Nov. 20

- Excision & Exact sequence of pair (or triple)
- local homology - invariance of dimension.
- Equivalence of simplicial & singular homology:

$X =$ Δ-complex, A a subcomplex,

$$H_n^{\Delta}(X, A) \xrightarrow{\cong} H_n(X, A)$$

$h : U \longrightarrow V$ $\quad h(x)$

$$H_*(U, U-x) \xrightarrow{\cong} H_*(V, V-h(x))$$

$$\text{only } * = n \qquad \text{only } * = m$$

with $H_*(U, U-x) \neq 0$ only for $* = n$, and $H_*(V, V-h(x)) \neq 0$ only for $* = m$

$\Rightarrow n = m$.

$X =$ Δ-complex

$\Delta^n(X) = \{$linear combinat. of simplices

$\Delta_n(X,A) = \Delta_n(X) / \Delta_n^n(A)$

$C_n(X,A) =$ singular chains

Same homology ?

$\Delta_*(\Delta^n, \partial\Delta^n) = \mathbb{Z}$ in $* = n$

$H_*(\Delta^n, \partial\Delta^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & * \neq n \end{cases}$

X $\cup$ A $\quad X^n =$ n-skeleton

assume $X = X^n$ for some n

Thm $H_*^\Delta(X,A) \xrightarrow[\cong]{i} H_*(X,A)$ is iso.

Pf $X^k =$ k-skeleton

$$H_{n+1}(X^{k-1}) \to H_{n+1}^{k}(X^{k}) \to H_{n+1}(X^k, X^{k-1}) \to H_n$$

Five Lemma

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

$\alpha, \beta, \delta, \varepsilon$ iso's $\Rightarrow$ γ is iso.

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a) β, δ onto, ε injective $\Rightarrow$ γ onto

Pf. $c' \in C'$. $k'(c') = \delta(d)$ (δ onto)

$\varepsilon l(d) = l'\delta(d) = 0$ (exactness)

$\Rightarrow l(d) = 0 \Rightarrow d = k(c)$

$k'(c' - \gamma(c)) = 0$ since $k'(c') - k'\gamma(c)$

$= k'(c) - \delta k(c) = 0$

$\therefore$ $c' - \gamma(c) = j'(b')$. $b' = \beta(b)$

$\gamma(c + j(b)) = \gamma(c) + j'\beta(b)$

$= \gamma(c) + c' - \gamma(c) = c'$

Nov 27

- Five Lemma
- Thm $H_n^{\Delta}(X,A) \to H_n(X,A)$ is iso

Pf Case 1. $\dim X < \infty$ & $A=\emptyset$.

$$\begin{array}{ccc} H_{n+1}^{\Delta}(X^k, X^{k-1}) \to & & \to H_{n-1}^{\Delta}(X^{k-1}) \\ \downarrow & & \downarrow \\ H_{n+1}(X^k, X^{k-1}) \to & & \to H_{n-1}(X^{k-1}) \end{array}$$

Five Lemma + Induction on k.

Case 2 $\dim X$ can be ∞, $A=\emptyset$

Based on compact set C meets only finitely many open simplices

$\Rightarrow$ singular cycle z supported on some k-skeleton

Also, if $z=\partial y$ y is supported on compact set

General: $A \neq \emptyset$, Use Five Lemma

- Remarks on Euler characteristic.

Section 2.2. Degree of a map

$f: S^n \to S^n$. $f_*: \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{Z} \qquad\qquad \mathbb{Z}$

So f_* is mult by m.

$\deg f = m$

(a) $\deg(\mathbb{1}) = 1$

(b) f not surjective $\Rightarrow \deg f = 0$

(c) $f \simeq g \Rightarrow \deg f = \deg g$

(d) $\deg(fg) = \deg f \deg g$

(e) f = reflection r across equator

$\deg(f) = -1$ (simplicial) homology

(f) antipodal map $= -\mathbb{1}$ has $\deg = (-1)^{n+1}$

(g) f has no fixed pt

$\Rightarrow \deg f = (-1)^{n+1}$

(h) $\Sigma f : \Sigma S^n \to \Sigma S^n$

$\deg(\Sigma f) = \deg f$

<u>Pf of (g)</u> $f \simeq -\mathbb{1}$.

$t \to (1-t) f(x) - tx$

<u>Hairy Ball Thm</u> S^n has continuous family of nonzero tangent vectors $\iff$ n is odd

<u>Pf</u> homotopy between $\mathbb{1}$ & $-\mathbb{1}$

<u>Prop</u> n even. Then $\mathbb{Z}_2$ is only nontrivial gp which acts freely on S^n.

<u>Pf</u> $G \hookrightarrow \mathrm{Homeo}(S^n)$

$\deg : G \to \{\pm 1\}$

Nontrivial elements $\to (-1)^{n+1}$

Nov 28

π_1 and H_1.

Thm. X path connected. Then

$$\pi_1(X, x_0)^{ab} = H_1(X)$$

Idea: $f : (I, \partial I) \to (X, x_0)$ a loop, then f is a singular 1-cycle

Ex. $X = S^{2n+1} / \mathbb{Z}/m$ is a lens spac.
Then $\pi_1(X) = \mathbb{Z}/m \Rightarrow H_1(X) = \mathbb{Z}/m$

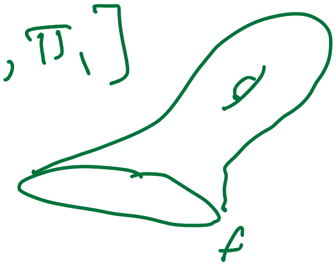
Consider paths $f : I \to X$
$f \simeq g$ means homotopic rel endpoints
$f \sim g$ means homologous

1) $f = \text{constant} \Rightarrow f \sim 0$
2) $f \simeq g \Rightarrow f \sim g$
3) $f \cdot g \sim f + g$
4) $\bar{f} \sim -f$

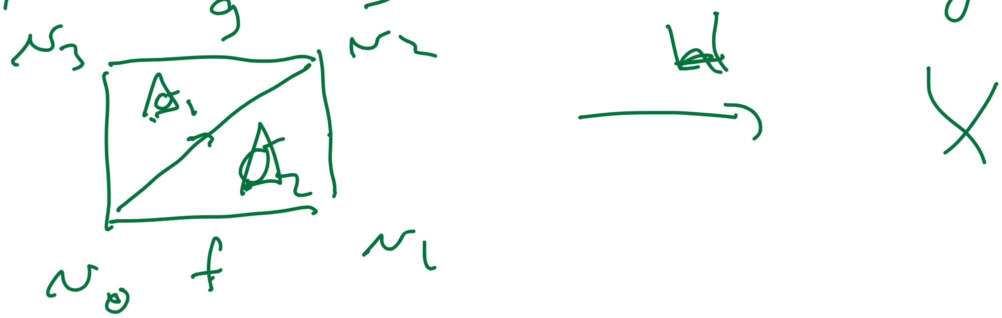
Define $h : \pi_1(X, x_0) \to H^1(X)$
h factors through $\pi_1(X, x_0)^{ab}$

Show

- h is surjective
- $\ker h \subset [\pi_1, \pi_1]$



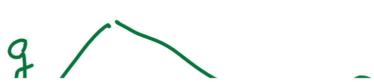
2) $f \simeq g \implies f \sim g$

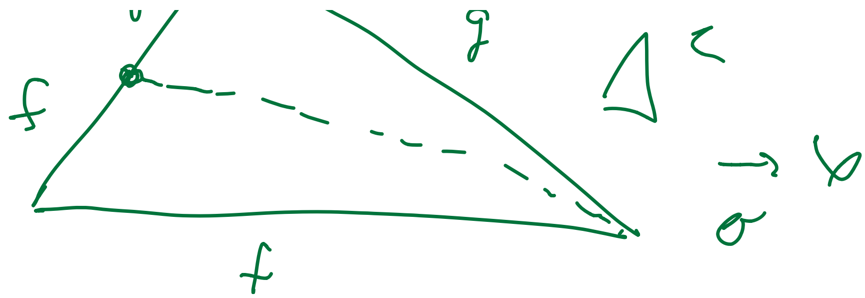


$$\partial(\sigma_2 - \sigma_1) = f + * - H_{[v_0, v_2]} - (* + g - H)_{[v_0, v_2]}$$

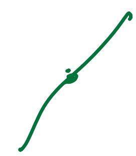
$$\implies f \sim g$$

3) $f \cdot g \sim f + g$





project Δ^2 onto edge



Do $f \cdot g$ on edge

$$\partial\sigma = f \cdot g - (f + g)$$

$\overline{f}$ = run f in the other direction

$$\overline{f} \sim -f$$

PS $f \cdot \overline{f} \simeq$ constant ~ 0

$$f \cdot \overline{f} \sim f + \overline{f}$$

Define

$$h : \pi_1(X, x_0) \longrightarrow H_1(X)$$

$h: [f]_{hom} \longmapsto [f]$

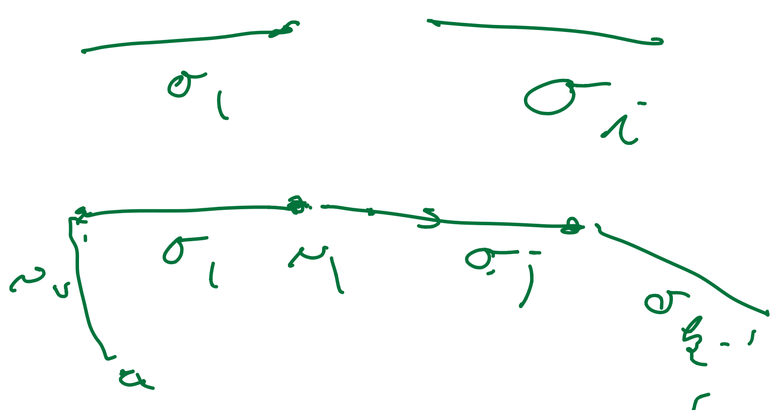
well-defined homo

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr] \arrow[dr, bend right] & & H_1(X) \\
& \tilde{h}: \pi_1^{ab} \arrow[ur, "i"'] &
\end{tikzcd}$$

Side Remarks

Any m-cycle in X for $m = 1, 2$ can be represented from m-mfld into X (mfld = Δ-complex)

$m = 1$ $\qquad \sum n_i \sigma_i = \sum \sigma_i$



$\sigma_i : \Delta^1 \to X$

$\longrightarrow X$

— — — .

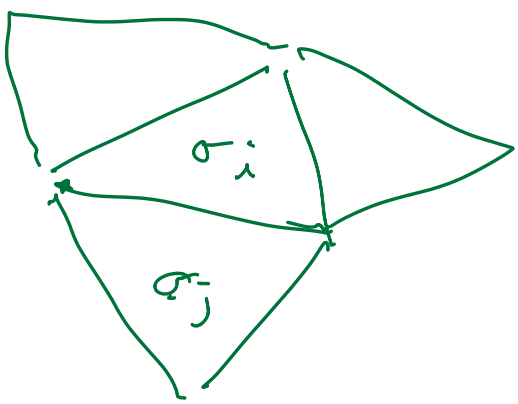
2) $\sum n_i \sigma_i = x$

$\sigma_i : \Delta^m \to X$

$\partial(\sum n_i \sigma_i) = 0$

$n\sigma_a = \underbrace{\sigma + \ldots + \sigma}_{n}$

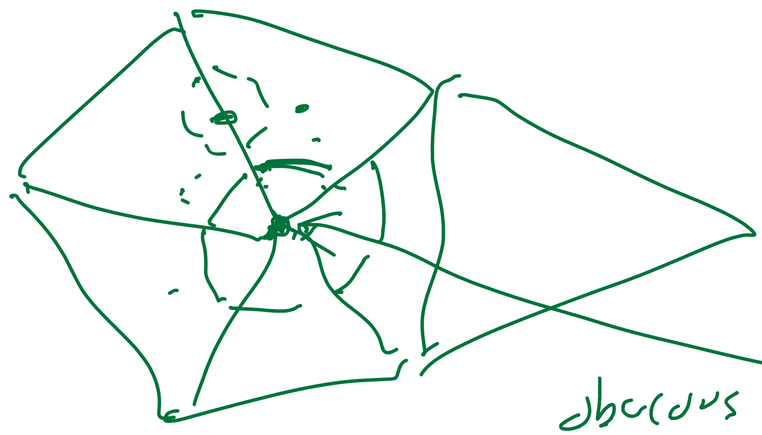
Assume all $n_i = \pm 1$



K = Δ-complex ~~formed~~ formed by gluing copies Δ^m ~~corresp~~ where σ_i & σ_j cancel in pairs

$K = \coprod \Delta_{\alpha}(k)$

glued along codim 1 face



Nbhd of vertex = cone on S^1

Comment This works
m-cycle
$f \quad K \longrightarrow X$

K is mfld on complement of K^{n-2}

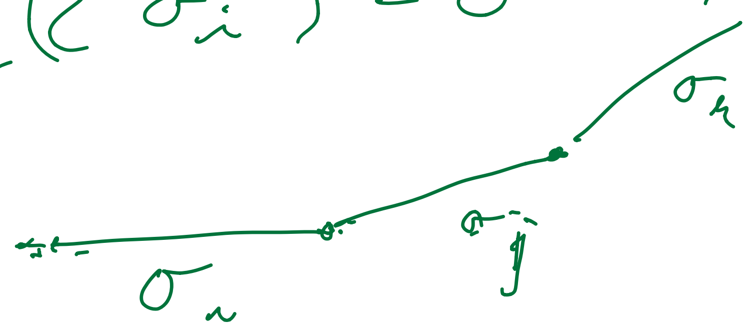
$h : \pi_1 \longrightarrow H_1$ is onto

Pf $\sum n_i \sigma_i$ = 1-cycle

as remarked can assume all $n_i = \pm 1$

$-\sigma \sim \bar{\sigma}$ so all n_i are $= 1$.

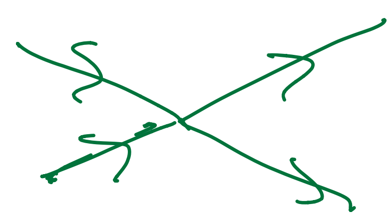
$\partial\left(\sum \sigma_i\right) = 0$



$\sigma_i - \sigma_j$

cancelling in pairs

can each σ_i is a loop.



join endpoint σ_i by path γ_i to x_0

$\gamma_i \sigma_i \bar{\gamma}_i \sim \sigma_i$

$\therefore$ each σ_i is a loop at x_0 with same base

$\sum \sigma_i \sim$ single loop σ of f

$\therefore$ h is onto.

$\bar{h} : \pi_1^{ab} \longrightarrow H_1$

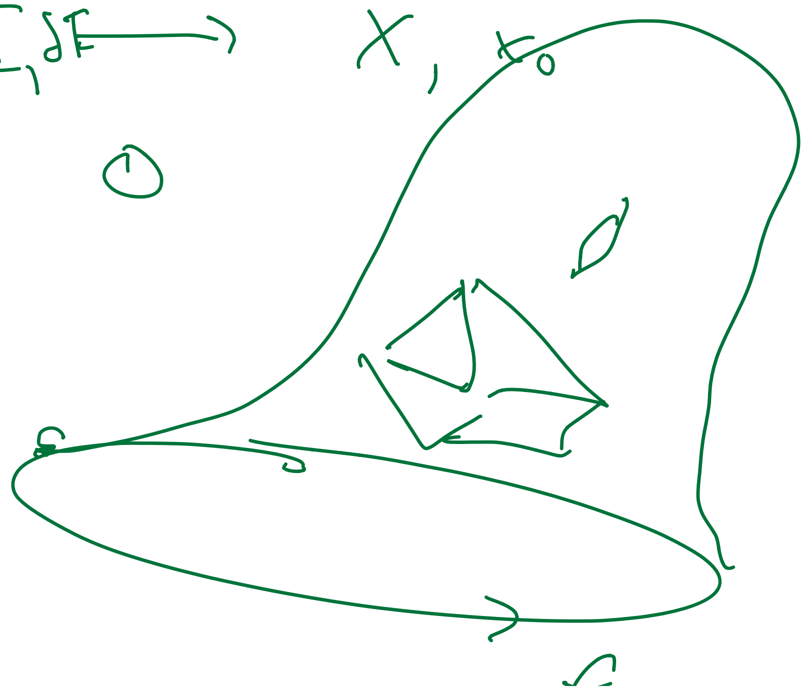
Show injection

$\ker h \supseteq [\pi_1, \pi_1]$

$f : I, \partial I \longrightarrow X, x_0$

$f \sim 0$

Show

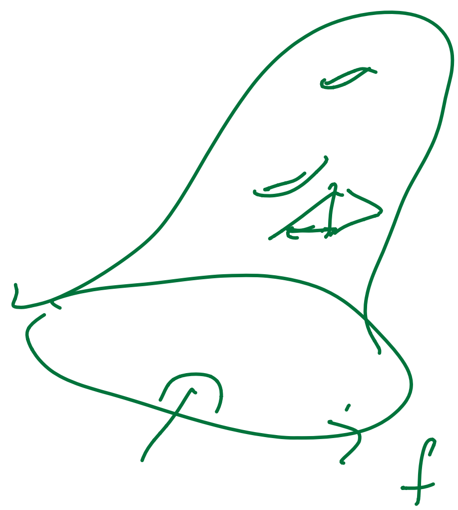


Pf $\quad f = \partial\left(\sum n_i \sigma_i\right)$

as before we construct

Δ-complex K

= 2-dim surface
with one one
∂-circle where
it is f on ∂

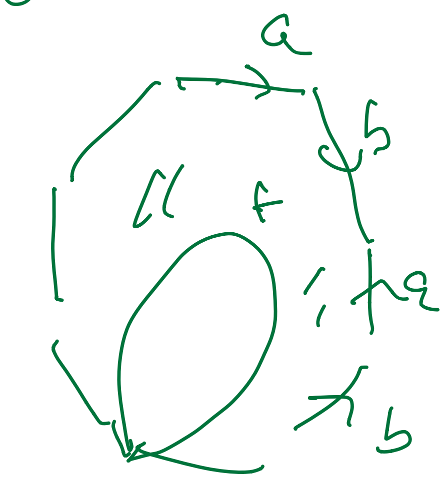


missing disk.

Key fact Orientable

remove disk, the

∂ is a commutator



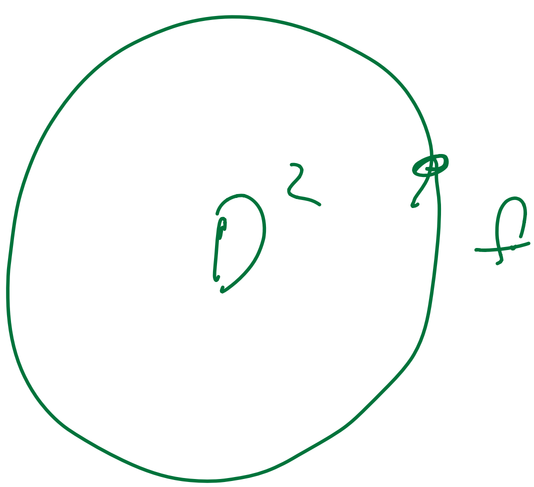
π_1(closed surface) $= \langle a_1, b_1 \dots a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g]$

f''

$f \in [\pi_1, \pi_1]$

Proves injectivity.

Difference between π_1 & H.



π_1

$$\int_{\gamma c} \omega = \int_c d\omega \qquad f$$