Feb. 21

<u>Poincaré duality</u>

M^n = n-dim, closed mfld
(closed " compact)

"orientable"

<u>Thm</u>

$$H_k(M) \cong H^{n-k}(M)$$

<u>Recall</u> $\dfrac{H^{n-k}(M)}{\text{Tors}} = \dfrac{H_{n-k}(M)}{\text{Tors}}$

UCT

$$\text{Tor } H^{n-k} = \text{Tor } H_{n-k-1}(M)$$

<u>Cor</u> Coefficients in $\mathbb{F}$.

$$H^{n-k}(M) = H_{n-k}(M)$$

k^{th} Betti number = $(n-k)^{th}$ Betti number

Ex. M^7 $\beta_0 = \beta_7$.

$$\beta_1(M^3) = \beta_2(M^3)$$

Orig Poincare Conj: M^3 has same homology $S^3 \Rightarrow M^3 = S^3$.

New Version: Simply connected $M^3 \Rightarrow M^3 = S^3$.

$H_0 = \mathbb{Z}$

$H_1 = \pi_1^{ab} = 0$

$\Rightarrow [H^3 = \mathbb{Z}, \quad H^2 = 0$

$\Rightarrow H_3 = \mathbb{Z}, H_2$

Need M^n to be PL mfld

P = simplices in some train.

e^k "Link of (e^k)" should be S^{n-k-1}

$e^* = \text{Cone}(S^{n-k-1})$

P' = barycentric subdivision.

$e \in \mathcal{P}$, v_e = barycenter of e = { chains of $\mathcal{P}$ of $\mathcal{P}$ }

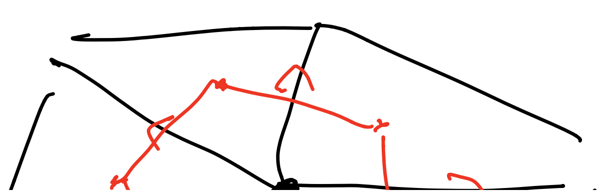
$\mathcal{P}_{>e^k} = \{ f \mid f > e^k \}$

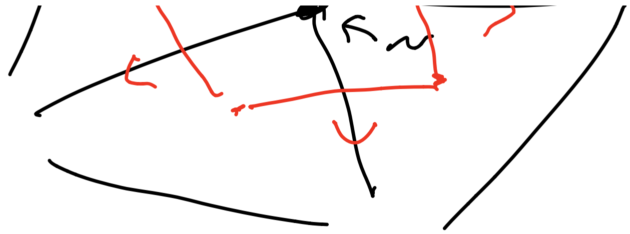
$Lk \subset \mathcal{P}'$, spanned by vertices v_f, $f > e$.

Claim main

$$\delta(e^i) = \partial((e^*)^{n-i})$$

i.e. cochain complex of cells is iso to chain complex of dual $n-i$-cells

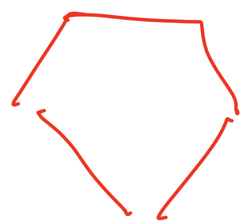




$\delta(v)$ = edges going out from v

v^* = Pentagon

$\partial(v^*)$ =



$$\begin{array}{ccc} C^i(P) & \longrightarrow & C_{n-i}(P^*) \\ \downarrow & & \downarrow \partial \, ? \\ C^i(P) & \longrightarrow & C_{n-i-1}(P^*) \end{array}$$

Feb 26

<u>Def'n</u> M^n a n-mfld = Hausdorff space locally homeo to $\mathbb{R}^n$.

$\forall x \in M^n$ has nbhd homeo to $\mathbb{R}^n$

$\mathbb{Z}$ coefficients

$H_n(M|_x) = H_n(M, M-x) = H_n(\mathbb{R}^n, \mathbb{R}^n - x)$

$= \mathbb{Z}.$

local orientation μ_x = gen for $H_n(M|_x)$

A ~~closed~~ subset $H_n(M|_A) = H_n(M, M-A)$

μ_x & μ_y are compatible if

$x, y \in B$. Def orientable

<u>Def</u>^{n} orientation cover $\tilde{M} \to M$

$\tilde{M} = \{\mu_x \mid x \in M, \mu_x$ is local orientation$\}$

<u>Prop</u> M connected. Then $\tilde{M} \to M$

is 2-sheeted cover.

$\tilde{M}$ has 2 components $\Leftrightarrow$ M orientable

" 1 " $\Leftrightarrow$ M not orientable

Pf Obvious

Note $\tilde{M}$ is orientable

Orientation Sheaf

$$M_{\mathbb{Z}} = \{ \alpha_x \mid \alpha_x \in H_n(M|_x) \}$$

$\downarrow p$

M 0-section $\alpha_x = 0$

otherwise

$M_{\mathbb{Z}}$ - 0-section $= \coprod$ copies of orientation cover

---

R-orientation

$$H_n(M|_x; R) = R$$

Require μ_x to be a unit

Note

$$H_n(M|_x; R) = H_n(M|_x) \otimes R$$

$M_R \to M$ subcovers

$\sim$

$\tilde{M}_r \to M$ where $\pm\mu_x \otimes r$

$\tilde{M}_r = \tilde{M}_{-r}$ = copy of orientation cover

M is R orientable $\Leftrightarrow$ $2 = 0$ in R

Thm M^n closed connected

(a) M^n R-orientable then

$H_n(M;R) \to H_n(M|_x;R) = R$

is iso $\forall x$

(b) not R-orientable then
$H_n(M;R) \to H_n(M|_x;R)$
injective image $\{r \in R \mid 2r = 0\}$

(c) $H_i(M;R) = 0$ $i > n$

Lemma $A \subset M^n$, A compact

(a) $x \to \alpha_x$ section of $M_R \to M$

$\exists!$ $\alpha_A \in H_n(M|A;R)$ ~~s.t.~~

image in $H_n(M|_x;R) = \alpha_x$

ⓑ $H_i(M|_A; R) = 0 \quad i > n$

Feb 26

Recall relative M-V sequence (p.152)

$(X, Y) = (A \cup B, C \cup D)$ interiors A & B cover X & same for Y

$\to H_n(A \cap B, C \cap D) \to H_n(A, C) \oplus H_n(B, D)$

$\to H_n(X, Y) \to$

In particular, if $A = B = X$

Then $\to H_n(X, C \cap D) \to H_n(X, C) \oplus H_n(X, D)$

$\to H_n(X, C \cup D) \to$

Pf of Lemma 4 steps

Step 1 True for $A, B, A \cap B \Rightarrow$

True for $A \cup B$. Relative ...

$$0 \to H_n(M|_{A\cup B}) \xrightarrow{\Phi} H_n(M|_A) + H_n(M|_B)$$

$$\xrightarrow{\psi} H_n(M|_{A \cap B})$$

Recall ‖

$$H_n(M, M - A\cap B) = H_n(M, M - A \cup M - B)$$

Because $H_i(M|_{A\cap B}) = 0 \quad i > n$

Terms $H_i(M|_{A\cup B}) = 0 \quad i > n$

∴ (b) holds

$$\Phi(\alpha) = (\alpha, -\alpha) \qquad \psi(\alpha, \beta) = \alpha - \beta$$

For (a), suppose $\exists !$ classes α_A, α_B, $\alpha_{A\cap B}$ restricting to $\alpha_x \in H_n(M|x)$. Classes α_A, α_B satisfy defining property of $\alpha_{A \cap B}$

∴ Images of α_A and α_B in $H_n(M|_{A\cap B})$ must both $= \alpha_{A\cap B}$

i.e. $\psi(\alpha_A, \alpha_B) = 0 \implies \exists$ class $\alpha_{A\cup B} \in H_n(M|_{A\cup B})$. which maps to α_x. Claim $\alpha_{A\cup B}$ is unique. Because if it maps to $\alpha_x = 0$ for each x, then same for α_A & α_B ($\alpha_{A \cap B} = 0$)

i.e. $\alpha_A = 0$ & $\alpha_B = 0$. Then apply observation to difference of 2 choices

---

<u>Step 2</u> Reduce to $M^n = \mathbb{R}^n$.

Any compact $A \subset M$ can be written as $A = A_1 \cup \cdots \cup A_m$ each $A_i \subset \mathbb{R}^n$. Apply step 1 to $(A_1 \cup \cdots \cup A_{m-1}) \cup A_m$

where $(\qquad) \cap A_m = (A_1 \cap A_m) \cup \cdots (A_{m-1} \cap A_m)$ is union of $(m-1)$ compact sets in $\mathbb{R}^n$. Induction on m reduces to $m = 1$. Case $m = 1$: $A \subset \mathbb{R}^n$ replace $H_n(M|A)$ by $H_n(\mathbb{R}^n|A)$.

$M = \mathbb{R}^n$.

Step 3 $A = A_1 \cup \dots \cup A_m$

convex compact sets. As in Step 2 can reduce to $m=1$, i.e

A is convex.

Then $\mathbb{R}^n - A$, $\mathbb{R}^n - x$ both deformation retract onto S^{n-1}

Step 4 $M = \mathbb{R}^n$, $A \subset \mathbb{R}^n$ arbitrary compact set. Show higher homology $= 0$ + extension property

$\alpha \in H_i(\mathbb{R}^n|_A)$ represented by singular cycle z. $C \subset \mathbb{R}^n - A$ be union images of simplices in ∂z.

$d(C, A) = \delta$ cover A by finitely many balls of radius δ centered at pts of A. $K = \bigcup$ balls.

$K \cap C = \emptyset$. relative cycle it

defines class $\alpha_K \in H_i(\mathbb{R}^n | K)$

mapping to α. If $i > n$, then

$H_i(\mathbb{R}^n | K) = 0$ so $\alpha = 0$

$\Rightarrow H_i(\mathbb{R}^n | A) = 0$.

If $i = n$ and $\alpha_x = 0 \in H_n(\mathbb{R}^n |_x)$

$\forall x$ in A, then true for all $x \in K$

(if α_x = image of α_K). This is

because $K = \bigcup$ finite # of Balls

& $H_n(\mathbb{R}^n | B) \xrightarrow{\cong} H_n(\mathbb{R}^n |_x)$. $\forall x \in B$

So $\alpha_x = 0$ $\forall x \in K \Longrightarrow$

$\alpha_K = 0$. This gives uniqueness.

For existence take α_A = image of

α_K

$\square$

March 2

Cap product:

Bilinear pairing

$$\cap : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

Coefficients in Ring R (omit)

<u>Def</u>: $\sigma : \Delta^k \to X$ $\varphi \in C^\ell(X)$

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0 \cdots v_\ell]})\ \sigma|_{[v_\ell \cdots v_k]}$$

Show this induces

$$\cap : H_k(X) \times H^\ell(X) \to H_{\ell-k}(X).$$

Assuming this, can define PD iso

for $X = M^n$ closed R-orientable mfld

Fundamental class $[M] = \mu_M$

$\in H_n(M)$. Take cap product with

$[M]$

P.D Thm Define $D: H^\ell(M) \to H_{n-\ell}(M)$ by $D(\alpha) = [M] \cap \alpha$. Then D is iso.

Cor n odd. Then $\chi(M^n) = 0$

Return to cap product.

Key Formula:

$$\partial(\sigma \cap \varphi) = (-1)^\ell \left[\partial\sigma \cap \varphi - (\sigma \cap \delta\varphi)\right]$$

$$\varphi(\partial\sigma|\ ) \partial\sigma|_{[\ ]}$$

Pf

- $$\partial\sigma \cap \varphi = \sum_{i=0}^{\ell} (-1)^i \varphi(\sigma|_{[v_0 \ldots \hat{v}_i \ldots v_\ell]})\, \sigma|_{[v_{\ell+1} \ldots v_k]}$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \varphi(\sigma|_{[v_0 \ldots v_\ell]})\, \sigma|_{[v_{\ell+1}\ \hat{v}_i\ v_k]}$$

$\delta\varphi(\sigma|_{[\ ]})\, \sigma|_{[\ ]}$

- $$\sigma \cap \delta\varphi = \sum_{i=0}^{\ell+1} (-1)^i \varphi(\sigma|_{[v_0\ \hat{v}_i \ldots v_{\ell+1}]})\, \sigma|_{[v_{\ell+1}\ v_k]}$$ ← extra term

$\cdot\ \partial(\sigma \frown \varphi) = \sum_{i=l}^{k} (-1)^{i-l} \varphi(\sigma|_{[v_0 \ldots v_l]})\, \sigma|_{[v_l \ldots \hat{v}_i \ldots v_k]}$

↖ 1 extra term $i=l$.

$\square$

. cycle $\frown$ cocycle = cycle

. If $\partial\sigma = 0 \Rightarrow \partial(\sigma \frown \varphi) = \pm(\sigma \frown \delta\varphi)$

i.e. cycle $\frown$ cobdry = bdry

. $\delta\varphi = 0 \Rightarrow \partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi)$

i.e. bdry $\frown$ cocycle = bdry.

$\therefore\ H_k(X) \times H^l(X) \longrightarrow H_{k-l}(X).$

Also relative formulas

$$H_k(X,A) \times H^l(X) \to H_{k-l}(X,A)$$

$$H_k(X,A) \times H^l(X,A) \longrightarrow H_{k-l}(X)$$

Also

$$H_k(X, A \cup B) \times H^l(X,A) \to H_{k-l}(X,B)$$

Naturality $\quad f : X \to Y$

$$
\begin{array}{ccccc}
\alpha \in H_k(X) & \times & H^{\ell}(X) & \longrightarrow & H_{k-\ell}(X) \\
\downarrow f_* & & \uparrow f^* & & \downarrow f_* \\
H_k(Y) & \times & H^{\ell}(Y) & \longrightarrow & H_{k-\ell}(Y) \\
 & & \ni \varphi & &
\end{array}
$$

$$f_*\alpha \cap \varphi = f_*(\alpha \cap f^*\varphi)$$

$$(\varphi \cup \psi)(\alpha) = \psi(\alpha \cap \varphi)$$

$$\alpha \in C_{k+\ell} \qquad \varphi \in C^k \qquad \psi \in C^{\ell}$$