Homework < Friday — Feb 5

Cohomology of Spaces

$$C^n(X;G) = \mathrm{Hom}(C_n(X);G) = C_n(X)^*$$

$= \{$functions from singular n-simplices to $G\}$

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X) \arrow[d] \\ C^n(X) & C^{n-1}(X) \arrow[l, "\delta"'] \end{tikzcd}$$

$$\frac{\ker \delta}{\operatorname{Im} \delta} = H^n(X)$$

$f : X \longrightarrow Y$ $\qquad f_* : H_n(X) \to H_n(Y)$

$f^* : H^n(Y) \longrightarrow H^n(X)$ $\qquad$ contravariant functor

Exact of pair: (X, A)

$$0 \to C_n(A) \to C_n(X) \to C_n(X)/C_n(A) \to 0$$

$$0 \leftarrow C^n(A) \leftarrow C^n(X) \leftarrow C^n(X,A) \leftarrow 0$$

$=$ functions which $= 0$ on A

Cup products

$C^k(X;R)$ coefficients in commutative ring R, eg $R=\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p$

$\varphi \in C^k(X)$, $\psi \in C^l(X)$

$\sigma : \Delta^{k+l} \longrightarrow X$ singular simplex

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0, \ldots v_k]})\, \psi(\sigma|_{[v_k \ldots v_{k+l}]})$$

front k-face — back l-face

<u>Lemma</u>: $\delta(\varphi \smile \psi) = (\delta\varphi \smile \psi) + (-1)^k(\varphi \smile \delta\psi)$

<u>Pf</u> $(\delta\varphi \smile \psi)(\sigma)$ $\quad \sigma : \Delta^{k+l+1} \to X$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0 \, \hat{v}_i \ldots v_{k+1}]})\, \psi(\sigma|_{[v_{k+1} \ldots v_{k+l+1}]})$$

$(-1)^k(\varphi \smile \delta\psi)(\sigma) =$

$$\sum_{i=k}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0 \ldots v_k]})\, \psi(\sigma|_{[v_k \ldots \hat{v}_i \ldots v_{k+l+1}]})$$

last and ... terms cancel

$= \delta(\varphi \smile \psi)(\sigma)$. $\square$

$$\boxed{\delta(\varphi \smile \psi) = (\delta\varphi \smile \psi) \pm \varphi \smile \delta\psi}$$

$\Rightarrow$ (cocycle) $\smile$ (cocycle) = (cocycle)

(coboundary) $\smile$ (cobdry) = (cobdry)

$\Rightarrow$ $\exists$ bilinear map

$$\smile : H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$$

Ex Orientable surface of genus g

$$\frac{\ker \delta}{\operatorname{Im} \delta} \times \frac{\ker \delta}{\operatorname{Im} \delta} \to \frac{\ker \delta}{\operatorname{Im} \delta}$$

Coefficients in R

$$\smile : H^k(X,A) \times H^\ell(X) \to H^{k+\ell}(X,A)$$

$$\smile : H^k(X,A) \times H^\ell(X,A) \to H^{k+\ell}(X,A)$$

$$H^k(X,A) \times H^\ell(X,B) \to H^{k+\ell}(X, A\cup B)$$

if A, B are open or subcomplexes

Pf $C^k(X,A) \times C^\ell(X,B) \to C^{k+\ell}(X, A+B)$

has cohomology of $H^{k+\ell}(X, A \cup B)$

Naturality

Prop $f: X \to Y$, then

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

Pf Chain level

External $\times$ product

$$H^k(X) \times H^\ell(Y) \to H^{k+\ell}(X \times Y)$$

$$(\alpha, \beta) \to \alpha \times \beta$$

$\alpha \times \beta = p_1^*\alpha \cup p_2^*\beta$ where

p_i's are projections

$$H^k(X,A) \times H^\ell(Y,B) \to H^{k+\ell}(X \times Y, X \times B \cup A \times Y)$$

Notation "

$(X,A) \times (Y,B)$

<u>Lemma</u> Sequence of pair $(X,A)\times Y$

Connecting homo is natural

$$\begin{array}{ccc} H^k(A)\times H^\ell(Y) & \xrightarrow{\delta\times 1} & H^{k+1}(X,A)\times H^\ell(Y) \\ \downarrow \times & & \downarrow \times \\ H^{k+\ell}(A\times Y) & \xrightarrow{\delta} & H^{k+\ell+1}(X\times Y, A\times Y) \end{array}$$

Remark on

<u>Künneth Formula:</u> cross product

induces

$$H^k(X) \otimes_R H^\ell(Y) \to H^{k+\ell}(X\times Y)$$

Kunneth formula says something like

$$\bigoplus_{\substack{k,\ell \\ k+\ell=n}} H^k(X)\otimes H^\ell(Y) \to H^n(X\times Y)$$

is iso.

TRUE when R is a field

or if R is a P.I.D &

either $H^*(X)$ or $H^*(Y)$ is free

R- module

Special Case $T^n = S^1 \times \dots \times S^1$

$H^k(T^n)$ is free R-module

with basis $\alpha_{i_1} \cup \dots \cup \alpha_{i_k}$, $i_1 < \dots < i_k$

where $\alpha_i = p_i^*(\alpha)$

$\alpha =$ generator of $H^1(S^1)$

(That is, $H^*(T^n) = \bigoplus H^k(T^n)$
is exterior algebra $\Lambda[\alpha_1, \dots, \alpha_n]$)

Step 1

$$H^n(Y) \xrightarrow{\cong} H^{n+1}(I \times Y, \partial I \times Y)$$
$$\beta \longrightarrow \alpha \times \beta$$

$\alpha =$ gen of $H^1(I, \partial I)$

Pf Split short exact sequence

$$0 \to H^n(I \times Y) \xrightarrow{\text{diag}} H^n(\partial I \times Y) \xrightarrow{\delta} H^{n+1}((I, \partial I) \times Y)$$

δ is iso on $0 \times Y$

$H^n(0 \times Y) = \{1_0 \times \beta\}$ $\quad \beta \in H^n(Y)$

By Lemma $\quad \delta(1_0 \times \beta) = \delta(1_0) \times \beta$

$\delta(1_0) =$ generator of $H^1(I, \partial I)$

any other generator α is multiple of $\delta(1_0)$ by unit of R

So $\quad \beta \to \alpha \times \beta \quad$ is iso.

Step 1 is equiv.

$$H^n(Y) \to H^{n+1}(S^1 \times Y,\ a_0 \times Y) \quad \text{is iso}$$
$$\| $$
$$H^{n+1}(I \times Y,\ \partial I \times Y)$$

$H^*(S^1 \times Y) = H^*(S^1 \times Y,\ a_0 \times Y) + H^*(Y)$
($a_0 \times Y$ is a retract)

Conclusion:

$$H^{n+1}(Y) \times H^n(Y) \to H^{n+1}(S^1 \times Y)$$

$$(\beta_1, \beta_2) \longrightarrow 1 \times \beta_1 + \alpha \times \beta_2$$

$$T^n = S^1 \times T^{n-1}$$

$$H^k(T^n) = H^k(T^{n-1}) + H^{k-1}(T^{n-1})$$

$\square$

Cor $\quad I^n = I^i \times I^j \qquad j = n-i$

① $H^i(I^i, \partial I^i) \times H^j(I^j, \partial I^j) \xrightarrow{\times} H(I^n, \partial I^n$

$(I^n, I^i \times \partial I^j \cup \partial I^i \times I^j)$

① is iso

$\Longleftrightarrow$

② $H^i(T^i) \times H^j(T^j) \to H^n(T^n)$ is iso

$(1 + \quad$ is$)$