Jan 8

First Homework: Due ~~Wed~~ Friday Jan 17

2.2: 1, 2, 3, 8, 11, 20, 21, 23, 24

<u>Degree Theory</u>: $f : S^n \to S^n$

$$\begin{array}{ccc} f_* : H_n(S^n) & \longrightarrow & H_n(S^n) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow[\times m]{} & \mathbb{Z} \end{array}$$

$\deg f = m$

- $\deg(fg) = \deg f \deg g$
- $f =$ reflection $\Rightarrow \deg f = -1$
- $f(x) = -x \Rightarrow \deg f = (-1)^{n+1}$
- f has no fixed pts $\Rightarrow \deg f = (-1)^{n+1}$

Local degree: Suppose $\exists y \in S^n$

$\# f^{-1}(y)$ is finite

$= \{x_1, \dots x_m\}$

$$\begin{array}{ccc} \coprod U_i - x_i & \longrightarrow & V - y \\ \cap & & \cap \\ S^n - f^{-1}(y) & & S^n - y \end{array}$$

$$f_* : H_n(U_i, U_i - x_i) \to H_n(V, V-y)$$
$$\quad\ \cong \mathbb{Z} \qquad\qquad\qquad \cong \mathbb{Z}$$

$$\deg f\big|_{x_i} = m$$

Cellular homology

$$C_n^{CW}(X) = H_n(X^n, X^{n-1})$$
$$= \text{free abelian gp on } \{n\text{-cells}\}$$

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_n(X^{n-1}) \arrow[ur, "j_n"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X^n/X^{n-1} \arrow[r] \arrow[d, phantom, "\cong"] & X^{n-1}/X^{n-2} \arrow[d, phantom, "\cong"] \\
\bigvee_\alpha S^n_\alpha \arrow[r] & \bigvee S^{n-1}_\beta \arrow[d] \\
S^n_\alpha \arrow[u] \arrow[r, "\deg = d_{\alpha\beta}"'] & S^n_\beta
\end{tikzcd}$$

Formula: $d_n(e^n_\alpha) = \sum d_{\alpha\beta}\, e^{n-1}_\beta$

Local degree

$$
\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V-y) \arrow[d, "\cong"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j", "\cong \oplus \mathbb{Z}"'] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}
$$

$$j(1) = (1, 1, \dots, 1) = \sum k_i(1) \longrightarrow \sum \deg f|_{x_i}$$

$$
\begin{tikzcd}[column sep=small]
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[r] & 0 & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
{} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & {} \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}
$$

Jan 9

Euler characteristic

$X =$ finite CW complex

$C_n(X) = C_n^{CW}(X)$

$c_n = \operatorname{rk} C_n(X) = \#$ n-cells

Def $\chi(X) = \sum (-1)^n c_n$

Thm $\chi(X) = \sum (-1)^n \operatorname{rk} H_n(X)$

Pf $0 \to B_n \to Z_n \to H_n \to 0$

$B_n = \operatorname{Im} \partial_{n+1}$, $Z_n = \ker \partial_n$, $H_n = Z_n / B_n$

$0 \to Z_n \to C_n \to B_{n-1} \to 0$

$c_n = \operatorname{rk} C_n \quad z_n = \operatorname{rk} Z_n \quad h_n = \operatorname{rk} H_n$

$z_n = b_n + h_n$

$c_n = z_n + b_{n-1} = b_n + b_{n-1} + h_n$

- $\tilde{X} \to X$ m-sheeted cover

Then $\chi(\tilde{X}) = m\chi(X)$

- (Homework): $\chi(X \times Y) = \chi(X)\chi(Y)$

<u>Hint</u>: $c_n(X) = \#\, n$ cells

$f^X(t) = \sum c_n(X)\, t^n$

Then $f^{X \times Y}(t) = \, ?$

- Long exact sequence

$0 \to A_n \to \cdots \to A_0 \to A_{-1} \to 0$

$$\sum (-1)^k \, \mathrm{rk}\, A_k = 0$$

- <u>Cor</u>: $\chi(X) = \chi(A) + \chi(X, A)$

<u>Ex</u> M_g = closed orientable surface of genus g

Then

$$\chi(M_g) = 2 - 2g$$

<u>Cor</u> $M_{g'} \to M_g$ m-fold cover

Then $g' = \, ?$

Mayer-Vietoris Sequences

$A, B \subset X \qquad X = \operatorname{int} A \cup \operatorname{int} B$

$$X = A \cup B$$

<u>Excision</u> $\quad H_n(A, A \cap B) \xrightarrow{\cong} H_n(X, B)$

<u>Thm</u> MV sequence, $\exists$ long exact sequence

$$\to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B)$$

What is ~~[illegible]~~ def'n ∂

$$\begin{tikzcd}
C_n(A \cap B) \arrow[r] & \oplus \arrow[r] & C_n(X) \supset Z_n \arrow[d] \\
C_{n-1}(A \cap B) & \oplus & C_{n-1}(X) \arrow[from=2-2, to=2-3]
\end{tikzcd}$$

$C_n(A+B)$

Excision $\Rightarrow$ MV sequence

$$H_*(A, A\cap B) \xrightarrow[\cong]{} H_*(X, B)$$

$$\begin{tikzcd}
H_n(A\cap B) \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(A, A\cap B) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(A\cap B) \arrow[d] \\
H_{n-1}(B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, B) \arrow[r] & H_{n-1}(B)
\end{tikzcd}$$

Homological Algebra $\Rightarrow$ MV
(Diagram Chase)

__Pf__ of Excision

$C_n(A+B) =$ had basis singular simplices
$\sigma : \Delta^n \longrightarrow X \quad \mathrm{Im}(\sigma) \subset A$ or B

$$H_*(C_*(A+B)) \cong H_*(X)$$

SES

$$0 \to C_n(A\cap B) \to C_n(A) \oplus C_n(B) \to C_n(A+B) \to 0$$

$$x \longrightarrow (x, -x)$$

$$(x, y) \longrightarrow x+y$$

$\therefore \exists$ LES in homology

The MV-sequence

$z \in C_n(A+B)$ a cycle

$\cancel{z}$ $(x, y) \longrightarrow z = x+y$

$\partial(x+y) = 0$

$\partial x = -\partial y$

$H_n(X) \longrightarrow H_{n-1}(A \cap B)$

$z \longrightarrow \partial x$

Main Use: 95% use

5-Lemma

<u>Remark</u> $X = A \cup B$

A, B subcomplexes

Same MV-sequence

<u>Pf</u> Replace $A, B, A\cap B$

by open U_A U_B $U_{A\cap B}$

$$\supseteq \qquad \supseteq \qquad \supseteq$$
$$A \qquad B \qquad A\cap B$$

Jan 12

Homology of $\mathbb{RP}^n$

- 1 cell e^k in each dim
- Attaching map : $\partial e^k \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}$

  $\deg q\varphi = (-1)^k$

$$\xrightarrow{2} C_3 \xrightarrow{0} \underset{\mathbb{Z}}{\overset{\shortparallel}{C_2}} \xrightarrow{2} \underset{\mathbb{Z}}{\overset{\shortparallel}{C_1}} \xrightarrow{0} \underset{\mathbb{Z}}{\overset{\shortparallel}{C_0}} \to$$

$$\tilde{H}_i(\mathbb{RP}^n) = \mathbb{Z}/2 \qquad i \text{ odd},\ i<n$$

$$H_{2k+1}(\mathbb{RP}^{2k+1}) = \mathbb{Z} \qquad ,\ 0 \text{ otherwise}$$

Lens space $L_m = S^{2n-1}/\mathbb{Z}/m$

$$\tilde{H}_i(L_m^{2n-1}) = \begin{cases} \mathbb{Z}/m & i \text{ odd} \\ \mathbb{Z} & i = 2n-1 \end{cases}$$

- Def $K(G,1)$
- Ex $K(\mathbb{Z}/2, 1) = \mathbb{RP}^\infty$

$$K(\mathbb{Z}/m, 1) = L_m^\infty$$

Homology of groups

<u>Def</u> $H_*(G) = H_*(K(G,1))$

<u>Thm</u> (P.A. Smith). If $K(G,1)$ finite dim'l $\Rightarrow$ G is torsion free

<u>Pf</u>. If not $\mathbb{Z}/m \subset G$

$X = K(G,1) \Rightarrow \tilde{X}$ contractible

$\tilde{X}/\mathbb{Z}/m$ is a model for $K(\mathbb{Z}/m, 1)$

but this is infinite dim'l. $\square$

Homology with Coefficients

G = abelian gp

eg $G = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}/2$, etc

Singular homology: X = top space

$$C_n(X;G) \overset{C_n(X)\otimes G}{=} \left\{ \sum n_i \sigma_i \;\middle|\; n_i \in G \right\}$$

∂ maps, relative homology as before

Similarly, $C_n^{CW}(X;G) = \bigoplus_{n\text{-cells}} G$

Ex $\mathbb{RP}^n$ coefficients in field F

$C_*^{CW}(\mathbb{RP}^n;F)$

$C_2 \to C_1 \to C_0$

$\xrightarrow{0} F \xrightarrow{2} F \xrightarrow{0} F$

char $F = 2$ $\quad$ $F = \mathbb{Z}/2$

$H_i(\mathbb{RP}^n;F) = \{ F$ all $i \le n$

char $F = 0$, $\tilde{H}_i(\mathbb{RP}^n;F) = \begin{cases} F & i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$

X is $K(G,1)$ $\quad$ $G =$ gp

$\pi_1(X) = G$

$\tilde{X} =$ univ cover is contractible

$\forall G \quad \exists\, K(G,1)$

unique upto homotopy

$S^\infty = \bigcup S^n$ is contractibl

$K(\mathbb{Z}/2, 1) = S^\infty/\mathbb{Z}/2 = \mathbb{R}P^\infty$

$K(\mathbb{Z}/m, 1) = S^\infty/\mathbb{Z}/m = L^\infty_m$

$H_k(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \forall k \leq n \end{cases}$

Eventually, we get formula for

$H_k(X; G)$ in terms of $H_k(X; \mathbb{Z})$

Univ. Coefficient Thm

1st Guess:

$H_k(X;G) = H_k(X) \otimes G$

$\exists$ short exact

$0 \to H_k(X) \otimes G \to H_k(X; G)$

$$\longrightarrow \mathrm{Tor}(H_{k-1}(X), G) \to 0$$

to be defined