- π_1 of a product
- retractions

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van Kampen's Theorem

- Free groups - universal property - words
- Any gp is quotient of a free gp
- Free products
- Amalgamated free products
- van Kampen's Thm (Statement)
- $\pi_1(X \vee Y)$
- π_1(connected graph) = free gp.

- Hatcher's Example with linked circles
- Indication of Proof of van Kampen's Thm

Prop $\pi_1(X \times Y, (x_0, y_0))$

$$\cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Pf $I = [0,1]$

$$F : (I, \partial I) \longrightarrow (X \times Y, (x_0, y_0))$$

$$\shortparallel$$

(f, g) , $f = p_X F : I \to X$

$[F] \to ([f], [g])$ $\square$

$\pi_1(\underbrace{S^1 \times \cdots \times S^1}_{n}) = \mathbb{Z} \times \cdots \times \mathbb{Z}$

---

retraction $\quad A \underset{r}{\overset{i}{\rightleftarrows}} X$

$$\pi_1(A, x_0) \underset{r_*}{\overset{i_*}{\rightleftarrows}} \pi_1(X, x_0)$$

$r_* i_* = \mathrm{id}$ $\quad$ i_*, inj

r_* surjection

Free gps.

Def A free gp on set ~~set~~ S is gp F_S

& $S \hookrightarrow F_S$ $\quad$ $s \to x_s$

s.t given any group G & function

$$\begin{tikzcd} F_S \arrow[rr, dashed, bend left, "\varphi"] & & \\ S \arrow[u, phantom, "\cup" description] \arrow[rr, "f"'] & & G \end{tikzcd}$$

$\exists!$ extension $\varphi : F_S \to G$

This defines F_S up to unique iso

$$\begin{tikzcd} F_S \arrow[rr, bend left, "\varphi"] & & F'_S \arrow[ll, bend left, "\varphi'"] \\ & S \arrow[ul, phantom, "\supset" description] \arrow[ur, phantom, "\subset" description] & \end{tikzcd}$$

$$\begin{tikzcd} F_S \arrow[rr, "\varphi'\varphi"] & & F_S \\ & S \arrow[ul, phantom, "\supset" description] \arrow[ur, phantom, "\subset" description] & \end{tikzcd}$$

$\varphi'\varphi = id$

& $\varphi\varphi' = id$

Free product of $(G_\alpha)_{\alpha \in S}$

Def $\ast\, G_\alpha$ = free product

universality property, $\forall H$

& any $\varphi_s : G_s \to H$

$$\begin{tikzcd} G \arrow[dr, dashed, "\varphi"] & \\ G_s \arrow[u, hook] \arrow[r] & H \end{tikzcd}$$

$\exists !\ \varphi$.

<u>Remark</u> Any gp H is the quotient of a free gp. S any set of gen $H = \langle S \rangle$

$$\begin{tikzcd} F_S \arrow[dr, "\varphi"] & \\ S \arrow[u, hook] \arrow[r] & H \end{tikzcd}$$

φ is onto

$$F_S / \ker \varphi \cong H$$

<u>Construction of F_S</u>

Consider "reduced words" in $\{x_s\}$

$$w = x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

<u>Notation</u>

$x_i = x_s \qquad m_i \neq 0$

reduced means $x_{i+1} \neq x_i$

$$x_i^{m_i} \in \mathbb{Z}_i = \langle x_i \rangle$$

$$F_S = \cancel{\mathbb{Z}} \ \ast_S \mathbb{Z}$$

Similarly free product (G_α)

$$\ast G_\alpha = \{ g_1 \cdots g_k \text{ reduced} \}$$

$$g_i \in G_\alpha - e$$

Multiplication

$$(h_1 \cdots h_\ell)(g_1 \cdots g_k)$$
$$= h_1 \cdots h_\ell g_1 \cdots g_k \longrightarrow \text{reduce}$$

<u>Associativity</u>: Trick

$G = \{\text{reduced words}\}$

Define injection

$$\ast G_\alpha \hookrightarrow \mathrm{Perm}(G)$$

$\{ G \to G \mid \text{bijection} \}$

Say S G_n acts on G

$g \in G_n$

$g \cdot (g_1 \cdots g_k) = g g_1 \cdots g_k$ reduced

$L_g : g_1 \cdots g_k \to g g_1 \cdots g_k$

$G_n \longrightarrow \operatorname{Perm} G$

Defn $L_{g_1 \cdots g_k} = L_{g_1} L_{g_2} \cdots L_{g_k}$

* G_n injects

* G_n acts transitively on G

$L_{g_1 \cdots g_k}(e) = g_1 \cdots g_k$

& injective.

$\therefore$ $\operatorname{Perm}(G)$ the is associative

mult on G_n is associative

$\square$

Ex F_2 = on $\{1, 2\}$

$a = x_1$ $b = x_2$

m_1 m_2 m_3

$F_2 \neq \mathbb{Z} + \mathbb{Z}$

$a^{\cdot} b^{\cdot} a^{\cdot} \cdots$

or $b^{m_1} a^{m_2} \cdots$

Ex $\mathbb{Z}/2 * \mathbb{Z}/2$ generators as before: a & b

word of $abab\ldots$

$|\mathbb{Z}/2 * \mathbb{Z}/2| = \infty$

$abab\ldots$

$\langle ab \rangle = \mathbb{Z}$ $(ab)^m$

$(ab)^{-1} = ba$

Amalgamated free product

$G_1 \quad G_2$ $H \xrightarrow{\varphi_1} G_1$

$H \xrightarrow{\varphi_2} G_2$

$$G_1 *_H G_2 = \frac{G_1 * G_2}{N}$$

N = normal subgp generated by $\varphi_1(h)^{-1}\varphi_2(h)$

Universal prop...

$G_i \xrightarrow{\psi_i}$

$\varphi_i \uparrow$

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{G} \end{array}$$

$\Rightarrow \varphi_1 = \varphi_j = \varphi_2 \ \varphi_2$

$\exists!\ G_1 *_H G_2 \longrightarrow \mathcal{G}$

More generally $\{G_\lambda\}_{\lambda \in \Lambda}$

& gp

$H \xrightarrow[\varphi_\lambda]{} G_\lambda$

Define $\underset{H}{*} G_\lambda$ by universal property

Statement of van Kampen's Thm

$$X = U_1 \cup U_2 \quad \text{path conn.}$$

& $U_1 \cap U_2$ is path connected

$x_0 \in$

$$\pi_1(U_i^{*}) \xrightarrow{(\varphi_i)_*} \pi_1(X, x_0)$$

$$\pi_1(X, x_0) \cong \pi_1(U_1) \underset{H}{*} \pi_1(U_2)$$

$H = \pi_1(U_1 \cap U_2, x_0)$

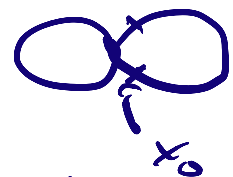
More generally: U_i are open, $x_0 \in U_i$

$$X = \bigcup U_i$$

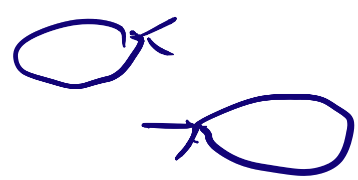
$$U_i \cap U_j = V \quad \forall i \neq j$$

Then
$$\pi_1(X, x_0) = \mathop{\ast}_H \pi_1(U_i, x_0)$$

$$H = \pi_1(V, x_0).$$

Ex $\quad S^1_1 \vee S^1_2 =$ 

$V =$  contractible open nbhd of x_0

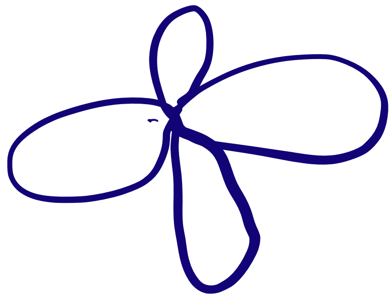
$U_i = S^1_i \cup V$ 

$U_i \sim S^1$

$$\pi_1(S^1 \vee S^1, x_0) = \pi_1(U_1, x_0) \mathop{\ast}_H \pi_1(U_2, x_0)$$

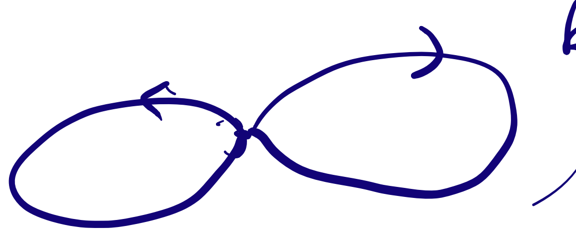
$$H = e = \pi_1(V, x_0)$$

$\pi_1(S^1 \vee S^1) = F_2$



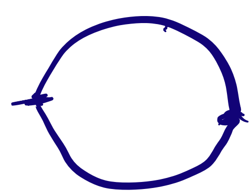
k copies

$\pi_1\left(\bigvee_k S^1\right) = F_k$



$a^{n_1} b^{m_2} a b \dots$

$aba^{-1}b^{-1} \neq e$



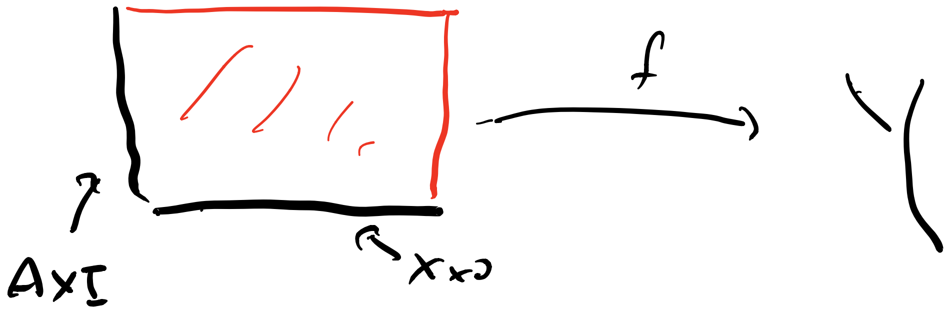
Sept. 15

- van Kampen's Thm
- π_1(wedge)
- π_1(connected graph) [$T \subset X$ a max tree, then X/T is $\simeq$ wedge of circles]
- Homotopy Extension Property (= cofibration)
- Prop: (X, A) a CW pair then $A \hookrightarrow X$ is a cofibration
- A contractible $\Rightarrow$ $X/A \sim X$
- Sketch of proof of van Kampen's Thm
- Hatcher's Linked Circles (Borromean rings)

Notes on HEP (Homotopy extension prop)

<u>**Def**</u> $A \subset X$ has HEP if for any map $f_0 : X \times 0 \to Y$ and any homotopy $h : A \times I \to Y$ extending $f_0|_{A \times 0}$ $\exists$ homotopy f_t or $f : X \times I \to Y$

extending h and f_0



<u>Fact</u>: $A \subset X$ has HEP iff $A\times I \cup X\times 0$ is a retract of $X\times I$.

<u>Prop</u> (X,A) CW pair then $X\times I$ retracts onto $X\times 0 \cup A\times I$ (so (X,A) has HEP)

<u>Prop</u> (X,A) CW pair, A contractible then X is h.e. to X/A.

<u>Pf</u> $f_t : X \to X$ homotopy extending contraction of A to

a pt. $q: X \to X/A$

$\bar{f}_t = f_t \circ q$. f_1 induces

$g: X/A \to X$. q & g are homotopy inverses

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_t"'] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_1"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

(X_α, x_α) path conn-

$$\pi_1\left(\bigvee_\alpha X_\alpha\right) = \mathop{\ast}_\alpha \pi_1(X_\alpha, x_\alpha)$$

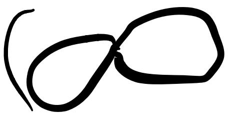
Proviso: x_α has an contractible open nbhd V_α

$$\vee X_\alpha \cup \bigcup_\alpha V_\alpha$$

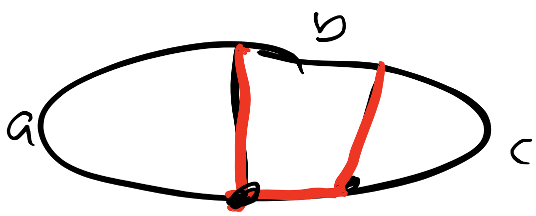
Use open sets $X_\alpha \cup V$

<u>Ex</u> $X_\alpha = S^1$ $\quad \alpha \in A$

$\pi_1\left(\bigvee_\alpha S^1\right) =$ Free gp on A.

$\pi_1\Big($  $\Big) = F_2$

$X =$ connected graph



$\exists$ maximal tree $T \subset X$

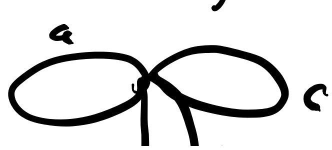
T is contractible

<u>**Fact:**</u> (X, A) CW - pair & A is contractible

$X/A \simeq X$.

$X =$ Graph $\quad X/T \simeq X$

1 vertex, 1 edge for $E(X-T)$

$X/T =$ 

U6

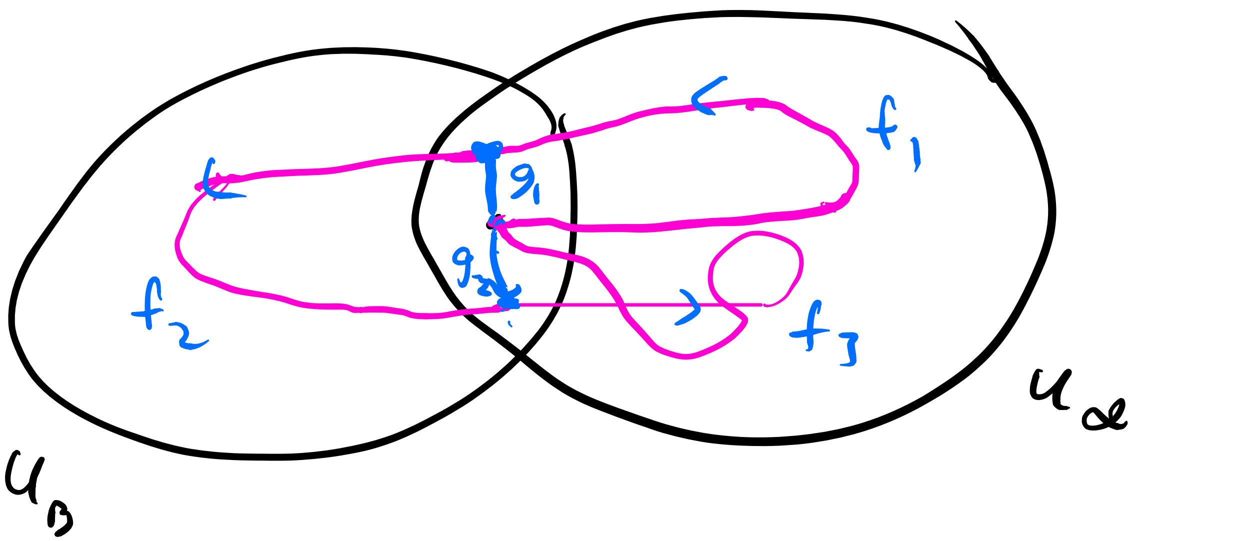
<u>Conclusion</u> $\pi_1(X) \cong \pi_1(X/T)$ is a free group.

Pf of van Kampen's Thm

$$\underset{\alpha}{*}\, \pi_1(U_\alpha) \xrightarrow{\Phi} \pi_1(X)$$

ker Φ is normal subgp generated by $\pi_1(U)$

intersection



$$f \sim \underbrace{f_1 \bar{g}_1}\ \underbrace{g_1 f_2 \bar{g}_2}\ \underbrace{g_2 f_3}$$

$$[f] = [f_1'][f_2'][f_3']$$

Φ is onto.

Sept 18

- Knot theory, linking, Borromean rings
- π_1 of space union 2-cell(s).
- Surface groups
- Presentation of a group

__Def__ A __knot__ in S^3 (or $\mathbb{R}^3$) is a (oriented) simple closed curve, i.e., an embedded S^1 in S^3.

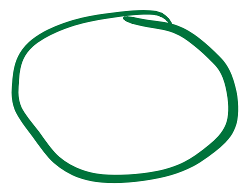
$(S^3, \underset{\cong S^1}{K})$ and (S^3, K') are <u>equivalent</u> if $\exists$ homeo $\varphi: S^3 \to S^3$ s.t. $\varphi(K) = K'$.

---

equivalent $\Rightarrow S^3 - K \cong S^3 - K'$

$\Rightarrow \pi_1(S^3 - K) \cong \pi_1(S^3 - K')$

<u>Ex</u> K = unknot



$\pi_1(S^3 - K) \cong \mathbb{Z}$

<u>Fact</u>: S^3 is the union of 2 solid tori. $(= S^1 \times D^2)$

$$S^3 = \partial(D^2 \times D^2)$$
$$= \partial D^2 \times D^2 \cup D^2 \times \partial D^2$$
$$= S^1 \times D^2 \underset{T^2}{\cup} D^2 \times S^1$$

Another way:

$$S^3 = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1$$

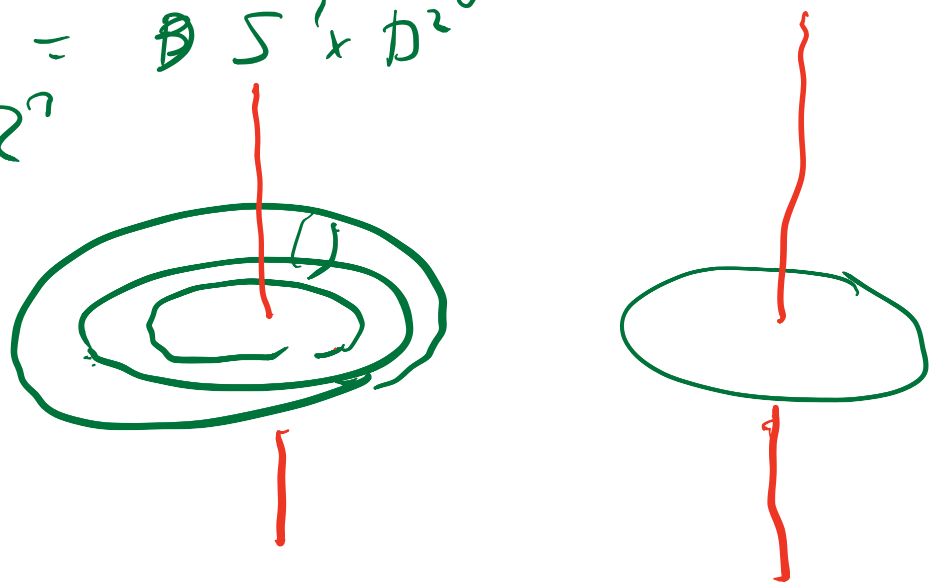
$A = \{(x,y) \mid |x| \leq \frac{\sqrt{2}}{2}\}$

$= D^2 \times S^1$

$B = \{(x,y) \mid |y| \leq \frac{\sqrt{2}}{2}\}$

$= S^1 \times D^2$

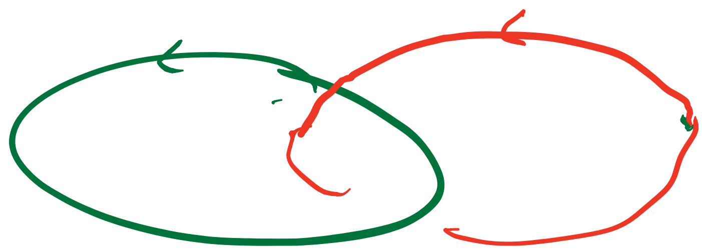
In $\mathbb{R}^3$



$\therefore \pi_1(S^3 \setminus \text{unknot}) = \pi_1(D^2 \times S^1)$

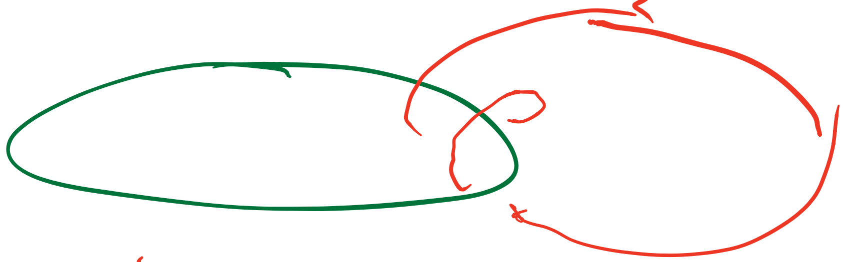
$= \mathbb{Z}$

Def <u>link = disjoint of circles</u>



$\pi_1(\text{red}) \longrightarrow \pi_1(S^1 \times D^2) = \pi_1(S^1)$

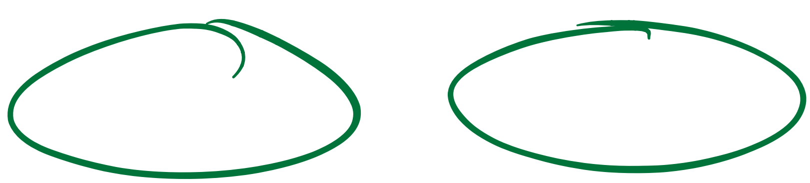
linking number is ± 1 $\mathbb{Z}$



linking number = 2

complement $\sim T^2$, $\pi_1 = \mathbb{Z} + \mathbb{Z}$

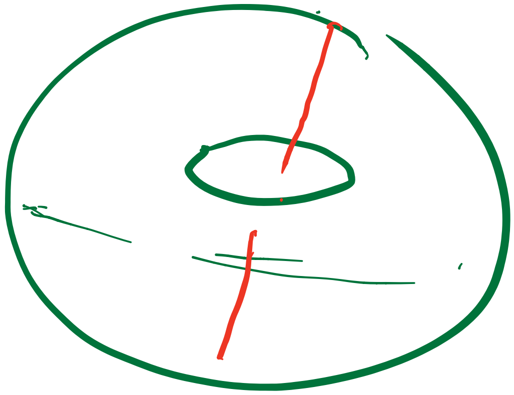
L =

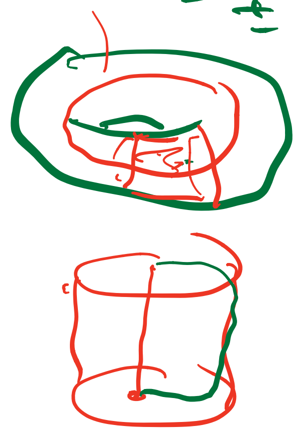


π_1(complement) = F_2 = free gp

hemisphere ?

Each component lies in $D^3_\pm$





$\simeq S^2 \vee S^1$

$\simeq$

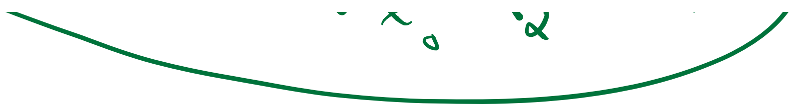
$\leftarrow X_\pm$

$X_+ \cup_{S^2} X_-$

$\pi_1(\text{complement}) = \pi_1(X_+) * \pi_1(X_-)$

Adding D^2's to X

$Y =$



~~Prop~~ $\varphi_\alpha : \partial D^2_\alpha \longrightarrow X$

$$\psi_\alpha = \bar{h}_\alpha \, \varphi_\alpha \, h_\alpha$$

$$[\psi_\alpha] \in \pi_1(X, x_0)$$

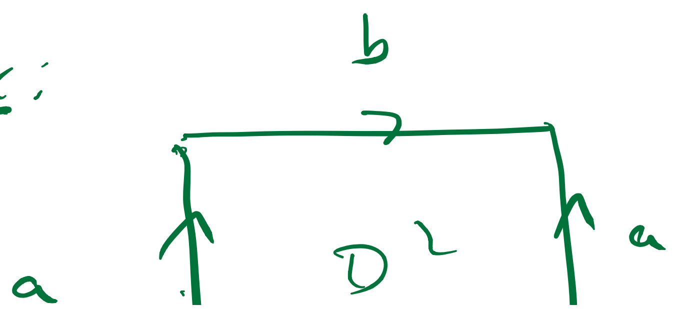
$$X \hookrightarrow Y$$

~~Prop~~ $\pi_1(X) \xrightarrow[i_*]{} \pi_1(Y)$

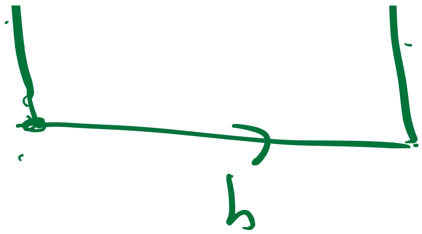
<u>Prop</u> i_* is onto & kernel $= N =$ normal subgp generated by $[\psi_\alpha]$

$$\pi_1(Y) \cong \pi_1(X)/N$$

<u>Ex</u>:



$= T^2$



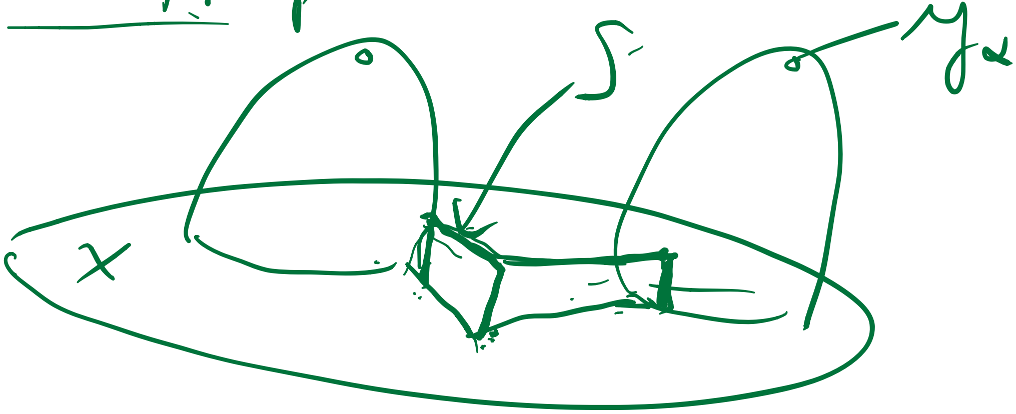
$\cup D^2$

$\varphi : S^1 \longrightarrow$ 

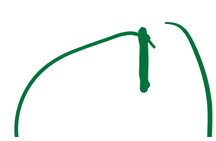
$[\varphi] = aba^{-1}b^{-1}$

$F_2^*/N = F_2^{ab} = \mathbb{Z}+\mathbb{Z}$

<u>Pf of Prop</u>



$Z = Y \cup S \sim Y$

$A = Z - \coprod y_\alpha \sim X$ 

$B \sim Z - X = \coprod D^2_\alpha \cup (S - \text{bottom})$ 

$\cap = \text{intersects} = \coprod (D^2_\alpha - y_\alpha) \cup \text{bottom}$

$\simeq \cup (\varphi_\alpha(S^1))$

$$\pi_1(Y) = \pi_1(A) *_N \pi_1(B)$$

$$\begin{array}{ccc} \| & & \| \\ \pi_1(Z) & & \pi_1(A)/N \end{array}$$

$$= \pi_1(X)/N.$$

$\square$
