0 *and* M_1 *are non-singular, it is topological. If, further,* $n \geqq 2m + 3$ *and* M_0 *and* M_1 *do not intersect, it is completely topological.*[24]

We shall not consider the question of proper maps, but merely note that any map of a closed manifold is proper. The following lemma is necessary.

Lemma 8. *Let* M *be a local* C^r*-m-manifold in the* C^r*-n-manifold* N, $n \geqq 2m + 1$. *Then there is a vector function* $v(p)$ *of class* C^r *in* M *such that for each* p, $v(p)$ *is independent of the directions in* M *at* p.[25]

The lemma may be proved most simply by imbedding N in E_ν (Lemma 19), triangulating M, and defining $v(p)$ successively over the 0-cells, 1-cells, $\cdots$ of M. It may also be proved easily by the methods in this paper.

Let $v_i(p)$ be a vector function in M_i as in the lemma, $i = 0, 1$. Let us C^r-imbed N in some E_ν (Lemma 19); then $v_0(p)$ becomes a vector $v_0'(p)$ in E_ν parallel to N but independent of M at p. Set

$$g_0(p \times t) = p + tv_0'(p);$$

this gives a C^r-map of $M' = M \times I'$ into E_ν. For $t <$ some $\zeta(p)$, $g_0(p \times t)$ lies in $R(N)$ (see Lemma 23); set $g_0'(p \times t) = Hg_0(p \times t)$. Let $f = g_0'$ in M_0; g_0' also defines the derivatives of f of order $\leqq r$ in the subset M_0 of M'. By the choice of v_0, f is a regular map of the subset M_0 of M' into N (see §2). Define f similarly in M_1.

By hypothesis, there is a continuous map f' of M' into N which agrees with f in M_0 and in M_1. We now apply Theorem 5 with A, B, t replaced by M', $M_0 + M_1$, r. This gives a regular C^r-map of M' into N, and hence a regular C^r-deformation of M_0 into M_1. The other statements in the theorem follow at once from Theorem 5, except for the statement on topological deformations; we leave the proof of this to §36.

13. **Spheres bounding differentiable cells.** Let S be the unit m-sphere in E_{m+1}, let Q' be the interior of S, and let Q be the interior of a larger concentric m-sphere. Let f be a regular C^r-map of S into N; we shall call $S_1 = f(S)$ *a local* C^r*-m-sphere* in N; we leave out "local" if f is (1-1). If f may be extended throughout Q so as to be regular and of class C^r, we say S_1 C^r*-bounds regularly* the $(m + 1)$-cell $f(Q')$. If f is completely regular, or (1-1), we call the bounding *completely regular*, or *topological*. If there exists an f which is merely continuous, we say S *bounds* a cell in N.

Theorem 7. *Any local* C^r*-m-sphere* S *in the* C^r*-n-manifold* N ($r \geqq 1$, $n \geqq 2m + 2$) *which bounds a cell in* N, C^r*-bounds regularly a cell in* N. *If* S *has at*

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[23] This is of course always the case if $N = E_n$.

[24] If N is an analytic manifold in E_ν, we may make the map of M' analytic except over M_0 and M_1 (see Theorem 2). We shall strengthen the theorem in §36.

[25] The lemma holds if $N = E_n$ and $n \geqq 2m$, as will be shown in a paper on "sphere-spaces." "Vector function" means here "direction function."

most regular singularities, the bounding is completely regular; if $n \geqq 2m + 3$ and S is non-singular, the bounding is topological.[26]

The proof runs almost exactly like that of Theorem 6.

14. **Examples** (A) Let M be the interval $(-\infty, +\infty)$. Let $a_0, a_1, \cdots$ be a sequence of points dense in E_n, $n \geqq 2$. For each integer $i \geqq 0$, set $f(i) = a_i$, and let f map the interval $(i, i+1)$ into the segment $\overline{a_i a_{i+1}}$. Set $f(-t) = f(t)$. This is a continuous map of M into a subset of E_n. Set $\eta(t) = 1/(1 + |t|)$. Theorem 2 then gives, if $n = 2$, an analytic curve everywhere dense in the plane, and if $n \geqq 3$, a non-singular analytic curve dense in E_n. On applying (C) of §9, we may (say in E_3) make the curve avoid all rational lines; we may also replace the curve by a denumerable number of such curves which are non-intersecting.

(B) Let B be the unit circle in the plane, with interior M, and set $A = B + M$. Map B into the whole of the sphere S_n of dimension $n \geqq 4$. Any continuous map of B into S_n may be extended continuously over M; hence, by Theorem 2, there is an analytic regular map F of M into S taking on the given boundary values; if $n \geqq 5$, $F(M)$ is non-singular. As in (A), we may make $F(M)$ avoid sets of $(n - m)$-extent zero, may find a denumerable number of non-intersecting surfaces of this sort if $n \geqq 5$, etc.

III. The Imbedding Theorem

In this part we shall prove Lemma 19; this is Theorem 1, except for the analyticity condition. The proof of Theorem 1 will be completed in Part V. The present proof falls into two parts. We first show, in Lemma 12, practically, that Theorem 2 with the conclusions (a) and (e′) alone holds; we then show that (B) and part of (A), §9, hold. This, with the lemma, gives the imbedding theorem. We first give some lemmas of a general nature.

15. **Some general lemmas.** The lemmas which follow are mostly simple extensions of results from AE.

Lemma 9.[27] *Let $f(p)$ be a C^r-map $(r \geqq 0)$ of the open set R in E_m into E_n, and let $\eta(p)$ be positive and continuous in R. Then there is an analytic map $F(p)$ in R which approximates (f, R, r, η).*

If $n = 1$, this follows at once from AE Lemma 6. We define open sets $R_1, R_2, \cdots$ as in that lemma, and let ϵ_i be the lower bound of $\eta(p)$ for p in R_{i+1}. For the general case, we apply the lemma separately for each coördinate in E_n.

Lemma 10. *Let A be a closed subset of the C^r-manifold M, let $\eta(p)$ be positive and continuous in $M - A$, let $\eta(p) \to 0$ as p approaches any point of A, and let*

[26] Compare footnote[24].

[27] See also Lemmas 22 and 26. We may make $F(p)$ approximate to $f(p)$ to higher and higher orders as p approaches the boundary of R; see AE Lemma 6.

$f(p)$ be a C^r-map of M into the C^r-manifold N. If F approximates $(f, M - A, r, \eta)$ and $F = f$ in A, then F is of class C^r in M.

Suppose first $M = E_m$, $N = E_1$; the theorem then follows from AE Lemma 1 (see AE, end of §11). The general case is an immediate consequence of this.

LEMMA 11.[28] *Let A be a subset of the open set R in E_m, with $\bar{A} \cdot R = A \cdot R$, and let $f(p)$ be of class C^r $(r \geqq 0)$ in the subset A of R. Then the definition of f may be extended throughout R so it is of class C^r there. The values of $f(p)$ may be points of E_n.*

If $R = E_m$ and f is real-valued, the proof is given by AE Lemma 2. If $R \neq E_m$, the proof needs only a slight alteration; or we may use AE Theorem III. If the values of f are points of E_n, we apply the lemma to each coördinate separately.

16. **Maps of a manifold with given properties.** The first of the three lemmas giving the imbedding theorem is the following.

LEMMA 12. *Let f be a C^r-map of the C^r-m-manifold M into the C^r-n-manifold N, let $\eta(p)$ be positive and continuous in M, and let $\Omega_1, \Omega_2, \cdots$ be (f, r, η)-properties. Then there is a C^r-map F of M into N which approximates (f, M, r, η) and has these properties.*

It is clear that there is a sequence of positive continuous functions $\eta_i''(p)$ such that if $f_0 = f$ and f_i approximates $(f_{i-1}, M, r, \eta_i'')$, then $F = \lim f_i$ exists and approximates (f, M, r, η). We shall choose functions $\eta_{1j}(j \geqq 1)$, f_1, $\eta_{2j}(j \geqq 2)$, $f_2, \cdots$ in that order so that if $\eta_i(p)$ for each p is the smallest of $\eta_{1i}(p), \cdots, \eta_{ii}(p), \eta_i''(p)$, then f_i approximates (f_{i-1}, M, r, η_i), and $f_j(j \geqq i)$ and F have the property $\Omega_i(i = 1, 2, \cdots)$. Suppose we have found these functions through f_{i-1}. For each p in M, let $\eta_i'(p)$ be the smallest of the numbers $\eta'(p)$ of (c), §7, for the properties $\Omega_1, \cdots, \Omega_{i-1}$; then if f' approximates (f_{i-1}, M, r, η_i'), f' will have the properties $\Omega_1, \cdots, \Omega_{i-1}$. Choose $\eta_{ij}(p)(j \geqq i)$ so that if $f_{i-1}' = f_{i-1}$ and f_j' approximates $(f_{j-1}', M, r, \eta_{ij})(j \geqq i)$, then $F' = \lim f_j'$ approximates (f_{i-1}, M, r, η_i'). Define η_i, and let $f_i(p)$ approximate (f_{i-1}, M, r, η_i) and have the property Ω_i, by (d), §7. We thus find all the above functions, and the function F has the required properties.

17. **The k-extent of a set.** Let A be a subset of E_m. We shall say the *k-extent of A is finite* if there is a number G such that if $0 < \epsilon < 1$, then there are sets $A_1, \cdots, A_\nu$ of diameter $< \epsilon$ which cover A, and $\nu\epsilon^k < G$. If M is a C^1-m-manifold with admissible maps θ_i etc., and A is a subset of M, we say the *k-extent of A is finite* if A is compact (and hence is in a finite number of the U_i), and each $\theta_i^{-1}(A \cdot U_i)$ is of finite k-extent in E_m. The subset A of E_m is of *k-extent zero* if for any $\epsilon' > 0$ there is a $\delta > 0$ such that if $0 < \epsilon < \delta$, then there are sets $A_1, \cdots, A_\nu$ of diameter $< \epsilon$ covering A, and $\nu\epsilon^k < \epsilon'$. Similarly

[28] This lemma is not needed in the proof of the fundamental theorems, though it is useful in Lemma 17. When we have proved the imbedding theorem, we may show easily that the lemma holds with E_m and E_n replaced by a C^r-manifolds (see §4).

if A is a subset of M. A set of zero k-extent ($k \leqq 0$) or of finite l-extent ($l < 0$) is vacuous. These definitions (in M) are independent of the admissible set of maps defining M (see Lemma 15). The sum of a finite number of sets of finite [zero] k-extent is of finite [zero] k-extent.

LEMMA 13. *A bounded subset A of E_k has finite k-extent; if A contains inner points, then it has not zero k-extent.*

This is obvious on cutting up E_k into equal cubes of any desired size. Equally obvious is

LEMMA 14. *If A has finite k-extent, it has zero $(k + 1)$-extent.*

LEMMA 15. *If M and N are C^1-m- and C^1-n-manifolds, f is a C^1-map of M into N, and A is a subset of M[29] of finite [zero] k-extent, then $f(A)$ is of finite [zero] k-extent.*

Consider first the case of finite k-extent. A is in a finite number of the U_i, and $f(A)$ is in a finite number of the V_j; hence we can put $A = A_1 + \cdots + A_t$, A_s in $U_{i(s)}$, $f(A_s)$ in $V_{j(s)}$. Set $B_s = \theta_{i(s)}^{-1}(A_s)$; then B_s is of finite k-extent in E_m. It is sufficient to show that if $g_s = \chi_{j(s)}^{-1} f \theta_{i(s)}$, then $g_s(B_s)$ is of finite k-extent in E_n. g_s is of class C^1 in the compact set B_s; hence, for some number μ, any subset B' of B_s is mapped by g_s into a set $g_s(B')$ whose diameter is at most μ times that of B'. Set $G^* = \mu^k G$. (G corresponds to B_s.) Given ϵ, $0 < \epsilon < 1$, set $\epsilon_1 = \epsilon/\mu^k$, and divide B_s into sets $B_{s1}, \cdots, B_{s\nu}$ of diameter $< \epsilon_1$, so that $\nu\epsilon_1^k < G$. Then $g(B_{s1}), \cdots g(B_{s\nu})$ cover $g(B_s)$, these sets are of diameter $< \epsilon$, and $\nu\epsilon^k < G^*$, as required.

Consider now the case of zero k-extent. Define the B_s etc. as before. Given $\epsilon' > 0$, set $\epsilon_1' = \epsilon'/\mu^k$, and choose $\delta > 0$ so that if $0 < \epsilon_1 < \delta$, then there are sets $B_{s1}, \cdots, B_{s\nu}$ of diameter $< \epsilon_1$ covering B_s, and $\nu\epsilon_1^k < \epsilon_1'$. Now take any ϵ, $0 < \epsilon < \delta$, and set $\epsilon_1 = \epsilon/\mu$. Define the sets B_{si}; then $g(B_{s1}), \cdots, g(B_{s\nu})$ have the required properties.

18. **Transformations of one set away from another.** Let R and R' be open subsets of E_m and E_h, and for each $\alpha = (\alpha_1, \cdots, \alpha_h)$ in R' let T_α be a C^r-map of R into E_n. Let $x' = T_\alpha(x)$ be of class C^1 in terms of the $n + h$ variables $(x, \alpha)(x \text{ in } R)$. If for each x in R and α in R' the vectors

$$\partial x'/\partial\alpha_1, \cdots, \partial x'/\partial\alpha_h$$

are independent, we shall say the T_α form an *h-parameter family* of C^r-maps. We may also define such families of maps of one manifold into another in an obvious manner; Lemma 16 holds for such families also.

LEMMA 16. *Let $T_\alpha(x)$ be an h-parameter family of C^1-maps of R into E_n, and let A and B be closed sets in R and E_n of finite k-extent and zero $(h - k)$-extent respectively. Then for some α in R', $T_\alpha(A)$ does not intersect B.*[29a]

We may suppose that $0 \leqq k \leqq h$. For some α^0 in R' and $\eta > 0$, all points α

[29] Note that M may be replaced by any open subset R of M which contains A. It is evidently sufficient that the map satisfy a Lipschitz condition.

[29a] We may evidently interchange "finite" and "zero".

within η of α^0 are in R'. Take $\xi \geqq 1$ so that if A^*, in A, is of diameter $< \epsilon$, then $T_{\alpha^0}(A^*)$ is of diameter $< \xi\epsilon$. The condition that the $\partial x'/\partial \alpha_i$ are independent for x in A and $\alpha = \alpha^0$ is equivalent to the condition that for some $\beta > 0$, any directional derivative for x in A and $\alpha = \alpha^0$ is a vector of length $\geqq \beta$. Hence we may take ζ, $0 < \zeta < \eta$, such that for any $\epsilon > 0$, if A^* and B^* are subsets of A and B of diameter $< \epsilon$, α and α' are points of R' within ζ of α^0, and the distance from α to α' is $> 4\xi\epsilon/\beta$, then either $T_\alpha(A^*) \cdot B^* = 0$ or $T_{\alpha'}(A^*) \cdot B^* = 0$.

Let G be the number corresponding to A, and take $\epsilon' \leqq (\beta\zeta)^h/[2^h(4\xi)^h G]$. Find $\delta < 1$ corresponding to B and ϵ'. Choose an ϵ, $0 < \epsilon < \delta$. Then we may set

$$A = A_1 + \cdots + A_\nu, \qquad B = B_1 + \cdots + B_\sigma,$$

so that the A_i cover A, the B_j cover B, the diameter of each A_i and each B_j is $< \epsilon$, and so that

$$\nu\epsilon^k < G, \qquad \sigma\epsilon^{h-k} < \epsilon'.$$

Let Z_{ij} be the set of all points α within ζ of α^0 such that $T_\alpha(A_i) \cdot B_j \neq 0$; then, by the choice of ζ, the diameter of Z_{ij} is $< 4\xi\epsilon/\beta$, and hence its ordinary h-volume is $< (4\xi\epsilon/\beta)^h$. Therefore the ordinary h-volume of all the Z_{ij} is less than

$$\nu\sigma(4\xi\epsilon/\beta)^h = (4\xi)^h(\nu\epsilon^k)(\sigma\epsilon^{h-k})/\beta^h < (4\xi)^h G\epsilon'/\beta^h \leqq \zeta^h/2^h.$$

Hence there is an α within ζ of α^0 which is in no Z_{ij}; for this α, $T_\alpha(A)$ does not intersect B.

19. **The transformation $T_{p,P}$.** Let P_0 be a fixed k-plane through a fixed point p_0 in E_n. Corresponding to any k-plane P not perpendicular to P_0 and any point p of P we may let correspond a non-homogeneous orthogonal transformation of space $T_{p,P}$ which carries p_0 and P_0 into p and P, and is analytic in p and P (see §24), for instance as follows. Let P_0' and P' be the parallel planes through the origin. Let $v_1, \cdots, v_n$ be mutually orthogonal unit vectors such that $v_1, \cdots, v_k$ are in P_0'. The following rules determine Tv_i. The Tv_i are mutually orthogonal unit vectors. If $1 \leqq i \leqq k$ and P_i' is the plane in P' determined by the projections $v_1', \cdots, v_i'$ of $v_1, \cdots, v_i$ into P', then $Tv_1, \cdots, Tv_i$ are in P_i' and determine the same orientation in P_i' as $v_1', \cdots, v_i'$; for each j $(j = k+1, \cdots, n)$, if P_j' is the plane determined by P' and $v_{k+1}, \cdots, v_j$, then $Tv_{k+1}, \cdots, Tv_j$ are parallel to this plane, and $Tv_1, \cdots, Tv_j$ determine the same orientation in it as $Tv_1, \cdots, Tv_k, v_{k+1}, \cdots, v_j$. $T_{p,P}$ is the unique linear transformation carrying p_0 into p and v_i into Tv_i.

20. (1-1) **maps and properties.** We can now prove

LEMMA 17. *If f is a regular C^r-map ($r \geqq 1$) of the C^r-m-manifold M into E_n, $n \geqq 2m + 1$, and η is a sufficiently small positive continuous function in M, then the property of maps f' which approximate (f, M, r, η) of being* (1-1) *and avoiding* *a fixed point b_0 is the logical sum of a denumerable number of (f, r, η)-properties.*[30]

Let $\theta_1, \theta_2, \cdots$ be a completely admissible set of maps defining M; we choose them so that if $\bar{U}_i$ and $\bar{U}_j$ have common points, then f is (1-1) in $\bar{U}_i + \bar{U}_j$. (This is simple; see §5.) Consider all sets $U_i + U_j$ such that $\bar{U}_i \cdot \bar{U}_j = 0$; we arrange these in a sequence $W_1, W_2, \cdots$. Take $\eta(p)$ so small that any map f' which approximates (f, M, r, η) is regular and (1-1) in all $\bar{U}_i + \bar{U}_j$ for which $\bar{U}_i \cdot \bar{U}_j \neq 0$. Take any k, and say $W_k = U_i + U_j$; let Ω_k hold for the map f' if $f'(\bar{U}'_i) \cdot f'(\bar{U}'_j) = 0$ and b_0 is not in $f'(\bar{U}'_i)$. As the U'_i cover M, the property of f' being (1-1) and avoiding b_0 is the sum of the properties $\Omega_1, \Omega_2, \cdots$.

It remains to show that Ω_k is an (f, r, η)-property. (a) of §7 holds; (b) holds with W' replaced by $W'_k = U'_i + U'_j$. As $f'(\bar{U}'_i)$ and $f'(\bar{U}'_j) + b_0$ are bounded closed subsets of E_n, (c) is obvious; it remains to prove (d). Let f' approximate (f, M, r, η). Let $\lambda'(x)$ be a function of class C^r in E_m which $= 1$ in $\bar{Q}'_m$ and $= 0$ in $E_m - Q_m$. (Such a function is given by Lemma 11, replacing A by $\bar{Q}'_m + (E_m - Q_m)$; or the function may be constructed directly without great difficulty.) Then $\lambda(p) = \lambda'(\theta_i^{-1}(p))$ in U_i and $= 0$ in $M - U_i$, is of class C^r in M, and $= 1$ in $\bar{U}'_i$. For any vector v in E_n, set

$$f(p, v) = f'(p) + \lambda(p)v.$$

Given an arbitrary $\eta'(p)$, we may choose $\beta > 0$ so that if[31] $\| v \| < \beta$, then $f(p, v)$ approximates (f', M, r, η'). By Lemmas 13, 14 and 15, $f'(\bar{U}'_j) + b_0$ is of zero $(m + 1)$-extent and $f'(\bar{U}'_i)$ is of finite m-extent. The transformations $T_v q = q + v$ with $\| v \| < \beta$ form an n-parameter family in E_n; by Lemma 16, there is a v_0 such that $T_{v_0} f'(\bar{U}'_i)$ does not intersect $f'(\bar{U}'_j) + b_0$. $f'' = f(p, v_0)$ with this v_0 is then the required approximation; for $f'' = f' + v_0$ in $\bar{U}'_i$ and $= f'$ in $\bar{U}'_j$.

21. **Regular maps and properties.** The final lemma is

LEMMA 18. *If f is a C^r-map $(r \geqq 1)$ of the C^r-m-manifold M into E_n, $n \geqq 2m$, and η is a positive continuous function in M, then the property of maps f' which approximate (f, M, r, η) of being regular is the logical sum of a denumerable number of (f, r, η)-properties.*

Let $\theta_1, \theta_2, \cdots$ be completely admissible maps defining M, and let Ω_i hold for f' if f' is regular in $\bar{U}'_i$; then if $\Omega_1, \Omega_2, \cdots$ hold, f' is regular in M. Each Ω_i satisfies the conditions (a), (b) and (c) of §7; to show that it is an (f, r, η)-property, we must show that it satisfies (d). Set $g(x) = f'(\theta_i(x))$ in Q_m; this is a C^r-map of Q_m into E_n. As in the last §, it is sufficient to show that for an arbitrary $\zeta > 0$ there is a map $g'(x)$ of Q_m into E_n which approximates (g, Q_m, r, ζ) and is regular in Q'_m. We then set

$$f''(p) = f'(p) + \lambda(p)[g'(\theta_i^{-1}(p)) - f'(p)] \text{ in } U_i,$$

and $f''(p) = f'(p)$ in $M - U_i$; as $f''(p) = g'(\theta_i^{-1}(p))$ in $\bar{U}'_i$, it is regular in $\bar{U}'_i$.

[30] In §34 we shall express the property as a sum of $[f, r, \eta, \Delta, \zeta]$-properties.

[31] $\| v \|$ is the length of v.

If $r = 1$, let $g_0(x)$ be a function of class C^2 which approximates (g, Q_m, r, ζ') (Lemma 9); otherwise, set $g_0(x) = g(x)$. We shall find functions $g_1(x), \cdots, g_m(x) = g'(x)$ such that $g_i(x)$ approximates $(g_{i-1}, Q_m, r, \zeta')$, and so that the vectors $\partial g_i/\partial x_1, \cdots, \partial g_i/\partial x_i$ are independent in $\bar{Q}'_m$; if ζ' is small enough, g' is the required function.

Suppose we have found g_{i-1}. For any vector v in E_n, set

$$g_i(x, v) = g_{i-1}(x) + x_i v;$$

then

$$\frac{\partial g_i(x, v)}{\partial x_i} = \frac{\partial g_{i-1}(x)}{\partial x_i} + v, \qquad \frac{\partial g_i(x, v)}{\partial x_j} = \frac{\partial g_{i-1}(x)}{\partial x_j} \qquad (j \neq i).$$

As the $\partial g_{i-1}(x)/\partial x_j$, $j = 1, \cdots, i-1$, are independent in $\bar{Q}'_m$, we need merely show that there is an arbitrarily small v_0 such that $\partial g_i(x, v_0)/\partial x_i$ is independent of these vectors at each point of $\bar{Q}'_m$; we then set $g_i(x) = g_i(x, v_0)$. By Lemma 13, it is sufficient to show that the vectors v, $\| v \| \leqq 1$, which do not have the required property, form a set of zero n-extent in E_n. Given the point x^0 of $\bar{Q}'_m$, it is sufficient to show that the vectors not having the required property in a closed neighborhood S of x^0 are of zero n-extent; for a finite number of such sets S cover $\bar{Q}'_m$.

Let $P(x)$ be the $(i-1)$-plane through the origin O in E_n determined by $\partial g_{i-1}(x)/\partial x_1, \cdots, \partial g_{i-1}(x)/\partial x_{i-1}$, and set $P_0 = P(x^0)$. (If $i = 1$, $P(x) = O$.) Choose S so that $P(x)$ is not perpendicular to P_0 for x in S. Let $y_1, \cdots, y_n$ be rectangular coördinates (with origin O) in E_n such that P_0 is the $(y_1, \cdots, y_{i-1})$-plane. Let $E = E_{m+i-1}$ be the space with coördinates $(x_1, \cdots, x_m, y_1, \cdots, y_{i-1})$. Set

$$v(x) = \frac{\partial g_{i-1}(x)}{\partial x_i}, \qquad K = \max \| v(x) \| \ (x \text{ in } S).$$

Let D be the subset of E with x in S, $\| \bar{y} \| \leqq K + 1$, where $\bar{y} = (y_1, \cdots, y_{i-1})$. Let $T_x = T_{O,P(x)}$ be the transformation of §19 leaving O fixed and carrying P_0 into $P(x)$. Given $\bar{y}$, set $y = (y_1, \cdots, y_{i-1}, 0, \cdots, 0)$ and

$$w(x, \bar{y}) = T_x(y) - v(x).$$

as g_{i-1} is of class C^2, this is a C^1-map of D into E_n. By Lemma 15, $w(D)$ is of finite $(m + i - 1)$-extent, and hence of zero n-extent. Now let v be any vector, $\| v \| \leqq 1$, such that for some x in S, $v(x) + v$ is in $P(x)$; we shall show that v is in $w(D)$. As $P(x) = T_x(P_0)$, there is a $\bar{y}$ such that

$$v(x) + v = T_x(y), \qquad v = w(x, \bar{y});$$

as $\| v(x) + v \| \leqq K + 1$, $\| y \| \leqq K + 1$, and v is in $w(D)$. Hence there is an arbitrarily small v_0 such that no $v(x) + v_0$ is in $P(x)$; $v(x) + v_0 = \partial g_i(x, v_0)/\partial x_i$ is independent of $\partial g_{i-1}/\partial x_1, \cdots, \partial g_{i-1}/\partial x_{i-1}$ in S, and the lemma is proved.

22. **Proof of the imbedding theorem.** The last three lemmas lead at once to

LEMMA 19. *Any C^r-m-manifold M may be C^r-imbedded in E_{2m+1}.*[32]

Let $\eta(p)$ be a positive continuous function in M such that if $p_1, p_2, \cdots$ is a sequence of points of M with no limit in M, then $\lim \eta(p_i) = 0$.[33] Let f map M into the origin O in E_{2m+1}; f is of class C^r. Applying Lemma 12 with the properties of Lemma 18 gives a regular C^r-map f' approximating (f, M, r, η). We now apply Lemma 12 again, this time with the properties of Lemma 17, setting $b_0 = O$. (The new η may have to be smaller than the last.) The resulting function F is (1-1) regular and of class C^r in M, and $F(p) \neq O$ in M. By the choice of η, the limit set $LF(M)$ either is void or equals O; hence F is proper. Therefore $F(M)$ is a C^r-manifold in E_{2m+1} C^r-homeomorphic with M, and the proof is complete.

IV. THE NEIGHBORHOOD OF A MANIFOLD IN E_n

Suppose M is a C^r-m-manifold in E_n, $r \geqq 2$, $n > m$. To each point p of M there is a normal plane $P'(p)$; this family of planes fills out a neighborhood of M in E_n in a (1-1) way. If $r = 1$, this may not hold; for the $P'(p)$ depend on the first derivatives of functions defining M. Our object in this part is to find an approximating family of planes $P(p)$ of class C^r. §24 is necessary in this proof, and also directly in the next part.

23. **Projective spaces in Euclidean spaces.** The following lemma will be needed in the next §.

LEMMA 20. *Projective n-space E_n^* may be imbedded analytically in Euclidean space E_{2n+1}.*

The points of E_n^* are the sets of numbers $(x_1, \cdots, x_{n+1}) \neq (0, \cdots, 0)$, proportional sets being the same point. Let S_n be the unit n-sphere $\Sigma x_i^2 = 1$ in E_{n+1}. To each pair of "opposite" points p, $-p$ of S_n corresponds a point $\gamma(p) = \gamma(-p)$ of E_n^*. S_n is an analytic manifold with obvious neighborhoods; mapping these neighborhoods into E_n^* under γ defines E_n^* as an analytic manifold. By Lemma 19, E_n^* is C^1-homeomorphic with a manifold M' in E_{2n+1}. Let ϕ denote this homeomorphism. Then $\psi(p) = \phi(\gamma(p))$ is a (2-1) regular map of S_n into E_{2n+1}.

For any map g of S_n into E_{2n+1}, set[34]

$$Ag(p) = [g(p) + g(-p)]/2; \text{ then } A\psi(p) = \psi(p).$$

It is easily seen that if g approximates $(\psi, S_n, 1, \delta)$, then the same is true of Ag. As $Ag(p) = Ag(-p)$, we may let correspond to g a map f of E_n^* into E_{2n+1}; this

[32] Note that we may let $F(M)$ have no (finite) limit set in E_{2m+1}, by applying a transformation with reciprocal radii at the end of the proof.

[33] Let θ_i be a set of admissible maps in M. Let $\eta'(x)$ be continuous in E_m, > 0 in Q_m, and $= 0$ in $E_m - Q_m$. Set $\eta_i(p) = \eta'(\theta_i^{-1}(p))$ in U_i and $= 0$ in $M - U_i$. If $\alpha_1, \alpha_2, \cdots$ are small enough, we may set $\eta(p) = \Sigma \alpha_i \eta_i(p)$.

[34] Compare footnote[17].

might be written $f(q) = Ag(\gamma^{-1}(q))$. Take $\epsilon > 0$ such that any map approximating $(\phi, E_n^*, 1, \epsilon)$ is (1-1) regular. Choose $\delta > 0$ so that if g approximates $(\psi, S_n, 1, \delta)$, then the corresponding f approximates $(\phi, E_n^*, 1, \epsilon)$. Extend ψ through a neighborhood R of S_n in E_{n+1}, say by letting it be constant on any half ray from the origin; it is then of class C^1 in R. By Lemma 9 (or the Weierstrass approximation theorem) there is an analytic map g approximating $(\psi, R, 1, \delta)$; considering g on S_n alone, the corresponding function f has then the required properties.

24. **k-planes in n-space.** Let $\mathfrak{S}$ be the space whose points are the k-planes in n-space through the origin. We shall express this space as an analytic manifold $M(n, k)$ in a Euclidean space $E(n, k)$. Given the plane P, let $v_1, \cdots, v_k$ be a set of independent vectors in P; their coördinates form a matrix, with k-rowed determinants $D_{i_1 \cdots i_k}(P)$. These determinants, arranged in a sequence $D_1^*(P), \cdots, D_{\gamma_{nk}}^*(P)$, $\gamma_{nk} = \binom{n}{k}$, form the homogeneous coördinates of a point $D^*(P)$ in projective space $E_{\gamma_{nk}-1}^*$. $D^*(P)$ is independent of the vectors $v_1, \cdots, v_k$, and $D^*(P) \neq D^*(P')$ if $P \neq P'$; thus we have a (1-1) map of $\mathfrak{S}$ into a subset $\mathfrak{S}'$ of $E_{\gamma_{nk}-1}^*$. By Lemma 20, we may imbed $E_{\gamma_{nk}-1}^*$ analytically in Euclidean space $E(n, k)$; this carries $\mathfrak{S}'$ into a subset $M(n, k)$ of $E(n, k)$.

We may show that $\mathfrak{S}'$ and hence $M(n, k)$ is an analytic manifold by taking any P_0, choosing a determinant, say $D_{1 \cdots k}(P_0)$, which is $\neq 0$, and expressing each $D_{i_1 \cdots i_k}$ in terms of the determinants $D_{1, \cdots, s-1, s+1, \cdots, k, t}$ by Vahlen's relations, which are analytic; this determines maps of the required nature in $\mathfrak{S}'$. An analytically equivalent set of maps may be given as follows: Given P_0, let $p_1^0, \cdots, p_k^0$ be points of P_0 which form linearly independent vectors from the origin, and let $L_1, \cdots, L_k$ be $(n-k)$-planes (or analytic $(n-k)$-manifolds) through $p_1^0, \cdots, p_k^0$ orthogonal to P_0. If P (through the origin) is near P_0, it intersects each L_i in a point p_i; the positions of the p_i determine a map of part of $E_{k(n-k)} = E_{n-k} \times \cdots \times E_{n-k}$ (k factors) into part of $M(n, k)$.

Another important space is the space $\mathfrak{S}^*$ of all k-planes in E_n; this also forms an analytic manifold. $M(n, k)$ is closed; the present manifold is open. We may map $\mathfrak{S}^*$ into $M(n, k)$ by letting any plane correspond to the parallel plane through the origin. We shall use the symbol P for points of either space; it will always be clear which space is meant.

By an *analytic function of k-planes* we shall mean a function which, when considered in $M(n, k)$, is analytic.

We shall say two planes of any dimensions (in either space) are *orthogonal* if any vector in (or perhaps better, parallel to) one is orthogonal to any vector in the other; *independent* if they have no common vector $\neq 0$; *perpendicular* if some vector $\neq 0$ in one is orthogonal to each vector of the other. If P, P' are points of $\mathfrak{S}$, we may let $\| P' - P \|$ be the distance between the corresponding points of $M(n, k)$.

25. The neighborhood of a manifold in space. Two more lemmas lead up to the main result of this part, Lemma 23.

LEMMA 21. *Let M be a C^r-m-manifold in E_n ($r \geqq 1$ finite or infinite), and let $P(p)$ be a function of class C^r in M satisfying* (a) *of Lemma* 23. *Then there is a function $\xi(p)$ in M satisfying the remaining conditions.*

Take any p_0 in M. A neighborhood U of p_0 may be determined by functions (3.1) of class C^r. Define the transformation $T_{p,P}$ in terms of p_0 and $P(p_0)$ as in §19. Set $w_i(p) = T_{p,P(p)}(v_i)$; then the points of $P(p)$ for p in U are given by

$$q = p + \sum_{i=1}^{n-m} \alpha_i w_i(p). \tag{25.1}$$

Using (3.1), we may express p in terms of $y_1, \cdots, y_m$: $p = \psi(y_1, \cdots, y_m)$. Putting in (25.1) gives q as a function of $y_1, \cdots, y_m, \alpha_1, \cdots, \alpha_{n-m}$:

$$\begin{aligned} q &= \psi(y_1, \cdots, y_m) + \Sigma \alpha_i w_i(\psi(y_1, \cdots, y_m)) \\ &= g(y_1, \cdots, y_m, \alpha_1, \cdots, \alpha_{n-m}). \end{aligned} \tag{25.2}$$

g is of class C^r. Consider the vectors $\partial g/\partial y_i$, $\partial g/\partial \alpha_j$ at $q = p_0$ in U. The $\partial g/\partial y_i$ are independent vectors in the tangent plane T to M at p_0, as the α_i vanish there, and the $\partial g/\partial \alpha_j = w_j(p_0)$ are independent vectors in $P(p_0)$; as $P(p_0)$ is independent of T, the whole set of vectors is independent. In other words, the Jacobian of (25.2) is $\neq 0$ at p_0, and hence in a neighborhood R' of p_0. Solving for $y_1, \cdots, y_m$ gives p in terms of q in R': $p = H(q)$. H is of class C^r.

We may cover M by such neighborhoods R' so that any bounded closed subset of M has points in but a finite number of the R'. It is easy to construct a positive function $\xi(p)$ in M such that if $R(p)$ is that part of $P(p)$ within $\xi(p)$ of p, then $R(p)$ lies in some R'. (c) and hence (b) of Lemma 23 now hold.

LEMMA 22. *Lemma 9 holds with R and E_n replaced by analytic manifolds M and N in E_μ and E_ν respectively.*[35]

Let $P(p)$ be the normal plane to p in M. Then $P(p)$ is analytic, and hence we may define $H(p)$ by the last lemma. H is analytic. Similarly we define P' and H' for N. Set $f(q) = f(H(q))$ in $R(M)$; this is a C^r-map of $R(M)$ into the subset N of E_ν (see Lemma 4). If the analytic function F' approximates to f closely enough and $R'(M)$ is a small enough neighborhood of M in $R(M)$, then F' maps $R'(M)$ into $R(N)$, and $F = HF'$ is an analytic map of $R'(M)$ into N. F, considered on M alone, is analytic (see Lemma 3).

LEMMA 23. *Let M be a C^r-m-manifold in E_n($r \geqq 1$ finite or infinite). Then there is a positive continuous function $\xi(p)$ and a function $P(p)$ of class C^r in M, such that:* (a) *$P(p)$ is an $(n - m)$-plane through p independent of the tangent plane to M at p.* (b) *If $R(p)$ is that part of $P(p)$ within $\xi(p)$ of p, then the $R(p)$ fill out a neighborhood $R(M)$ of M in a (1-1) way.* (c) *If $H(q) = p$ for q in $R(p)$, then H is of class C^r in $R(M)$. Moreover, if M is analytic, so are $P(p)$ and $H(p)$.*

[35] See also Lemma 27.

We have just considered the analytic case. If $n = m$, the lemma is trivial (then H is the identity); suppose $n > m$. Let $P'(p)$ be the normal plane to M at p; there is a corresponding point $D'(p)$ in $M(n, n - m)$. D' is of class C^{r-1} and is thus continuous in M. Extended D' so as to be continuous throughout a neighborhood R of M in E_n. (Almost any method in use will do this; or we may use Lemma 11.) If R is small enough, $D'(R)$ is in $R(M(n, n - m))$; then $D'' = H'D'$ is a continuous map of R into $M(n, n - m)$, and $D'' = D'$ in M. (H' is defined in $R(M(n, n - m))$ as in the last lemma.) By Lemma 22 we may approximate D'' in R by an analytic function D so closely that if $P(p)$ is the plane through p in M parallel to the plane defined by $D(p)$, then $P(p)$ is independent of the tangent plane to M at p. $P(p)$, considered in M alone, is of class C^r, by Lemma 3. The lemma now follows from Lemma 21.

V. Analytic Manifolds

26. **The lemma and method of proof.** Our object in this part is to prove

Lemma 24. *Let M be a C^r-m-manifold in E_n ($r \geqq 1$ finite or infinite). Then there is a C^r-homeomorphic analytic manifold M^* in E_n.*

This, together with Lemma 19, completes the proof of Theorem I. Actually, M^*, as constructed, will approximate to M to any desired degree, but it is easier to find an approximating analytic manifold after a homeomorphic analytic one is found. (See Lemma 22.) We may suppose that $n > m$; if $n = m$, then M is analytic.

To prove the lemma, we first construct an analytic $(n - 1)$-manifold S "surrounding" M, and then find in an analytic fashion a "center" M^* of S. The proof is most easily visualized for $n = 3$, $m = 1$. The construction of S is straightforward. We determine a function positive and analytic near M and vanishing in M, subtract a very small positive analytic function, and let S be the set of points where the resulting function vanishes. The inside of S is filled up by $(n - m)$-planes $P(p)$ approximately normal to M (see Lemma 23). The resulting function $D(P)$ with values in $M(n, n - m)$ (see §24) is of class C^r inside of S. We approximate this function by an analytic function, and thus determine an analytic family of planes $P^*(p)$. (These planes, unlike the $P(p)$, intersect each other inside S.) A point p inside S is in M^* if and only if p is at the center of mass of that connected part of $P^*(p)$ inside S which contains p.

The following lemma is necessary.[36]

Lemma 25. *Given an open set R in E_n, a positive continuous function $\eta(p)$ in R, and $r \geqq 0$, there is an analytic function $\omega(p)$ in R such that*

$$(26.1) \qquad \omega(p) > 0, \qquad |D_k\omega(p)| < \eta(p) \text{ in } R \qquad (\sigma_k \leqq r).$$

Let $C_1, C_2, \cdots$ be a denumerable set of overlapping cubes covering R, and let

[36] This lemma, except for analyticity, is practically equivalent to a theorem of Ostrowski, Bull. des Sciences Math., 1934, pp. 64–72. See also our lemma 10. The theorem was known to the author in 1933. Note that we may make $r = \infty$ in a manner similar to that in AE Lemma 6.

$\phi_i(x)$ be a function of class C^∞ in R which is > 0 within C_i and $= 0$ in $R - C_i$ (see for instance AE, §9). If the a_i are small enough and positive, then $\phi(x) = \Sigma a_i \phi_i(x)$ is a positive function of class C^∞ in R satisfying the inequality (26.1). If $\omega(p)$ is an analytic function approximating (ϕ, R, r, ζ) for small enough $\zeta(p) > 0$ (see Lemma 9), then (26.1) holds.

27. **The manifold S and spheres $S^*(p, P)$.** In this section we shall find S, and shall show that certain $(n - m)$-planes P through points p near M intersect S in $(n - m - 1)$-spheres $S^*(p, P)$, which are analytic and vary analytically with p and P. In the next section we shall find the analytic manifold M^*.

Define the planes $P(p)$ and the projection $H(p)$ in the neighborhood $R(m)$ as in Lemma 23. We extend the definitions of $P(p)$ and $\xi(p)$ through $R(M)$ by setting

$$P(p) = P(H(p)), \qquad \xi(p) = \xi(H(p)).$$

Define the function $\Phi(p)$ in $R(M)$ by

$$\Phi(p) = \| p - H(p) \|. \tag{27.1}$$

As $H(p)$ is of class C^r in $R(M)$, $\Phi(p)$ is of class C^r in $R(M) - M$. By Lemmas 9 and 10 there is a function $\Phi'(p)$ continuous in $R(M)$ and analytic in $R(M) - M$ such that $\Phi' = 0$ in M, and it and its gradient satisfy

$$| \Phi'(p) - \Phi(p) | < \tfrac{1}{3}\, \xi(p), \qquad \| \nabla\Phi'(p) - \nabla\Phi(p) \| < \tfrac{1}{3} \tag{27.2}$$

in $R(M) - M$. By Lemma 25, there is a positive analytic function $\omega(p)$ in $R(M)$ such that

$$| \omega(p) | < \tfrac{1}{3}\, \xi(p), \qquad \| \nabla\omega(p) \| < \tfrac{1}{3}. \tag{27.3}$$

Set

$$\Phi^*(p) = \Phi'(p) - \omega(p); \tag{27.4}$$

then $\Phi^*(p)$ is continuous in $R(M)$ and is analytic in $R(M) - M$, and $\Phi^* < 0$ in M. S is determined by the vanishing of Φ^*.

To prove the existence of and properties of S, we shall introduce some auxiliary functions. Let p_0 be any point of M. Some neighborhood U of p_0 in M is defined by equations (3.1). Given any subset K of M, let $R(K)$ be the set of all points p of $R(M)$ such that $H(p)$ is in K. $P(p)$ is independent of T (see Lemma 23); hence there is a neighborhood U' of p_0 in M, U' in U, and a $\delta > 0$, such that if P is an $(n - m)$-plane through a point p of $R(U')$ and

$$\| P - P(p) \| < \delta, \tag{27.5}$$

then P is independent of T and hence intersects T in a unique point $H^*(P)$. (T is the tangent plane to M at p_0.) H^* is analytic. Set

$$H'(p) = H^*(P(p)), \qquad u(p) = \frac{p - H(p)}{\| p - H(p) \|}, \qquad u'(p) = \frac{p - H'(p)}{\| p - H'(p) \|}. \tag{27.6}$$

We may choose U' and δ so small that for any p in $R(U')$ and any P through p satisfying (27.5),

$$\Phi^*(H^*(P)) < 0, \qquad \| H'(p) - H(p) \| < \xi(p)/6, \tag{27.7}$$

and

$$\text{if } \Phi^*(p) \geqq 0, \quad \text{then} \quad \| u'(p) - u(p) \| < \tfrac{1}{3}. \tag{27.8}$$

For any P satisfying (27.5), let T_P be the transformation $T_{H^*(P),P}$ of §19, using the fixed point p_0 and plane $P(p_0)$, and let $S(P)$ be the unit $(n - m - 1)$-sphere in P about $H^*(P)$. Given any point $\bar{q}$ of $S_0 = S(P(p_0))$, there is a corresponding point

$$q' = T_P(\bar{q}) = \mu(P, \bar{q}) \text{ in } S(P); \tag{27.9}$$

μ is analytic. To each P satisfying (27.5), each $\bar{q}$ in S_0, and any $\alpha > 0$, let correspond points p', q', q by

$$q = p' + \alpha(q' - p') = H^*(P) + \alpha[\mu(P, \bar{q}) - H^*(P)]. \tag{27.10}$$

For such values of $\alpha > 0$ which make q lie in $R(U) - U$ we define the analytic function

$$\sigma(P, \bar{q}, \alpha) = \Phi^*(q). \tag{27.11}$$

We shall show next that for some γ, $0 < \gamma < \delta$, if P is a plane through a point p of $R(U')$, $\| P - P(p) \| < \gamma$, and $\bar{q}$ is in S_0, then there is unique number

$$\alpha = \rho(P, \bar{q}) > 0 \tag{27.12}$$

which, put in (27.10) and (27.11), makes σ vanish (with q in $R(U)$); moreover, ρ is analytic. Set

$$\sigma'(p, \bar{q}, \alpha) = \sigma(P(p), \bar{q}, \alpha); \tag{27.13}$$

it is sufficient to show that, using $P(p)$, there is a unique point q of the line segment $\overline{p'q'}$ in $R(U')$ such that $\Phi^*(q) = 0$, and $\partial\sigma'/\partial\alpha > 0$ at this point.

By definition of $R(M)$, $R(U')$ contains all points of $P(p)$ within $\xi(p)$ of $H(p)$. As $p' = H'(p)$ for $P = P(p)$, (27.7) gives

$$\| p' - H(p) \| < \xi(p)/6. \tag{27.14}$$

Hence, if q'' is the point q for which $\alpha = 5\xi(p)/6$ (keeping $\bar{q}$ fixed), all of $\overline{p'q''}$ lies in $R(U')$. Moreover, as $H(q'') = H(p)$, (27.1) through (27.4) with (27.14) give

$$\Phi^*(q'') > \Phi(q'') - \tfrac{2}{3}\,\xi(p) > 0. \tag{27.15}$$

By (27.7), $\Phi^*(p') < 0$; hence there is a point of $\overline{p'q''}$ for which $\Phi^* = 0$.

Now take any q on $\overline{p'q''}$ such that $\Phi^*(q) \geqq 0$, keeping $P = P(p)$. As

$$\| q' - p' \| = 1, \qquad p' = H'(q'),$$

and $u'(q') = u'(q)$, differentiating (27.13) and using (27.10) gives

$$\frac{\partial \sigma'}{\partial \alpha} = |\nabla\Phi^*(q)\cdot(q' - p')|_{P=P(p)} = \nabla\Phi^*(q)\cdot u'(q). \tag{27.16}$$

The projection of $\nabla\Phi$ into any plane P equals the gradient of Φ as a function defined in P; hence, by (27.1) and (27.6),

$$\text{Proj}_{P(p)}\, \nabla\Phi(q) = u(q),$$

and if u' is any vector parallel to $P(p)$, then

$$\nabla\Phi(q)\cdot u' = u(q)\cdot u'.$$

Hence, by (27.2) through (27.4) and (27.8),

$$\left|\frac{\partial \sigma'}{\partial \alpha}\right| = |\, [\nabla\Phi^*(q) - \nabla\Phi(q)]\cdot u'(q) + u(q)\cdot[u'(q) - u(q)] + 1\, | \tag{27.17}$$
$$> -\tfrac{2}{3} - \tfrac{1}{3} + 1 = 0.$$

This shows that Φ^* vanishes at a unique point q of $\overline{p'q''}$, and the existence and analyticity of ρ is proved.

Take any P satisfying (27.5) with δ replaced by γ; putting (27.12) in (27.10) gives q as a function of $\bar{q}$. As $\bar{q}$ ranges over S_0, q ranges over an analytic $(n - m - 1)$-sphere $S^*(P)$; this sphere varies analytically with P. It is the intersection of P and S. A finite or denumerable number of neighborhoods U' cover M; for each there is a corresponding $\gamma > 0$. Let $\gamma(p)$ be a positive continuous function in $R(M)$ such that $\gamma(p) = \gamma(H(p))$, and if p is in any U', then $\gamma(p)$ is less than the corresponding γ. Now if p is any point of $R(M)$, P contains p, and

$$\| P - P(p) \| < \gamma(p), \tag{27.18}$$

then $R(p)$ intersects S in an analytic sphere $S^*(p, P)$ which varies analytically with p and P.

28. **The analytic manifold M^*.** For any p in $R(M)$ and any plane P through p satisfying (27.18), let $Q^*(p, P)$ be that part of P inside $S^*(p, P)$. Let $g(p, P)$ be the center of mass of $Q^*(p, P)$. We shall show that if $P^*(p)$ is any analytic function in $R(M)$ approximating to $P(p)$ closely enough in $R(M)$, then the set M^* of points in $R(M)$ satisfying

$$g(p, P^*(p)) = p \tag{28.1}$$

is an analytic manifold in $R(M)$, C^r-homeomorphic with M.

We shall first show that $g(p, P)$ is analytic. Consider a point p_0 of M and a neighborhood U of p_0 in M etc. as before. If $V(p, P)$ is the $(n - m)$-volume of $Q^*(p, P)$ and $|\, dp\,|$ denotes the volume element, then for p in $R(U')$ and any P

through p within $\gamma(p)$ of $P(p)$,[37]

$$(28.2)\qquad g(p, P) = \frac{1}{V(p, P)} \int_{Q^*(p, P)} q \mid dq \mid.$$

We shall express this integral in a different form. The points of $Q^*(p, P)$ are given by the pairs

$$(\bar{q}, \alpha); \qquad \bar{q} \text{ in } S_0, \qquad 0 \leqq \alpha \leqq \rho(P, \bar{q}).$$

Letting W_0 be the $(n - m - 1)$-volume of S_0 and noting that T_P preserves volume, (28.2) may be written

$$(28.3)\quad g(p, P) = \frac{1}{W_0} \int_{S_0} \left[\frac{n - m}{\rho^{n-m}} \int_0^{\rho} \alpha^{n-m-1} \{p' + \alpha(q' - p')\}\, d\alpha \right] \mid d\bar{q} \mid,$$

where $\rho = \rho(P, \bar{q})$. This expression is easily seen to be analytic.

Let M' be the set of points in $R(M)$ satisfying $g(p, P(p)) = p$. Each $Q^*(p, P(p))$ has exactly one point in M', namely, its center of mass. Taking p_0 etc. again, set

$$(28.4)\qquad \tau(p, P) = T_P^{-1}(p) - T_P^{-1}(g(p, P)), \qquad \tau(p) = \tau(p, P(p)).$$

$\tau(p, P)$ is a point (or vector) of $E_{n-m} = P(p_0)$. $\tau(p) = 0$ if and only if p is in M'. We shall show that if $P^*(p)$ is an analytic function approximating $P(p)$ closely enough in $R(U')$ through the first order, and

$$(28.5)\qquad \tau^*(p) = \tau(p, P^*(p)),$$

then the vanishing of $\tau^*(p)$ determines an analytic manifold through p_0. To this end, let $\tau_1(p), \cdots, \tau_{n-m}(p)$ and $\tau_1^*(p), \cdots, \tau_{n-m}^*(p)$ be the components of $\tau(p)$ and $\tau^*(p)$ in the directions of fixed mutually orthogonal vectors in $P(p_0)$; then $\tau(p) = 0$ if and only if the $\tau_i(p) = 0$, and similarly for $\tau^*(p)$. The $\tau_i(p)$ vanish at a unique point of each $Q^*(p_1, P(p_1))$, and the $\nabla\tau_i(p)$ are independent as functions in $P(p_1)$; hence the same is true of the $\tau_i^*(p)$ and the $\nabla\tau_i^*(p)$, if the approximation of $P^*(p)$ is close enough. Therefore $\tau^*(p) = 0$ defines an analytic manifold M_1^* in $R(U')$, which cuts each $Q^*(p, P(p))$ in a unique point p_1', and such that the tangent plane to M_1^* at p_1' is independent of $P(p_1)$.

If $P^*(p)$ approximates to $P(p)$ closely enough in $R(M)$, then the above will hold near each point of M, and the vanishing of $\tau^*(p)$ will determine an analytic manifold M^* cutting each $Q^*(p, P(p))$ as noted. As is seen from (28.4) and (28.5), the points of M^* satisfy (28.1). The map $p' = H(p)$ of M^* into M is (1-1) and of class C^r. As the tangent plane to M^* at p is independent of $P(p)$, the inverse is also of class C^r (see Lemma 21); hence the map is a C^r-homeomorphism and the proof is complete.

[37] Compare footnote[17].

VI. Proof of Theorem 2

We shall first prove the existence of analytic linear functionals as defined in §7; we will then be able to prove Theorem 2 and the properties in §9. Finally, we shall prove the unproved statement in Theorem 6.

29. Real-valued analytic linear functionals. We shall generalize AE Lemma 7 as follows:

Lemma 26. *Let R be an open set in E, let $\Delta(p, q)$ and $\zeta(p)$ be as in §7 in R, and let r and s be finite, $s \leqq r$. Then there is an analytic linear $(R, E_1, r, \Delta, \zeta)$-functional $\mathfrak{L}$; moreover, $\mathfrak{L}$ is defined for any polynomial P of degree $\leqq s$, and $\mathfrak{L}P = P$.*

Let $R_1, R_2, \cdots$ be bounded open subsets of R such that $\bar{R}_i$ is in R_{i+1} and $R_1 + R_2 + \cdots = R$, and let ϵ_i be the minimum of $\zeta(x)$ for x in $\bar{R}_{i+1}$. Let $\Delta_i'(\rho)$ be the maximum of $\Delta(x, x')$ for points x and x' of $\bar{R}_i$ whose distance apart is ρ; these functions are easily seen to satisfy the requirements in AE Lemma 7. Let a be a fixed point of R. Given any function f of class C^r in R, set

$$\mathfrak{L}^*f(x) = \sum_{\sigma_k \leqq s} \frac{D_k f(a)}{k!} (x - a)^k; \tag{29.1}$$

This is the polynomial of degree $\leqq s$ approximating to f most closely at a. (See AE for the notation.) $\mathfrak{L}^*$ is a linear functional, and for any polynomial P of degree $\leqq s$, $\mathfrak{L}^*P = P$. As seen in AE, footnote on p. 78, for each i there is a number K_i such that if f satisfies (7.1) and hence

$$| D_k f(x') - D_k f(x) | \leqq \Delta_i'(\| x' - x \|) \text{ in } \bar{R}_i \qquad (\sigma_k \leqq s), \tag{29.2}$$

then

$$| D_k f(a) | < K_i \qquad (0 < \sigma_k \leqq s). \tag{29.3}$$

Let $\Delta_i''(\rho)$ be the maximum in $\bar{R}_i$ of

$$| D_k P(x') - D_k P(x) | \quad \text{for} \quad \| x' - x \| = \rho, \qquad \sigma_k \leqq r,$$

for polynomials $P(x)$ of degree $\leqq s$ whose derivatives at a are $\leqq K_i$, and set $\Delta_i(\rho) = \Delta_i'(\rho) + \Delta_i''(\rho)$. Now if f is any function of class C^r in R satisfying (29.2), then $f - \mathfrak{L}^*f$ satisfies the same equation with Δ_i' replaced by Δ_i.

Let $\mathfrak{L}'$ be the linear functional given by AE Lemma 7 with $M = 0$, and set

$$\mathfrak{L}f = \mathfrak{L}'(f - \mathfrak{L}^*f) + \mathfrak{L}^*f. \tag{29.4}$$

$\mathfrak{L}$ is defined for all $f = f' + P$, where f' satisfies (7.1) and hence (29.2), and P is a polynomial of degree $\leqq s$; for

$$f - \mathfrak{L}^*f = f' - \mathfrak{L}^*f',$$

and this function satisfies (29.2) with Δ_i, and is 0 at a. As both $\mathfrak{L}^*$ and $\mathfrak{L}'$ are linear, $\mathfrak{L}$ is linear. As $\mathfrak{L}'$ is analytic and $\mathfrak{L}^*f$ is a polynomial, $\mathfrak{L}$ is analytic.

Obviously $\mathfrak{L}P = P$ for polynomials P of degree $\leqq s$. Finally, $\mathfrak{L}f$ approximates (f, R, r, ζ), and the proof is complete.

30. **Analytic linear functionals.** We replace R and E_1 in the last lemma by M and E_ν, as follows.

LEMMA 27. *Let M be an analytic m-manifold in E_μ, let $\Delta(p, q)$ and $\zeta(p)$ be as in* §7, *and let r be finite. Then there is an analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional.*[38]

It is sufficient to prove this for real-valued functions; the general case then follows on applying it to each coördinate separately. Given any f in M, define f' in $R(M)$ by $f'(p) = f(H(p))$. We shall show that the functional of the last lemma, which we now call $\mathfrak{L}'$, may be applied to f'; then $\mathfrak{L}f$ is $\mathfrak{L}'f'$ considered in M alone.

Suppose f is of class C^r in M; then if $f'(p') = f(H(p'))$, f' is of class C^r in $R(M)$ (see the proof of Lemma 4); we let $P(p)$ be the normal plane to M at p. Differentiating $f'(p') = f'(H(p'))$ shows that $D_kf'(p')$ is a polynomial of degree $\leqq \sigma_k$ in the derivatives of order $\leqq \sigma_k$ of f' at $p = H(p')$ and of $H(p')$ at p'. Say $p = \theta_i(x)$. Then $D_kf'(p)$ is determined by the $D_sf_i(x) = D_sf(\theta_i(x))$ and the $D_t\theta_i(x)$. (The latter determine $P(p)$.) Hence

$$D_kf'(p') = \Phi[D_sf_i(x), D_t\theta_i(x), D_uH(p')] \qquad (\sigma_s, \sigma_t, \sigma_u \leqq \sigma_k), \tag{30.1}$$

for $\sigma_k \leqq r$. (k, s, t and u have respectively ν, m, m and ν components.) As the θ_i are admissible, there are but a finite number of such expressions for $D_kf'(p')$. Let a be a fixed point of M. Given any f in M or $R(M)$, set $\bar{f}(p) = f(p) - f(a)$. For any compact subset A of M there is a number K such that if f satisfies (7.1) in M, then

$$| D_k\bar{f}(p) | < K \text{ in } A \qquad (\sigma_k \leqq r), \tag{30.2}$$

(see AE, footnote on p. 78). Hence, by (30.1), for any two points p' and q' of $R(M)$ there is a number Δ such that for any such f, $(\bar{f})' = \bar{f}'$ and hence f' satisfies

$$| D_kf'(q') - D_kf'(p') | \leqq \Delta \qquad (\sigma_k \leqq r). \tag{30.3}$$

Let $\Delta^*(p', q')$ be the minimum of such numbers Δ. There are several (but a finite number of) choices for $D_kf(p)$ in (7.1); we take $\Delta^*(p', q')$ large enough for all these.

We show now that if $p'_h \to p'_0$ and $q'_h \to p'_0$, then $\Delta^*(p'_h, q'_h) \to 0$. Suppose not. Then there are a k $(\sigma_k \leqq r)$, sequences $\{p'_h\}$ and $\{q'_h\}$ approaching p'_0, and functions f_h in M satisfying (7.1), such that

$$| D_kf'_h(q'_h) - D_kf'_h(p'_h) | > \alpha > 0. \tag{30.4}$$

We may suppose that $p_0 = H(p'_0)$, $p_h = H(p'_h)$, and $q_h = H(q'_h)$ are in some U_i; (30.1) then applies. Replace f_h, f'_h by $\bar{f}_h$, $\bar{f}'_h$ as before. Then the $D_s\bar{f}_{hi}(x_h)$ are

[38] If $M = R$, we may have $\mathfrak{L}P = P$ as in the last lemma; P is a polynomial with values in E_ν if each of its coördinates is.

bounded ($x_h = \theta_i^{-1}(p_h)$), and we may suppose $D_s\bar{f}_{hi}(x_h) \to D_{is}$. As $y_h = \theta_i^{-1}(q_h) \to \lim x_h$, (7.1) shows that $D_s\bar{f}_{hi}(y_h) \to D_{is}$ also. Therefore all the variables in (30.1) approach the same limit when p is replaced by p_h as when it is replaced by q_h. As Φ is continuous, $D_k\bar{f}'_h(p_h)$ and $D_k\bar{f}'_h(q_h)$ approach the same limit; but this contradicts (30.4). It is now easy to construct a continuous function $\Delta'(p', q')$ for p', q' in $R(M)$ of the required nature, such that $\Delta'(p', q') \geqq \Delta^*(p', q')$.[39] Now if f is any function of class C^r in M satisfying (7.1), and $f'(p) = f'(H(p))$ in $R(M)$, then f' satisfies

$$(30.5) \qquad | D_k f'(q) - D_k f'(p) | \leqq \Delta'(p, q) \qquad (\sigma_k \leqq r).$$

Applying Lemma 26 to f' with $s = 0$ gives an analytic function $\mathfrak{L}'f'$ approximating $(f', R(M), r, \zeta')$; then $\mathfrak{L}f = \mathfrak{L}'f'$ in M approximates (f, M, r, ζ), if ζ' is sufficiently small.

31. **Proof of Theorem 2 with** (b) **and** (c) **omitted.** The proof of Theorem 2 with just (a) and (e′) is given by Lemma 12. We shall prove it with (a), (d), (e) and (f); the proof will be complete when we have proved (A) and (B) of §9.

We first apply Lemma 12 to find a function F' of class C^r which approximates (f, M, r, η) and has the (f, r, η)-properties $\Omega_1, \Omega_2, \cdots$. If $n \geqq 2m$, we include in these properties those of §21, to make F' regular. This is permissible, as the finiteness condition of (e) of the theorem is satisfied for these properties. Only a slight change in the proof of Lemma 18 is necessary because of E_n being replaced by N. Let W_i^* and η_i^* be the neighborhoods and functions of §7(b) and (c) corresponding to Ω_i and F'. Because of the finiteness condition, there is a positive continuous function ζ in M such that if F approximates (F', M, r, ζ), then it approximates (f, M, r, η), and for each i, it approximates $(F', \overline{W}_i^*, r, \eta_i^*)$; F then has the properties $\Omega_1, \Omega_2, \cdots$. It remains to show that the analytic function F may be chosen so as to approximate (F', M, r, ζ) and have the properties $\Omega_1', \Omega_2', \cdots$.[40]

Replace the $\Delta(p, q)$ of the theorem if necessary by a larger Δ so that (7.1) is satisfied with f' and Δ replaced by F' and $\frac{1}{2}\Delta$. Let $\mathfrak{L}$ be the analytic linear $(M, E_\nu, r, \Delta, \frac{1}{2}\zeta)$-functional given by Lemma 27; we suppose ζ is so small that if F'' approximates (F', M, r, ζ), then $F''(M)$ is in $R(N)$. For some ζ', if F'' satisfies (7.1) and approximates (F', M, r, ζ'), then $\mathfrak{L}F''$ approximates (F', M, r, ζ). We must now choose F'' so that it approximates (F', M, r, ζ') and satisfies (7.1), and so that $F = \mathfrak{L}F''$ has the properties $\Omega_1', \Omega_2', \cdots$.

[39] Let $\rho(p)$ be the smaller of 1 and half the distance from p to $E_\nu - R(M)$, and let $\rho(p, q)$ be the smaller of $\rho(p)$, $\rho(q)$, $\| q - p \|$. Take p and q in $R(M)$, let $\delta_\alpha(p, q)$ be the upper bound of $\Delta^*(p', q')$ for $\| p' - p \| \leqq \alpha$, $\| q' - q \| \leqq \alpha$, and set

$$\Delta'(p, q) = \| q - p \| + \frac{1}{\rho(p, q)} \int_0^{\rho(p, q)} \delta_\alpha(p, q)\, d\alpha.$$

[40] *If* $n \geqq 2m + 1$ *and* f *is proper*, we may find an F which is analytic, regular, (1-1) and proper, and has the properties $\Omega_1, \Omega_2, \cdots$, by including in these properties those of §§20 and 21, and then applying Lemma 22 with its $\eta(p)$ sufficiently small. (See the end of §6.)

Let W_i, W'_i, G^i_j be the open sets and functions corresponding to Ω'_i $(i = 1, 2, \cdots)$. We shall choose sets of numbers α^i so that if

$$f_0 = F', \qquad f_i = f_{i-1} + \sum_j \alpha^i_j G^i_j, \qquad F'' = \lim f_i,$$

then F'' is the required function. As the W'_i are bounded, if f' is a function satisfying an inequality of the nature of (7.1), and α is small enough, then $f' + \sum_{i,j} \alpha^i_j G^i_j$ approximates to f' as closely as we please and satisfies (7.1) with a new Δ as near the old as we wish. Hence we may choose numbers $\bar{\alpha}^1, \bar{\alpha}^2, \cdots$ such that if $\alpha^1, \alpha^2, \cdots$ is any sequence with $|\alpha^i_j| \leqq \bar{\alpha}^i$, then F'', using these α^i, satisfies (7.1) and approximates (F', M, r, ζ'). The proof now runs exactly like that of Lemma 12, except that $H\mathfrak{L}f_j$ $(j \geqq i)$ and $H\mathfrak{L}F''$ will have the property Ω'_i. The existence of α^i at each step such that $H\mathfrak{L}f_i$ has the property Ω_i is given by (d′) of §7.

32. **Certain maps of M into N like translations near a point.** Let f be a C^r-map of M into $N (r \geqq 0)$, take p_0 in M, and set $q_0 = f(p_0)$. Given any $\delta > 0$, we shall find neighborhoods W', W of p_0 in M such that $\bar{W}'$ is in W and $\bar{W}$ is within δ of p_0 (measuring in E_μ), and we shall find C^r-maps $G_1, \cdots, G_n$ of M into E_ν such that $G_i(p) = O$ for p in $M - W$, and such that

$$(32.1) \qquad f_\alpha(p) = H\mathfrak{L}[f(p) + \sum \alpha_i G_i(p)] = H[\mathfrak{L}f(p) + \sum \alpha_i \mathfrak{L}G_i(p)]$$

for $|\alpha_i| \leqq 1$ is an n-parameter family of C^r-maps in $\bar{W}'$. $\mathfrak{L}$ is any $(M, E_\nu, r, \Delta, \zeta)$-functional for Δ large enough and ζ small enough near p_0.

Let θ be a C^r-map of Q_m into a neighborhood W of p_0; we will determine the size of W later. Set $W' = \theta(Q'_m)$. Let $\lambda(p)$ be of class C^r in M, $= 1$ in $\bar{W}'$, and $= 0$ in $M - W$ (see §20). Let P be the tangent plane to N at q_0, let $y_1, \cdots, y_\nu$ be rectangular axes in E_ν such that P is the $(y_1, \cdots, y_n)$-plane, and let v_i be the unit vector in the direction of y_i. Set

$$(32.2) \qquad G_i(p) = \lambda(p) v_i \text{ in } M \qquad (i = 1, \cdots, n).$$

In using Lemma 23, let $P(q)$ be the normal plane to q in N. Then obviously

$$(32.3) \qquad \frac{\partial H(q_0)}{\partial y_j} = v_j \ (j = 1, \cdots, n), \text{ and } = 0 \qquad (j = n + 1, \cdots, \nu).$$

Putting (32.2) in (32.1) and differentiating gives therefore

$$(32.4) \qquad \frac{\partial f_\alpha(p_0)}{\partial \alpha_i} = \sum_{j=1}^{n} [\mathfrak{L}G_i(p_0)]_j v_j = \text{Proj } \mathfrak{L}G_i(p_0),$$

where Proj v is the projection of a vector v in E_ν into P. If we leave out $\mathfrak{L}$ in (32.4), we may choose W so that the resulting vectors $\partial f_\alpha/\partial \alpha_i$ are independent in $\bar{W}'$. We then choose Δ and ζ so that $\mathfrak{L}$ is defined for the terms in (32.1) with $|\alpha_i| \leqq 1$, and so that the vectors $\partial f_\alpha/\partial \alpha_i$ are independent in $\bar{W}'$; then $f_\alpha(p)$ *is an n-parameter family in $\bar{W}'$.*

33. **Certain maps of M into N like rotations plus translations near a point.** Let f be a C^r-map ($r \geqq 1$) of M into N. Take p_0, q_0, etc. as before; we shall choose W, W', and C^r-maps G_{ij}. Let $\bar{E} = E_{(m+1)n}$ be a Euclidean space with coördinates

$$z_{ij} \qquad (i = 1, \cdots, m+1; j = 1, \cdots, n).$$

The subscripts i, j will, in this section, always range over the values shown, unless otherwise stated. Let $\bar{v}_{ij}$ be the unit vector in the direction of z_{ij}. Let θ map Q_m into W. We may consider $(x_1, \cdots, x_m)$ (in Q_m) as coördinates in W, and write

$$x \text{ for } p = \theta(x), \qquad \frac{\partial g(p)}{\partial x_i} \text{ for } \left.\frac{\partial g(\theta(x))}{\partial x_i}\right|_{x=\theta^{-1}(p)} \text{ in } W.$$

Corresponding to any C^r-map g of W into E_ν, define the C^{r-1}-map $\bar{g}$ of W into $\bar{E}$ by

$$\bar{g}(p) = \sum_{j;i \leqq m} \frac{\partial g_j(p)}{\partial x_i} \bar{v}_{ij} + \sum_j g_j(p) \bar{v}_{m+1,j}\,; \tag{33.1}$$

g_j is the j^{th} component of g in E_ν. (Note that j runs to n only.) Define

$$G'_{ij}(p) = \lambda(p) x_i v_j \ (i \leqq m), \qquad G'_{m+1,j}(p) = \lambda(p) v_j\,. \tag{33.2}$$

For these G, (33.1) gives in $\overline{W}'$

$$\bar{G}'_{ij}(p) = \bar{v}_{ij} + x_i \bar{v}_{m+1,j} \ (i \leqq m), \qquad \bar{G}'_{m+1,j}(p) = \bar{v}_{m+1,j}\,. \tag{33.3}$$

These vectors are obviously independent for p in $\overline{W}'$.

The family of maps $f_\beta(p)$ will be defined by

$$f_\beta(p) = H\mathfrak{L}[f(p) + \sum_{i,j} \beta_{ij} G'_{ij}(p)]. \tag{33.4}$$

As before, we find

$$\frac{\partial}{\partial \beta_{ij}} f_\beta(p_0) = \text{Proj } \mathfrak{L} G'_{ij}(p_0).$$

As $\text{Proj } G'_{ij}(p) = G'_{ij}(p)$, the vectors $\overline{\text{Proj } \mathfrak{L} G'_{ij}(p_0)}$ are linearly independent for small enough ζ. It is easily seen that the operations of passing from g to $\bar{g}$ and of differentiating are permutable; hence, as before, we may take W and W', Δ, and ζ so that the vectors $\partial \bar{f}_\beta(p)/\partial \beta_{ij}$ are linearly independent in $\overline{W}'$. Hence $\bar{f}_\beta$ *is an* $(m+1)n$-*parameter family of* C^{r-1}-*maps of* $\overline{W}'$ *into* $\bar{E}$.

34. **Proof of** (B) **and** (C), §9. In the hypothesis of (B), $n \geqq 2m+1$. As regularity is taken care of by (e) of the theorem, we may first replace the given map by a regular C^r-map; let the new map be f. As f is locally (1-1), we may find a positive continuous function $\delta(p)$ in M such that $f(p)$ is (1-1) for p within $\delta(p_0)$ of p_0, for any p_0. For each p_0 in M there is a neighborhood W as in §32; moreover, these may be taken arbitrarily small. Hence we may choose such

neighborhoods $W_1, W_2, \cdots$ such that $W'_1 + W'_2 + \cdots$ covers M, any compact subset of M has points in common with but a finite number of the $\bar{W}_s$, and if p is in $\bar{W}'_s + \bar{W}'_t$, $\bar{W}_s \cdot \bar{W}_t \neq 0$, then $\bar{W}_s + \bar{W}_t$ lies within $\delta(p)$ of p. Next define the maps $G^s_1, \cdots, G^s_n$ $(s = 1, 2, \cdots)$ as in §32. We may suppose that the η of the theorem is so small that any f' approximating $(f, M, 1, \eta)$ is (1-1) in all $\bar{W}_s + \bar{W}_t$ with $\bar{W}_s \cdot \bar{W}_t \neq 0$.

Arrange the pairs of numbers (s, t) for which $\bar{W}_s \cdot \bar{W}_t = 0$ in a sequence. For any k, let (s, t) be the k^{th} member of the sequence, and let Ω_k be the property of maps f' which holds if $f'(\bar{W}'_s)$ does not intersect $f'(\bar{W}'_t)$. The property of f' of being (1-1) is the sum of these properties. We shall show that Ω_k is an $[f, r, \eta, \Delta, \zeta]$-property. (a), (b) and (c) of §7 hold; we shall prove (d′), with W, W', G_i replaced by $W_s + W_t$, $W'_s + W'_t$, G^s_i. Given f' and $\mathfrak{L}$, set

$$f'_{\alpha^s}(p) = H\mathfrak{L}\left[f'(p) + \sum_{i=1}^{n} \alpha^s_i G^s_i(p)\right]. \tag{34.1}$$

Applying §32, we now see that f'_{α^s} is an n-parameter family of C^r-maps $(r \geqq 1)$ in $\bar{W}'_s$. As $f'(\bar{W}'_t)$ is of zero $(m + 1)$-extent in N, $\bar{W}'_s$ is of finite m-extent in M, $(m + 1) + m \leqq n$, and $f'_{\alpha^s} = f'$ in W_t, there is an arbitrarily small α^s such that $f_{\alpha^s}(\bar{W}'_s) \cdot f_{\alpha^s}(\bar{W}'_t) = 0$ (Lemma 16), and (d′) is proved.

To prove (C), §9 we proceed as above. Let $K = K_1 + K_2 + \cdots$ be the subset of N, each K_s being of zero $(n - m)$-extent. Arrange the pairs (s, t) in a sequence, and let Ω_k hold if $f'(\bar{W}'_s)$ does not intersect K_t. The proof runs now exactly as above.

35. **Proof of** (A) **and** (D), §9. As before, we may suppose that the given map is regular. If $r = 0$ or 1, we may at the beginning replace the map f by a C^2-map (Lemma 22). Hence we suppose that $r \geqq 2$. Define $\delta(p)$ and the W_s, W'_s exactly as in §34, and define the G'_{ij} as in §33. Again, let k correspond to (s, t), and let Ω_k be the property of maps f' which holds if f' has at most regular singularities in $\bar{W}'_s + \bar{W}'_t$; complete regularity is the sum of these properties. We must prove (d′) for Ω_k.

Before proceeding, consider (D). Let $V_1, V_2, \cdots$ be admissible neighborhoods in N', and let Ω^*_k hold if $\bar{W}'_s$ intersects $\bar{V}_t$ only in the proscribed manner. If $n' = m$, this is the same as stating that $f'(\bar{W}'_s) + \bar{V}_t$ has at most regular singularities. If we show how to transform $\bar{W}'_s$ with reference to $\bar{V}_t$, the same process transforms $\bar{W}'_s$ with reference to $\bar{W}'_t$; hence we need merely prove (d′) for (D).

Given f' and $\mathfrak{L}$, set

$$f'_{\beta^s}(p) = H\mathfrak{L}\left[f'(p) + \sum_{i,j} \beta^s_{ij} G'^s_{ij}(p)\right], \tag{35.1}$$

To each f' corresponds an $(m + 1)n$-parameter family of maps $\bar{f}'_{\beta^s}$ of $\bar{W}'_s$ into $\bar{E}$, by §33. We shall show that if $\bar{f}'_{\beta^s}(\bar{W}'_s)$ avoids a certain set S, then $f'_{\beta^s}(\bar{W}'_s)$ intersects $\bar{V}_t$ in the proper manner. S will be the sum of a denumerable number of sets of finite $(m + 1)(n - 1)$-extent. As the number of parameters in $\bar{f}'_{\beta^*}$ is $(m + 1)n$ and the dimension of $\bar{W}_s$ is m, we may apply Lemma 16 and make $\bar{W}'_s$ avoid any of these. Applying the process in the proof of Lemma 12, we may make $\bar{W}'_s$ avoid S, and thus complete the proof.

We shall leave out the indices s, t in what follows. The vector $\partial' f'_\beta(p)/\partial x_i$ in P (see §32) with components $\partial f'_{\beta j}(p)/\partial x_i$ $(j = 1, \cdots, n)$, by definition, (see §33) is the projection of the vector $\partial f'_\beta(p)/\partial x_i$ in E_ν into P. As f'_β is regular (for small β^s_{ij}), the latter vectors $(i = 1, \cdots, m)$ are independent. As the W'_s may be taken arbitrarily small, we may suppose that the $\partial' f'_\beta(p)/\partial x_i$ are independent in $\bar{W}'$. As N' may be cut into arbitrarily small pieces, we may suppose that $\bar{V}$ projects in a (1-1) regular manner into P (if $\bar{V}$ intersects $f'_\beta(\bar{W}')$ for any β). As $f'_\beta(\bar{W}')$ and $\bar{V}$ are both in N, and the projection of N into P is regular near q_0, they intersect in an allowable manner in N if and only if their projections do in P. Let $P\bar{V}$, Pq etc. denote the projections of $\bar{V}$, q, etc. into P.

Let p be a point of $\bar{W}'$, and q, a point of $\bar{V}$; we shall consider under what conditions $Pf'_\beta(p) = Pq$, the intersection being of an unallowable character. This is so if the vectors $\partial' f'_\beta(p)/\partial x_i$ determine a plane P_m in P which has a plane P_h of dimension $h > k = m + n' - n$ in common with the plane $P_{n'}$ tangent to $P\bar{V}$ at $q' = Pq$. Hence the set S in $\bar{E}$ which $\bar{f}'_\beta(\bar{W}')$ must avoid is the set of points z with the following property. For some q in $\bar{V}$, some $h > k$ and plane P_h in the tangent plane $P_{n'}$ to $P\bar{V}$ at $q' = Pq$, and some plane P_m which contains q' and has exactly P_h in common with $P_{n'}$, the last n coördinates of z determine (in P) the point q' and the first mn coördinates determine the direction of P_m. Let z_0 be that point of S we have just described; we shall consider that part of S_h near z_0, S_h being those points of S with this corresponding h.

A point z of S_h is determined by the set

$$(q, P_h, P_m, v), \quad \text{where} \quad v = (v_1, \cdots, v_m), \qquad v_i = \frac{\partial' f'_\beta(p)}{\partial x_i},$$

q, P_h, P_m, v being chosen in the order given. (The last n coördinates of z are then determined.) q runs over a set of dimension n'. Now keep q fixed, and vary P_h. P_h lies in $P_{n'}$ and contains q'; the dimension of such a set of planes is $h(n' - h)$. Next vary P_m. It is determined by naming a plane P_{m-h} through q' in the plane P_{n-h} through q' orthogonal to P_h; hence the set of planes P_m is of dimension $(m - h)(n - m)$. Finally, vary the vectors v_i. Each one may vary freely as long as it remains in P_m; hence the dimension of this set of positions of the set of vectors is m^2. Consequently, if we set $h = k + h'$ and note that $n' - h = n - m - h'$, the above set runs over a part of a Euclidean space of dimension

$$\begin{aligned} d_h &= n' + h(n' - h) + (m - h)(n - m) + m^2 \\ &= n' - hh' + (h + m - h)(n - m) + m^2 \\ &= mn + n' - hh' \leqq mn + n' - h \leqq mn + n - m - 1. \end{aligned}$$

The map of this set into the corresponding part of S_h is of class C^1, and hence the $(m + 1)(n - 1)$-extent of the latter is finite, as required.

36. **Completion of the proof of Theorem** 6. Let $\theta_1, \theta_2, \cdots$ be a completely admissible set of maps in that part M^* of M' for which $0 < t < 1$. We take them so that if $\bar{U}_i \cdot \bar{U}_j \neq 0$, then f is (1-1) in $\bar{U}_i + \bar{U}_j$. In applying Theorem 2 in the proof of Theorem 6, let us introduce (f, r, η)-properties as follows. Arrange the pairs of numbers (i, j) in a sequence. Let the k^{th} member be (i, j). Then Ω_k holds for $f(p \times t) = \phi_t(p)$ if the following is true. If $0 < t < 1$, $p \times t$ is in $\bar{U}'_i$, and $q \times t$ is in $\bar{U}'_j$, then $f(p \times t) \neq f(q \times t)$. We shall show that each property is an (f, r, η)-property (for small enough η). It will follow that *each* $\phi_t(0 < t < 1)$ *may be made* (1-1), *irrespective of* M_0 *and* M_1. The new map of M' into N is of class C^r, by Lemma 10.

We may suppose the U_l are so small that if $\bar{U}_i \cdot \bar{U}_j \neq 0$, then $\bar{U}_i + \bar{U}_j$ is in some V_h in N, and so that vectors $v_1, \cdots, v_{n-m}$ may be chosen in E_n with the following property. Set $f'_h(p \times t) = \chi_h^{-1} f(p \times t)$, and let $U'_j(t)$ be the set of points $p \times t$ of U'_j whose second coördinate is t. If P and P' are the m- and $(n - m)$-planes through the origin in E_n, the first being orthogonal to and the second parallel to $v_1, \cdots, v_{n-m}$, then any $f'_h(\bar{U}_j(t))$ projects into a subset of P so that both the projection and its inverse are of class C^r. The points

$$q = p + \sum_{s=1}^{n-m} \alpha_s v_s, \qquad p \text{ in } f'_h(\bar{U}_j(t)),$$

fill out an open set R_t in E_n, and if solving this (see §3) gives

$$p = H(q, t), \qquad \alpha_s = \Phi_s(q, t),$$

then H and the Φ_i are of class C^r. To any C^r-map g of U_i into Q_n let $\bar{g}$ be the map of U_i into E_{n-m} whose s^{th} component is

$$\bar{g}_s(p \times t) = \Phi_s(g(p \times t), t).$$

Define the family of maps

$$g_\beta(p \times t) = f'_h(p \times t) + \sum_{s=1}^{n-m} \beta_s v_s \text{ in } \bar{U}_i;$$

then the $\bar{g}_\beta$ have the property that

$$\frac{\partial}{\partial \beta_s} \bar{g}_\beta(p \times t) = v_s .$$

Hence the $\bar{g}_\beta$ form an $(n - m)$-parameter family in E_{n-m}. As $\bar{U}_i$ is of dimension $m + 1 < n - m$, there is an arbitrarily small β such that $\bar{g}_\beta(p \times t) \neq 0$ in $\bar{U}_i$ (see Lemmas 14–16). For this β, no $g_\beta(\bar{U}'_i(t))$ intersects any $f'_h(\bar{U}'_j(t))$ (same t), for no $\Phi_s(g_\beta(p \times t), t) = 0$. Consequently, using $\lambda(p)$ etc. as in §20, we prove (d) of §7. The other properties are obvious, and the statement is proved.

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