Jan 24

Cohomology

$C_n(X) =$ n-chains on X (singular, simplicial or cellular)

$C^n(X;G) = \mathrm{Hom}(C_n(X); G)$
$=$ n-cochains

Leave out if $G = \mathbb{Z}$.

$$C_n \xrightarrow{\partial} C_{n-1}$$

$$C^n \xleftarrow{\delta} C^{n-1}$$

$\delta =$ "coboundary"

$H^n = \ker \delta / \mathrm{Im}\, \delta$

a

Contravariant Functor

$f: X \to Y$

$f_\#: C_n(X) \to C_n(Y)$

$f^\#: C^n(Y) \to C^n(X)$

Cup Product: On CW-cx

$$C^i(X) \times C^j(Y) \to C^{i+j}(X \times Y)$$

Rough Idea

$$\begin{tikzcd} H^i(X) \times H^j(X) \arrow[r] & H^{i+j}(X \times X) \arrow[d, "\Delta^*"] \\ & H^{i+j}(X) \end{tikzcd}$$

Δ = diagonal

Meaning of δ (cocycles, coboundaries)

$\Delta^n(X)$ = cochains on a Δ-complex X.

$\varphi \in \Delta^{n-1}$ is a function on $(n-1)$ simplices

$\delta\varphi \in \Delta^n$

$$\delta\varphi(\sigma) = \varphi(\partial\sigma)$$

0-coeycle $\varphi \in \Delta^0$ $\qquad v \xrightarrow{e} w$ edge

$$\delta\varphi(e) = \varphi(w) - \varphi(v)$$

$\delta\varphi = 0$ means φ is constant

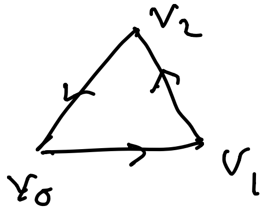
(on path components)

$\psi \in \Delta^1$ is coboundary means
$\psi = \delta \varphi$

<u>Ex</u> $X =$ tree then any ψ is a coboundary

$X =$ 2-complex, $\psi \in \Delta^1$

ψ is 1-cocycle means it sums to 0 around any 2-simplex



$$\psi([v_0, v_2]) = \psi([v_0, v_1]) + \psi([v_1, v_2])$$

<u>Remarks</u> Can also think of a 1-cycle as function on Edges.

Kirkhoff's Laws:

. Current = cycle
. voltage = cocycle

Given a 1-cocycle ψ, find a function on vertices φ s.t. $\psi = \delta\varphi$.

Analogy with 1-forms and path integrals

Jan 26

Homework 3.2 pp 204 - 206:
1, 2, 3, 4, 5, 11, 13 Feb ~~7~~ ?

Universal Coefficient Thm

$$\to C_n \xrightarrow{\partial} C_{n-1} \leftarrow \quad \text{chain complex}$$

$$C_n^* \xleftarrow{\delta} C_{n-1}^* \leftarrow \quad \text{dual cochain complex}$$

$$C_n^* = \mathrm{Hom}(C_n, G)$$

$H^n(C;G)$ = homology of C^* = $\ker \delta / \mathrm{Im}\, \delta$

<u>UCT</u> $\exists$ split short exact sequence $H_n(C)^*$

$$0 \to \mathrm{Ext}(H_{n-1}(C),G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G) \to 0$$

• Define $h: H^n(C;G) \to \mathrm{Hom}(H_n(C),G))$

$Z_n = \ker \partial_n \subset C_n \supset B_n = \mathrm{Im}\, \partial_{n+1}$

$\varphi : C_n \to G$ a cocycle, i.e. $\delta\varphi = \varphi\partial = 0$ ✓

$\varphi_0 = \varphi|_{Z_n}$ induces $\bar{\varphi}_0 : H_n(C) \to G$

$\varphi = \delta\psi \Rightarrow \varphi_0|_{B_n} = 0$

$\varphi(\partial y)$ = $\delta\varphi(y) = 0$

$h: [\varphi] \longrightarrow \bar{\varphi}_0$

• h is surjective

$0 \to Z_n \overset{i}{\underset{p}{\rightleftarrows}} C_n \to B_{n-1} \to 0$ is split

$\Rightarrow$ $\exists\ p: C_n \to Z_n$

if $\varphi_0 : Z_n \to G$ vanishes on B_n

$\varphi = \varphi_0 p : C_n \to G$

i.e. $\bar{\varphi}_0$ extends to $\varphi \in \ker \delta$.

So $\quad H_n(C)$

$$0 \to \ker h \to H^n(C;G) \to \mathrm{Hom}(H_n(C),G) \to 0$$

is split

short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "i_n"] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Dualizing get short exact seq.

$$0 \leftarrow Z_{n+1}^* \leftarrow C_{n+1}^* \leftarrow B_n^* \leftarrow 0$$

(Since previous sequence split)

long exact sequence in (co) homology

$$B_n^* \leftarrow Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \leftarrow Z_{n-1}^*$$

connecting homo $= (i_n)^*$

Breaks up into exact sequen

$$0 \leftarrow \ker i_n^* \leftarrow H^n(C;G) \leftarrow \operatorname{coker} i_{n-1}^* \leftarrow 0$$

$$\ker i_n^* = \mathrm{Hom}(H_n(C);G)$$

So Q. What is $\operatorname{coker} i_n^*$.

If $H_{n-1}(C)$ were free abelian

then $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$

dualize

$$0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow H_{n-1}(C)^* \leftarrow 0$$

So coker i_{n-1}^* would $= 0$

$A \to B \to C \to 0$ exact the

$A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is exact

Free resolution of abelian gp H

exact sequen

$$\cdots \to F_2 \to F_1 \to F_0 \to H \to 0$$

F_i = free abelian. Dualizing

we get a chain complex

$$F_2^* \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$$

Take homology

We had free resolution H

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$$

$$0 \leftarrow B_{n-1}^* \leftarrow Z_{n-1}^* \leftarrow H_{n-1}^* \leftarrow 0$$

$B_{n-1}^* = F_1^*$, $Z_{n-1}^* = F_0^*$

We want $\operatorname{coker} i_{n-1}^* = H^1(F^*)$?

$$\operatorname{Ext}(H_{n-1}, G) = \text{"}\operatorname{Ext}\text{"}$$

Lemma: $\alpha : H \to H'$ &

2 free resolutions

$$\begin{array}{ccccccc} \to F_1 & \to & F_0 & \to & H & \to & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow \alpha & & \\ F_1' & \to & F_0' & \to & H' & \to & 0 \end{array}$$

$\exists$ chain maps $\alpha_k : F_k \to F_k'$

and any 2 such are

chain homotopic.

b) F & F' are 2 free resolutions of H Then

$$H^n(F; G) \cong H^n(F', G)$$

Def'n $\operatorname{Ext}(H, G) = H^1(F; G)$

Appendix 3A

Feb 2

Universal Coefficient Thm for Homology

$$H_n(C;G) = H_n(C_*\otimes G)$$

$$\begin{array}{ccccccccc} 0 \to & Z_n & \to & C_n & \to & B_{n-1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & Z_{n-1} & \to & C_{n-1} & \to & B_{n-2} & \to 0 \end{array}$$

$$B_n\otimes G \xrightarrow{i\otimes 1} Z_n\otimes G \to H_n(C,G) \to B_{n-1}\otimes G \xrightarrow{i_{n-1}\otimes 1}$$

$$0 \to \operatorname{coker}(i_n\otimes 1) \to H_n(C,G) \to \ker(i_{n-1}\otimes 1) \to 0$$

Claim $\operatorname{coker}(i_n\otimes 1) = \dfrac{Z_n\otimes G}{\operatorname{Im}(i\otimes 1)} = H_n\otimes G$

Lemma $A \to B \to C \to 0$ exact

$\Rightarrow$ $A\otimes G \to B\otimes G \to C\otimes G \to 0$

Apply this $0 \to B_n \xrightarrow{i_n} Z_n \to H_n \to 0$

$$Z_n\otimes G / \operatorname{Im}(A\otimes G) \cong H_n(C)$$

Thm

$$\boxed{0 \to H_n \otimes G \to H_n(C;G) \to \ker(i_{n-1}\otimes 1) \to 0}$$

$$\ker(i_{n-1}\otimes 1) \cong \mathrm{Tor}(H_{n-1}, G)$$

Free resolutions ass.

Pf of Lemma $(j\otimes 1)\circ(i\otimes 1) = 0$

exactness at $B\otimes G \iff$

$$\frac{B\otimes G}{\mathrm{Im}(i\otimes 1)} \xrightarrow{\cong} C\otimes G$$

$\xleftarrow{\psi}$

$$\psi : C\times G \longrightarrow \frac{B\otimes G}{\mathrm{Im}}$$

$\psi(c,g) = b\otimes g$ where $j(b) = c$.

Well-defined $j(b') = c$

$$b\otimes g - b'\otimes g = (b-b')\otimes g$$

$$j(b\otimes g) - j(b'\otimes g) = j(b-b')\otimes g$$

$$= 0$$

$j(b) = j(b')$ $\quad b - b' \in \operatorname{Im} i$

φ is well-defined hom

$$\bar{\varphi} : C\otimes G \rightleftarrows \frac{B\otimes C}{\operatorname{Im}(i\otimes 1)}$$

$\blacksquare$

$$\begin{array}{ccccccc} & \text{free} & & \text{free} & & & \\ 0 \to & B_n & \xrightarrow{i} & Z_n & \to & H_{n-1} & \to 0 \\ & \| & & \| & & & \\ & F_1 & \to & F_0 & \to & H & \end{array}$$

$$F_1\otimes G \to F_0\otimes G \to H\otimes G \to 0$$

$$\ker(i\otimes 1) = H\,(F\otimes G$$

Free resolution

$$\to F_2 \to F_1 \to F_0 \to H \to 0$$

$$F_2 \otimes G \to F_1 \otimes G \to F_0 \otimes G \to H \otimes G \to 0$$

$$\mathrm{Tor}_i(H;G) = H_i(F \otimes G)$$

Independent of resolution

$$0 \to F_1 \to F_0 \to H \to 0$$

for abelian gps

only $\mathrm{Tor}_1(H,G) = \mathrm{Tor}(H,G)$