Formal View point. (= Eilenberg-Steenrod)
Axioms for reduced homology theory)
for CW complexes
Axioms for homology!

0) Functor $(fg)_* = f_* g_*$, $1\!\!1_* = 1\!\!1$

1) $f \sim g : X \to Y \Rightarrow f_* = g_*$

2) $\exists\ \partial : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A$
s.t long exact sequence of pai-

3) $X = \bigvee_\alpha X_\alpha \quad i_\alpha : X_\alpha \to X$

$\oplus (i_\alpha)_* : \bigoplus \tilde{h}_*(X_\alpha) \xrightarrow{\cong} \tilde{h}_*(X)$

E-S Axioms for unreduced
$h_n(X, A)$

- Add excision:

$$\tilde{h}_n(X/A) = h_n(X/A, A/A)$$

- Go from one to other

$$\tilde{h}_n(X) = \ker(h_n(X) \to h_n(x_0))$$

$h_n(X) = \tilde{h}_n(X_+)$, where
$X_+ = X \amalg x_0$

$$\tilde{H}_n(X,A) \overset{\sim}{\underset{\nearrow}{=}} \tilde{H}_n(X/A, A/A)$$

$\quad \parallel \quad$ excision $\quad \parallel$

$$H_n(X,U) = H_n(X-A, U-A)$$

Classical Applications $\qquad$ Jan 17

__Prop__ a) If $D \subset S^n$ is homeo to D^k, $k \geq 0$, then $\tilde{H}_i(S^n - D) = 0$

b) If $S \subset S^n$ homeo to S^k, $0 \leq k < n$, then

$$\tilde{H}_i(S^n - S) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise} \end{cases}$$

Jordan Curve Thm

Pf of a) : $h: I^k \to D$ homeo.

Induction on k.

$k=0$ ✓ $S^n - pt$ ✓

$A = S^n - h(I^{k-1} \times [0, \frac{1}{2}])$

$B = S^n - h(I^{k-1} \times [\frac{1}{2}, 1])$

$A \cap B = S^n - D$, $A \cup B = S^n - h(I^{k-1} \times \frac{1}{2})$

Apply MV Sequence

$\tilde{H}_i(S^n - D) \neq 0 \Rightarrow$ either $H_i(A) \neq 0$ or $H_i(B) \neq 0$

Continue subdividing . . .

$I_0 \supset I_1 \supset \ldots \quad I_m$

"length $\frac{1}{2^m}$

b) Induction on k, $k=0$ ✓

$S = D_1 \cup D_2 \qquad D_1 \cap D_2 = S^{k-1}$

Apply MV sequence.

$\hookrightarrow S^n - h(I^{k-1} \times I_m)$ each with non zero ~~H~~

homology

$$\bigcap I_m = p$$

<u>Claim</u> $\tilde{H}_i(S^n - D) \neq 0 \Rightarrow$

$$\tilde{H}_i(S^n - h(I^{k-1} \times p)) = 0$$

contradicts

α is a cycle in $S^n - D$

$\alpha = 0$ in larger

$\alpha = \text{bdry}\ \cancel{\partial \beta \quad \beta}$ singular chain

$$\beta \in C_{i+1}(S^n - h(I^{k-1} \times I_m))$$

$$\Rightarrow \alpha = \partial \beta \qquad [\alpha] = 0$$

in $\tilde{H}_i(S^n - h(I_{k-1} \times I_m))$

<u>Pf</u> of b) induction on k $\quad S^k \subset S^n$

$$S = D_1 \cup D_2$$

$$D_1 \cap D_2 = \partial D_i = S^{k-1}$$

$$S^n - D = (S^n - D_1) \cup (S^n - D_2)$$

$$\tilde H_i(S^n - S) \to \tilde H_i(S^n - D_1) \oplus \tilde H_i(S^n - D_2) \to \tilde H_i(S^n - (\underbrace{D_1 \cap D_2}_{S^{k-1}})) \xrightarrow[\partial]{\cong} \tilde H_{i-1}(S^n - S)$$

QED

Jan 21

Invariance of Domain

Thm Suppose $X \subset \mathbb{R}^n$ is homeo to an open subset of $\mathbb{R}^n$. Then X is open in $\mathbb{R}^n$.

Pf: $U \subset \mathbb{R}^n$ open. $h : U \to X \subset S^n$ homeo

Let $D = f(D^n)$ be nbhd of x in X

$S = f(S^{n-1})$. Then $S^n - D$ is open and connected (since $\tilde H_*(S^n - D) = 0$)

$S^n - S$ has 2 components. Since $S^n - S = S^n - D \amalg D - S$ These must be 2 components. So $D - S$ is open. in $\mathbb{R}^n$

Remark: Argument shows $h: U \to X \subset \mathbb{R}^n$ a continuous bijection, then h is homeo (h is open map).

Cor: If M is closed n-mfld & N is connected n-mfld & $f: M^n \hookrightarrow N^n$ is embedding (= homeo onto its image) then f is onto.

Pf $f(M)$ is closed in N & also open.

Division Algebras

Thm $\mathbb{R}$ and $\mathbb{C}$ are the only finite dim'l real division algebras which are commutative and have 1 (identity)

Algebra means bilinear mult $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

"division algebra" means $ax = b$, $xa = b$ are solvable for $a \neq 0$

Remark Classical examples are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. ($\mathbb{O}$ = octonions)

Algebraic Topology Thm: Any real division algebra has dim 1, 2, 4 or 8.

Pf of Thm $f: S^{n-1} \to S^{n-1}$, $S^{n-1} \subset \mathbb{R}^n$. $f(x) = x^2/|x|^2$

So $f(-x) = f(x)$. Defines $\bar{f} : \mathbb{RP}^{n-1} \to S^{n-1}$ "$|x|$"

Claim $\bar{f}$ is injective: $\bar{f}(x) = \bar{f}(y)$

$$x^2 = \alpha^2 y^2 \qquad (x - \alpha y)(x + \alpha y) = 0$$

$$x = \pm \alpha y$$

$\therefore\ \bar{f}$ is homeo onto its image

$\therefore\ \bar{f}$ is surjective; $n \geq 2$ gives contradiction

Finally, show commutative & 2 dim'l $\Rightarrow \cong \mathbb{C}$.

(simple algebra) Finding i, $j \notin \mathbb{R}$

$j^2 = a + bj \qquad (j - b/2)^2 = j^2 - bj + b^2/4 = a + \underset{\in \mathbb{R}}{b^2/4}$

$\therefore\ j^2 = a$. If $a \geq 0$ $a = c^2 \Rightarrow j \neq \pm c$

so $j^2 = -c^2$

Borsuk-Ulam Thm. $f : S^n \to S^n$, $\mathbb{Z}_2$-equiv i.e. $f(-x) = -f(x)$. Then f has odd degree.

Cor $g : S^n \to \mathbb{R}^n$ Then $\exists\ x \in S^n$ s.t. $g(x) = g(-x)$.

Pf $f(x) = g(x) - g(-x)$. If $f(x) \neq 0$

set $F = f(x)/|f(x)| : S^n \to S^{n-1}$

$F|_{S^{n-1}}(-x) = -(F|_{S^{n-1}})$ so odd degree.

Remark: If $f: S^n \to S^n$ is even ($f(-x) = f(x)$), ~~then~~ f has even degree:

$P^n = \mathbb{R}P^n$

$\bar{f}: P^n \to S^n$ induced map

Then

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "p"'] & S^n \\ P^n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Pf of Borsuk-Ulam. Transfer sequence

$$0 \to C_k(P^n) \xrightarrow{\tau} C_k(S^n) \to C_k(P^n) \to 0$$

$\mathbb{Z}_2$ - coefficients

$$0 \to H_n(P^n) \xrightarrow{\cong} H_n(S^n) \xrightarrow{0} H_n(P^n) \xrightarrow[\partial]{\cong} H_{n-1}(P^n) \to$$

$$\cdots H_i(P^n) \xrightarrow{\cong} H_{i-1}(P^n)$$

$$\cdots H_1(P^n) \xrightarrow{\cong} H_0(P^n) \xrightarrow{0} H_0(S^n) \xrightarrow{\cong} H_0 P^n$$

f & $\bar{f}$ induce commutative diagram of long exact sequence

$$\begin{tikzcd} H_i(S^n) \arrow[r] \arrow[d] & H_i(P^n) \arrow[r] \arrow[d, "\bar{f}_*"] & H_{i-1}(P^n) \arrow[d, "\bar{f}_*"] \\ {} & {} & {} \end{tikzcd}$$

$$H_i(S^n) \longrightarrow H_i(P^n) \longrightarrow H_{i-1}(P^n)$$

By Induction starting $i=1$

$\bar{f}_* : H_i(P^n) \to H_i(P^n)$ is iso

$f_* : H_n(S^n; \mathbb{Z}_2) \longrightarrow H_n(S^n, \mathbb{Z}_2)$

is iso $\quad f_*$ is mult

by $\deg f \Rightarrow \deg f$ odd