

Oct. 23. HW 131 1-4

• Categories (Def'n) \mathcal{C}

- A collection of objects, $\text{Ob}(\mathcal{C})$
- For any 2 objects X, Y , a set of Morphisms $\text{Hom}(X, Y)$
- $\mathbb{I}_X \in \text{Hom}(X, X)$
- Compositions $\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$ s.t
 $f \circ \mathbb{I}_X = f$, $\mathbb{I}_Y \circ f = f$ +
associative.

Functors (Def'n)

A functor F from \mathcal{C} to \mathcal{D}
assigns to each object X of \mathcal{C}
an object $F(X)$ of \mathcal{D} and
to each $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ a
morphism

$$F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

s.t $F(\mathbb{I}_X) = \mathbb{I}_{F(X)}$ &

$$F(f \circ g) = F(f) \circ F(g)$$

This is a covariant functor

There are also contravariant functors
 (Sometimes covariant functor = functor
 + contra variant functor = cofunctor)

Axioms for a homology theory

Section 2.3

- A sequence of functors H_n
 from CW pairs (X, A) & continuous
 maps } to abelian groups,
 indexed by $n \in \mathbb{N} (= \mathbb{Z}_{\geq 0})$.
Note: $H_n(X) = H_n(X, \emptyset) =$ abelian
 Also $H_n(f) = f_*$ s.t. $\circ f_* = f \circ$
- $f \sim g : (X, A) \rightarrow (Y, B)$
 then $f_* = g_* : H_n(X, A) \rightarrow H_n(Y, B)$
- \exists boundary homomorphisms δ giving long exact sequence
 $\delta : H_n(A) \rightarrow H_{n-1}(A)$ giving long exact sequence of pair
 $\xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xleftarrow{\delta} H_{n-1}(A) \rightarrow H_{n-1}(X) \rightarrow \dots$

a.t following commutes

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) \\ \downarrow f_* & & \downarrow f_* \\ H_n(Y, B) & \xrightarrow{\partial} & H_{n-1}(B) \end{array}$$

• If $X = \bigvee_\alpha X_\alpha$ &

$i_\alpha: X_\alpha \rightarrow X$ is inclusion

then $\bigoplus [i_\alpha]_*: \bigoplus_\alpha H_n(X_\alpha, p^+_\alpha) \rightarrow H_n(X, p^+)$
is isomorphism.

• (Dimension Axiom)

$$H_n(p^+) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases} .$$

usually

Intuitive discussion of
homology & homotopy groups
graphs, 2 dim simplicial complexes

Historically: H_n defined for

cell complexes

Singular Homology theory
defined all spaces &
continuous

maps X

ase 0 No edges only
vertices

$$H_0(X) = \bigoplus_{\text{vertices}} \mathbb{Z}$$

$$\Delta_0(X) = \text{group of } \partial\text{-chains}$$

edge = ordered pair of
vertices

$$\text{edge} = [n_0, n_1] \xrightarrow{n_0 \rightarrow n_1}$$

$$\partial[n_0, n_1] = [n_1] - [n_0] \in \Delta_0$$

-cycles

$\Delta_1(X)$ = free gp on edges of X .

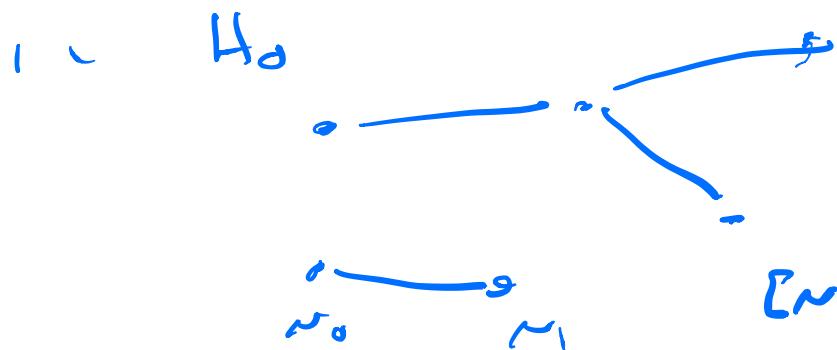
Extend ∂ to a homomorphism

$$\partial: \Delta_1(X) \rightarrow \Delta_0(X)$$

$$\sigma = [n_0, n_1] \quad \partial\sigma = [n_1] - [n_0]$$

$$\partial \sum n_\alpha \sigma = \sum n_\alpha \partial \sigma$$

$$H_n = \frac{\ker \partial: \Delta_n \rightarrow \Delta_{n-1}}{\text{Im } \partial: \Delta_{n+1} \rightarrow \Delta_n}$$

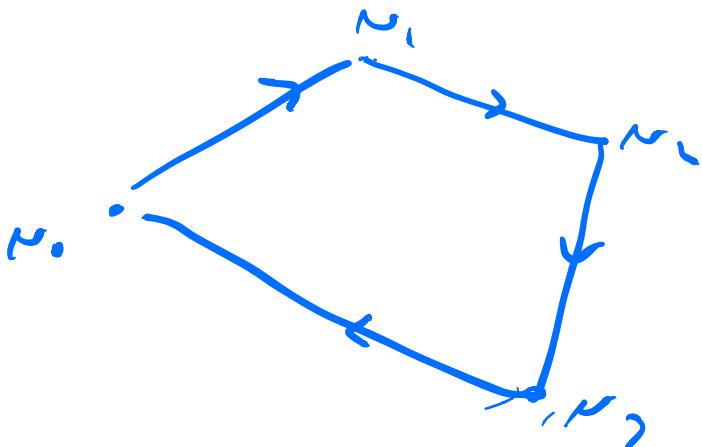


$$\begin{aligned} & [n_1] - [n_0] \in \text{Im } \partial \\ & [n_1] - [n_0] = 0 \\ & \text{In } H_0(X) \end{aligned}$$

Conclusion $H_0(X)$ = free abelian gp on connected

components

1 example of 1-cycle



$$= [n_0, n_1] + [n_1, n_2] \\ + \cancel{[n_2, n_3]} - [n_0, n_3]$$

$$\partial = \cancel{[n_1]} - [n_0] + \cancel{[n_2]} - \cancel{[n_1]} \\ \cancel{[n_3]} - \cancel{[n_2]} = \cancel{([n_3] - [n_0])}$$

$$= 0$$

is cycle

provided $\delta_1 : \Delta_2(x) \rightarrow \Delta_1(x)$

s o map

class

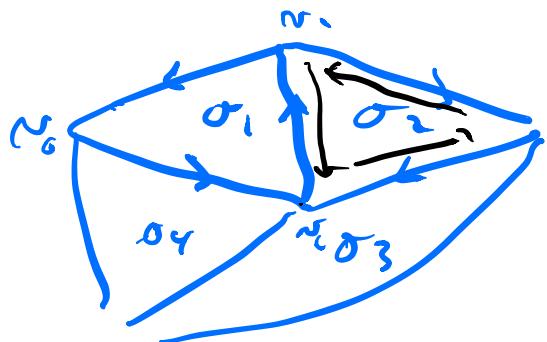
ι represents a
in $H_1(X)$

fact $H_1(X) =$ abelianization
of $\pi_1(X, x_0)$

implex = higher dim'l
triangle

simplicial complexes, Δ -complex

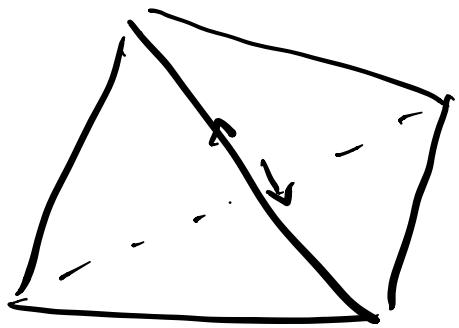
intuitive idea of γ -cycle



$$\text{-cycle } x = \sum n_i \sigma_i$$

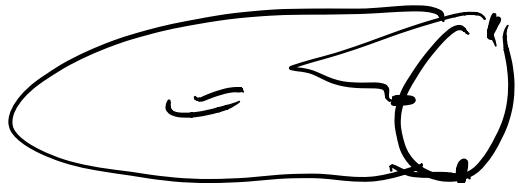
$x = \sum_{\text{over edges}} n_i \sigma_i$
 \Rightarrow edge between $\sigma_1 \& \sigma_2$
coefficients from $\sigma_1 + \sigma_2$
1 edge should occur
ith opposite sign.

link of 2-dim'l
= orientable closed surface.

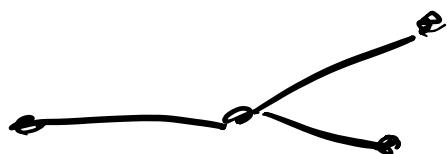


is 4 triangles $\sigma_1, \dots, \sigma_4$

$$\sum \pm \sigma_i = 0$$



not a 2-cycle



not 1-cycle

O t.25: Simplices and Simplicial
complexes. (HW 2.2)
Next Friday (1-4)
Prove Lemma 2.1

- A few preliminaries.
- affine space, convex combinations of points, affine maps, affinely independent points
- an n -simplex spanned by $p_0, \dots, p_n = \left\{ \sum t_i p_i \mid \sum t_i = 1, t_i \in [0, 1] \right\}$
- the standard n -simplex in \mathbb{R}^{n+1}
- lemma: Given a function from vertices of $[p_0, \dots, p_n]$ to another affine space $P_i \rightarrow \mathbb{R}^{n+1}$
by affine map F s.t.
 $F(p_i) = z_i$
- def'n of Simplicial complex is topological space. A special case of CW-complex = a space formed by gluing together simplices
- Def'n of Abstract simplex

plex \mathcal{R} . It is a set of vertices V and subset $\mathcal{R} \subset P(V)$ s.t
 i) $\forall n \in V, \{n\} \in \mathcal{R}$
 ii) $\sigma \in \mathcal{R}, z \subset \sigma \Rightarrow z \in \mathcal{R}$.

• 6 metric Realization of simplicial complex.

• Hatcher's Δ -complexes.

• 1 Rips Complex

A fine space versus vector space

Def An affine space \mathbb{A}

is a set \mathbb{A} together with free, transitive action of vector space V on it

$$I \cap A = \underbrace{\bigcap}_{n=1}^{\infty} \underbrace{A_n}_{\text{neighborhoods}} \cap p + V$$

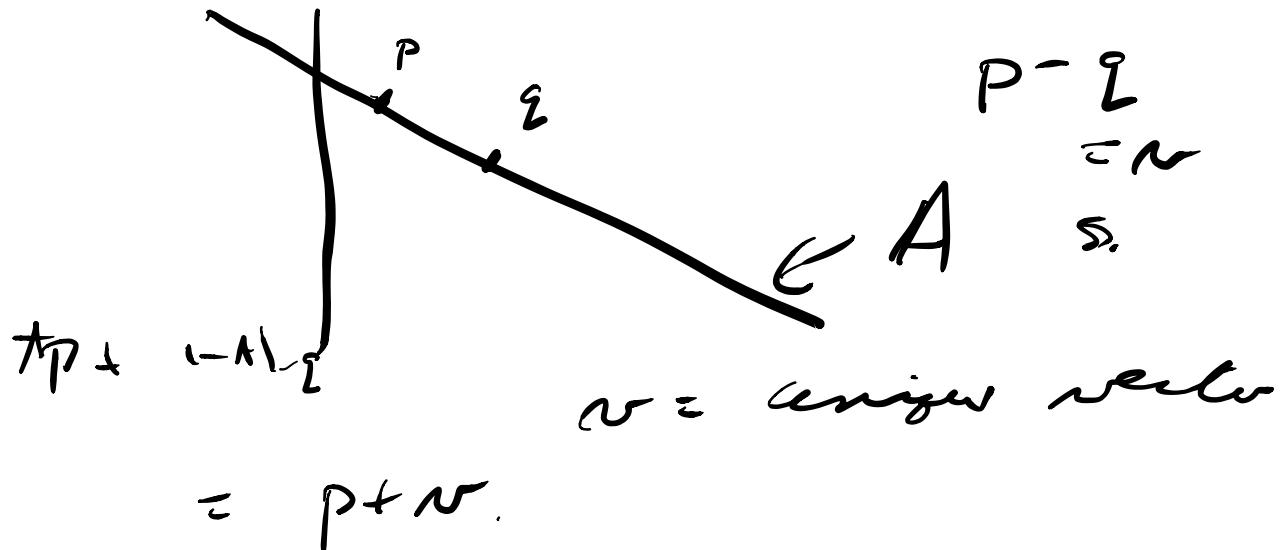
$$V \rightarrow PTV$$

$$A \leftrightarrow V$$

$$P \in \mathbb{R}^{n \times n}$$

pick a point p

E. translate of linear subspace



Affine combination of p_0, \dots, p_n

$$\Leftrightarrow \sum_{i=0}^n t_i p_i := p_0 + t_1(p_1 - p_0) + \dots + t_n(p_n - p_0)$$

$$t_0 = 1 - \sum_{i=1}^n t_i$$

affine map $f: A \rightarrow A'$

$$f(p+v) - f(p) = F(v)$$

F is linear transform.

Def P_0, \dots, P_n are
affinely independent

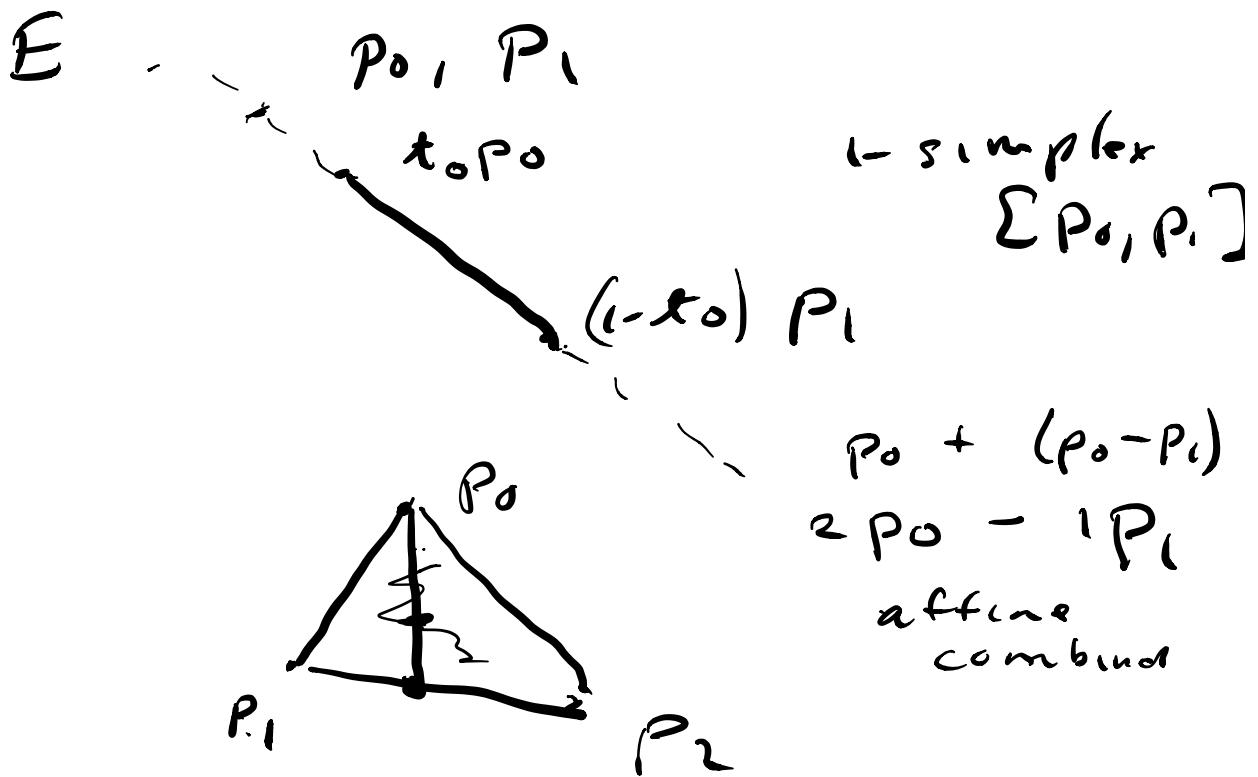
if vectors $P_1 - P_0, P_2 - P_0, \dots, P_n - P_0$
are linearly ind.

Exercise Doesn't depend
choice on P_0

Def $\sum t_i p_i$, $\sum t_i = 1$
is convex combination
(all $t_i \in [0, 1]$).

Def P_0, \dots, P_n are affinely
ind. the n -simplex
spanned by $\{P_0, P_1, \dots, P_n\}$,

$$\left\{ \sum x_i p_i \mid \sum x_i = 1, x_i \in [0, 1] \right\}$$



All errecte def in is

$[P_0, \dots, P_n] = \text{convex hull}$

= Smallest convex
set containing
 P_0, \dots, P_n

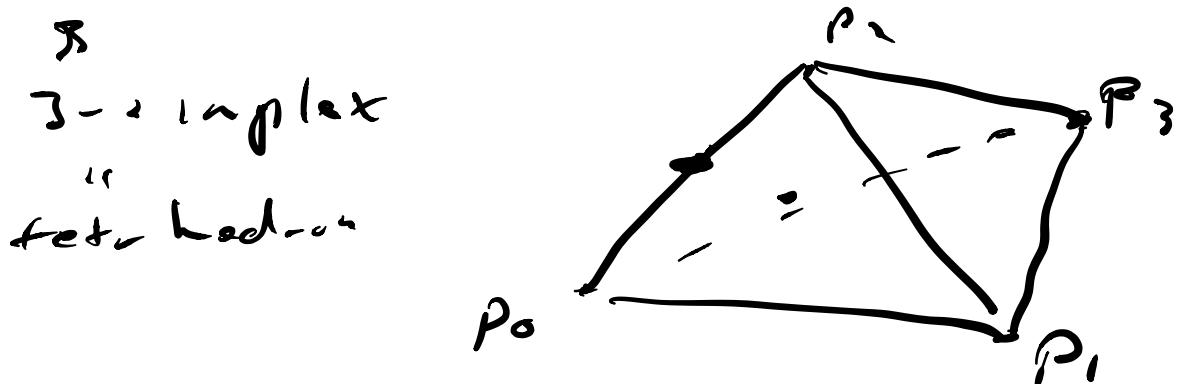
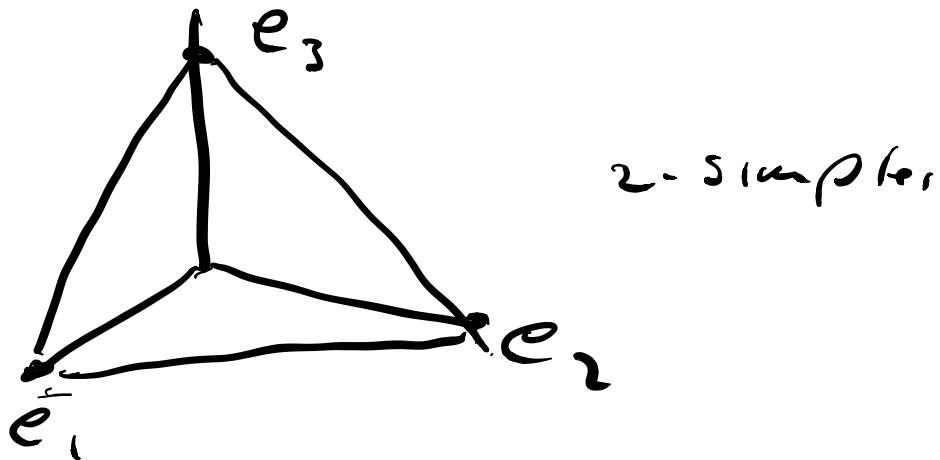
Ter . logy:

ays $p \in [P_0, \dots, P_n]$
has unique expression of

\mathbf{P} - V für n
 $P = \sum_{i=0}^n t_i P_i$
 t_i ($i=1$) are ~~barycentric~~ coordinates

Def the standard n -simplex

in \mathbb{R}^{n+1} is the
n-simplex $[e_1, \dots, e_{n+1}]$



Kre Lemma, If p_0, \dots, p_n
 are affinely independent,

$\& z_i \in A'$ are pre-specified

\exists a affine isomorphism
 $F: A \rightarrow A'$

s.t. $F(p_i) = z_i$

5 placial Cox

R sh def'n - Space formed
by gluing together simplices,
($\nu \in$ affine iso's w/ of
area)

$$\begin{aligned} \text{oriented } [\nu] &= \frac{1}{n+1} \sum p_i \\ [\nu]_{[p_0, \dots, p_n]} &= \sum \frac{1}{n+1} p_i \end{aligned}$$

Require $\sigma \cap \sigma'$
is common faces of $\sigma + \sigma'$

Rank (Δ)-complex almost
same but intersection is
union of faces)



A complex $\overset{\sim}{\longrightarrow}$ glue to
 glue Σ_1 2 simplices 

Def 'n An abstract simplex

cx consists of set V

of vertices &

$$\subset P(V)_{+ \neq \emptyset}$$

\mathcal{R} is a set of finite
empty subsets of V

s.t.

① $\forall n \in V, \{n\} \in \mathcal{R}$.

② If $\sigma \in \mathcal{R}$ $\sigma \neq \emptyset \subset \tau \subset \sigma$,
then $\tau \in \mathcal{R}$.

free Realization as
to . space

1 V = vector space by
basis $(e_n)_{n \in V}$

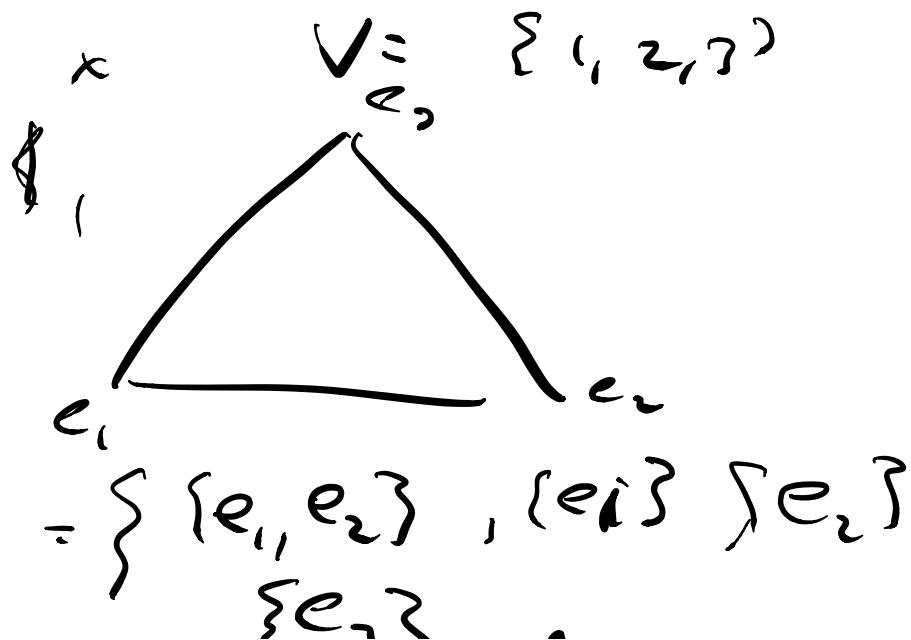
- $\left\{ \sum t_n e_n \mid \begin{array}{l} \text{only finitely} \\ \text{many } t_n \neq 0 \end{array} \right\}$

Δ = Simplex spanned
by the e_i .

$V = \{1, 2, 3\}$, Previous picture

$$\begin{aligned} \text{= abstract complex} \\ \text{g om. realization } & \cancel{\text{for } f = f_{n_1}} \\ \mathbb{E}_1 \quad I = \bigcup_{\sigma \in \mathcal{S}} [\sigma] \end{aligned}$$

$$\begin{aligned} \sigma = n_0, \dots, n_k \quad [\sigma] \\ = [n_0, \dots, n_k] \end{aligned}$$



Th is the top. space.

\mathcal{C} - category
 $= \text{Ob}(\mathcal{C})$

dge = morphism

ℓ_1, ℓ_2 are composed

ℓ_1, ℓ_2 $\ell_1 \circ \ell_2$
 full ~~is not~~ gives a
 -simplex

Oct 27

- n-simplex : $[n_0, \dots, n_n] = \sum t_i n_i$
- face : set one or more of $t_i = 0$
- open simplex = $\text{int}[n_0, \dots, n_n]$, i.e
all $t_i > 0$.

- Geometric def'n of Simplicial complex (union of closed simplices in \mathbb{R}^N) or any space homeo to such a union (weak topology)
- Abstract simplicial ex
- Abstract simplicial ex associated to a poset.
- Rips complex of a subset of a metric space.
- Hatcher's def'n of Δ -complex
 - faces of a single simplex can be identified with each other
 - This allows use of fewer simplices
 - Important to order vertices
- Official def'n (Hatcher p. 101)

A Δ -complex structure on a space X is a collection of maps $\sigma: \Delta^n \rightarrow X$ s.t.

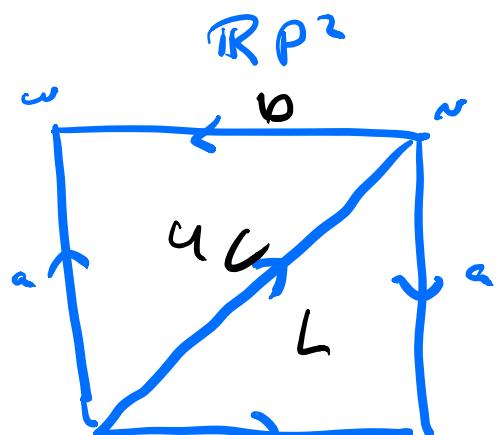
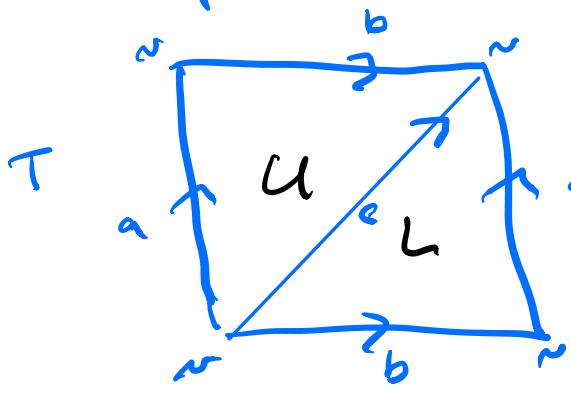
1) σ_α is an embedding on each $\overset{\circ}{\Delta}{}^{n_\alpha}$ (open simplex) and each pt of X lies in unique $\sigma_\alpha(\overset{\circ}{\Delta}{}^{n_\alpha})$

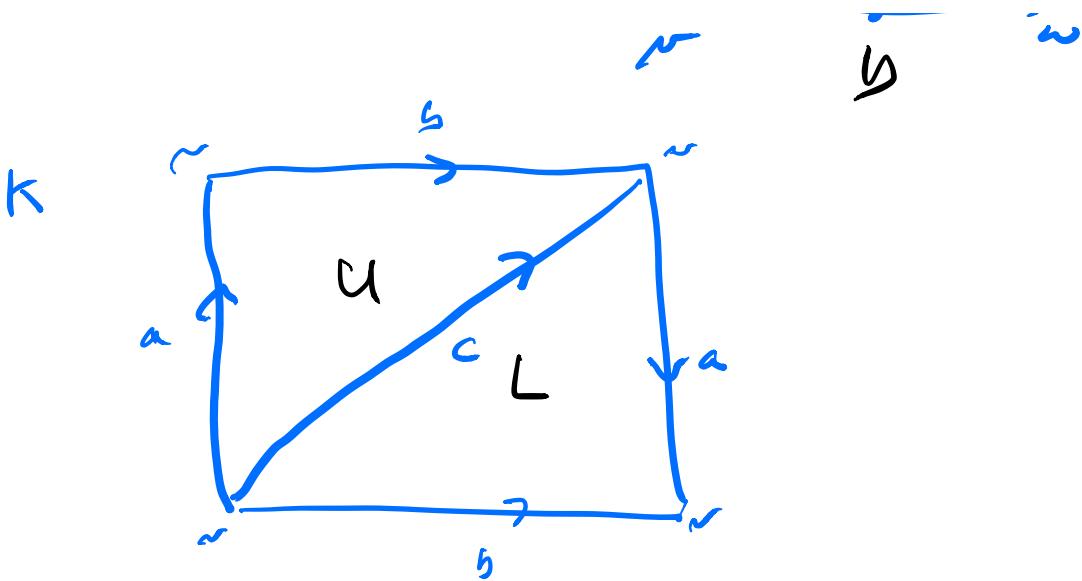
2) $\sigma_\alpha|_{\text{open face}} \rightarrow$ another map in collection $\sigma_\beta : \overset{\circ}{\Delta}{}^{n-1} \rightarrow X$

(Here we are identifying faces of Δ^n with images of canonical affine map $\Delta^{n-1} \rightarrow \Delta$)

3) weak topology: $A \subset X$ is open $\Leftrightarrow \sigma_\alpha^{-1}(A)$ is open $\forall \alpha$.

Examples





$$\partial ([v_0, \dots, v_n]) = \sum (-1)^i [v_0, \hat{v}_i, \dots, v_n]$$

Chain gps: If X is given by collection $\sigma_\alpha : \Delta^{n_\alpha} \rightarrow X$

Then define

$\Delta_n(X) =$ free abelian gp on
n-simplices of X

= free abelian gp on $\{\sigma_\alpha : \Delta^n \rightarrow X\}$

$$n_2 = n.$$

Define $\partial_n : \Delta_n(X) \rightarrow \Delta_{n-1}(X)$

as in s above

$\ker \partial_n = n\text{-cycles}$

$\text{Im } \partial_{n+1} = n\text{-boundaries}$

$$H_n(X) = \frac{\ker \partial_n}{\text{Im } \partial_{n+1}}$$

Oct 20

- Calculate examples

$S^1, T^2, \mathbb{RP}^2, H_n(S^n)$

- Problems with ~~singular~~ homology
 - Depends on Δ -structure
 - Not functorial in continuous maps
(for simplicial maps it can be made to work for simplicial maps)
-

Singular homology. X = top spc. (Eilenberg)

- A singular n -simplex on X
 - I) a continuous map $\sigma: \Delta^n \rightarrow X$
- Singular chain group $C_n(X)$

$C_n(X)$ = free abelian group on
set of singular n -simplices

$$= \left\{ \sum n_\sigma \sigma \mid \begin{array}{l} \sigma \text{ is singular} \\ n\text{-simplex} \\ n_\sigma \in \mathbb{Z} \end{array} \right\}$$

$\partial: C_n(X) \rightarrow C_{n-1}(X)$ defined
as before

$$H_n(X) = \frac{\ker \partial}{\text{Im } \partial}.$$

Prop : If X is path
connected then $H_0(X) = \mathbb{Z}$

$$\begin{aligned} \partial: C_n(X) &\rightarrow C_{n-1}(X) \\ \sigma: [v_0, \dots, v_n] &\xrightarrow{\Delta^n} X \\ \partial \sigma &= \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \end{aligned}$$

$$\partial_{n+1} \partial_n \sigma = 0$$

$$\underline{\text{Def}}: H_n(X) = \frac{\ker \partial_n}{\text{Im } \partial_n}$$

π_1 im ∂_{n+1}

$$\overbrace{H_n(S^1)} = ?$$
$$C_n(X) \xrightarrow{\epsilon} \mathbb{Z}$$
$$\sum n_{\sigma} \rightarrow \sum n_{\sigma}$$

HW 5, 6, 8, 11, 12, 14, 15, 18
20, 22 Nov. 1

- Def'n of $H_n(X)$
- X path connected $\Rightarrow H_0(X) = \mathbb{Z}$
- $\{X_\alpha\}$ the path components of X
then $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$
- Remark: $H_1(X) = \pi_1(X)^{ab}$
- Reduced homology $\tilde{H}_n(X)$
- Factoriality. $f: X \rightarrow Y$ continuous
gives chain map $f_*: C_*(X) \rightarrow C_*(Y)$

$$\therefore f_* \partial = \partial f^*.$$

\therefore Induced map on homology

$$f_* : H_n(X) \rightarrow H_n(Y)$$

$X \xrightarrow{f} Y \xrightarrow{g} Z$, Then

$$(gf)_* = g_* f_* \Rightarrow (gf)_* = g_* f_*$$

- Cor homology gp's are homeomorphism invariants.

- Relative homology gp's
 $H_n(X, A)$.

- Exact sequences, short exact sequences

- Snake Lemma.

$$\sigma : [n_0, \dots, n_k] \rightarrow X$$

$$\partial \sigma = \sum_{i=1}^k \sigma|_{\{n_0, \dots, \hat{n}_i, \dots, n_k\}}$$

Extend linearly to $C_n \quad \partial \partial = 0$

$$\ker \partial_n = \{n\text{-cycles}\}$$

$$\text{Im } \partial_{n+1} = \{n\text{-boundaries}\}$$

$$H_n(X) = \frac{\ker \partial}{\text{Im } \partial}$$

Start with H_0

$$\sigma_0 : \Delta^0 \xrightarrow{\quad} X \quad \sigma([v_0]) = p + \infty \in X$$

$$\Delta^1 \xrightarrow{\sigma_1^*} X$$

$\sigma([v_0, v_1])$

Prop X path connected
 $\Rightarrow H_0(X) = \mathbb{Z}$

Pf Define "augmentation"
 $\varepsilon : C_0(X) \rightarrow \mathbb{Z}$

$$\varepsilon \left(\sum n_\alpha \sigma \right) = \sum n_\alpha$$

$\delta: C_*(X) \rightarrow C_0(X)$

$$\operatorname{Im} \delta = \ker \varepsilon$$

$$\therefore H_0(X) = \frac{C_0(X)}{\operatorname{Im} \delta} \xrightarrow[\sim]{\varepsilon} \mathbb{Z}$$

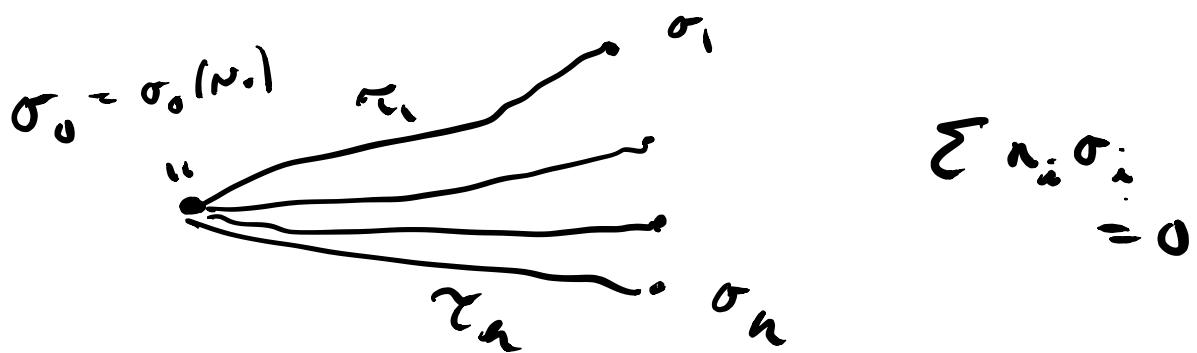
$$\operatorname{Im} \delta \subset \ker \varepsilon ?$$

$$\sigma: [n_0, n_1] \rightarrow X$$

$$\delta \sigma = \sigma(n_1) - \sigma(n_0)$$

$$\varepsilon(\delta \sigma) = 1 - 1 = 0 \quad \checkmark$$

$$\ker \varepsilon \subset \operatorname{Im} \delta \quad \text{"}\sigma_i(n_0)\text{"} \quad \checkmark$$



$$\tau_i: [n_0, n_1] \rightarrow X$$

$$\delta \tau_i = \sigma_i - \sigma_{i-1}$$

Show $\sum n_i \sigma_i \in \text{Im } \partial$

$$\begin{aligned}\partial(\sum n_i z_i) &= \sum n_i \partial z_i \\ &= \sum n_i \sigma_i - (\sum n_i) \sigma_0\end{aligned}$$

□

Prop $\{X_\alpha\}$ are path components of X .

$$H_n(X) = \bigoplus_{\alpha} H_n(X_\alpha)$$

Pf

$$\begin{aligned}C_n(X) &= \bigoplus_{\alpha} C_n(X_\alpha) \\ C_{n-1}(X) &= \bigoplus_{\alpha} C_{n-1}(X_\alpha)\end{aligned}$$

QED.

Prop $X = p +$

$$H_n(\rho^+) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n=0 \end{cases}$$

\mathbb{P}^1 Only map $\Delta^n \rightarrow \rho^+$

$$C_n(\rho^+) = \mathbb{Z}$$

$$\partial: C_1(\rho^+) \rightarrow C_0(\rho^+) = 0$$

$$C_2 \rightarrow C_1$$

$$\sigma: \Delta \rightarrow X \quad \partial\sigma = 1 \text{ (constant)}$$

$$C_n(\rho^+) = \mathbb{Z} \quad n \text{ even}$$

$$\partial: C_n \rightarrow C_{n-1} \quad n \text{ is odd}$$

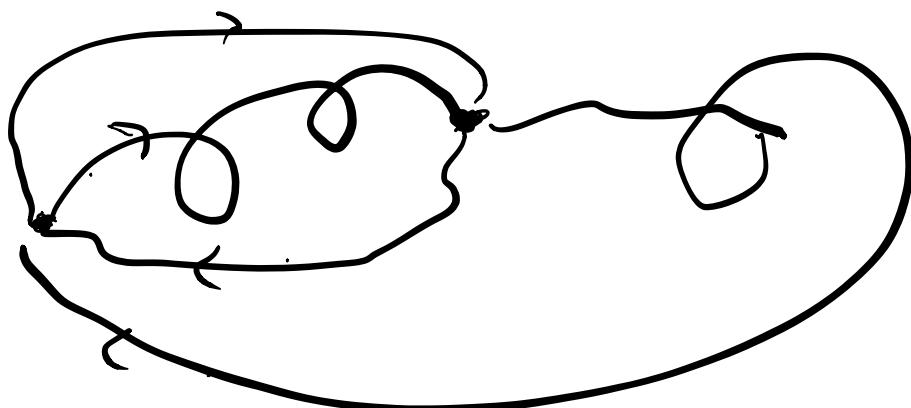
$$\text{n odd} \quad \partial: C_n \rightarrow C_{n-1} \quad \sigma\text{-map}$$

$$\text{get } \bigoplus_{n>0}$$

$$H_n(\rho^+) = 0 \quad \square$$

Remark

$$H_1(x) = \pi_1(x, x_0)^{ab}$$



Functionality:

$$f: X \rightarrow Y \quad \text{continuous}$$

Induces map homomorphism

$$C_n(f) = f_{\#} : C_n(X) \rightarrow C_n(Y)$$
$$\sigma: \Delta^n \rightarrow X$$

$$f_{\#}\sigma = f\sigma: \Delta^n \rightarrow Y$$

This is chain map

$$C_n(X) \xrightarrow{f_{\#}} C_n(Y)$$

$$\begin{array}{ccc} \downarrow \delta & & \downarrow \delta \\ C_{n-1}(X) & \xrightarrow{f_*} & C_{n-1}(Y) \\ \delta f_* = f_* \delta & \leftarrow \boxed{\text{chain map}} \end{array}$$

$$\begin{aligned} f_* \delta \alpha &= f_* \sum (-1)^i c_i |_{C_{n-1}(X)} \\ &= \sum (-1)^i (f_* \alpha) |_{\Sigma} \\ &= \delta (f_* \alpha) \end{aligned} \quad \square$$

$\therefore f_*$ induces a homomorphism $f_*: H_n(X) \rightarrow H_n(Y)$

Pf

Show $f_*(\ker \delta) \subset \ker(\delta)$

& $f_*(\text{Im } \delta) \subset \text{Im } \delta$

Suppose ~~α~~ $\alpha = \sum n_i \alpha$

$$\partial \alpha = 0 \quad \text{a "cycle"}$$

$$\Rightarrow (f_* \alpha) = f_*(\partial \alpha) = f_*(0) = 0 \quad \checkmark$$

$$\alpha = \partial \beta$$

$$f_*(\partial \beta) = f_*(\alpha)$$

$$\begin{matrix} \text{“} & & \uparrow \\ \partial f_*(\beta) & & \text{Thus it is bdry.} \end{matrix}$$

□

Lemma $X \xrightarrow{f} Y \xrightarrow{g} Z$

Then

$$(gf)|_{\#} = g_{\#} f_{\#}$$

on $H_n(X) \rightarrow H_n(Z)$

$$(gf)_* = g_* f_*$$

Cor $H_n(X)$ is a homeo.
invariant, i.e.)

$$X \cong X' \Rightarrow H_n(X) \cong H_n(X')$$

ps

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & \underbrace{\quad}_{f^{-1}} & \end{array}$$

$$\begin{aligned} f \circ f_x &= (f \circ f)_x \\ &= \text{id}(1_X)_x \\ &= \text{id} \end{aligned}$$

□

Reduced ~~relative~~ homology
Augmented chain c.c.

$$C_n(X) \rightarrow \cdots \rightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}^{>0}$$

$$\varepsilon(\sum a_\sigma \sigma) = \sum a_\sigma$$

$$\text{Im } \partial_1 \subset \ker \varepsilon$$

Reduced homology $\tilde{H}_n(X)$
The homology of augmented
chain complex

$$\tilde{H}_n(X) = H_n(X) \quad n > 0$$

$$0 \rightarrow \tilde{H}_0(X) \rightarrow H_0(X) \longrightarrow \mathbb{Z} \rightarrow 0$$

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$$

or X is path conn.

$$\tilde{H}_0(X) = 0$$

Exact sequence of pair

Relative Homology

$A \subset X$ subspace

$$C_n(A) \subset C_n(X)$$

$$\downarrow \qquad \text{dingular chain}$$

$$C_{n-1}(A) \qquad C_{n-1}(X)$$

$$\text{Def } C_n(X, A) = \frac{C_n(X)}{C_n(A)}$$

↓
 $C_{n-1}(X, A)$

$$H_n(X, A) = \frac{\ker \delta_n}{\text{Im } \delta_n}$$

Exact sequence -

$$D_{m+1} \xrightarrow{\delta_{m+1}} D_m \xrightarrow{i_m} D_{m-1} \rightarrow D_{m-2} \rightarrow \dots$$

$$\text{Def } \text{Im } i_{m+1} = \ker i_m$$

Special case Short exact

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$$

\uparrow

$\ker i = 0$ $i = \text{inj ect.}$
 $j \neq 0 \neq 0$ $\text{Im } j = C$

$$0 \rightarrow A \rightarrow A \oplus C \xrightarrow{\quad P \quad} C \rightarrow 0$$

Counter example

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$$

$$0 \rightarrow \text{Cu}(A) \xrightarrow{x_2} \text{Cu}(X) \rightarrow \text{Cu}(X/A) \rightarrow 0$$

Thm \exists exact seqn.

$$\begin{array}{c} H_n(X, A) \\ \hookrightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A^{-1}) \\ \downarrow \delta \\ H_{n-1}(A) \rightarrow H_{n-1}(X) \rightarrow \end{array}$$

Folows , Snake lem.

$$0 \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow 0$$

$$0 \rightarrow A_{n-1} \xrightarrow{\downarrow} B_{n-1} \xrightarrow{\downarrow} C_{n-1} \rightarrow 0$$

short exact

long exact of homolo
as above