<u>Homework</u>: 1.2: 2, 5, 6, 9, 10, 15, 16, 18, 20
Sep. 20

Topics:

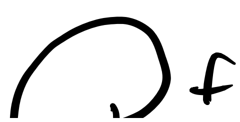
- Review $\pi_1(X, x_0)$
- Change of Base point
- Simply connected Space
- Fundamental groupoid
- induced homomorphisms
- $\pi_1(S^1, 1) \cong \mathbb{Z}$
  - Covering space $\mathbb{R} \to S^1$
  - Unique path lifting
  - Homotopy lifting property

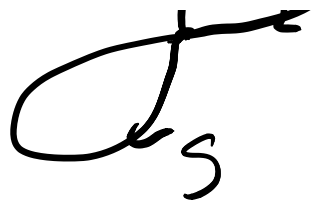
$\pi_1(S^1, 1) \cong \mathbb{Z}$

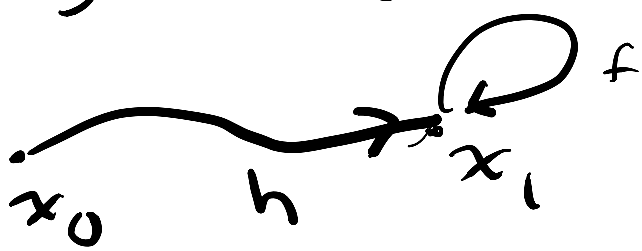
$$S^1 = \{ e^{2\pi i \theta} \in \mathbb{C} \mid \theta \in [0,1] \}$$

$$p: \mathbb{R}^1 \to S^1 \qquad p(s) = e^{2\pi i s}$$

$$\pi_1(X, x_0) = \left\{ \begin{array}{c} \text{homotopy classes} \\ \text{of loops} \\ \text{at } x_0 \end{array} \right\}$$



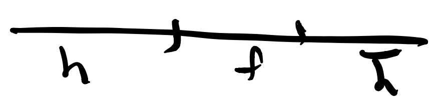




Prop: $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

$\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$

$\beta_h([f]) = h \cdot f \cdot \bar{h}$



Claim β_h is iso.

$(\beta_h)^{-1} = \beta_{\bar{h}}$

$$\underbrace{\bar{h}\ h}_{x_0}\ f\ \underbrace{\bar{h}\ h}_{x_0} \sim f$$

$\beta_h[f \cdot g] = \beta_h[f]\, \beta_h(g)$

$$h f \cdot g \bar{h} \simeq h f \cdot \underbrace{\bar{h} \cdot h}_{x_0} g \bar{h}$$

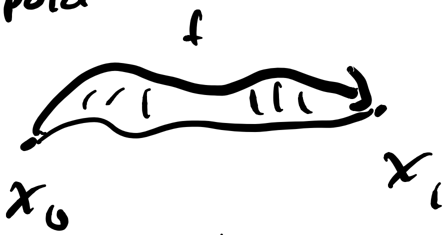
Fundamental Groupoid

Gp = Category with one object + every morphism

Groupoid = Small category where morphism is invertible

$\Pi_1(X) = \{$path homotopy classes of paths$\}$

" fund groupoid



$\pi_1(X, x_0) \qquad \pi_1(X, x_1)$

<u>Def</u> $\varphi : (X, x_0) \to (Y, \varphi(x_0))$

$\varphi(x_0) = y_0$

$\varphi_*: \pi_1(X, x_0) \xrightarrow[\varphi_*]{} \pi_1(Y, y_0)$

$f \longrightarrow f \circ \varphi$

$\pi_1 : \{\text{Pointed spaces}\} \rightsquigarrow \{\text{groups}\}$

Defn:

$$\Phi : \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, 1)$$

$$S^1 = \{ e^{2\pi i \theta} \in \mathbb{C} \}$$

$$\omega_n : [0,1] \longrightarrow S^1$$

$$\omega_n(s) = e^{2\pi i n s}$$

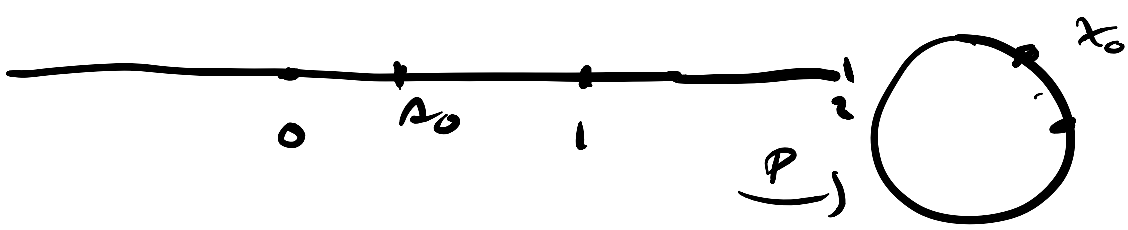
$$\Phi : n \to [\omega_n]$$

Define $p : \mathbb{R} \longrightarrow S^1$

$$p(s) = e^{2\pi i s}$$

$$e^{2\pi i s_0} = x_0 \in S^1$$

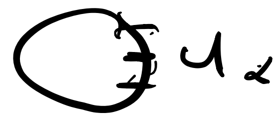
$$p^{-1}(x_0) = s_0 + \mathbb{Z}\ s_0$$



S^1 has open cover by $\{U_\alpha\}$

p maps each component

of $\pi^{-1}(U_\alpha)$ homeo to U_α



1) Unique path liftings:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & S^1 \end{tikzcd} \qquad f(0)=1$$

$\exists$ unique "lifting" $\tilde{f} : [0,1] \to \mathbb{R}$

$\tilde{f}(0) \in p^{-1}(1)$, say $\tilde{f}(0) = 0$

2) (Unique) homotopy lifting property

$f = f_0$, f_t is homotopy

Given lift $\tilde{f}_0 : [0,1] \to \mathbb{R}$ S^1

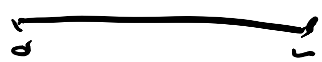
$\exists!$ lift $\tilde{f}_t$ extending $\tilde{f}_0$

$$F(s,t) = f_t(s)$$

$$\omega_n : [0,1] \longrightarrow S^1$$

$$\tilde{\omega}_n : [0,1] \longrightarrow \mathbb{R}$$

$$\tilde{\omega}_n(s) = ns \qquad \boxed{p(\tilde{\omega}_n(s)) = \omega_n}$$



Remark $\tilde{f}$ be another path from 0 ~~to~~ n

$$\tilde{f} \sim \tilde{\omega}_n \quad \text{rel endpoints}$$

$$p\tilde{f} \sim \omega_n$$

$$\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$$

~~given $n \in \mathbb{Z}$ take~~

any path from

$$f : [0,1] \to S^1$$

lift to $\tilde{f} : [0,1] \to \mathbb{R}$

$\tilde{f}(0) = 0$

$\tilde{f}(1) \in p^{-1}(1) = \mathbb{Z}$

$\cancel{\Phi(n) =}$

$\cancel{n \mapsto \Phi}$

Φ is surjective

$f : [0,1] \to S^1$

$n = \tilde{f}(1) \in \mathbb{Z}$

$\Phi(n) = [p \circ \tilde{f}]$

well-defined

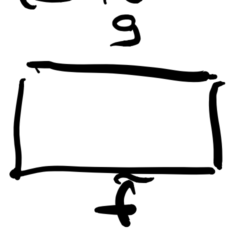
$f \sim g$

$\tilde{f} \sim \tilde{g}$

$\tilde{f}(0) = \tilde{g}(0) = 0$

$p(\tilde{g}(1)) \in$ constant

$= \tilde{f}(1) = n$

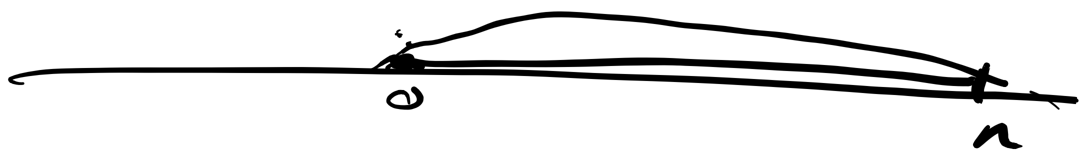


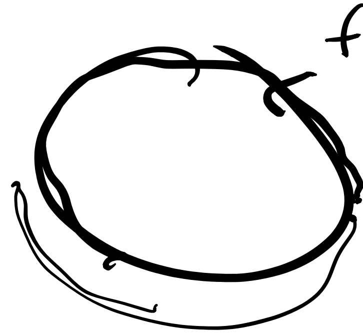
$\tilde{f}_t(1)$ is constant

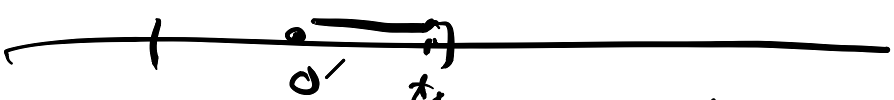
lift of homotopy is

path homotopy

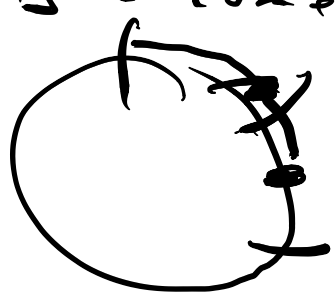
$\Rightarrow$ well-defined







$p\,|$ interval = homeo



$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ Y \times I \arrow[r] \arrow[ur, dashed] & S^1 \end{tikzcd}$$

Sept. 8

- $\pi_1(S^1) = \mathbb{Z}$

- Homotopy Lifting Property:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}_0"] \arrow[r, "f_0"'] & S^1 \end{tikzcd}$$

Given $f_0 : Y \longrightarrow S^1$, and a homotopy $F : Y \times [0,1] \longrightarrow S^1$, and a lift of f_0 to $\tilde{f}_0 : Y \to \mathbb{R}$, $\exists!$ lift $\tilde{F} : Y \times [0,1] \longrightarrow \mathbb{R}$ of F.

Sept 11

Topics:

Classical Applications of $\pi_1(S^1) = \mathbb{Z}$

- S^1 is not a retract of D^2
- Brouwer Fixed Point Thm.
- Borsuk Ulam Thm
- Fundamental Thm of Algebra.

- homotopy equivalence induces iso on π_1.

Other calculations

- $\pi_1(S^n) = 0$, $n \geq 2$
- $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$
- Ex: $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ $(= \mathbb{Z} + \mathbb{Z})$

<u>Brouwer Fixed Point Theorem</u>: Any map $h: D^2 \to D^2$ has a fixed point, i.e., $\exists\, x \in D^2$ s.t. $h(x) = x$.

<u>Borsuk-Ulam Thm</u>: Given $f: S^2 \to \mathbb{R}^2$, $\exists\, x \in S^2$ s.t. $f(x) = f(-x)$.

<u>Fundamental Thm of Algebra</u>: Any nonconstant polynomial p with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Lemma about homotopies: $\varphi_t : X \to Y$ a homotopy between φ_0 & φ_1. Let $h: [0,1] \longrightarrow Y$ be the path traced out by $\varphi_t(x_0)$. The following diagram commutes

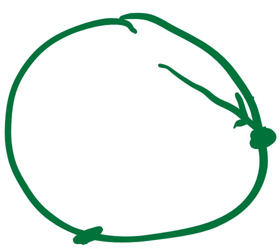
$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\
& \pi_1(Y, \varphi_0(x_0))
\end{tikzcd}$$

Thm S^1 is not retract of D^2.

Pf

$$\begin{tikzcd}
S^1 \arrow[r, hook, bend left, "i"] & D^2 \arrow[l, bend left, "r"]
\end{tikzcd}$$

suppose r exist



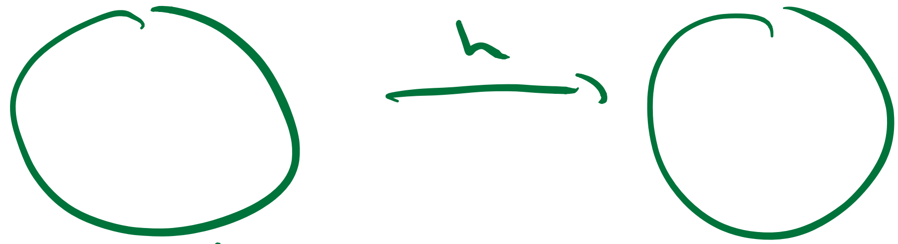
x_0

$r \circ i : S^1 \to S^1$
$= id$

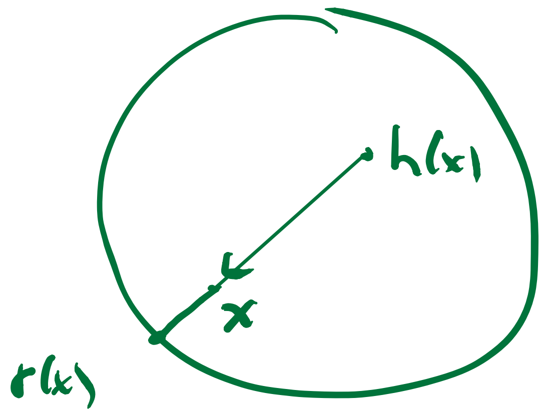
$\pi_1(D^2, x_0) = 0$

$$\begin{tikzcd}
\pi_1(S^1, x_0) \arrow[r, "i"] \arrow[d, equal] & \pi_1(D^2, x_0) \arrow[r, "r"] \arrow[d, equal] & \pi_1(S^1, x_0) \arrow[d, equal] \\
\mathbb{Z} \arrow[rr, bend right, "id"] & 0 & \mathbb{Z}
\end{tikzcd}$$

Brouwer Fixed Pt Thm



Suppose $\nexists$ fixed pt.



$r: D^2 \longrightarrow S^1$ is retraction.

Pf of Borsuk-Ulam

$f: S^2 \longrightarrow \mathbb{R}^2$ $\exists\, x$ st $f(x) = f(-x)$

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

$g: S^2 \longrightarrow S^1$

$g(-x) = -g(x)$ Equivariant map.

$S^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$

$g = g|_{S^1}$ $\qquad$ $S^1 \subset \mathbb{C}$ is equator

$$g : S^1 \longrightarrow S^1 \qquad g(-x) = -g(x)$$

$$\begin{array}{ccc} \pi_1(S^1, x_0) & \longrightarrow & \pi_1(S^1, g(x_0)) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow{\times q} & \mathbb{Z} \end{array}$$

$$\omega : [0,1] \longrightarrow S^1$$ standard generator

$$\omega(s) = e^{2\pi i s}$$

$$g\omega = \varphi$$

$$\boxed{\varphi(s + \tfrac{1}{2}) = -\varphi(s)}$$

Lift φ to $\tilde{\varphi} : [0,1] \to \mathbb{R}$

$$\begin{tikzcd} {[0,1]} \arrow[r] \arrow[dr] & \mathbb{R} \arrow[d, "p"] \\ & S^1 \end{tikzcd}$$

$$\tilde{\varphi}(s + \tfrac{1}{2}) = \tilde{\varphi}(s) + \frac{q}{2}$$

q odd integer

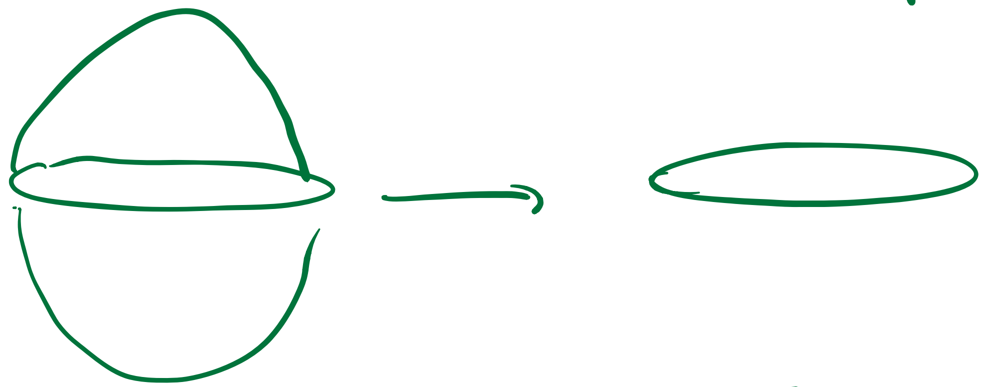
$q/2$ is constant (q integer)

$$\tilde{\varphi}(\tfrac{1}{2}) = \tilde{\varphi}(0) + q/2$$

$$\tilde{\varphi}(\tfrac{1}{2} + \tfrac{1}{2}) = (\tilde{\varphi}(0) + q/2) + q/2$$

$$\tilde{\varphi}(1) = q = \text{odd}$$

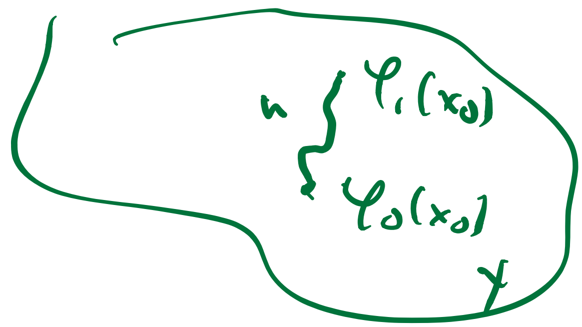
g is not null-homotopic



$g|_{S^1}$ extends to $D^2 \to S^1$

Contradiction. 

X, x_0



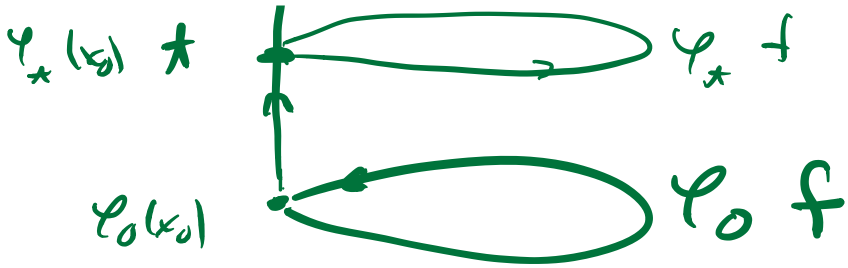
Φ

$h(s) =$ path traced out by homotopy between $\Phi(s, t)$

Pf of Lemma

$f : [0,1] \to (X, x_0)$

$\varphi_1(x_0$  $\varphi_1 f$



$$h_t(s) = h(ts)$$

$$h_t \cdot (\varphi_t f) \cdot \bar{h}_t \sim \varphi_0 f$$

$$\Rightarrow t = 1 \qquad h(\varphi_1 f)\bar{h} = \varphi_0 f$$

$$\beta_h((\varphi_1)_*[f]) = (\varphi_0)_*[f] \quad \square$$

__Prop__: $\varphi : (X, x_0) \to (Y, \varphi(x_0))$

$\psi : (Y, \varphi(x_0)) \to (X, \psi\varphi(x_0))$

homotopy inverses

$$(\psi\varphi \sim 1\!\!1_X \qquad \varphi\psi \sim 1\!\!1_Y$$

__Claim__ φ_* is isomorphism.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\varphi_*"'] & \pi_1(Y, \varphi(x_0)) \arrow[r, "\psi_*"] & \pi_1(X, \psi\varphi(x_0)) \arrow[d, "\varphi_*"] \\ & & \pi_1(Y, \varphi\psi\varphi(x_0)) \end{tikzcd}$$

$\pi_1(X, \ldots$

$\psi_* \circ \varphi_* = \beta_n$ an iso.

$\Rightarrow \varphi_*$ is inj.

$\varphi_* \circ \psi_*$ is iso so ψ_* is injective

& ψ_* is surjective

$\therefore \varphi_*$ is isomorphism

Q.E.D

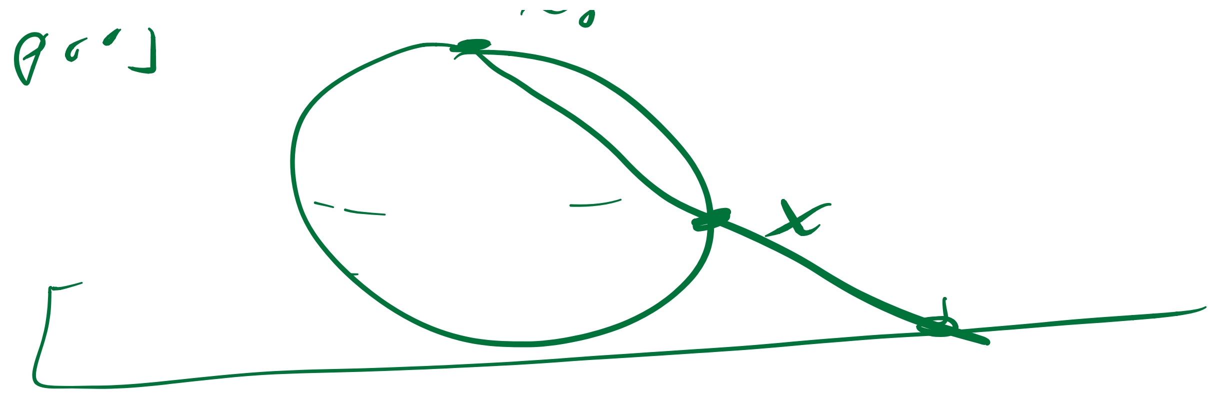
<u>Cor</u> X is contractible $\Leftrightarrow \pi_1(X, x_0) = 0$.

<u>Facts</u>

<u>Lemma</u> $\pi_1(S^n) = 0$ for $n \geq 2$.

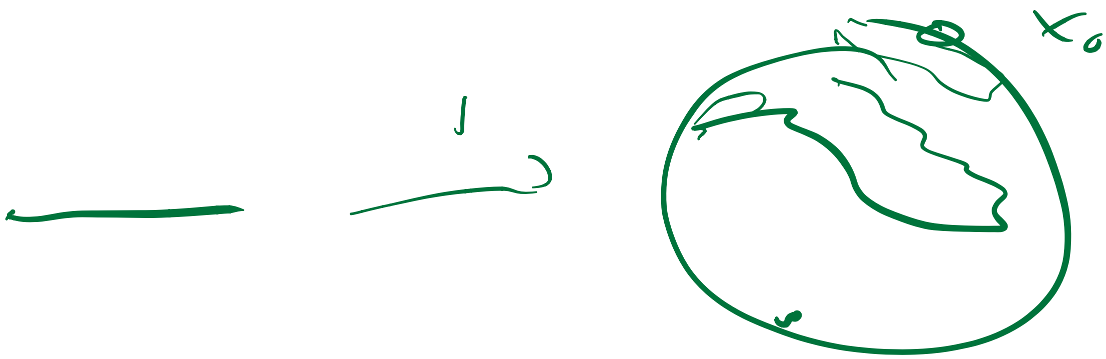
<u>Pf</u> $S^n - \{x_0\}$ is homeo to $\mathbb{R}^n$ (Stereographic

x_0



$f : [0,1] \longrightarrow S^2$ is loop

Claim $\operatorname{Im}(f)$ misses a pt x_0



$\downarrow$

$\mathbb{R}^2 \sim$ contr

$\pi_1(X \times Y, (x_0, y_0))$

$= \pi_1(X, x_0) \times \pi_1(Y, y_0)$

$$= \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

$$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$$