signature theorem. A central objective was to show that the bilinear pairing

$$(1)\qquad \pi_m(\mathrm{SO}_n)\otimes\pi_n(\mathrm{SO}_m)\ \to\ \pi_0\mathrm{Diff}^+(S^{m+n})\ \to\ \Gamma_{m+n+1}$$

described above can be used to construct exotic spheres which can be detected by a diffeomorphism invariant

$$\lambda : \Gamma_{4k-1} \to \mathbf{Q}/\mathbf{Z}$$

in many cases with $m \equiv n \equiv 3 \pmod 4$. However, I ran out of time, so that the lectures didn't actually get that far. For that reason, I have extended these lectures, translated back into English, by adding a final section which completes the argument.

Further Developments

The bilinear pairing of Equation (1) and other similar pairings have been studied or applied by several authors. See for example P. Kahn [1965], A. Kosinski [1967], and T. Lawson [1973]. Perhaps the most important result which has been obtained by such methods is the following statement which was proved by Antonelli, Burghelea and Kahn in [1972]:

Theorem. *If $n \geq 7$, then the group $\mathrm{Diff}^+(S^n)$ of all orientation preserving diffeomorphisms of the sphere does not have the homotopy type of a finite complex.*

By way of contrast, it is known that the inclusion $\mathrm{SO}_{n+1} \hookrightarrow \mathrm{Diff}^+(S^n)$ is a homotopy equivalence for $n \leq 3$. The case $n = 1$ is quite easy, and for $n = 2$ this was proved by Smale [1959b]. The proof for $n = 3$ by Hatcher [1983] is much more difficult.

Remark. The statement that $\mathrm{Diff}^+(S^2)$ has the homotopy type of SO_3 can also be proved by using quasiconformal methods. The ***Beltrami differential*** of an orientation preserving diffeomorphism f of a Riemann surface is C^∞-smooth, being given, in terms of a local coordinate z, by the expression

$$(2)\qquad \left(\frac{\partial f/\partial\bar z}{\partial f/\partial z}\right)\frac{d\bar z}{dz}\qquad\text{with}\qquad \left|\frac{\partial f/\partial\bar z}{\partial f/\partial z}\right| < 1 .$$

This vanishes if and only if f is a conformal diffeomorphism. If our Riemann surface is the Riemann sphere $\mathbf{C}\cup\infty$, then there is a converse statement: Following Ahlfors and Bers, any smooth Beltrami differential satisfying Equation (2) can be integrated to yield a diffeomorphism f which is unique up to composition with a Möbius automorphism. If we normalize by requiring that f must fix three points, say 0, 1 and ∞, then f is uniquely determined. Thus the subgroup of $\mathrm{Diff}^+(\mathbf{C}\cup\infty)$ consisting of diffeomorphisms which fix these three points is homeomorphic to the convex set consisting of all such Beltrami differentials, and hence is contractible. But any element of $\mathrm{Diff}^+(\mathbf{C}\cup\infty)$ can be written uniquely as the composition of a Möbius automorphism with an element of this contractible subgroup. This proves that there is a deformation retraction from the group $\mathrm{Diff}^+(\mathbf{C}\cup\infty)$ onto the subgroup $\mathrm{PSL}_2(\mathbf{C})$ of Möbius automorphisms, which in turn deformation retracts onto its maximal compact subgroup SO_3. □

Lectures on Differential Topology

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Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under diffeomorphism (differentiable homeomorphism). Typical problems falling under this heading are the following:

(1) Given two differentiable manifolds, under what conditions are they diffeomorphic?

(2) Given a differentiable manifold, is it the boundary of some differentiable manifold with-boundary?

(3) Given a differentiable manifold, is it parallelizable?

All of these problems concern more than the topology of the manifold, yet they do not belong to differential geometry, which usually assumes additional structure (e.g., a connection or a metric).

The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, whereby one passes from the manifold M to its tangent bundle, and thence to a cohomology class in M which depends on this bundle.

These notes are intended as an introduction to the subject; we will try to go as far as possible without bringing in algebraic topology. Our two main goals are Whitney's theorem that a differentiable n-manifold can be embedded as a closed subset of the euclidean space $\mathbb{R}^{2n+1}$ (see Corollary 1.32); and Thom's theorem that the non-orientable cobordism group $\mathcal{N}_n$ is isomorphic to a certain stable homotopy group (see Theorem 3.15).

Chapter 1 is mainly concerned with approximation theorems. First the basic definitions are given and the inverse function theorem is exploited (§1.1–1.12). Next two local approximation theorems are proved, showing that a given map can be approximated by one of maximal rank (§1.13–1.21). Finally locally finite coverings are used to derive the corresponding global theorems: namely Whitney's embedding theorem and Thom's transversality lemma (§1.28–1.36).

Chapter 2 is an introduction to the theory of vector space bundles, with emphasis on the tangent bundle of a manifold.

Chapter 3 makes use of the preceding material in order to study the cobordism groups $\mathcal{N}_n$.

1. Embeddings and Immersions of Manifolds.

Notation. If x is in the euclidean space $\mathbf{R}^n$, the coordinates of x are denoted by $(x^1, \dots, x^n)$. Let $\|x\| = \max |x^i|$; let $C(r) = C^n(r)$ denote the set of x such that $\|x\| < r$; and $C^n(x_0, r)$ the set of x such that $\|x - x_0\| < r$. The closure of a cube C is denoted by $\overline{C}$.

A real valued function $f(x^1, \dots, x^n)$ is *differentiable* if the partials of f of all orders exist and are continuous (i.e. "differentiable" means C^∞). A map $f : U \to \mathbf{R}^p$ (where U is an open set in $\mathbf{R}^n$) is differentiable if each of the coordinate functions $(f^1, \dots, f^p)$ is differentiable. Df denotes the $p \times n$ Jacobian matrix $(\partial f^i / \partial x^j)$ of f; one verifies that the Jacobian of a composition of two differentiable functions is given by the rule $D(fg) = Dg \cdot Df$. The notation $\partial(f^1, \dots, f^p)/\partial(x^1, \dots, x^n)$ is also used. If $n = p$, $\det(Df)$ denotes the determinant.

Definition 1.1. A *topological n-manifold* M^n is a Hausdorff space with a countable basis which is locally homeomorphic to $\mathbf{R}^n$.

A *differentiable structure* $\mathcal{D}$ on a topological manifold M^n is a collection of real-valued functions, each defined on an open subset of M, such that:

(1) For every point p of M there is a neighborhood U of p and a homeomorphism h of U onto an open subset of $\mathbf{R}^n$ such that a function f, defined on the open subset W of U, is in $\mathcal{D}$ if and only if fh^{-1} is differentiable.

(2) If U_i are open sets contained in the domain of f with union U, then $f|_U \in \mathcal{D}$ if and only if $f|_{U_i}$ is in $\mathcal{D}$, for each i.

A *differentiable manifold* M^n is a manifold provided with a differentiable structure $\mathcal{D}$; the elements of $\mathcal{D}$ are called the differentiable functions on M. Any open set U and homeomorphism h which satisfy the requirements of (1) above are called a *coordinate system* on M. *Notation.* A coordinate system is sometimes denoted by the coordinate functions: $h(p) = (u^1(p), \dots, u^n(p))$.

Definition 1.2. (Alternate). Let a collection (U_i, h_i) be given, where h_i is a homeomorphism of the open subset U_i of M^n onto an open subset of $\mathbf{R}^n$, such that

(a) the U_i cover M

(b) $h_j h_i^{-1}$ is a differentiable map on the open set $h_i(U_i \cap U_j)$, for all i, j.

Define a *coordinate system* as an open set U and a homeomorphism h of U onto an open subset of $\mathbf{R}^n$ such that $h_i h^{-1}$ and hh_i^{-1} are differentiable on $h(U \cap U_i)$ and $h_i(U \cap U_i)$ respectively, for each i. Define the associated *differentiable structure* on M as the collection of all such coordinate systems. A function f, defined on the open set V, is *differentiable* if fh^{-1} is differentiable on $h(U \cap V)$, for all coordinate systems (U, h).

One shows readily that these two definitions are entirely equivalent.

Definition 1.3. Let M_1, M_2 be differentiable manifolds. If U is an open subset of M_1, $f : U \to M_2$ is *differentiable* if for every differentiable function g on an open set $V \subset M_2$, the composition gf is differentiable on $f^{-1}(V) \subset M_1$.

A function $f : M_1 \to M_2$ is a *diffeomorphism* if f and f^{-1} are defined and differentiable.

If $A \subset M_1$, a function $f : A \to M_2$ is *differentiable* if it can be extended to a differentiable function defined on a neighborhood U of A.

(A coordinate system (U, h) on M^n can then be defined as an open set U in M and a diffeomorphism h of U onto an open set in $\mathbf{R}^n$.)

If $A \subset M$, we have just defined the notion of differentiable function for subsets of A. Suppose that A is locally diffeomorphic to $\mathbf{R}^k$: this collection is easily shown to be a differentiable structure on A. In this case, A is said to be a *differentiable submanifold* of M.

The following lemma is familiar from elementary calculus.

LEMMA 1.4. *Let $f : C^n(r) \to \mathbf{R}^p$ satisfy the condition $|\partial f^i / \partial x^j| \le b$, for all i, j and all $x \in C^n(r)$. Then $\|f(x) - f(y)\| \le bn\|x - y\|$, for all $x, y, \in C^n(r)$.*

THEOREM 1.5. (Inverse Function Theorem). *Let U be an open subset of $\mathbf{R}^n$, let $f : U \to \mathbf{R}^n$ be differentiable, and let Df be non-singular at x_0. Then f maps some neighborhood of x_0 diffeomorphically onto some neighborhood of $f(x_0)$.*

Proof. We may assume $x_0 = f(x_0) = 0$, and that $Df(x_0)$ is the identity matrix.

Let $g(x) = f(x) - x$, so that $Dg(0)$ is the zero matrix. Choose $r > 0$ so that $x \in U$ and $Df(x)$ is non-singular and $|\partial g^i / \partial x_j| \le \frac{1}{2n}$, for all x with $\|x\| < r$.

Assertion. *If $y \in C(r/2)$, there is exactly one $x \in C(r)$ such that $f(x) = y$.* In fact, by the previous lemma,

$$(1) \qquad \|g(x) - g(x_0)\| \le \tfrac{1}{2}\|x - x_0\|$$

on $C(r)$. Let us define a sequence $x_0, x_1, \dots$ by $x_0 = 0$, $x_1 = y$, $x_{n+1} = y - g(x_n)$. This sequence is defined, since $x_n - x_{n-1} = g(x_{n-2}) - g(x_{n-1})$, so that

$$\|x_n - x_{n-1}\| \le \frac{1}{2}\|x_{n-2} - x_{n-1}\| \le \frac{1}{2^{n-1}}\|y\|;$$

and thus $\|x_n\| \le 2\|y\|$ for each n. Hence the sequence x_n converges to a point x with $\|x\| \le 2\|y\|$, so that $x \in C(r)$. Then $x = y - g(x)$, so that $f(x) = y$. This proves the existence of x. To show uniqueness, note that if $f(x) = f(x_1) = y$, then $g(x_1) - g(x) = x - x_1$, contradicting (1).

Hence $f^{-1} : C(r/2) \to C(r)$ exists. Note that

$$\|f(x) - f(x_1)\| \ge \|x - x_1\| - \|g(x) - g(x_1)\| \ge \frac{1}{2}\|x - x_1\|$$

so that $\|y - y_1\| \ge 1/2\|f^{-1}(y) - f^{-1}(y_1)\|$. Hence f^{-1} is continuous; the image of $C(r/2)$ under f^{-1} is open because it equals $C(r) \cap f^{-1}(C(r/2))$, the intersection of two open sets.

To show that f^{-1} is differentiable, note that

$$f(x) = f(x_1) + Df(x_1) \cdot (x - x_1) + h(x, x_1),$$

where $(x - x_1)$ is written as a column matrix and the dot stands for matrix multiplication. Here $h(x, x_1)/\|x - x_1\| \to 0$ as $x \to x_1$. Let A be the inverse matrix of $DF(x_1)$. Then

$$A \cdot (f(x) - f(x_1)) = (x - x_1) + A \cdot h(x, x_1), \qquad \text{or}$$
$$A \cdot (y - y_1) + A \cdot h_1(y, y_1) = f^{-1}(y) - f^{-1}(y_1),$$

where $h_1(y, y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Now

$$\frac{h_1(y,y_1)}{\|y-y_1\|} = -\frac{h(x,x_1)}{\|x-x_1\|}\,\frac{\|x-x_1\|}{\|y-y_1\|}.$$

Since $\|x-x_1\|/\|y-y_1\| \le 2$, $h_1(y,y_1)/\|y-y_1\| \to 0$ as $y \to y_1$. Hence

$$D(f^{-1}) = A = (D(f))^{-1}.$$

This means that $D(f^{-1})$ is obtained as the composition of the following maps:

$$C(r/2) \xrightarrow{f^{-1}} C(r) \xrightarrow{Df} \mathrm{GL}(n) \xrightarrow{\text{matrix inversion}} \mathrm{GL}(n);$$

where $\mathrm{GL}(n)$ denotes the set of non-singular $n\times n$ matrices, considered as a subspace of n^2-dimensional euclidean space. Since f^{-1} is continuous and Df and matrix inversion are C^∞-functions, $D(f^{-1})$ is continuous, i.e., f^{-1} is C^1. In general, if f^{-1} is C^k, then by this argument $D(f^{-1})$ is also, i.e., f^{-1} is of class C^{k+1}. This completes the proof. □

LEMMA 1.6. *Let U be an open subset of $\mathbf{R}^n$, let $f : U \to \mathbf{R}^p$ $(n \le p)$, $f(0) = 0$, and let $Df(0)$ have rank n. Then there exists a diffeomorphism g of one neighborhood of the origin in $\mathbf{R}^p$ onto another so that $g(0) = 0$ and $gf(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$, in some neighborhood of the origin.*

Proof. Since $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^n)$ has rank n, we may assume that

$$\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)$$

is the submatrix which is non-singular. Define $F : U \times \mathbf{R}^{p-n} \to \mathbf{R}^p$ by the equation

$$F(x^1, \ldots, x^p) = f(x^1, \ldots, x^n) + (0, \ldots, 0, x^{n+1}, \ldots, x^p).$$

F is an extension of f, since $F(x^1, \ldots, x^n, 0, \ldots, 0) = f(x^1, \ldots, x^n)$. The matrix DF is non-singular at the origin, since its determinant is equal to

$$\det\left(\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)\right),$$

which is non-zero. Hence F has a local inverse g. Thus g maps one neighborhood of the origin in $\mathbf{R}^p$ onto another with

$$gF(x^1, \ldots, x^p) = (x^1, \ldots, x^p),$$

and hence

$$gf(x^1, \ldots, x^n) = gF(x^1, \ldots, x^n, 0, \ldots, 0) = (x^1, \ldots, x^n, 0, \ldots, 0). \quad □$$

COROLLARY 1.7. *Let $A = A^k$ be a differentiable submanifold of M^n. Given $x \in A$, there is a coordinate system (U, h) on M about x, such that $h(U \cap A) = h(U) \cap \mathbf{R}^k$ (where $\mathbf{R}^k$ is considered as the subspace $\mathbf{R}^k \times 0$ of $\mathbf{R}^k \times \mathbf{R}^{n-k} = \mathbf{R}^n$).*

Proof. Let (U_1, h_1) be a coordinate system on M about x; by hypothesis, there is a differentiable map f of a neighborhood V of x in M into $\mathbf{R}^k$ such that $f|_{V\cap A} = f_1$ is a diffeomorphism whose range is an open set W in $\mathbf{R}^k$. We may assume $U_1 = V$, and $h_1(x) = f(x) = 0$.

Now $fh_1^{-1}h_1f_1^{-1}$ is the identity on W, so that its Jacobian, which equals $D(fh_1^{-1}) \cdot D(h_1f_1^{-1})$ is non-singular. Hence $D(h_1f_1^{-1})$ has rank k, so that by the previous lemma, there is a diffeomorphism g of some neighborhood $V_1 \subset h_1(U_1)$ of the origin onto another such that $g(0) = 0$ and $gh_1f_1^{-1}(x^1, \ldots, x^k) = (x^1, \ldots, x^k, 0, \ldots, 0)$. Then $U = h_1^{-1}(V_1)$ and $h = gh_1$ will satisfy the requirements of the lemma. □

LEMMA 1.8. *Let U be an open subset of $\mathbf{R}^n$, let $f : U \to \mathbf{R}^p$, $f(0) = 0$, $(n \ge p)$, and let $Df(0)$ have rank p. Then there is a diffeomorphism h of some neighborhood of the origin in $\mathbf{R}^n$ onto another such that $h(0) = 0$ and $fh(x^1, \ldots, x^n) = (x^1, \ldots, x^p)$.*

Proof. We may assume $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^p)$ is non-singular at 0, since $Df(0)$ has rank p. Define $F : U \to \mathbf{R}^n$ by the equation

$$F(x^1, \ldots, x^n) = (f^1(x), \ldots, f^p(x), x^{p+1}, \ldots, x^n).$$

Then $DF(0)$ is non-singular; let h be the local inverse of F. Let g project $\mathbf{R}^n$ onto the subspace $\mathbf{R}^p$; $f = gF$. Then

$$fh(x^1, \ldots, x^n) = gFh(x^1, \ldots, x^n) = g(x^1, \ldots, x^n) = (x^1, \ldots, x^p). \quad □$$

Exercise 1.9. Let U be an open subset of $\mathbf{R}^n$, $f : U \to \mathbf{R}^p$, $f(0) = 0$; and let $Df(x)$ have rank k for all x in U. Then there are local diffeomorphisms h and g of $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively such that

$$gfh(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

throughout some neighborhood of the origin.

Definition 1.10. If $f : M_1 \to M_2$, the *rank* of f at x is the rank of $D(h_2fh_1^{-1})$ at $h_1(x)$, where (U_1, h_1) and (U_2, h_2) are coordinate systems about x and $f(x)$, respectively. The differentiable map $f : M^n \to M^p$ is an *immersion* if rank $f = n$ everywhere $(n \le p)$. It is an *embedding* if it is also a homeomorphism into.

If $f : M^n \to M^p$, then $y \in M^p$ is a *regular value* of f if rank $f = p$ on the entire set $f^{-1}(y)$. Otherwise, y is a *critical value*. (If $y \notin f(M^n)$, then by definition y is a regular value of f.)

Exercise 1.11. If A is a differentiable submanifold of M, the inclusion $A \to M$ is an embedding; and conversely if $f : M_1 \to M$ is an embedding then $f(M_1)$ is a differentiable submanifold.

Exercise 1.12. If y is a regular value of $f : M^n \to M^p$, then $f^{-1}(y)$ is a differentiable submanifold of M^n of dimension $n - p$ (or is empty).

Definition 1.13. A subset A of $\mathbf{R}^n$ has *measure zero* if it may be covered by a countable collection of cubes $C(x, r)$ having arbitrarily small total volume. In such a case, $\mathbf{R}^n - A$ is everywhere dense (i.e., it intersects every non-empty open set).

LEMMA 1.14. *Let U be an open subset of $\mathbf{R}^n$; let $f : U \to \mathbf{R}^n$ be differentiable. If $A \subset U$ has measure 0, so does $f(A)$.*

Proof. Let C be any cube with $\overline{C} \subset U$. Let b denote the maximum of $|\partial f^i/\partial x^j|$ on $\overline{C}$ for all i, j. By Lemma 1.4, $\|f(x) - f(y)\| \le bn\|x - y\|$ for $x, y \in \overline{C}$.

Now $A\cap C$ has measure zero; let us cover $A \cap C$ by cubes $C(x_i, r_i)$, with closures contained in U, such that $\sum_{i=1}^{\infty} r_i^n < \epsilon$. Then $f(C(x_i, r_i)) \subset C(f(x_i), bnr_i)$, so that $f(A \cap C)$ is covered by cubes of total volume $b^n n^n \sum r_j^n < b^n n^n \epsilon$. Hence $f(A \cap C)$ has measure zero.

Since A can be covered by countably many such cubes C, $f(A)$ has measure zero. □

COROLLARY 1.15. *If $f : U \to \mathbf{R}^p$ is differentiable, where U is an open subset of $\mathbf{R}^n$ and $n < p$, then $f(U)$ has measure zero.*

Proof. Project $U \times \mathbf{R}^{p-n}$ onto U and apply f. Since $U \times 0$ has measure zero in $\mathbf{R}^p$, so does $f(U)$. □

Definition 1.16. If $A \subset M$, A has *measure zero* if $h(A \cap U)$ has measure zero for every coordinate system (U, h).

COROLLARY 1.17. *If $f : M^n \to M^p$ is differentiable and $n < p$, then $f(M^n)$ has measure zero.*

Definition 1.18. Let $\mathcal{M}(p,n)$ denote the space of $p \times n$ matrices, with the differentiable structure of the euclidean space $\mathbf{R}^{pn}$. Let $\mathcal{M}(p,n;k)$ denote the subspace consisting of matrices of rank k. Thus $\mathcal{M}(p,n;n)$ is an open subset of $\mathcal{M}(p,n)$ if $p \geq n$; the determinental criterion for rank proves this. More generally, we have:

LEMMA 1.19. *$\mathcal{M}(p,n;k)$ is a differentiable submanifold of $\mathcal{M}(p,n)$ of dimension $k(p+n-k)$, where $k \leq \min(p,n)$.*

Proof. Let $E_0 \in \mathcal{M}(p,n;k)$; we may assume that E_0 is of the form $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$, where A_0 is a non-singular $k \times k$ matrix. There is an $\epsilon > 0$ such that if all the entries of $A - A_0$ are less than ϵ, A must also be nonsingular. Let U consist of all matrices in $\mathcal{M}(p,n)$ of the form $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with all the entries of $A - A_0$ less than ϵ.

Then E is in $\mathcal{M}(p,n;k)$ if and only if $D = CA^{-1}B$; for the matrix

$$\begin{pmatrix} I_k & 0 \\ X & I_{p-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ XA + C & XB + D \end{pmatrix}$$

has the same rank as E. If $X = -CA^{-1}$, this matrix is

$$\begin{pmatrix} A & B \\ 0 & -CA^{-1}B + D \end{pmatrix}.$$

If $D = CA^{-1}B$, this matrix has rank k. The converse also holds, for if any element of $-CA^{-1}B + D$ is different from zero, this matrix has rank $> k$.

Let W be the open set in euclidean space of dimension

$$(pn - (p-k)(n-k)) = k(p+n-k)$$

consisting of matrices $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, with all the entries of $A - A_0$ less than ϵ. The map

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \to \begin{pmatrix} A & B \\ C & -CA^{-1}B \end{pmatrix}$$

is then a diffeomorphism of W onto the neighborhood $U \cap \mathcal{M}(p,n;k)$ of E_0. □

THEOREM 1.20. *Let U be an open set in $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^p$ be differentiable, where $p \geq 2n$. Given $\epsilon > 0$, there is a $p \times n$ matrix $A = (a_j^i)$ with each $|a_j^i| < \epsilon$, such that $g(x) = f(x) + A \cdot x$ is an immersion. (Here x is written as a column matrix.)*

Proof. $Dg(x) = Df(x) + A$; we would like to choose A in such a way that $Dg(x)$ has rank n for all x. I.e., A should be of the form $Q - Df$, where Q has rank n.

We define $F_k : \mathcal{M}(p,n;k) \times U \to \mathcal{M}(p,n)$ by the equation

$$F_k(Q,x) = Q - Df(x).$$

Now F_k is a differentiable map, and the domain of F_k has dimension $k(p+n-k)+n$. As long as $k < n$, this expression is monotonic in k (its partial with respect to k is $p + n - 2k$). Hence the domain of F_k has dimension not greater than

$$(n-1)(p+n-(n-1)) + n = (2n-p) + pn - 1$$

for $k < n$. Since $p \geq 2n$, this dimension is strictly less than $pn = \dim \mathcal{M}(p,n)$.

Hence the range of F_k has measure zero in $\mathcal{M}(p,n)$, so that there is an element A of $\mathcal{M}(p,n)$, arbitrarily close to the zero matrix, which is not in the range of F_k for $k = 0, \ldots, n-1$. Then $A + Df(x) = Dg(x)$ has rank n, for each x. □

THEOREM 1.21. *Let U be an open subset of $\mathbf{R}^n$; and let $f : U \to \mathbf{R}^p$ be differentiable. Given $\epsilon > 0$, there is a $p \times n$ matrix A and a $p \times 1$ matrix B, with entries less than ϵ in absolute value, such that the map*

$$g(x) = f(x) + A \cdot x + B$$

has the origin as a regular value.

Remark. The following much more delicate result has been proved by [Sard, A.]: The set of critical values of any differentiable map has measure zero.

Proof of Theorem 1.21. Note that the theorem is trivial if $p > n$, since then $f(U)$ has measure zero, and we may choose $A = 0$ and B small in such a way that 0 is not in the image of g.

Assume $p \leq n$. We wish $Dg(x_0) = Df(x_0) + A$ to have rank p, where x_0 ranges over all points such that

$$g(x_0) = 0 = f(x_0) + A \cdot x_0 + B.$$

Hence A is of the form $Q - Df(x)$, and B is of the form $-f(x) - A \cdot x$, where Q is to have rank p.

We define $F_k : \mathcal{M}(p,n;k) \times U \to \mathcal{M}(p,n) \times \mathbf{R}^p$ by the equation

$$F_k(Q,x) = (Q - Df(x), -f(x) - (Q - Df(x)) \cdot x).$$

Then F_k is differentiable. If $k < p$, the dimension of its domain is not greater than $(p-1)(p+n-(p-1)) + n = p + pn - 1$. Hence the image of F_k, $k = 0, \ldots, p-1$ has measure zero; so that there is a point (A, B) arbitrarily close to the origin which is not in any such image set. This completes the proof. □

Definition 1.22. A covering of X is *locally finite* if every point has a neighborhood which intersects only finitely many elements of the covering. A *refinement* of a covering of X is a second covering each element of which is contained in an element of the first covering. A Hausdorff space is *paracompact* if every open covering has a locally finite open refinement.

If X is paracompact, and U_α is an open covering, there is a locally-finite indexed open covering V_α with $V_\alpha \subset U_\alpha$ for each α. (Each point has a neighborhood that intersects V_α for only finitely many α.) For let W_β be a locally-finite refinement of U_α such that distinct indices β correspond to distinct sets; choose $\alpha(\beta)$ so that $W_\beta \subset U_{\alpha(\beta)}$ for each β. Set $V_{\alpha_0} = \bigcup_{\alpha(\beta)=\alpha_0} W_\beta$. Given a neighborhood intersecting only finitely many W_β, it intersects only finitely many V_α as well.

THEOREM 1.23. *If X is locally compact and Hausdorff, having a countable basis, X is paracompact.*

Thus every "manifold", in the sense of Definition 1.1, is automatically paracompact.

Proof of Theorem 1.23. Let $U_1, U_2, \ldots$ be a basis for X with $\overline{U}_i$ compact for each i. We first construct a sequence $A_1, A_2, \ldots$ of compact sets whose union is X, such that $A_i \subset \mathrm{Int}A_{i+1}$. Start with $A_1 = \overline{U}_1$. Given A_i compact, let k be the smallest integer such that $k \geq i$, and such that A_i is contained in $U_1 \cup \cdots \cup U_k$; and let A_{i+1} be the closure of this union.

Let $\mathcal{O}$ be an open covering of X. Cover the compact set $A_{i+1} - \mathrm{Int}A_i$ by a finite number of open sets $V_1, \ldots, V_n$ where each V_i is contained in some element of $\mathcal{O}$, and in the open set $\mathrm{Int}A_{i+2} - A_{i-1}$. Let P_i denote the collection $(V_1, \ldots, V_n)$, and let $P = P_0 \cup P_1 \cup \cdots$. P refines $\mathcal{O}$, and since any compact closed neighborhood C is contained in some A_i, C can intersect only finitely many elements of P. □

Exercise 1.24. Prove that every paracompact space is normal (disjoint closed subsets have disjoint neighborhoods). First prove that it is regular (the special case where one of the two closed sets is a point).

THEOREM 1.25. *Let M^n be a differentiable manifold, $\{U_\alpha\}$ an open covering of M. There is a collection (V_j, h_j) of coordinate systems on M such that*

(1) $\{V_j\}$ is a locally-finite refinement of $\{U_\alpha\}$.
(2) $h_j(V_j) = C^n(3)$.
(3) If $W_j = h_j^{-1}(C^n(1))$, then $\{W_j\}$ covers M.

Proof. The proof proceeds along lines similar to the previous one. The only difference is that one chooses the V_j to satisfy (2), and makes sure that the sets $h_j^{-1}(C(1))$ also cover $A_{i+1} \setminus \mathrm{Int}A_i$. □

LEMMA 1.26. *There exists a C^∞ function $\varphi(x^1, \ldots, x^n)$ such that $\varphi = 1$ on $\overline{C}(1)$, $0 < \varphi < 1$ on $C(2) \setminus \overline{C}(1)$, $\varphi = 0$ on $\mathbf{R}^n \setminus C(2)$.*

In fact, this function may be defined by the equation $\varphi(x^1, \ldots, x^n) = \prod_1^n \psi(x_i)$, where

$$\psi(x) = \frac{\lambda(2+x)\cdot\lambda(2-x)}{\lambda(2+x)\cdot\lambda(2-x) + \lambda(x-1) + \lambda(-x-1)}$$

and

$$\lambda(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

Note that the denominator in the expression for ψ is always positive, and that

$$\begin{array}{rcl} \psi(x) &=& 1 \text{ for } |x| \leq 1 \\ 0 < \psi(x) &<& 1 \text{ if } 1 < |x| < 2 \\ \psi(x) &=& 0 \text{ if } |x| \geq 2. \end{array}$$ □

Definition 1.27. Let $f, g : X \to Y$, where Y is a metric space, and let $\delta(x)$ be a positive continuous function defined on X. Then g is a *δ-approximation* to f if $d(f(x), g(x)) < \delta(x)$ for all x. [One can impose a topology on the space $F(X, Y)$ of all differentiable maps by taking the δ-approximations to f as basic neighborhoods of f. This topology is independent of the choice of metric on Y, provided that X, Y are paracompact.]

THEOREM 1.28. *Given a differentiable map $f : M^n \to \mathbf{R}^p$ where $p \geq 2n$, and a continuous positive function δ on M^n, there exists an immersion $g : M^n \to \mathbf{R}^p$ which is a δ-approximation to f. If $\operatorname{rank} f = n$ on the closed set N, we may choose $g|_N = f|_N$.*

Proof. Clearly the rank of f is equal to n on a neighborhood U of N. Cover M^n by U and $M^n - N$. Let (V_i, h_i) be a refinement of this covering, constructed as in Theorem 1.25. As before, $h_i(W_i) = C(1)$ and $h_i(V_i) = C(3)$. Let $h_i(V_i') = C(2)$, so that $V_i \supset V_i' \supset W_i$. Let the V_i be indexed with positive and negative integers so that those V_i with $i \leq 0$ are the ones contained in U. Let $\epsilon_i = \min \delta(x)$ on the compact set $\overline{V}_i'$.

Set $f_0 = f$. Given $f_{k-1} : M^n \to \mathbf{R}^p$, having rank n on $N_{k-1} = \bigcup_{j<k} \overline{W}_j$, consider $f_{k-1}h_k^{-1} : C(3) \to \mathbf{R}^p$. Let A be a $p \times n$ matrix; let $F_A : C(3) \to \mathbf{R}^p$ be defined by the equation

$$F_A(x) = f_{k-1}h_k^{-1}(x) + \varphi(x)\, A \cdot (x),$$

where (x) is written as usual as an $n \times 1$ column matrix; A is yet to be chosen; and $\varphi(x)$ is the function defined in Lemma 1.26.

First, we want $F_A(x)$ to have rank n on the set $K = h_j(N_{k-1} \cap \overline{U}_k)$; we are given that $f_{k-1}h_k^{-1}$ has rank n on K. Now

$$D(F_A(x)) = D(f_{k-1}h_k^{-1}(x)) + A \cdot (x) \cdot D\varphi(x) + \varphi(x)A.$$

(Here $D\varphi$ is a $1 \times n$ matrix.) The map of $K \times \mathcal{M}(p,n)$ into $\mathcal{M}(p,n)$ which carries (x, A) into $D(F_A(x))$ is continuous. It carries $K \times (0)$ into the open subset $\mathcal{M}(p,n;n)$ of $\mathcal{M}(p,n)$. Hence if A is sufficiently small, this map will carry $K \times A$ into $\mathcal{M}(p,n;n)$; our first requirement is that A be this small.

Secondly, we require A to be small enough that $\|A \cdot (x)\| < \epsilon_k/2^k$ for all $x \in C(3)$.

Finally, by Theorem 1.20, A may be chosen arbitrarily small so that

$$f_{k-1}h_k^{-1}(x) + A \cdot (x)$$

has rank n on $C(2)$. Let A be chosen to satisfy this requirement. We then define $f_k : M^n \to \mathbf{R}^p$ by the equation:

$$f_k(y) = \begin{cases} f_{k-1}(y) + \varphi(h_k(y))A \cdot (h_k(y)), & \text{for} \quad y \in V_k \\ f_{k-1}(y), & \text{for} \quad y \in M - \overline{U}_k. \end{cases}$$

These definitions agree on the overlapping domains, so that f_k is differentiable. By the first condition on A, it has rank n on N_{k-1}; by the third condition it has rank n on $\overline{W}_k$. By the second condition, f_k is a $\delta/2^k$ approximation to f_{k-1}.

We define $g(x) = \lim_{k\to\infty} f_k(x)$. Since the covering V_i is locally finite, all the f_k agree on a given compact set for k sufficiently large; it follows that g is differentiable and has rank n everywhere. It is also a δ-approximation to f. □

LEMMA 1.29. *If $p > 2n$, any immersion $f : M^n \to \mathbf{R}^p$ can be δ-approximated by an injective immersion g. If f is injective in a neighborhood U of the closed set N, we may choose $g|_N = f|_N$.*

Proof. Choose a covering $\{U_\alpha\}$ of M such that each $f|_{U_\alpha}$ is an embedding. Let (V_i, h_i) be the locally finite refinement constructed in Theorem 1.25, and let $\varphi(x)$ be the function constructed in Lemma 1.26. Then we can define a differentiable function from M to $\mathbf{R}$ by the formula

$$\varphi_i(y) = \begin{cases} \varphi(h_i(y)), & \text{if} \quad y \in V_i \\ 0, & \text{otherwise.} \end{cases}$$

As before, we assume (V_i, h_i) refines the covering $\{U, M - N\}$ and that those V_i with $i \leq 0$ are the ones contained in U. Let $f_0 = f$. Given the immersion $f_{k-1} : M^n \to \mathbf{R}^p$, we define f_k inductively by the equation

$$f_k(y) = f_{k-1}(y) + \varphi_k(y)b_k\,,$$

where b_k is a point of $\mathbf{R}^p$ yet to be chosen. By the argument of the previous theorem, if b_k is sufficiently small, f_k will have rank n everywhere. The first requirement is that b_k be this small, and the second requirement is that b_k be small enough that f_k be a $\delta/2^k$ approximation to f_{k-1}.

Finally, let N^{2n} be the open subset of $M^n \times M^n$ consisting of pairs (y, y'), with $\varphi_k(y) \neq \varphi_k(y')$. Consider the differentiable map

$$(y, y') \mapsto -\frac{f_{k-1}(y) - f_{k-1}(y')}{\varphi_k(y) - \varphi_k(y')}$$

from N^{2n} into $\mathbf{R}^p$. Since $2n < p$, the image of N^{2n} has measure 0, so that b_k may be chosen arbitrarily small and *not* in this image. For every $k > 0$, it follows easily that $f_k(y) = f_k(y')$ if and only if both $\varphi_k(y) - \varphi_k(y') = 0$ and $f_{k-1}(y) - f_{k-1}(y') = 0$.

Define $g(y) = \lim_{k\to\infty} f_k(y)$. This limit exists and is differentiable since the covering $\{V_i\}$ is locally finite. If $g(y) = g(y')$ with $y \neq y'$, it would follow that $f_{k-1}(y) = f_{k-1}(y')$ and $\varphi_k(y) = \varphi_k(y')$ for all $k > 0$. The former condition implies that $f(y) = f(y')$, so that y and y' cannot belong to any one set V_i. Because of the latter condition, this means that neither is in any set V_i' for $i > 0$. Hence, they lie in U, contradicting the fact that f is injective on U. □

Definition 1.30. Let $f : M^n \to \mathbf{R}^p$. The *limit set* $L(f)$ is the set of $y \in \mathbf{R}^p$ such that $y = \lim f(x_n)$ for some sequence $\{x_1, x_2, \dots\}$ which has no subsequence converging to a point on M^n.

Exercise. Show the following:

(1) $f(M)$ is a closed subset of $\mathbf{R}^p$ if and only if $L(f) \subset f(M)$.

(2) f is a topological embedding if and only if f is injective and $L(f) \cap f(M)$ is vacuous.

LEMMA 1.31. *There exists a differentiable map $f : M^n \to \mathbf{R}$ with $L(f)$ empty.*

Proof. Let (V_i, h_i) and φ be chosen as in Theorem 1.25 and Lemma 1.26, with i ranging over positive integers; and again let

$$\varphi_i(y) = \begin{cases} \varphi(h_i(y)), & \text{if} \quad y \in V_i \\ 0, & \text{otherwise.} \end{cases}$$

Define $f(y) = \sum_j (j\,\varphi_j(y))$. This sum is finite, since V_i is a locally finite covering. If $\{x_i\}$ is a set of points of M having no limit point, only finitely many lie in any compact subset of M. Given m, there is an integer i such that x_i is not in $\overline{W}_1 \cup \cdots \cup \overline{W}_m$. Hence $x_i \in \overline{W}_j$ for some $j > m$, whence $f(x_i) > m$. Thus the sequence $f(x_m)$ cannot converge. □

COROLLARY 1.32. *Every M^n can be differentiably embedded in $\mathbf{R}^{2n+1}$ as a closed subset.*

Proof. Let $f : M^n \to \mathbf{R} \subset \mathbf{R}^{2n+1}$ differentiably, with $L(f) = 0$. Set $\delta(x) \equiv 1$, and let g be an injective immersion which is a δ-approximation to f. Then $L(g)$ is empty, so that g is a homeomorphism. □

Definition 1.33. Consider a differentiable map $f : M^n \to N^p$, together with a codimension q differentiable submanifold $N_1 = N_1^{p-q} \subset N^p$. Given a point $x \in f^{-1}(N_1) \subset M$, let $(u, \dots, u^n)$ be a coordinate system about x; and let $(v^1, \dots, v^p)$ be a coordinate system about $f(x)$ chosen so that the intersection of N_1 with the associated coordinate neighborhood is defined by the equations $v^1 = \cdots = v^q = 0$. (Compare Lemma 1.6.) By definition, the *transverse regularity condition* for f and N_1 is satisfied at x if the $q \times n$ matrix

$$(\partial v^i/\partial u^j) \qquad \begin{matrix} i = 1, \dots, q \\ j = 1, \dots, n \end{matrix}$$

has rank q at x.

Remark 1.34. This condition is independent of the particular choice of local coordinates. In terms of the first derivative map from the tangent space $T_x(M) \to T_y(N)$ where $y = f(x)$, and the quotient map from $T_y(N)$ to $T_y(N)/T_y(N_1)$, it is just the condition that the composition maps $T_x(M)$ onto $T_y(N)/T_y(N_1)$. Compare the discussion in Definition 2.6 below.

Note that the set of points on which this transverse regularity condition is satisfied is open as a subset of $f^{-1}(N_1)$. The map f is said to be *transverse regular on N_1* if the condition is satisfied for every x in $f^{-1}(N_1)$.

LEMMA 1.35. *If $f : M^n \to N^p$ is transverse regular on N_1^{p-q} then $f^{-1}(N_1)$ is a differentiable submanifold of dimension $n-q$ (or is empty).*

Proof. Let π project $\mathbf{R}^p$ onto its first q components; $\pi : \mathbf{R}^p \to \mathbf{R}^q$. If $(V, h) = (v^1, \ldots, v^p)$ is the coordinate system hypothesized in Definition 1.33, then

$$N_1 \cap V = h^{-1}\pi^{-1}(0),$$

where 0 denotes the origin in $\mathbf{R}^q$, and $f^{-1}(N_1 \cap V) = (\pi h f)^{-1}(0)$. Since $\pi h f$ has rank q at $x \in f^{-1}(N_1 \cap V)$, the origin is a regular value of $\pi h f$. Hence $(\pi h f)^{-1}(0)$ is a differentiable submanifold of M of dimension $n-q$ (see Exercise 1.12). □

THEOREM 1.36. *Let $f : M^n \to N^p$ be differentiable; let $N_1 = N_1^{p-q}$ be a closed differentiable submanifold of N. Let A be a closed subset of M such that the transverse regularity condition for f and N_1 holds at each x in $A \cap f^{-1}(N_1)$. Let δ be a positive continuous function on M. There exists a differentiable map $g : M^n \to N^p$ such that*

(1) *g is a δ-approximation to f,*
(2) *g is transverse regular on N_1, and*
(3) *$g|_A = f|_A$.*

Proof. There is a neighborhood U of A in M such that f satisfies the transverse regularity condition on $U \cap f^{-1}(N_1)$. Cover N by $Y_0 = N \setminus N_1$, together with coordinate systems (Y_i, η_i) for $i > 0$ each of which has coordinate functions $(v^1, \ldots, v^n)$ such that $v^1 = \cdots = v^q = 0$ on N_1. Now the open sets $f^{-1}(Y_i)$ cover M, as do the open sets U and $M \setminus A$. Let $\{(V_j, h_j)\}$ be a refinement of both coverings, constructed as in Theorem 1.25. Recall that $h_j(V_j) = C(3)$, $h_j(V_j') = C(2)$, $h_j(W_j) = C(1)$, and the W_j cover M. The V_j are to be indexed with positive and negative integers so that those V_j which are contained in U are the ones with $j \le 0$.

Let φ be as in Lemma 1.26, and define $\varphi_i(x) = \varphi(h_i(x))$ for $x \in V_i$ and $\varphi_i(x) = 0$ elsewhere. For each j choose $i(j) \ge 0$ so that $f(V_j)$ is contained in $Y_{i(j)}$.

Set $f_0 = f$. Suppose $f_{k-1} : M \to N$ is defined and satisfies the transverse regularity condition for N_1 at each point of the intersection of $f_{k-1}^{-1}(N_1)$ with $\bigcup_{j<k} \overline{W}_j$. Furthermore suppose that $f_{k-1}(\overline{V}_j') \subset Y_{i(j)}$ for each j. Setting $i = i(k)$, it follows in particular that $f_{k-1}(\overline{V}_k') \subset Y_i$.

Consider the composition

$$\pi\eta_i f_{k-1} h_k^{-1} : C(2) \to \mathbf{R}^q,$$

where again π projects $\mathbf{R}^p$ onto $\mathbf{R}^q$. By Theorem 1.21, there is an arbitrarily small affine function $L(x) = A \cdot (x) + B$ from $\mathbf{R}^p$ to $\mathbf{R}^q$ such that when added to the previous function, the resulting map has the origin as a regular value. Consider $\mathbf{R}^q$ as the first q coordinates in $\mathbf{R}^p$, and define

$$f_k(x) = \begin{cases} \eta_i^{-1}\left(\eta_i f_{k-1}(x) + L(h_k(x))\,\varphi_k(x)\right), & \text{for } x \text{ in a neighborhood of } \overline{V}_k' \\ f_{k-1}(x), & \text{for } x \text{ in } M \setminus V_k'. \end{cases}$$

Here L is yet to be chosen. Of course, we must choose L small enough that

$$\eta_i f_{k-1} + L\varphi_k$$

lies in $C(1)$ for $x \in \overline{V}_k'$, in order that η_i^{-1} may be applied to it. This is the first requirement on L. Secondly, we choose L small enough that f_k is a $\delta/2^k$ approximation to f_{k-1}. Thirdly choose L small enough so that $f_k(\overline{V}_j')$ is contained in $Y_{i(j)}$ for each j. This is possible since only a finite number of the sets $\overline{V}_j'$ can intersect $\overline{V}_k'$.

Now f_k by definition satisfies the transverse regularity condition for N_1 at each point of $f_k^{-1}(N_1) \cap \overline{W}_k$. We want to choose L small enough that the condition is satisfied at each point of the intersection of $f_k^{-1}(N_1)$ with $\bigcup_{j<k} \overline{W}_j$. It is sufficient to consider the intersection of the latter set with $\overline{V}_k'$; let this intersection be denoted by K. Consider the function which maps each pair (x, L) with $x \in K$ to the pair

$$\left(f_k(x)\,,\ D(\pi\eta_i f_k h_k^{-1})\cdot(h_k(x))\right) \in N \times \mathcal{M}(q, n).$$

(Here the dot means that the derivative matrix is to be evaluated at the point $h_k(x)$.) This function is continuous and carries $K \times (0)$ into the set

$$\left((N - N_1) \times \mathcal{M}(q, n)\right) \cup \left(N \times \mathcal{M}(q, n; q)\right),$$

which is open in $N \times \mathcal{M}(q, n)$. Hence for L sufficiently small, (x, L) is carried into this set, so that f_k satisfies the transverse regularity condition for N_1 at each point of $f_k^{-1}(N_1) \cap \left(\bigcup_{j \le k} \overline{W}_j\right)$.

We define $g(x) = \lim_{k\to\infty} f_k(x)$, as usual. □

2. Vector Space Bundles

Definition 2.1. An n-dimensional real *vector space bundle* ξ is a triple (π, a, s). Here π is a continuous map of E onto B, where E and B are Hausdorff spaces; and each set $F_b = \pi^{-1}(b)$ is called a *fibre*. The map $s : \mathbf{R} \times E \to E$ must carry each $\mathbf{R} \times F_b$ onto F_b; while the map a is defined on $\bigcup_b (F_b \times F_b) \subset E \times E$ and carries each $F_b \times F_b$ onto F_b.

The following must be satisfied:

(1) Each F_b is an n-dimensional real vector space with s and a as scalar product and vector addition, respectively.

(2) (Local triviality.) For each b in B, there is a neighborhood U of b and a homeomorphism $\varphi : U \times \mathbf{R}^n \to \pi^{-1}(U)$ such that φ is a vector space isomorphism of $b' \times \mathbf{R}^n$ onto $F_{b'}$ for each b' in U.

If in (2) the neighborhood U may be taken as all of B, the bundle is said to be the *trivial bundle*.

If ξ, η are n-dimensional and p-dimensional vector space bundles, respectively, we define the *product bundle* $\xi \times \eta$ as follows:

$$\begin{aligned} E(\xi \times \eta) &= E(\xi) \times E(\eta) \\ B(\xi \times \eta) &= B(\xi) \times B(\eta) \\ (\pi \times \lambda)(x, y) &= (\pi(x), \lambda(y)) \end{aligned}$$

where π, λ are the projections in ξ, η respectively and $F_b(\xi \times \eta)$ has the usual product structure for vector spaces.

If U is a subset of $B(\xi)$, then $\xi|_U$ denotes the bundle $\pi : \pi^{-1}(U) \to U$. It is called the *restriction* of the bundle ξ to U.

Definition 2.2. Let M^n be a differentiable manifold and let x_0 be in M. A *tangent vector at* x_0 is an operation X which assigns to each differentiable function f defined in a neighborhood of x_0, a real number. The following conditions must be satisfied:

(1) If g is the restriction of f, $X(g) = X(f)$.
(2) $X(cf + dg) = cX(f) + dX(g)$ (where c, d are real numbers).
(3) $X(f \cdot g) = X(f) \cdot g(x_0) + f(x_0) \cdot X(g)$, where the dot means ordinary real multiplication.

For the constant function 1, we have $X(1) = X(1 \cdot 1) = X(1) + X(1)$, by (3); hence $X(1) = 0$. It follows by (2) that $X(c) = 0$ for any constant function c.

If one thinks of a tangent vector as being the velocity vector of a parameterized curve lying in the manifold, then $X(f)$ is merely the derivative of f with respect to the parameter of the curve. This is made more precise below.

LEMMA 2.3. *Let* $(u^1, \dots, u^n)$ *be a coordinate system about* x. *Let* X *be a tangent vector at* x. *Then* X *may be written uniquely as a linear combination of the operators* $\partial/\partial u^i$ *evaluated at* x,

$$X = \sum \alpha^i \frac{\partial}{\partial u^i}.$$

Proof. We assume $u(x)$ is the origin. Given any differentiable $f(u^1, \dots, u^n)$ define

$$g(u^1, \dots, u^n) = \begin{cases} (f(u^1, \dots, u^n) - f(0, u^2, \dots, u^n))/u^1, & \text{if } u^1 \neq 0 \\ \partial f(u^1, \dots, u^n)/\partial u^1, & \text{if } u^1 = 0. \end{cases}$$

To see that g is differentiable, note that

$$g(s, u^2, \dots, u^n) = \int_0^1 \frac{\partial f}{\partial u^1}(st, u^2, \dots, u^n)\, dt.$$

(Then $f(u^1, \dots, u^n) = u^1 g_1(u^1, \dots, u^n) + f(0, u^2, \dots, u^n)$.) Similarly,

$$f(0, u^2, \dots, u^n) = u^2 g_2(u^2, \dots, u^n) + f(0, 0, u^3, \dots, u^n),$$

where $g_2(0) = \partial f/\partial u^2(0)$. Finally, we have

$$f(u^1, \dots, u^n) = \sum u^i g_i + f(0), \quad \text{where} \quad g_i(0) = \frac{\partial f}{\partial u^i}(0).$$

Thus

$$\begin{aligned} X(f) &= \sum X(u^i) g_i(0) + 0 \cdot X(g_i) \\ &= \sum \alpha^i \frac{\partial f}{\partial u^i}(0), \quad \text{where} \quad \alpha^i = X(u^i). \qquad \square \end{aligned}$$

Remark. The α^i are called the *components* of the vector X with respect to the coordinate system $(u^1, \dots, u^n)$. If $(v^1, \dots, v^n)$ is another coordinate system about x, and $X = \sum \beta^j \partial/\partial v^j$, then $\alpha^i = X(u^i) = \sum \beta^j \partial u^i/\partial v^j$.

Definition 2.4. (Alternate). A tangent vector at x is an assignment to every coordinate system $(u^1, \dots, u^n)$ about x of an element $(\alpha^1, \dots, \alpha^n)$ of $\mathbf{R}^n$, with the requirement that if (β^j) is assigned to the system $(v^1, \dots, v^n)$, then $\alpha^i = \sum \beta^j \partial u^i/\partial v^j$. The derivation operator X is then defined as $\sum \alpha^i \partial/\partial u^i$. One checks readily that

(a) $X(f)$ is independent of the coordinate system used, and
(b) $X(f)$ satisfies requirements (1), (2), and (3) for a tangent vector.

Definition 2.5. The *tangent bundle* τ of an n-dimensional differentiable manifold M is constructed as follows. For each x in M, the tangent vectors at x form an n-dimensional vector space. (The operations $\partial/\partial u^i$ form a basis, by Lemma 2.3.) Let the disjoint union of these be denoted $E(\tau)$; and define $\pi : E(\tau) \to M$ as mapping the tangent vector X at x_0 into x_0. The local product structure is given by $\varphi_u : U \times \mathbf{R}^n \to E$, where $(U, h) = (u^1, \dots, u^n)$ is a coordinate system on M, and where $\varphi_u(x_0, a^1, \dots a^n)$ is defined to be the tangent vector

$$X = \sum a^i \frac{\partial}{\partial u^i}$$

at the point x_0. One checks immediately that φ_u gives us a vector space isomorphism for each fibre. Since φ_u is to be a homeomorphism, this structure imposes a topology on E; and since each $\varphi_v^{-1}\varphi_u$ is a homeomorphism on $(U \cap V) \times \mathbf{R}^n$ this topology is unambiguously determined.

Indeed, $\varphi_v^{-1}\varphi_u$ is a C^∞ map on $(U \cap V) \times \mathbf{R}^n$, so that E is a differentiable manifold of dimension $2n$ (using Definition 1.2 of a differentiable manifold). The map π is differentiable of rank n.

Definition 2.6. If $f : M_1 \to M_2$, there is an induced map $df : E(\tau_1) \to E(\tau_2)$ defined as follows: $df(X) = Y$, where $Y(g) = X(gf)$. If X is a vector at x_0, Y is a vector at $f(x_0)$. Clearly df is linear on each fibre; it is called the *derivative map*.

If (U, h) and (V, k) are coordinate systems about x_0 and $f(x_0)$ respectively, and if (α^i), (β^j) are the respective components of X and $Y = df(X)$ with respect to these coordinate systems, then $(\beta^j) = D(kfh^{-1}) \cdot (\alpha^i)$ where the vector components are written as column matrices, as usual.

Definition 2.7. Let ξ, η be two n-dimensional vector space bundles. A *bundle map* $f : \xi \to \eta$ is a continuous map of $E(\xi)$ into $E(\eta)$ which carries each fibre isomorphically onto a fibre. The induced map $f_B : B(\xi) \to B(\eta)$ is automatically continuous.

If $B(\xi) = B(\eta)$ and the induced map is the identity, f is said to be an *equivalence*. Note that if f is an equivalence, it is a homeomorphism: Locally f is just a map $U \times \mathbf{R}^n \to V \times \mathbf{R}^n$, and can be described by an expression of the form

$$(x, (\alpha^1, \dots \alpha^n)) \mapsto (x, (\beta^1, \dots \beta^n)) \qquad \text{where} \qquad \beta^i = \sum_j A^i_j(x) \alpha^j$$

and where $x \mapsto (A^i_j(x))$ is a continuous function from $U \cap V$ to the group of non-singular $n \times n$ matrices. Since the operation of matrix inversion is continuous, it follows easily that f^{-1} is also continuous.

If there is an equivalence of ξ onto η, we write $\xi \simeq \eta$.

LEMMA 2.8. *Given a bundle η with projection map $\lambda : E(\eta) \to B(\eta)$, and a map $f : B_1 \to B(\eta)$, there is a bundle $\pi : E_1 \to B_1$ and a bundle map $g : E_1 \to E(\eta)$ such that $\lambda g = f\pi$. Furthermore, E_1 is unique up to an equivalence.*

This E_1 is called the *induced bundle* and is often denoted by $f^*\eta$.

Proof of Lemma 2.8. Let E_1 be the subset of $B_1 \times E(\eta)$ consisting of points (b, e) such that $f(b) = \lambda(e)$. Define $\pi(b, e) = b$ and $g(b, e) = e$. The map g is an isomorphism on each fibre. To show that E_1 is a vector space bundle, let $\varphi : V \times \mathbf{R}^n \to E(\eta)$ be a product neighborhood in $E(\eta)$, and let $f(U) \subset V$. Then define $\varphi_1 : U \times \mathbf{R}^n \to E_1$ by $\varphi_1(b, x) = (b, \varphi(f(b), x))$. This is continuous and injective; and its image equals $\pi^{-1}(U)$. Its inverse carries (b, e) into $(b, p\varphi^{-1}(e))$ (where p projects $V \times \mathbf{R}^n$ onto $\mathbf{R}^n$), so that it is continuous.

Now suppose $g' : E' \to E(\eta)$ is another bundle map, where $\pi' : E' \to B_1$ is a bundle and $\lambda g' = f\pi'$. We map $E' \to E_1$ by the formula

$$e' \mapsto (\pi'(e'), g'(e')) \in E_1. \tag{2}$$

Because g' is an isomorphism on each fibre, this map (2) is also. It is an equivalence, since it induces the identity on the base space. □

Definition 2.9. Let ξ, η be two bundles over B. The *Whitney sum* $\xi \oplus \eta$ is a bundle defined as follows: Consider the product bundle $E(\xi) \times E(\eta) \to B \times B$; let d be the diagonal map $B \to B \times B$. The induced bundle $d^*(\xi \times \eta)$ is defined as the Whitney sum $\xi \oplus \eta$.

Note that the fibre over b in $\xi \oplus \eta$ is merely $F_b(\xi) \times F_b(\eta)$, so that

$$\dim(\xi \oplus \eta) = \dim \xi + \dim \eta.$$

Note also the commutativity and associativity of $\oplus$. I.e., $\xi \oplus \eta \simeq \eta \oplus \xi$ and $(\xi \oplus \eta) \oplus \zeta \simeq \xi \oplus (\eta \oplus \zeta)$. The proof is left as an exercise.

Definition 2.10. If ξ, η are bundles over B, then $g : E(\xi) \to E(\eta)$ is a *homomorphism* if

(1) it maps each fibre linearly into a fibre, and
(2) the induced map on B is the identity.

Note that an equivalence is both a bundle map and a homomorphism. An *embedding* of bundles is a injective homomorphism.

THEOREM 2.11. *If $f : E(\xi) \to E(\eta)$ maps each fibre linearly into a fibre, then f may be factored into a homomorphism followed by a bundle map.*

Proof. Let π_1, π_2 be the projections in ξ, η, respectively.

Let $f_B : B(\xi) \to B(\eta)$ be the map induced by f. Let $E_1 = f_B^*\eta$ be the bundle induced by f_B; let g be the bundle map $E_1 \to E(\eta)$ and π the projection $E_1 \to B(\xi)$.

Define $h : E(\xi) \to B(\xi) \times E(\eta)$ by the equation $h(e) = (\pi_1(e), f(e))$. The image of h actually lies in that subset of $B(\xi) \times E(\eta)$ which is E_1; then h is a

homomorphism. From the definition, $f = gh$.

$$\begin{array}{ccccc} E(\xi) & \xrightarrow{h} & E_1 & \xrightarrow{g} & E(\eta) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} \\ B(\xi) & \xrightarrow[i]{} & B(\xi) & \xrightarrow[f_B]{} & E(\eta) \end{array}$$

□

LEMMA 2.12. *Let ξ, η be bundles over B of dimensions n, p, respectively; let $g : \xi \to \eta$ be a homomorphism. If g is onto, then the kernel of g is a well defined vector bundle. If g is injective, then the cokernel of g, i.e., the quotient η/image g is a well defined vector bundle.*

Proof. Suppose g is injective (i.e., has rank n when restricted to each fibre). In $E(\eta)$, we define $e \sim e'$ if $e - e'$ exists and is in the image of g. We identify the elements of these equivalence classes.; the resulting identification space is defined to be $E(\eta/g(\xi))$. It is a bundle over B with projection naturally defined and each fibre is a vector space of dim $p - n$. We need only to show the existence of a local product structure.

Let U be an open set in B, with $\xi|_U$ equivalent to $U \times \mathbf{R}^n$ and $\eta|_U$ equivalent to $U \times \mathbf{R}^p$. Let g_0 denote the homomorphism of $U \times \mathbf{R}^n \to U \times \mathbf{R}^p$ induced by g. Now $(\eta/g(\xi))|_U$ is equivalent to the quotient $U \times \mathbf{R}^p/g_0(U \times \mathbf{R}^n)$, so that it suffices to show that this latter quotient is locally a product.

g_0 is given by a matrix $M(b) \in \mathcal{M}(p, n)$ which depends continuously on the point $b \in U$. Given b_0, we may assume that in a neighborhood U_0 of b_0, the first n rows are independent. We define $h : U_0 \times \mathbf{R}^n \times \mathbf{R}^{p-n} \to U_0 \times \mathbf{R}^p$ as the linear function on $\mathbf{R}^p$ whose matrix (non-singular) is

$$\left(M(b) \middle| \frac{0}{I_{p-n}} \right)$$

The image of $U_0 \times \mathbf{R}^n \times 0$ under h is just $g_0(U_0 \times \mathbf{R}^n)$; since h is an equivalence, it induces an equivalence of

$$U_0 \times \mathbf{R}^{p-n} \simeq \frac{U_0 \times \mathbf{R}^n \times \mathbf{R}^{p-n}}{U_0 \times \mathbf{R}^n \times 0} \quad \text{onto} \quad \frac{U_0 \times \mathbf{R}^p}{g_0(U_0 \times \mathbf{R}^n)}.$$

Secondly, suppose g is onto (i.e., it has rank p on each fibre). $E(g^{-1}(0))$ is defined as that subset of $E(\xi)$ consisting of points e with $g(e) = 0$. Again, we need to show the existence of a local product structure. Let U, g_0, and $M(b)$ be as above. Given b_0, we may assume that the first p columns of $M(b)$ are independent in the neighborhood U_0 of b_0. We define $h : U_0 \times \mathbf{R}^n \to U_0 \times \mathbf{R}^p \times \mathbf{R}^{n-p}$ by the matrix function

$$\left(\frac{M(b)}{0 \mid I_{n-p}} \right)$$

Now h followed by the natural projection of $U_0 \times \mathbf{R}^p \times \mathbf{R}^{n-p}$ onto $U_0 \times \mathbf{R}^p$ equals $g_0|_{U_0}$. Hence h^{-1} maps $U_0 \times 0 \times \mathbf{R}^{n-p}$ onto $g_0^{-1}(U_0 \times 0)$; since h is an equivalence, the restriction of h^{-1} to $U_0 \times 0 \times \mathbf{R}^{n-p}$ is also. □

Remark. If g is onto, $\xi/g^{-1}(0)$ is a bundle, being the quotient of the inclusion homomorphism $g^{-1}(0) \to \xi$. If g is injective, $g(\xi)$ is a bundle, being the kernel of the projection homomorphism $\eta \to \eta/g(\xi)$.

Definition 2.13. If φ is a nonnegative function on B, the ***support*** of φ is the closure of the set of x with $\varphi(x) > 0$. A ***partition of unity*** is a collection φ_α of continuous non-negative functions on B, such that the sets $C_\alpha = \text{support } \varphi_\alpha$ form a locally-finite covering of B, and $\sum \varphi_\alpha(x) = 1$ (this is a finite sum for each x).

LEMMA 2.14. *Let B be a normal space; U_α a locally-finite open covering of B. Then there is a partition of unity φ_α with support $\varphi_\alpha \subset U_\alpha$ for each α.*

Proof. First, we show that there is an open covering V_α of B with $\overline{V}_\alpha \subset U_\alpha$ for each α. Assume the U_α indexed by a set of ordinals (well-ordering theorem). Let V_α be defined for all $\alpha < \beta$ and assume that the sets V_α along with the sets U_α for $\alpha \geq \beta$ cover B. Consider the set $A(B) = B - \bigcup_{\alpha<\beta} V_\alpha - \bigcup_{\alpha>\beta} U_\alpha$. Then $A(\beta) \subset U_\beta$. Let V_β be an open set containing the closed set $A(\beta)$ with $\overline{V}_\beta \subset U_\beta$ (normality). This completes the construction of the V_α.

Now let g_α be a function which is positive on $\overline{V}_\alpha$ and 0 outside U_α (normality again). Define $\varphi_{\alpha_0}(x) = g_{\alpha_0}(x)/\sum g_{\alpha(x)}$. Since U_α is locally-finite, the sum in the denominator is finite and positive, so φ_α is well-defined. □

Remark. If B is a differentiable manifold, φ_α may be chosen to be differentiable: For fixed α, we can cover B with coordinate systems (V_i, h_i) as in Theorem 1.25 refining the covering $\{U_\alpha, B - \overline{V}_\alpha\}$. As in Lemma 1.31, let $\varphi_i(y) = \varphi(h_i(y))$ for $y \in V_i$, and $= 0$ otherwise (with φ as in Lemma 1.26). Let $g_\alpha(y) = \sum \varphi_i(y)$, where the sum extends over all i such that $V_i \subset U_\alpha$, and then proceed as above.

LEMMA 2.15. *Let B be paracompact and let $0 \to \xi \xrightarrow{i} \eta \xrightarrow{\varphi} \zeta \to 0$ be an exact sequence of homomorphisms of bundles. Then there is an equivalence $f : \eta \to \xi \oplus \zeta$, with fi the natural inclusion and φf^{-1} the natural projection.*

Proof. Let $\dim \xi = n$; $\dim \zeta = p$. We first construct a Riemannian metric on η (i.e., a continuous inner product in $E(\eta)$). Let U_α be a locally finite covering of B with $\eta|_{U_\alpha}$ trivial; let g_α be the corresponding projection of $\eta|_{U_\alpha}$ onto $\mathbf{R}^{n+p}$. Let φ_α be a partition of unity with support $\varphi_\alpha \subset U_\alpha$.

If e, e' are in $E(\eta)$ and $\pi(e) = \pi(e')$, define $e \cdot e' = \sum_\alpha \varphi_\alpha(\pi(e))\, g_\alpha(e) \cdot g_\alpha(e')$, where the dot on the right hand side is the ordinary scalar product in $\mathbf{R}^{n+p}$. This is a finite sum; it satisfies the axioms for a scalar product.

The way we use the Riemannian metric is to break η up into $iE(\xi)$ and its orthogonal complement. Let ξ' be the image of ξ in η and let $E(\zeta')$ be defined as that subset of $E(\eta)$ consisting of elements which are orthogonal to $i(E(\xi))$. In order to show that ζ' has a local product structure, consider the homomorphism

$$h : \eta \to \xi'$$

which sends each vector into its orthogonal projection in ξ'. [Verification that h is continuous. Over any coordinate neighborhood U we can choose a basis $a_1, \ldots, a_n$ for the fibre of ξ'. Then the function h carries $v \in E(\eta)$ into $\sum t_j a_j \in E(\xi') \subset E(\eta)$,

where $t_j = \sum B_{jk}(v \cdot a_k)$ and where (B_{jk}) denotes the inverse matrix to $(a_j \cdot a_k)$.] Since h is onto, its kernel ζ' is again a vector space bundle.

Now the bundle $i(\xi) = \xi'$ is equivalent to ξ. It remains to show that ζ' is equivalent to ζ and that η is equivalent to $\xi' \oplus \zeta'$. The former follows immediately from the fact that $\varphi|_{\zeta'}$ is a homomorphism; from rank considerations it must be injective and onto as well. The latter follows by noting that $E(\xi' \oplus \zeta')$ is defined as the subset of $E(\xi') \times E(\zeta')$ consisting of points (e_1, e_2) such that $\pi(e_1) = \pi(e_2)$. Consider the map f of $E(\xi' \oplus \zeta')$ into $E(\eta)$ obtained by taking (e_1, e_2) into their sum in $E(\eta)$ (this sum exists because e_1 and e_2 lie in the same fibre). This is clearly a homomorphism; from rank considerations, it must be injective and onto. □

Definition 2.16. Let M_1, M_2 be differentiable manifolds, and let $f : M_1 \to M_2$ be an immersion. The ***normal bundle*** ν_f is defined as follows. Let τ_1, τ_2 be the tangent bundles of M_1, M_2 respectively. By Theorem 2.11, the map $df : E(\tau_1) \to E(\tau_2)$ may be factored into a homomorphism h of $E(\tau_1)$ into $E(f^*\tau_2)$ followed by a bundle map g. Now h is an injective homomorphism because f is an immersion; hence by Lemma 2.12, $f^*\tau_2/\text{image } h$ is a bundle over M_1. It is called the normal bundle ν_f.

Then $0 \to \tau_1 \to f^*\tau_2 \to \nu_f \to 0$ is an exact sequence of homomorphisms, so that by Lemma 2.15, $f^*\tau_2$ is equivalent to $\tau_1 \oplus \nu_f$. Indeed, given a Riemannian metric on $f^*\tau_2$, ν_f is equivalent to the orthogonal complement of the image of τ_1.

Let us consider the case $M_2 = \mathbf{R}^{n+p}$, where $\dim M_1 = n$. Then τ_2 is the trivial bundle, so that $f^*\tau_2$ is as well. (Proof: If $f : B \to B(\eta)$ and η is trivial, so is $f^*\eta$. We have the diagram

$$\begin{array}{ccc} & & B \times \mathbf{R}^n \\ & & \downarrow{\scriptstyle \pi} \\ f : B_1 & \longrightarrow & B. \end{array}$$

$E(f^*\eta)$ is defined as that subset of $B_1 \times (B \times \mathbf{R}^n)$ consisting of points (b_1, b, x) such that $f(b_1) = \pi(b, x)$; i.e., of all points $(b_1, f(b_1), x)$. If we map this into (b_1, x), we obtain an equivalence of $f^*\eta$ with the bundle $B_1 \times \mathbf{R}^n \to B_1$.)

Thus $\tau_1 \oplus \nu_f$ is equivalent to a trivial bundle. In what follows, we investigate the following question: Given ξ, does there exist an η with $\xi \oplus \eta$ trivial? Using Theorem 1.28, this is always the case for ξ the tangent bundle of an n-manifold, and indeed η may be chosen also to have dimension n. A more general answer appears in Lemma 2.19.

Definition 2.17. Let $f : M_1 \to M_2$; let $\dim M_1 = n$, $\dim M_2 = p$. If f has rank p at every point of M_1, it is said to be ***regular***. If f is regular, the homomorphism $h : \tau_1 \to f^*\tau_2$ given by Theorem 2.11 is an onto map. By Lemma 2.12, the kernel of h is a bundle α_f. It is called the ***bundle along the fibre***.

Note that $f^{-1}(y)$ is a submanifold of M_1 of dimension $n - p$ (by Exercise 1.12 or Lemma 1.35). The inclusion i_y of $f^{-1}(y)$ into M_1 induces an inclusion di_y of its tangent bundle into τ_1. The kernel of h consists precisely of the vectors which are in the image of some di_y, i.e., the vectors tangent to the submanifolds $f^{-1}(y)$ are the ones carried into 0 by h.

One has the exact sequence $0 \to \alpha_f \to \tau_1 \xrightarrow{h} f^*\tau_2 \to 0$, so that by Lemma 2.15, τ_1 is equivalent to $\alpha_f \oplus f^*\tau_2$.

Definition 2.18. A bundle ξ is of *finite type* if B is normal and may be covered by a finite number of neighborhoods $U_1, \ldots, U_k$ such that $\xi|_{U_i}$ is trivial for each i.

LEMMA 2.19. *ξ is of finite type if B is compact, or paracompact finite dimensional.*

The former statement is clear; let us consider the latter. By definition, the dimension of B satisfies $\dim(B) \leq n$ if every open covering has an open refinement such that

(*) no point of B is contained in more than $n+1$ elements of the refinement.

It is a standard theorem of topology that an n-manifold has dimension n in this sense.

Cover B by open sets U, with $\xi|_U$ trivial; let $\{V_\alpha\}$ be an open refinement of this covering satisfying (*). Since B is paracompact (Definition 1.22), we may assume that $\{V_\alpha\}$ is locally-finite as well. Let φ_α be a partition of unity with support $\varphi_\alpha \subset V_\alpha$ for each α (Lemma 2.14).

Let A_i be the set of unordered $(i+1)$-tuples of distinct elements of the index set $\{\varphi_\alpha\}$. Given a in A_i, where $a = \{\alpha_0, \ldots, \alpha_i\}$, let W_{ia} be the set of all x such that $\varphi_\alpha(x) < \min[\varphi_{\alpha_0}(x), \ldots, \varphi_{\alpha_i}(x)]$ for all $\alpha \neq \alpha_0, \ldots, \alpha_i$. Each set W_{ia} is open, and W_{ia} and W_{ib} are disjoint if $a \neq b$. Also W_{ia} is contained in the intersection of the supports of $\varphi_{\alpha_0}, \ldots, \varphi_{\alpha_i}$, and hence in some set V_α. If we set X_i equal to the union of all sets W_{ia}, for fixed i, the result is that $\xi|_{X_i}$ is trivial. (For $\xi|_{W_{ia}}$ is trivial, and the W_{ia} are disjoint.)

Finally, the sets $X_0, \ldots, X_n$ cover B. Given x in B, x is contained in at most $n+1$ of the sets V_α, so that at most $n+1$ of the functions φ_α are positive at x. Since some φ_α is positive at x, x is contained in one of the sets W_{ia} for $0 \leq i \leq n$.

[The intuitive idea of the proof is as follows: Consider an n-dimensional simplicial complex, with φ_α the barycentric coordinate of x with respect to the vertex α. The sets W_{0a} will be disjoint neighborhoods of the vertices, the sets W_{1a} disjoint neighborhoods of the open 1-simplices, and so on.]

THEOREM 2.20. *If ξ is of finite type, there is a bundle η such that $\xi \oplus \eta$ is trivial.*

Proof. We proceed by showing that ξ may be embedded in a trivial bundle $B \times \mathbf{R}^m$, so that we have the exact sequence $0 \to \xi \xrightarrow{i} B \times \mathbf{R}^m \to B \times \mathbf{R}^m/i(\xi) \to 0$ by Lemma 2.12. The theorem then follows from Lemma 2.15. (Paracompactness is not needed since the trivial bundle clearly has a Riemannian metric.)

Cover B by finitely many neighborhoods $U_1, \ldots, U_k$ with $\xi|_{U_i}$ trivial for each i. Let $\varphi_1, \ldots, \varphi_k$ be a partition of unity with support $\varphi_i \subset U_i$ for each i (Lemma 2.14). Let f_i denote the equivalence of $E(\xi|_{U_i})$ onto $U_i \times \mathbf{R}^n$; let $f_i^1, \ldots, f_i^n$ denote the coordinate functions of its projection into $\mathbf{R}^n$.

We define $h : E(\xi) \to B \times \mathbf{R}^{nk}$ as follows:

$$h(e) = \left(\pi(e),\ \varphi_i(\pi(e)) \cdot f_i^j(e)\right) \qquad \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, n \end{matrix}$$

(no summation is indicated). This is well-defined, since $\varphi_i(\pi(e)) = 0$ unless $e \in E(\xi|_{U_i})$. It is clearly a homomorphism, since each f_i^j is linear on $E(\xi|_{U_i})$. To show that it is injective, let $e \neq 0$. Then for some i, $\varphi_i(\pi(e)) > 0$. Since f_i is an equivalence, $f_i^j(e) \neq 0$ for some j. Hence $h(e) \neq (\pi(e), 0)$, as desired. □

Definition 2.21. The bundle ξ is *s-equivalent* to η if there are trivial bundles o^p, o^n such that $\xi \oplus o^p \simeq \eta \oplus o^n$.

Here $o^p = B \times \mathbf{R}^p$. Symmetry and reflexivity are clear. To show transitivity, assume $\xi \oplus o^p \simeq \eta \oplus o^q$ and $\eta \oplus o^r \simeq \zeta \oplus o^s$. Then $\xi \oplus o^p \oplus o^r \simeq \zeta \oplus o^s \oplus o^q$.

Note that s-equivalence differs from equivalence. E.g., consider the two-sphere S^2 in $\mathbf{R}^3$. Then $\tau^2 \oplus \nu^1 = o^3$. The normal bundle ν^1 is easily seen to be trivial; but it is a classical theorem of topology that τ^2 is not (it does not admit a non-zero cross-section). Hence τ^2 is s-trivial, but not trivial.

THEOREM 2.22. *The set of s-equivalence classes of vector space bundles of finite type over B forms an abelian group*[1] *under $\oplus$.*

Proof. To avoid logical difficulties, we consider only subbundles of $B \times \mathbf{R}^m$, for all m. This suffices, since any bundle ξ of finite type may be embedded in some $B \times \mathbf{R}^m$, by Theorem 2.20. The class of trivial bundles o^p is the identity element, and the existence of inverses is the substance of Theorem 2.20. □

COROLLARY 2.23. *Given two immersions of the differentiable manifold M in euclidean space, their normal bundles are s-equivalent.*

Definition 2.24. M^n is a *π-manifold* if M may be immersed in some $\mathbf{R}^{n+p}$ so that its normal bundle is trivial.

This is equivalent to the requirement that τ^n be s-trivial: Let τ^n be s-trivial. If we take some immersion of M into $\mathbf{R}^{n+p}$, then $\tau^n \oplus \nu^p$ is trivial by Definition 2.16, so that ν^p is s-trivial, i.e., $\nu^p \oplus o^q = o^{p+q}$ for some q. Consider the composite immersion $M \to \mathbf{R}^{n+p} \subset \mathbf{R}^{n+p+q}$. The normal bundle of M in $\mathbf{R}^{n+p+q}$ is just $\nu^p \oplus o^q$, which is trivial.

Conversely, if ν^p is trivial for some immersion, then τ^n is s-trivial because $\tau^n \oplus \nu^p$ is trivial.

Definition 2.25. Let $G_{p,n}$ denote the set of all n-dimensional vector subspaces of $\mathbf{R}^{p+n}$ (i.e., all n-dim hyperplanes through the origin). It is called the *Grassmann manifold* of n-planes in $n+p$ space.

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[1](Added 2006.) In the language of K-theory, which was introduced into topology shortly afterwards by Atiyah and Hirzebruch, the group of s-equivalence classes of vector bundles over B is denoted by $\widetilde{K}_{\mathbf{R}}(B)$ (or $\widetilde{KO}(B)$), where $K_{\mathbf{R}}(B) \cong \mathbf{Z} \oplus \widetilde{K}_{\mathbf{R}}(B)$ is the Grothendieck ring of virtual real vector bundles over the compact connected space B, and $\mathbf{Z}$ is the subring generated by the trivial line bundle. See for example D. Husemoller, "Fibre Bundles", McGraw-Hill 1966.

Its topology is obtained as follows: Consider $\mathcal{M}(n, n+p; n)$ (Definition 1.18). We identify two elements of this set if the hyperplanes spanned by their row vectors are the same. $G_{p,n}$ is in injective correspondence with this identification space, and is given the identification topology. If ρ is the projection

$$\rho : \mathcal{M}(n, n+p; n) \to G_{p,n},$$

then $\rho(A) = \rho(B)$ if and only if $A = CB$ for some non-singular $n \times n$ matrix C. The hyperplane $\rho(A)$ consists of all points $(x^1, \ldots, x^{n+p}) \in \mathbf{R}^{n+p}$ which equal $(c^1, \ldots, c^n) \cdot A$ for some choice of constants c^i. If $\rho(A) = \rho(B)$, then

$$\begin{aligned}(1, 0, \ldots, 0) \cdot A &= (c_1^1, \ldots, c_1^n) \cdot B \\ (0, 1, \ldots, 0) \cdot A &= (c_2^1, \ldots, c_2^n) \cdot B, \text{ etc., for some choice of } c_i^j.\end{aligned}$$

Then $IA = CB$, where C has rank n because A does. The converse is clear.

(a) $G_{p,n}$ is locally euclidean. Let $A \in \mathcal{M}(n, n+p; n)$; after permuting the columns, we may assume $A = (P, Q)$ where P is $n \times n$ and non-singular. Let U be the set of all such A; it is an open set in $\mathcal{M}(n, n+p; n)$, being the inverse image of the non-zero reals under the continuous map $(P, Q) \to \det P$. If $\rho(P, Q) = \rho(R, S)$, where P is non-singular, then $(P, Q) = (CR, CS)$ for some non-singular C. Hence R is necessarily non-singular; it follows that $\rho^{-1}(\rho(U)) = U$, so that $\rho(U)$ is open in $G_{p,n}$ (by definition of the identification topology).

We show $\rho(U)$ homeomorphic with $\mathbf{R}^{pn}$. Define $\varphi : U \to \mathbf{R}^{pn}$ by $\varphi(P, Q) = P^{-1}Q$. If $\rho(P, Q) = \rho(R, S)$ then $(P, Q) = (CR, CS)$, so that

$$P^{-1}Q = (CR)^{-1}(CS) = R^{-1}S.$$

Hence φ induces a continuous map $\varphi_0 : \rho(U) \to \mathbf{R}^{pn}$. Define $\psi : \mathbf{R}^{pn} \to \rho(U)$ by $\psi(Q) = \rho(I, Q)$ where Q is an $n \times p$ matrix. One checks immediately that ψ and φ_0 are inverses of each other.

$$\begin{array}{ccc} \mathcal{M}(n, n+p; n) & \supset & U \xrightarrow{\ \varphi\ } \mathbf{R}^{pn} \\ \downarrow \rho & & \downarrow \rho \\ G_{p,n} & \supset & \rho(U) \underset{\psi}{\overset{\varphi_0}{\rightleftarrows}} \mathbf{R}^{pn} \end{array}$$

(b) To show that $G_{p,n}$ is Hausdorff, we show that ψ maps every compact set into a closed set (this will clearly suffice). Let K be a compact subset of $\mathbf{R}^{pn}$; we show $\varphi^{-1}(K)$ is closed in $\mathcal{M}(n, p+n; n)$. $\varphi^{-1}(K)$ consists of all matrices (P, Q) with P non-singular and $P^{-1}Q \in K$. Let $(P, Q) \in \mathcal{M}(n, n+p; n)$ be the limit of the sequence (P_i, Q_i) of elements of $\varphi^{-1}(K)$. Since K is compact some subsequence of the sequence $\varphi(P_i, Q_i) = P_i^{-1}Q_i$ converges to a point R of K. Then the corresponding subsequence of the sequence Q_i converges to PR, so that $(P, Q) = P(I, R)$. Since (P, Q) has rank n it follows that P is non-singular, so that $(P, Q) \in \varphi^{-1}(K)$, as desired.

Hence $G_{p,n}$ is a manifold of dimension pn.

(c) $G_{p,n}$ is a differentiable manifold and ρ is a differentiable map. A function f on the open set V in $G_{p,n}$ belongs to the differentiable structure $\mathcal{D}$ if $f\rho$ is differentiable. To show that this satisfies the conditions for a differentiable structure, we show that $(\rho(U), \varphi_0)$, as defined in (a), is a coordinate system. Let f be defined on $V \subset \rho(U)$. Given $Q \in \mathbf{R}^{pn}$, $f\varphi_0^{-1}(q) = f\rho(I, Q)$ so that $f\varphi_0^{-1}$ is differentiable if $f\rho$ is. Conversely, given $(P, Q) \in V$, $f\rho(P, Q) = f\varphi_0^{-1}\varphi_0\rho(P, Q) = f\varphi_0^{-1}(P^{-1}Q)$, so that $f\rho$ is differentiable if $f\varphi_0^{-1}$ is.

(d) $G_{p,n}$ is compact. Let L be the subset of $\mathcal{M}(n, n+p; n)$ consisting of matrices whose rows are orthonormal vectors. L is a closed and bounded subset of $\mathbf{R}^{n(n+p)}$. Since $\rho(L) = G_{p,n}$ (the Gram-Schmidt orthogonalization process proves this), $G_{p,n}$ is compact.

(e) $G_{p,n}$ is diffeomorphic to $G_{n,p}$. Geometrically, the homeomorphism h is defined as carrying each hyperplane into its orthogonal complement. It is clearly injective; to show it is differentiable we use the coordinate system $(\rho(U), \varphi_0)$ defined in (a). Let g map U into $\mathcal{M}(n, n+p; p)$ by carrying (P, Q) into $(-(P^{-1}Q)^\tau, I_p)$; it is differentiable (τ denotes transpose). The row space of (P, Q) is the same as that of $(I_n, P^{-1}Q)$, while the row vectors of this matrix are orthogonal to those of $(-(P^{-1}Q)^\tau, I_p)$ (multiply the one by the transpose of the other). Hence g induces $h|_{\rho(U)}$, so that the latter is differentiable.

Definition 2.26. Let $E(\gamma_p^n)$ be defined as that subset of $G_{p,n} \times \mathbf{R}^{n+p}$ consisting of pairs (H, x) where x is a vector lying in the hyperplane H. It is called the *universal bundle* (for reasons we shall see). The projection π maps (H, x) into H; the fibre is thus an n-dimensional subspace of $\mathbf{R}^{n+p}$.

To show that γ_p^n is an n-dimensional vector space bundle over $G_{p,n}$, we need to show the existence of a local product structure. Let $(\rho(U), \varphi_0)$ be a coordinate neighborhood on $G_{p,n}$, as in (a) above. We define $h : \rho(U) \times \mathbf{R}^n \to \pi^{-1}\rho(U)$ as carrying $(H, (x^1, \ldots, x^n))$ into $(H, (x^1, \ldots, x^n) \cdot (I_n, Q))$ where $Q = \varphi_0(H)$. This is a vector in the hyperplane H; h is clearly an isomorphism on each fibre. Its inverse is continuous, since it sends $(H, (y^1, \ldots, y^n))$ in $G_{p,n} \times \mathbf{R}^{p+n}$ into $(H, (y^1, \ldots, y^n))$ in $\rho(U) \times \mathbf{R}^n$.

Definition 2.27. ξ is a *differentiable vector space bundle* if $E(\xi)$ and $B(\xi)$ are differentiable manifolds, and if the homeomorphisms

$$U \times \mathbf{R}^n \to \pi^{-1}(U)$$

which specify the local product structure can be chosen as diffeomorphisms.

It follows that $\pi : E \to B$ is differentiable of maximum rank. Note that B can be differentiably embedded in E by mapping b into the 0-vector of F_b. The normal bundle of this embedding is just ξ.

Examples of differentiable bundles include the tangent bundle of a manifold, the normal bundle of an immersed manifold, and the universal bundle γ_p^n above. In the latter case, $E(\gamma_p^n)$ is embedded differentiably in $G_{p,n} \times \mathbf{R}^{p+n}$.

THEOREM 2.28. *Let ξ^n be an n-dimensional vector space bundle over a normal base space. The following conditions are equivalent:*

(a) *ξ is of finite type.*

(b) *There is a bundle η^p such that $\xi^n \oplus \eta^p$ is trivial.*

(c) *There is a bundle map $\xi^n \to \gamma_p^n$ for some p. (Thus the terminology "universal bundle" for γ_p^n.)*

Proof. We have already shown that (a) implies (b) Theorem 2.20 ; the bundle η^p there constructed has dimension $n(k-1)$, where k is the number of elements in the covering $U_1, \ldots, U_k$ of $B(\xi) = B$ such that $\xi|_{U_i}$ is trivial.

(b) implies (c): Condition (b) means that ξ^n may be embedded in the trivial bundle $B(\xi) \times \mathbf{R}^{p+n}$; let f be this embedding. We wish to define g and g_B in the following diagram:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{g} & E(\gamma_p^n) \\ \downarrow \pi & & \downarrow \\ B(\xi) & \xrightarrow[g_B]{} & G_{p,n} \end{array}$$

Since f is an injective homomorphism, $f(F_b)$ is the cartesian product of b and an n-dimensional hyperplane H^n in $\mathbf{R}^{p+n}$; let $g_B(b)$ equal this hyperplane H^n. If $e \in F_b$, then $f(e) = (b, x)$, where x is a vector in the hyperplane H^n; let $g(e) = (H^n, x)$ in $G_{p,n} \times \mathbf{R}^{p+n}$. Then $g(e)$ actually lies in the subset of $G_{p,n} \times \mathbf{R}^{p+n}$ which constitutes $E(\gamma_p^n)$. From rank considerations, g is automatically an isomorphism on each fibre.

It remains to show that g is continuous. Locally, g just looks like a map

$$U \times \mathbf{R}^n \to G_{p,n} \times \mathbf{R}^{p+n}.$$

We factor it into a continuous map $h : U \times \mathbf{R}^n \to \mathcal{M}(n, n+p; n) \times \mathbf{R}^{p+n}$ followed by the projection $\rho \times 1$ into $G_{p,n} \times \mathbf{R}^{p+n}$. Locally, f looks like a map $U \times \mathbf{R}^n \to B \times \mathbf{R}^{n+p}$. Let $e_1, \ldots, e_n$ be a basis for $\mathbf{R}^n$; we define $h(b, x)$ as $(A, p_2 f(b, x))$. Here p_2 projects $B \times \mathbf{R}^{n+p}$ onto its second factor and A is the matrix having $p_2 f(b, e_1), \ldots, p_2 f(b, e_n)$ as its rows. Then h is continuous, and $(\rho \times 1)h$ equals g.

(Note: the converse assertion, (c) implies (b), can be proved by the same argument.)

(c) implies (a): Being compact, $G_{p,n}$ is covered by finitely many neighborhoods U_i with $\gamma_p^n|_{U_i}$ trivial. (In fact $(n+p)!/n!p!$ neighborhoods will suffice.) If f is a bundle map $\xi^n \to \gamma_p^n$ then the sets $f_B^{-1}(U_i) = V_i$ cover B, and $\xi|_{V_i}$ is equivalent to the bundle induced by $f_B : V_i \to G_{p,n}$ (the uniqueness part of Lemma 2.8). Then $\xi|_{V_i}$ is trivial (since it is induced from a trivial bundle). □

3. The Cobordism Theory of Thom.

Definition 3.1. An *n-manifold-with-boundary* Q is a Hausdorff space with a countable basis which is locally homeomorphic with $\mathbf{H}^n$ (the subset of $\mathbf{R}^n$ such that $x^1 \geq 0$). The *boundary* ∂Q is that subset of Q corresponding to $\mathbf{R}^{n-1}$ under the local homeomorphism ($\mathbf{R}^{n-1}$ being the subset of $\mathbf{R}^n$ with $x^1 = 0$). ∂Q is well-defined, since the image of an open set in $\mathbf{R}^n$ under a homeomorphism of it into $\mathbf{R}^n$ must be open (Brouwer theorem on invariance of domain). It is clear that ∂Q is an $(n-1)$-manifold.

A *differentiable structure* $\mathcal{D}$ on Q is a collection of real-valued functions f defined on open subsets of Q such that

(1) every point of Q has an open neighborhood U and a homeomorphism h of U into an open subset of $\mathbf{H}^n$, such that f is in $\mathcal{D}$ if and only if fh^{-1} is differentiable. (f is defined on an open subset of U; fh^{-1} differentiable means that it may be extended to a neighborhood of $h(U)$ *in* $\mathbf{R}^n$ so as to be differentiable).

(2) If U_i are open sets contained in the domain of f and $U = \bigcup U_i$, then $f|_U \in \mathcal{D}$ if and only if $f|_{U_i} \in \mathcal{D}$ for each i.

As before, (U, h) is called a *coordinate system* on Q, and one can define differentiable structures alternatively by means of coordinate systems.

We impose an additional condition on $\mathcal{D}$ in Definition 3.2.

Definition 3.2. Let M_1, M_2 be compact differentiable n-manifolds. They are said to lie in the same *cobordism class* ($M_1 \sim M_2$) if there is a compact differentiable $n+1$ manifold-with-boundary Q such that ∂Q is diffeomorphic with the disjoint union of M_1 and M_2 (denoted by $M_1 + M_2$) .

Symmetry and reflexivity of this relation are clear. To show transitivity, we impose the additional condition on $\mathcal{D}$ that there is a neighborhood U of ∂Q in Q which is diffeomorphic with $\partial Q \times [0, 1)$, the diffeomorphism being the identity on $\partial Q \times 0$. This is redundant, but we assume it to avoid proving it. Transitivity follows:

Let $M_1 + M_2$ be diffeomorphic with ∂Q_1 and $M_2 + M_3$ diffeomorphic with ∂Q_2; let h_1, h_2 be the diffeomorphisms. We form a new space Q_3 from $Q_1 \cup Q_2$ by identifying each point of $h_1(M_2)$ with its image under $h_2 h_1^{-1}$. There is then a homeomorphism of $M_2 \times (-1, 1)$ into this space which equals h_1 when restricted to $M_2 \times 0$, and is a diffeomorphism of $M_2 \times [0, (-1)^i)$ into Q_i for $i = 1, 2$. (It is derived from the postulated "collar neighborhoods" $\partial Q_i \times [0, 1)$.) If this is taken to be a coordinate system on Q_3, then Q_3 becomes a differentiable manifold-with-boundary, and $M_1 + M_3$ is diffeomorphic with ∂Q_3, while Q_1 and Q_2 are diffeomorphic with subsets of Q_3.

Definition 3.3. As usual, there are logical difficulties involved in considering the collection of *all* manifolds. One way of avoiding them is to consider only manifolds-with-boundary embedded in some euclidean space $\mathbf{R}^p$: If Q_1 is a differentiable manifold-with-boundary and $Q_2 = \partial Q_1 \times [0, 1)$, then the space Q_3 constructed in the preceding paragraph is a differentiable manifold, so that it may be embedded in some euclidean space. Hence Q_1 may so be embedded.

With these restrictions, the set of cobordism classes of n-manifolds forms an abelian group (denoted by $\mathcal{N}_n$) under the operation $+$ (disjoint union). If $M_1 \sim M_1'$ and $M_2 \sim M_2'$, this means that $M_i + M_i'$ is diffeomorphic with ∂Q_i. Then $(M_1 + M_2) + (M_1' + M_2')$ is diffeomorphic with $\partial(Q_1 \cup Q_2)$, so that $M_1 + M_2 \sim M_1' + M_2'$ and the operation $+$ is well-defined on cobordism classes. The zero element is the vacuous manifold or the n-sphere (or ∂Q, where Q is any compact differentiable $(n+1)$-manifold-with-boundary). The remaining axioms are clear. Note that $M + M$ is diffeomorphic with $\partial(M \times [0,1])$, so that every element is of order 2.

The groups $\mathcal{N}_n$ are called the (non-orientable) *cobordism groups*. Let $\mathcal{N}$ denote the direct sum $\mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \cdots$. There is a bilinear symmetric pairing of $\mathcal{N}_i$, $\mathcal{N}_j$ into $\mathcal{N}_{i+j}$, i.e., a homomorphism of $\mathcal{N}_i \oplus \mathcal{N}_j$ into $\mathcal{N}_{i+j}$ induced by the operation of cartesian product.

First, $(M_1 + M_2) \times M_3 = (M_1 \times M_3) + (M_2 \times M_3)$ by definition of cartesian product. Second, if $M_1 \sim 0$, i.e., $M_1 = \partial Q$, then $M_1 \times M_2$ is diffeomorphic with $\partial(Q \times M_2)$, so that $M_1 \times M_2 \sim 0$.

Since $M_1 \times M_2 \sim M_2 \times M_1$, and since $M_1 \times p \sim M_1$ (where p is a point-manifold), this pairing makes $\mathcal{N}$ into a (graded) commutative ring with unit. Indeed, it is a graded algebra over the field $\mathbf{Z}/2$.

Remark 3.4. The general result of Thom is the following

Theorem. *$\mathcal{N}$ is a polynomial algebra over $\mathbf{Z}/2$ with one generator in each positive dimension except those of the form $2^m - 1$. If n is even, the real projective n-space is a generator.*

This theorem means that there are compact manifolds $M^2, M^4, M^5, \ldots$ such that every compact manifold is in the cobordism class of a disjoint union of products of these manifolds, and that there are no relations among the generators (except commutativity and associativity of products).

Thom's procedure is to show that $\mathcal{N}_n$ is isomorphic with the $(n+k)^{\text{th}}$ homotopy group of a certain space T_k, and then to compute these homotopy groups. We shall consider only the first of these two problems in the present notes.

Definition 3.5. Let h be an embedding of the differentiable manifold M^n in $\mathbf{R}^{n+k}$; consider the normal bundle of this embedding. Using the standard Riemannian metric for the tangent bundle to $\mathbf{R}^{n+k}$, this normal bundle is equivalent to the orthogonal complement of the image in the tangent bundle of $\mathbf{R}^{n+k}$ of the tangent bundle of M^n (Definition 2.16); this complement we denote by ν^k. Define e as the canonical map of $E(\nu^k)$ into $\mathbf{R}^{n+k}$ which maps the vector v normal to M^n at x into its end point. (Described differently, one maps the tangent bundle to $\mathbf{R}^{n+k}$ into itself canonically by mapping the vector v, based at x, into the point $x + v$ of $\mathbf{R}^{n+k}$. This map is differentiable; its restriction to $E(\nu^k)$ is the map e.)

Consider M^n as the set of zero vectors of $E(\nu^k)$.

THEOREM 3.6. *There is a neighborhood of M^n in $E(\nu^k)$ which is mapped diffeomorphically by e onto a neighborhood of M^n in $\mathbf{R}^{n+k}$.*

Proof. Note that e is differentiable, and that it has rank $n+k$ at points of $M^n \subset E(\nu^k)$. (This is easily checked by computing the derivative matrix of e with respect to a local coordinate system.) Hence e has rank $n+k$ in some neighborhood of M^n in $E(\nu^k)$, so that it is a local homeomorphism at points of M^n, that is, it maps a neighborhood of each $x \in M^n$ homeomorphically onto a neighborhood of $f(x)$. We then appeal to the topological lemma:

If $f : X \to Y$ is a local homeomorphism-onto and the restriction of f to a closed subset A is a homeomorphism, then f is a homeomorphism on some neighborhood V of A.

(X, Y are Hausdorff spaces with countable bases; X is locally compact.) This lemma is proved as follows:

(1) If A is compact, the lemma holds. For otherwise, there would be points x, y arbitrarily close to A such that $f(x) = f(y)$. Since A has a compact neighborhood, we may choose sequences x_n, y_n converging to x, y, respectively, in A such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Hence $f(x) = f(y)$ so that $x = y$, f being a homeomorphism on A. But then f is not a local homeomorphism at x.

(2) Let A_0 be a compact subset of A. Then there is a neighborhood U_0 of A_0 such that $\overline{U}_0$ is compact and f is a homeomorphism on $\overline{U}_0 \cup A$: It will suffice for f to be injective, since f is a local homeomorphism-onto. By (1), let V_0 be a neighborhood of A_0 so that $f|_{\overline{V}_0}$ is injective. If no neighborhood of A_0 in V_0 satisfies the requirements for U_0, there is a sequence of points x_n of $X - A$ converging to $x \in A_0$ with $f(x_n) \in f(A)$. Choose $y_n \in A$ with $f(x_n) = f(y_n)$. Since f is continuous, $f(y_n)$ converges to $f(x)$; since f is a homeomorphism on A, y_n converges to x. Since $x_n \neq y_n$, this contradicts the fact that f is a local homeomorphism at x.

(3) Express A as the union of an ascending sequence of compact sets $A_1 \subset A_2 \subset \cdots$. Let V_1 be a neighborhood of A_1 such that $\overline{V}_1$ is compact and f is a homeomorphism on $\overline{V}_1 \cup A$ (by (2)). Given V_i a neighborhood of A_i satisfying these conditions, consider the set $\overline{V}_i \cup A_{i+1}$. It is a compact subset of $\overline{V}_i \cup A$, and f is a homeomorphism on $\overline{V}_i \cup A$. Hence by (2) there is a neighborhood V_{i+1} of $\overline{V}_i \cup A_{i+1}$ with $\overline{V}_{i+1}$ compact, such that f is a homeomorphism on $\overline{V}_{i+1} \cup A$. We proceed by induction. f is injective on $V = \bigcup V_{i+1}$, so that it is a homeomorphism of V (being a local homeomorphism-onto). □

COROLLARY 3.7. *Any differentiable submanifold (without boundary) of $\mathbf{R}^{n+k}$ is a differentiable neighborhood retract.*

The projection of $E(\nu^k) \to M^n$ induces (under e) a differentiable map of a neighborhood of M^n in $\mathbf{R}^{n+k}$ onto M^n which is the identity on M^n.

Definition 3.8. Let ξ be a vector space bundle with compact base space $B(\xi)$. The one-point compactification $T(\xi)$ of the total space $E(\xi)$ is called the *Thom space* of ξ. The added point will be denoted by ∞.

Let ξ have a Riemannian metric. Let $T_\epsilon(\xi)$ be obtained from $E(\xi)$ by identifying all vectors of length greater than or equal to ϵ to a point. Let $\alpha(x)$ be a C^∞ function with $\alpha'(x) \geq 0$ which equals 1 in a neighborhood of $x = 0$ and $\to \infty$ as $x \to 1$. The map of $E(\xi)$ into $T(\xi)$ which carries the vector e into the vector $e\,\alpha(\|e\|/\epsilon)$ induces a homeomorphism of $T_\epsilon(\xi)$ onto $T(\xi)$ which is a diffeomorphism on the set $E_\epsilon(\xi)$, consisting of vectors of length less than ϵ. The fact that B is compact is used here.

Definition 3.9. Let the compact manifold M^n be embedded in $\mathbf{R}^{n+k}$. ν^k is given the Riemannian metric of $\mathbf{R}^{n+k}$; by Theorem 3.6 there is a neighborhood of M^n in $\mathbf{R}^{n+k}$ which is diffeomorphic to the subset $E_{2\epsilon}(\nu^k)$ of $E(\nu^k)$. Such a neighborhood is called a ***tubular neighborhood*** of M^n.

By Definition 3.8, we see that $T(\nu^k)$ is homeomorphic with the space obtained from $\mathbf{R}^{n+k}$ by collapsing the exterior of the tubular ϵ-neighborhood of M to a point.

We will need three lemmas concerning approximation by differentiable functions.

LEMMA 3.10. *Let A be a closed subset of the differentiable manifold M, let $f : M \to \mathbf{R}^m$ be differentiable on A. Let δ be a positive continuous function on M. There exists $g : M \to \mathbf{R}^m$ such that*

(1) *g is differentiable*
(2) *g is a δ-approximation to f*
(3) *$g|_A = f|_A$.*

Proof. It suffices to prove this lemma in the case $m = 1$.

Given $x \in A$, $f|_A$ may be extended to a differentiable function f_x in a neighborhood N_x of x. Let N_x be chosen small enough that $|f_x(y) - f(y)| < \delta(y)$ for all $y \in N_x$.

Given $x \in M \setminus A$, choose a neighborhood N_x of x small enough that $|f(y) - f(x)| < \delta(y)$ for all $y \in N_x$. Define $f_x(y) \equiv f(x)$ for $y \in N_x$.

Now let φ_α be a differentiable partition of unity with support φ_α contained in some N_x, say $N_{x(\alpha)}$, for each α. Define $g(y) = \sum \varphi_\alpha(y) f_{x(\alpha)}(y)$. One checks the conditions of the lemma easily. □

More generally:

LEMMA 3.11. *Let $f : M_1 \to M_2$ be a continuous map of differentiable manifolds which is differentiable on the closed subset A of M_1. Let $\epsilon(x) > 0$ be given; and give M_2 the metric determined by some embedding $M_2 \subset \mathbf{R}^p$. Then there exists $g : M_1 \to M_2$ such that*

(1) *g is differentiable*
(2) *g is an ϵ-approximation to f*
(3) *$g|_A = f|_A$.*

Proof. There is a neighborhood U of M_2 in $\mathbf{R}^p$ of which M_2 is a differentiable retract (Corollary 3.7). Let ρ be the differentiable retraction of U onto M_2. Let $\delta(x)$ be a positive function on M_2 so chosen that the cubical neighborhood of $f(x)$ of radius $\delta(x)$ lies in U, and so that its image under ρ has radius less than $\epsilon(x)$. Let $f_1 : M_1 \to \mathbf{R}^p$ be a differentiable map which is a δ-approximation to f, such that $f_1|_A = f|_A$ (by Lemma 3.10). Define $g(x) = \rho(f_1(x))$. □

LEMMA 3.12. *Let $f : M_1 \to M_2$ be a continuous map of differentiable manifolds; let the metric on M_2 be obtained by embedding it in some euclidean space. Given $\epsilon(x)$, there is a $\delta(x)$ such that if $g : M_1 \to M_2$ is a δ-approximation to f, g is homotopic to f under a homotopy $F(x,t)$ with*

(1) *$F(x,t) = f(x)$ for any x such that $g(x) = f(x)$ and*
(2) *$F(x,t)$ is an ϵ-approximation to f for any t.*

Proof. Let U, ρ and $\delta(x)$ be chosen as in Lemma 3.11. Let $g : M_1 \to M_2$ be a δ-approximation to f.

Then the line segment from $g(x)$ to $f(x)$ lies in U, so that

$$F(x,t) = \rho(tg(x) + (1-t)f(x))$$

is well-defined. Furthermore $F(x,t)$ is an ϵ-approximation to $f(x)$ for any t. □

Definition 3.13. Let ξ^k be a differentiable vector space bundle with $B(\xi)$ compact and n-dimensional; let $E(\xi^k)$ be given a metric by embedding it as a closed differentiable submanifold in some euclidean space (it is an $(m+k)$-manifold).

Given an element of $\pi_{n+k}(T(\xi^k), \infty)$, let it be represented by the map

$$f : (\overline{C}_{n+k}, \partial\overline{C}_{n+k}) \to (T(\xi^k), \infty), \tag{3}$$

where $\overline{C}_{n+k}$ is the closed cube $[0,1]^{n+k}$ and $\partial\overline{C}_{n+k}$ is its boundary. Let U denote the open subset $f^{-1}(E(\xi^k))$ of C_{n+k}. Let $g : U \to E(\xi^k)$ be a differentiable δ-approximation to $f|_U$, where δ is so chosen that $\delta < 1$ and g is homotopic to f, the homotopy F also being a 1-approximation to f. (This ensures that f will be continuous if we define $F(x,t) \equiv \infty$ for $x \in \overline{C}_{n+k} - U$.)

Now g may be approximated in turn by a differentiable map $h : U \to E(\xi)$ which is transverse regular on the submanifold $B(\xi)$ of $E(\xi)$. We choose the approximation close enough that g is homotopic to h, the homotopy H being a 1-approximation to g for each t. Extend h to $\overline{C}_{n+k}$ by defining $h(x) = \infty$ for $x \in \overline{C}_{n+k} - U$. Then h is in the homotopy class of f; and $h^{-1}(B(\xi))$ is a differentiable submanifold $M^n \subset U$ which is closed in $\overline{C}_{n+k}$, and thus compact.

THEOREM 3.14. *There is a well-defined homomorphism*

$$\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}_n$$

which assigns to each homotopy class of maps f the cobordism class of the manifold M^n constructed above.

Proof. Let $H : (\overline{C}_{n+k} \times I, \partial\overline{C}_{n+k} \times I) \to (T(\xi^k), \infty)$ be a homotopy between $h_0 = H(x,0)$ and $h_1 = H(x,1)$. Let h_0, h_1 satisfy the conditions

(1) h_i is differentiable on $h_i^{-1}(E(\xi))$
(2) h_i is transverse regular on $B(\xi)$, $\quad (i = 1, 2.)$

We wish to show that $h_0^{-1}(B)$ and $h_1^{-1}(B)$ belong to the same cobordism class.

We may assume that $H(x,t) = H(x,0)$ for $t \le 1/3$, and $H(x,t) = H(x,1)$ for $t \ge 2/3$. Let $U = h^{-1}(E(\xi)) \cap [\overline{C}_{n+k} \times (0,1)]$; then U is an open subset of $\mathbf{R}^{n+k+1}$. Let $G : U \to E(\xi)$ be a differentiable 1-approximation to H which equals H on the closed subset A, where $A = U \cap [\overline{C}_{n+k} \times ((0,1/4] \cup [3/4,1))]$. (See Lemma 3.11. H is differentiable on A.)

Now G satisfies the transverse regularity condition for $B(\xi)$ at points in A (since h_0 and h_1 are transverse regular on B) so that by Theorem 1.36 there is a differentiable map $F : U \to E(\xi)$ which equals G on A, is transverse regular on $B(\xi)$, and is a 1-approximation to G. Because F is a 2-approximation to H, it remains continuous if we define $F(x,t) = \infty$ for $(x,t) \in (\overline{C}_{n+k} \times (0,1)) - U$. Because F equals H on A, it remains continuous if we define $F(x,t) = H(x,t)$

for $t = 0, 1$. Hence $F^{-1}(B)$ is a compact subset of $\overline{C}_{n+k} \times I$, being closed and bounded.

Because $F|_U$ is transverse regular on B, $(F|_U)^{-1}(B)$ is a differentiable $(n+1)$-submanifold of $\overline{C}_{n+k} \times (0,1)$. Its intersection with $\overline{C}_{n+k} \times t$ equals $h_0^{-1}(B) \times t$ for $t \leq 1/4$ and $h_1^{-1}(B) \times t$ for $t \geq 3/4$. Hence $F^{-1}(B)$ is a differentiable manifold-with-boundary whose boundary is $h_0^{-1}(B) + h_1^{-1}(B)$. Thus λ is well-defined.

It is trivial to show that λ is a homomorphism, because the sum in $\mathcal{N}_n$ is derived from disjoint unions of representative manifolds. □

THEOREM 3.15. *If ξ^k is the universal bundle γ_m^k where $k \geq n+1$, $m \geq n$, then $\lambda : \pi_{n+k}(T(\gamma_m^k), \infty) \to \mathcal{N}_n$ is onto.*

Proof. Let M^n be a compact n-manifold; let $k \geq n+1$. Let M^n be embedded in C_{n+k} (Corollary 1.32); let ν^k be the normal bundle of this embedding. The Riemannian metric on $E(\nu^k)$ is that derived from the natural scalar product on the tangent bundle to $\mathbf{R}^{n+k}$, in which ν^k is contained.

By Theorem 3.6, for small ϵ the subset of $E_{2\epsilon}(\nu^k)$ of $E(\nu^k)$ is diffeomorphic with a tubular neighborhood of M^n in C_{n+k}; let U be the image of $E_\epsilon(\nu^k)$.

Let p_1 project $\overline{C}_{n+k}$ on the space obtained from $\overline{C}_{n+k}$ by identifying $\overline{C}_{n+k} - U$ to a point (denoted by $\overline{C}_{n+k}/\overline{C}_{n+k} - U$).

Let p_2 be the diffeomorphism of U onto $E_\epsilon(\nu^k)$, followed by the map of $E(\nu^k)$ into $T_\epsilon(\nu^k)$ which identifies all vectors of length $\geq \epsilon$ (Definition 3.8). p_2 is then extended by mapping $\overline{C}_{n+k} - U$ into ∞.

Let p_3 be the homeomorphism of $T_\epsilon(\nu^k)$ onto $T(\nu^k)$ constructed in Definition 3.8. The composite map $p_3 p_2 p_1$ is a diffeomorphism of U onto $E(\nu^k)$.

Finally, let p_4 be the bundle map of ν^k into γ_m^k induced from the embedding of M^n in $\mathbf{R}^{n+k} \subset \mathbf{R}^{m+k}$. Because both fibres have dimension k, this map satisfies the transverse regularity condition for $G_{k\,m}$ at each point of M^n. Extend p_4 in the obvious way to map $T(\nu^k)$ into $T(\gamma_m^k)$.

Let $g = p_4 p_3 p_2 p_1$. Then $g : \partial\overline{C} \to \infty$. Let $\mu(M^n)$ denote the homotopy class of g in $\pi_{n+k}(T(\gamma_m^k), \infty)$. Now g is transverse regular on $G_{k\,m}$ and $M^n = g^{-1}(G_{k\,m})$. By definition, the cobordism class of M^n is the image of $\mu(M^n)$ under λ, so that $\lambda\mu(M^n) = [M^n]$. □

THEOREM 3.16. *If ξ^k is the universal bundle γ_m^k with $k \geq n+2$, $m > n$, then λ is one-to-one.*

Proof. Given an element of $\pi_{n+k}(T(\gamma_m^k), \infty)$, we may suppose it represented by a map

$$f : (\overline{C}_{n+k}, \partial\overline{C}_{n+k}) \to (T(\gamma_m^k), \infty)$$

which is differentiable on $f^{-1}(E)$ and transverse regular on $G_{m\,k}$ (by Definition 3.13). Let

$$M^n = f^{-1}(G_{m\,k});$$

we wish to show that if M^n is the boundary of an $(n+1)$-manifold-with boundary Q, then f is homotopic to the constant map.

M^n is a submanifold of C_{n+k}; let its normal bundle be ν^k. Let ϵ be chosen so that $E_{2\epsilon}(\nu^k)$ is diffeomorphic with the 2ϵ-neighborhood of M^n; let U_ϵ be the image of the vectors $E_\epsilon(\nu^k)$. Impose a Riemannian metric on γ_m^k; let δ be chosen so that $\|x\| \geq \epsilon$ implies $\|f(x)\| \geq \delta$ for $x \in E(\nu^k)$.

Step 1. f is homotopic to a map f_1 such that

(1) f_1 is differentiable on $f_1^{-1}(E)$ and transverse regular on $G_{m\,k}$.
(2) $f = f_1$ on $M^n = f_1^{-1}(G_{m\,k})$.
(3) f_1 carries everything outside U_ϵ into ∞.

Define $F : E(\gamma_m^k) \times I \to T(\gamma_m^k)$ by the equation $F(e,t) = e\,\alpha(t\|e\|/\delta)$, where α is the function defined in Definition 3.8. Let $f_1(x) = F(f(x), 1)$.

Step 2. By the diffeomorphism of $U_{2\epsilon}$ with $E_{2\epsilon}$, f_1 induces a map $\overline{f}_1$ of $\overline{E}_\epsilon(\nu^k)$ into $T(\gamma_m^k)$ which carries $\partial(E_\epsilon)$ into ∞. Any homotopy of $\overline{f}_1$ which leaves $\partial(E_\epsilon)$ at ∞ induces a homotopy of f_1.

Now $\overline{f}_1$ is homotopic to a map $\overline{f}_2$ such that

(1) $\overline{f}_2$ is differentiable on $\overline{f}_2^{\,-1}(E)$ and transverse regular on $G_{m\,k}$.
(2) $\overline{f}_2 = \overline{f}_1$ on $M^n = \overline{f}_2^{\,-1}(G_{m\,k})$.
(3) $\overline{f}_2$ is locally a bundle map in some neighborhood of M^n.

The homotopy leaves $\partial(E_\epsilon)$ at ∞.

Consider $G : \overline{E}_\epsilon(\nu^k) \times I \to T(\gamma_m^k)$ defined by the equation $G(e,t) = \overline{f}_1(te)/t$. As $t \to 0$, $G(e,t)$ approaches a limit which is non-zero if $e \neq 0$ (since $\overline{f}_1$ is differentiable and t-regular). It is easily seen to be a bundle map. It will not suffice for our purposes, since it does not carry $\partial(E_\epsilon) \times I$ into ∞. Choose $\delta > 0$ so that $\|x\| \geq \epsilon$ implies $\|G(x,t)\| \geq \delta$ for $x \in E(\nu^k)$, $t \in I$, and define $H(e,t) = [G(e,t)]\,\alpha(\|G(e,t)\|/\delta)$. If we set $\overline{f}_2 = H(e,0)$, then $\overline{f}_2$ is a bundle map for $\|e\|$ small (since $\alpha(x) \equiv 1$ for x small). The map $E(e,1) = \overline{f}_1(e)\,\alpha(\|\overline{f}_1(e)\|/\delta)$ does not equal $\overline{f}_1$, but it is homotopic to $\overline{f}_1$, the homotopy leaving $\partial(E_\epsilon)$ at ∞. The homotopy is defined by the equation

$$K(e,t) = \overline{f}_1(e)\,\alpha(t\|\overline{f}_1(e)\|/\delta), \quad \text{as in Step 1.}$$

Step 3. Let Q be the $n+1$ manifold-with-boundary such that $M^n = \partial Q$. Let h be a diffeomorphism of $M^n \times [0,1)$ into Q which carries $M^n \times 0$ onto ∂Q. Define $h_1 : Q \to C_{n+k} \times I$ as follows:

If $x = h(y,t)$ where $y \in M^n$ and $0 \leq t \leq 1/2$, let $h_1(x) = (y,t)$.
If $x \notin \text{image}\, h$, let $h_1(x) = p$, where p is some fixed point interior to $C_{n+k} \times I$.
If $x = h(y,t)$ where $y \in M^n$ and $1/2 \leq t \leq 1$, let

$$h_1(x) = (1 - \beta(t))\,h_1(y, 1/2) + \beta(t)p, \quad \text{where}$$

$\beta(t)$ is a C^∞ function with $\beta'(t) \geq 0$, $\beta(t) \equiv 0$ in a neighborhood of $t = 1/2$ and $\beta(t) \equiv 1$ in a neighborhood of $t = 1$.

h_1 is a differentiable map of $\operatorname{Int} Q$ into $\operatorname{Int}(C_{n+k} \times I)$; and h_1 is an injective immersion in a neighborhood of ∂Q. Since $\dim(C_{n+k} \times I) > 2(n+1)$, h_1 may be approximated by an injective immersion h_2 which equals h_1 in a neighborhood of ∂Q (by Lemma 1.29). It may be extended to an embedding of Q into $C_{n+k} \times I$. (Since Q is compact, an injective immersion is automatically an embedding.) Let Q now be considered as this subset of $C_{n+k} \times I$.

Step 4. We have a map f_2 of $\overline{C}_{n+k} \times 0$ into $T(\gamma^k_m)$ which is a bundle map when restricted to a small tubular neighborhood of $M^n \times 0$ in $C_{n+k} \times 0$. We extend it to $\overline{C}_{n+k} \times [0, b)$ for b small in the trivial way. Suppose there exists a map g of the ϵ'-neighborhood N of Q in $C_{n+k} \times I$ into $T(\gamma^k_m)$ which equals f_2 in some neighborhood of ∂Q in $C_{n+k} \times I$ and maps each point of $N - Q$ into a non-zero vector in $E(\gamma^k_m)$. Our theorem then follows: Let δ be so chosen that, if the distance of x from Q is $\geq \epsilon 1/2$, then $\|g(x)\| \geq \delta$.
Define $g_1 : C_{n+k} \times I \to T(\gamma^k_m)$ by the equation

$$g_1(x, s) = \begin{cases} g(x,s)\,\alpha(\|g(x,s)\|/\delta), & \text{for } (x,s) \in N, \text{ and} \\ \infty, & \text{otherwise.} \end{cases}$$

The restriction of g_1 to $C_{n+k} \times 0$ does not equal the map f_2, but it is homotopic to f_2, by the same technique as used at the end of Step 2. Thus g_1 provides the homotopy required for our theorem.

To show that the extension g exists, we refer to Steenrod, "Fibre Bundles" (Princeton University Press, 1951). According to §19.4 and 19.7 of this book, the principal bundle associated with γ^k_m is an m-universal bundle. That is: given a vector space bundle ξ^k over a complex of dimension $\leq m$, any bundle map of ξ^k, restricted to a subcomplex, into γ^k_m can be extended throughout ξ^k. We will assume the well known result that Q can be triangulated. The dimension $n+1$ of Q is $\leq m$. Hence any bundle map of the normal bundle ν^k of Q, restricted to a polyhedral neighborhood of ∂Q, into γ^k_m can be extended throughout ν^k.

Applying this result to the map f_2, this completes the proof of Theorem 3.16. □

Letting T_k stand for the union of the Thom spaces $T(\gamma^k_m) \subset T(\gamma^k_{m+1}) \subset \cdots$, in the fine (direct limit) topology, Theorems 3.15 and Theorem 3.16 imply the following.

THEOREM 3.17. *The cobordism group $\mathcal{N}_n$ is canonically isomorphic to the stable homotopy group $\pi_{n+k}(T_k)$, for $k \geq n+2$.*

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Lectures on Differentiable Structures

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Princeton University, Spring 1961

1. Strong and Weak Diffeotopies: the diffeotopy extension problem.

We shall be concerned solely with *smooth*, i.e. C^∞, manifolds. Without making precise definitions for the moment, we can state our basic problem as follows: can every motion of a submanifold of a given manifold V be extended to all of V? Figure 1 illustrates a motion of a submanifold of $\mathbf{R}^3$ which cannot be extended to all of $\mathbf{R}^3$. The trefoil knot is moved so that the knotted portion approaches a point; at the final stage the knot is replaced by a circle. We will illustrate, after a few definitions, other motions of submanifolds which cannot be extended.

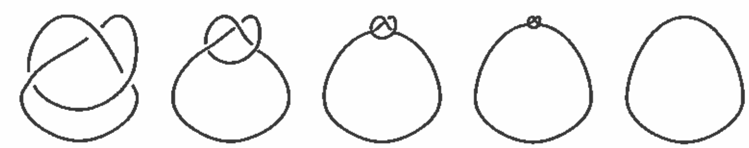

FIGURE 1.

Definition. Let M and V be two smooth manifolds. A smooth map $f : M \to V$ is an *embedding* iff.

(1) f is a homeomorphism of M into V

(2) the rank of f_* at $p \in M$ is the dimension of M at p [where f_* is the induced map on the tangent space of M].

If $f(M) = V$ then f^{-1} will be smooth and f is called a *diffeomorphism*.

Definition. A *diffeotopy from M into V* is a one-parameter family of embeddings which is smooth as a function of both variables, i.e. a smooth map $f : M \times I \to V$ (I denotes the unit interval $[0, 1]$) such that each $f_t : M \to V$, defined by $f_t(x) = f(x, t)$ is an embedding. The map f is called a diffeotopy between f_0 and f_1, which are called diffeotopic maps.

The family of embeddings of S^1 into $\mathbf{R}^3$ indicated in Figure 1 is not a diffeotopy because it is not smooth at $(x, 1)$, where x is the point towards whose image the knotted portion of the circle converges.

Although the unit interval I is strongly suggested as a parameter space in the definition of homotopic maps, it has several disadvantages for defining diffeotopic maps. For one thing, it is not immediately clear that diffeotopy is an equivalence relation. Moreover $M \times I$ is a manifold with boundary if M has no boundary, and $M \times I$ has corners if M has a boundary. These objections present no real difficulties because, without loss of generality, we may assume that

$$f_t = \begin{cases} f_0, & \text{for} \quad t \leq 1/3 \\ f_1, & \text{for} \quad t \geq 2/3. \end{cases}$$

To see this simply define $\overline{f}_t$ as $f_{\lambda(t)}$ where λ is smooth and

(1) $\lambda = 0$ for $t \leq 1/3$
(2) $\lambda = 1$ for $t \geq 2/3$
(3) $0 \leq \lambda \leq 1$.

It now becomes clear that the relation of diffeotopy is transitive. Furthermore we see that the open unit interval (or the full real line) can be used in place of I in the definition.

Definition. A *strong diffeotopy of V to itself* is a diffeotopy f from V into V such that each f_t is a diffeomorphism. If $f_0, f_1 : M \to V$ are two embeddings, f_0 is *strongly diffeotopic* to f_1 iff there is a strong diffeotopy F of V to itself carrying f_0 to f_1, i.e. such that F_0 is the identity and $F_1 \circ f_0 = f_1$.

Two maps f and g may be diffeotopic without being strongly diffeotopic. Figure 2 indicates a family of embeddings of $\mathbf{R}$ into $\mathbf{R}^2$; at the last stage the image of $\mathbf{R}$ is a circle minus a point.

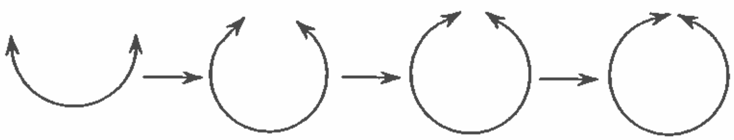

FIGURE 2.

Clearly there is not even a homeomorphism of $\mathbf{R}^2$ onto itself which will take the first embedding into the last. We might hope to avoid such situations by requiring that each $f_t(M)$ be a closed subset of V. However there is an embedding of $\mathbf{R}$ into $\mathbf{R}^3$ which shows that this requirement is not sufficient; it is indicated in Figure 3.

For $|u| \geq 1$ the embedding f_0 satisfies $f_0(u) = (u, 0, 0)$. The embedding f_0 is diffeotopic to the standard embedding of $\mathbf{R}$ in $\mathbf{R}^3$. This can be seen by sliding the knotted portion to the left to ∞, as follows: let

$$\begin{aligned} f_t(u) &= f_0(u + t/(1-t)) - (t/(1-t), 0, 0), \quad 0 \leq t < 1 \\ f_1(u) &= (u, 0, 0). \end{aligned}$$

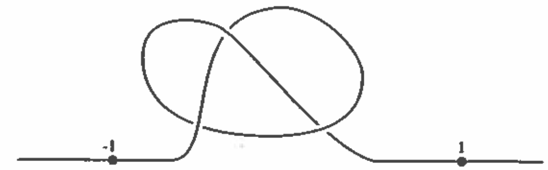

FIGURE 3.

It is again clear that f_0 and f_1 are not strongly diffeotopic. This example is really just that of Figure 1, with the bad point at ∞.

A somewhat more spectacular example of an embedding of $\mathbf{R}$ in $\mathbf{R}^3$ that is diffeotopic, but not strongly diffeotopic, to the usual embedding is illustrated in Figure 4.

FIGURE 4.

There are loops infinitely far to the left but there is a rightmost loop after which we have the usual embedding of a line. Clearly, by pulling out the last loop between $t = 0$ and $t = 1/2$, the second between $t = 1/2$ and $t = 3/4$, etc., we can get a diffeotopy with the standard embedding, which can certainly be extended to $\mathbf{R}^3$ between $t = 0$ and $t = t_0$ for any $t_0 < 1$. Nevertheless, the fundamental group of the complement of $\mathbf{R}^3$ under this embedding is non-abelian, so that the diffeotopy cannot be extended to $\mathbf{R}^3$ between $t = 0$ and $t = 1$, (see R. Fox, *A remarkable simple closed curve*, Annals of Math. 50 (1949), 264–265.)

These examples should justify all the hypothesis of the following theorem.

THEOREM 1.1. *Let M and V be smooth manifolds. Let $f_0, f_1 : M \to V$ be two smooth embeddings which are diffeotopic under a diffeotopy which leaves all points fixed outside a compact set $M_0 \subset M$. Then f_0 and f_1 are strongly diffeotopic, under a strong diffeotopy which leaves all points fixed outside a compact set $V_0 \subset V$.*

This theorem is due to R. Thom, *La classification des immersions*, Séminaire Bourbaki 157 (Dec. 1957). See also R. Palais, *Local triviality of the restriction map for embeddings*, Commentarii 34 (1960), 305–312.

Proof. We may assume that f_t is defined for all real t, and that

$$f_t = \begin{cases} f_0, & \text{for} \quad t \leq 1/3 \\ f_1, & \text{for} \quad t \geq 2/3. \end{cases}$$

Define a map $F : M \times \mathbf{R} \to V \times \mathbf{R}$ by $F(x, t) = (f_t(x), t)$. It is easy to see that F is an embedding. The map F is indicated in Figure 5 where V is a line and M a point.

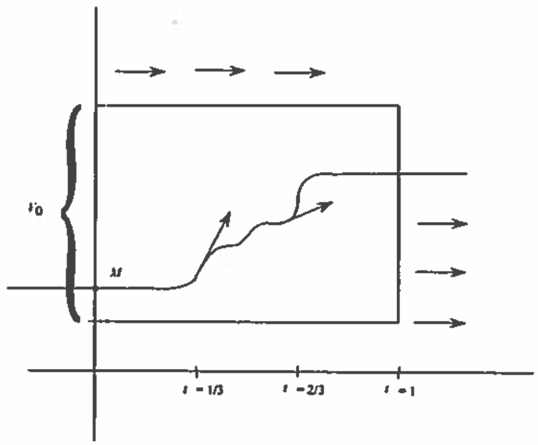

FIGURE 5.

Let V_0 be a compact neighborhood of $f(M_0 \times I)$.

We wish to define a vector field X on $V \times \mathbf{R}$ such that

(1) outside of $V_0 \times I$ the field X is $\frac{\partial}{\partial t}$.

(2) in $F(M \times \mathbf{R})$ the field X is $F_*(\frac{\partial}{\partial t})$.

(X is indicated by arrows in Figure 5).

If $y \notin (V_0 \times I)$ and $y \in F(M \times \mathbf{R})$ these requirements agree: if (2nd. coord y) $\notin [0, 1]$ they agree because f_t is constant for $t \notin [0, 1]$; if (1st. coord y) $\notin V_0$ then $y \notin f(M_0 \times I)$ so they agree since f_t is constant outside M_0. Therefore X has been defined on $F(M \times \mathbf{R})$ and outside $V_0 \times I$.

In order to extend X to all of $V \times \mathbf{R}$ it is sufficient to extend it locally (we then use partitions of unity). At points on the boundary of $V_0 \times I$ we can clearly extend X locally by using the field $\frac{\partial}{\partial t}$. If $y \in F(M \times I)$ there is, by the implicit function theorem, a local coordinate system $(u_1, \ldots, u_{n+1})$ for $V \times \mathbf{R}$ in a neighborhood U of y such that $U \cap F(M \times I) = \{p \in V \times I : u_1(p) = \ldots = u_{n-k}(p) = 0\}$ where $n = \dim V$, and $k = \dim M$. It is clear how to extend a vector field on a $(k+1)$-plane of $\mathbf{R}^{n+1}$ to all of $\mathbf{R}^{n+1}$, which gives an extension of X to the neighborhood of U. If y is in the interior of $V_0 \times I$ and $y \notin F(M \times I)$ then we can extend the vector field arbitrarily.

Thus we can get a vector field X on $V \times \mathbf{R}$ satisfying (1) and (2). We may also assume that

(3) the $\mathbf{R}$ component of X is always $\frac{\partial}{\partial t}$.

In fact we may replace X by X' where the V-component of X' is equal to the V-component of X but the $\mathbf{R}$ component is $\frac{\partial}{\partial t}$.

Conditions (1) and (2) remain valid.

Let $\phi_t : V \to V$ be given by $\phi_t(x) =$ first component of solution at time t of the vector field X, passing through $(x, 0)$. (The second component is clearly t, by condition (3)). The solutions of the vector field X are known to exist locally; their global existence, from time $t = 0$ to $t = 1$, follows from conditions (1) and (3) and the compactness of $V_0 \times I$. By condition (2), if $x \in M$ we have $\phi_1(x) = f_1(x)$. The family ϕ_t thus gives us the desired strong diffeotopy of V onto V. (Clearly ϕ leaves all points fixed outside V_0.) □

Remark. The above proof works just as well if M is a manifold with boundary. However, the theorem is false if V has a boundary, as is seen by pushing a small circle to the boundary of a large disk, (Figure 6).

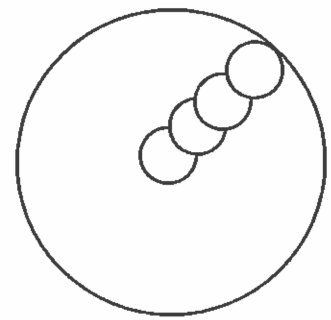

FIGURE 6.

2. Embedding a Cell in a Manifold.

Denote a point $(x_1, \ldots, x_n) \in \mathbf{R}^n$ by $\vec{x}$, for convenience. $\vec{0}$ denotes $(0 \ldots, 0)$.

LEMMA 2.1. *Let $\vec{f}$ be a smooth map of $\mathbf{R}^n$ into $\mathbf{R}^m$ such that $\vec{f}(\vec{0}) = \vec{0}$. Then there are smooth functions $\vec{g}_i$ such that*

$$\vec{f}(\vec{x}) = \sum_{i=1}^{n} x_i \vec{g}_i(\vec{x})$$

and

$$\vec{g}_i(\vec{0}) = \frac{\partial \vec{f}}{\partial x_i}(\vec{0}).$$

(This second condition actually follows from the first.)

Proof.

$$\vec{f}(\vec{x}) = \int_0^1 \frac{d\vec{f}(t\vec{x})}{dt}dt = \int_0^1 \sum_{i=1}^n x_i \frac{\partial \vec{f}(t\vec{x})}{\partial x_i}dt.$$

Therefore we can let

$$\vec{g}_i = \int_0^1 \frac{\partial \vec{f}(t\vec{x})}{\partial x_i}dt.$$

□

THEOREM 2.2. *Let M be a smooth, connected, orientable manifold of dimension n. Let $f,\ g : \mathbf{R}^n \to M$ be smooth, orientation preserving embeddings. Then f is (weakly) diffeotopic to g.*

Proof.

Case 1: $M = \mathbf{R}^n$. It is sufficient to show that f is diffeotopic to the standard embedding. We can assume that $\vec{f}(\vec{0}) = \vec{0}$, where we are now using the notation $\vec{f} = (f_1, \ldots, f_n)$, with $f_i : \mathbf{R}^n \to \mathbf{R}$.

Let

$$\vec{h}(\vec{x}, t) = t^{-1}\vec{f}(t\vec{x}) \quad \text{for} \quad t \neq 0.$$

To examine the behavior of $\vec{h} = (h_1, \ldots, h_n)$ as $t \to 0$ we write, using Lemma 2.1,

$$\begin{aligned} h_j(\vec{x}, t) &= t^{-1}f_j(t\vec{x}) \\ &= t^{-1}\sum_{i=1}^n tx_i g_{ij}(t\vec{x}) \\ &= \sum_{i=1}^n x_i g_{ij}(t\vec{x}). \end{aligned}$$

Therefore $\vec{h}$ is smooth if we define h_j by this formula on $\mathbf{R}^n \times \mathbf{R}$.

For $t \neq 0$ the map $\vec{x} \to h(\vec{x}, t)$ is clearly an embedding. At $t = 0$ we have

$$h_i(\vec{x}, 0) = \sum_{i=1}^n x_i g_{ij}(0) = \sum_{i=1}^n x_i \frac{\partial f_j}{\partial x_i}(\vec{0})$$

which is linear, non-singular, and of positive determinant, since the matrix

$$\left(\frac{\partial f_i}{\partial x_i}(\vec{0}) \right)$$

is non-singular and f is orientation preserving. Thus f is diffeotopic to a linear map which is diffeotopic to the identity.

Case 2: M unrestricted but $f(\mathbf{R}^n) \subset g(\mathbf{R}^n)$. Case 2 is clear from Case 1.

Case 3: $f(\mathbf{R}^n) \cap g(\mathbf{R}^n) \neq \emptyset$. We can choose an orientation preserving embedding h so that $h(\mathbf{R}^n)$ is a cell in $f(\mathbf{R}^n) \cap g(\mathbf{R}^n)$. By Case 2, we have $h \approx f$ and $h \approx g$, so that $f \approx g$.

Case 4: General case. Since M is connected there is a sequence of embeddings

$$f = h_0,\ h_1,\ \ldots,\ h_n = g.$$

such that $h_i(\mathbf{R}^n) \cap h_{i+1}(\mathbf{R}^n) \neq 0$, $(i = 0, \ldots, n-1)$. By Case 3

$$f = h_0 \approx h_1 \approx \ldots \approx h_n = g.$$

The following theorem is due to R. Palais and J. Cerf. (R. Palais, *Extending diffeomorphisms*, Proceedings Amer. Math. Soc. 11 (1960), 274–277.)

THEOREM 2.3. *Let D^n be the closed unit disk in $\mathbf{R}^n$. Let M be a smooth, connected, orientable manifold of dimension n. Let $f, g : D^n \to M$ be orientation preserving embeddings such that f and g can be extended to embeddings $F, G : \mathbf{R}^n \to M$. Then f and g are strongly diffeotopic.*

Actually, F and G will always exist if M has no boundary, so that this hypothesis is redundant.

Proof. This follows from Theorems 1.1 and 2.2. □

Definition. Let M be a smooth, compact, oriented manifold. The diffeotopy classes of all orientation preserving diffeomorphisms is a group under composition, denoted by $\pi_0 \operatorname{Aut} M$.

As an application of Theorem 2.3 we will prove the following.

THEOREM 2.4. *$\pi_0 \operatorname{Aut} S^n$ is abelian.*

Proof. Let $f : S^n \to S^n$ be a smooth map and let $i : D^n \hookrightarrow S^n$ be the "standard" embedding of D^n into the northern hemisphere on S^n, given by

$$i(t\vec{u}) = ((\sin t\pi/2)\vec{u}, \cos t\pi/2).$$

Assume that S^n is so oriented that i is orientation preserving. We have two maps i, $f \circ i : S^n \to S^n$. By Theorem 2.3 there is a family of diffeomorphisms $h_t : S^n \to S^n$ with $h_0 = I$ and $h_1 \circ f \circ i = i$, which means that $h_1 \circ f$ is the identity on $i(D^n)$. Therefore $f = h_0 \circ f$ is diffeotopic to $h_1 \circ f$, a map which is the identity on $i(D^n)$.

Now given two maps $f, g : S^n \to S^n$ let

$$\begin{aligned} f &\approx f' \\ g &\approx g' \end{aligned}$$

where f' leaves the northern hemisphere fixed and g' leaves the southern hemisphere fixed. Then

$$f \circ g \circ f^{-1} \circ g^{-1} \approx f' \circ g' \circ (f')^{-1} \circ (g')^{-1}.$$

This last map clearly leaves all points of S^n fixed.

3. The Connected Sum of Two Manifolds.

Definition. Let M_1 and M_2 be smooth, connected, oriented manifolds, both of dimension n. Let D^n be the unit n-disk in $\mathbf{R}^n$ and let rD^n be the n-disk of radius r. Let $i_q : 2D^n \to M_q$ be embeddings $(q = 1, 2)$, one preserving orientation, one reversing it. Form the disjoint sum

$$[M_1 - i_1(\tfrac{1}{2}D^n)] + [M_2 - i_2(\tfrac{1}{2}D^n)]$$

and identify $i_1(t\vec{u})$ with $i_2(t^{-1}\vec{u})$ for $\vec{u} \in S^{n-1}$ and $1/2 < t < 2$. The result is denoted $M_1 \# M_2$, and called the *connected sum* of M_1 and M_2. It has an obvious orientable smooth structure.

By Theorem 2.3, the manifold $M_1 \# M_2$ is well defined (up to orientation preserving diffeomorphism). Moreover, it is clear that $\#$ is commutative, associative, and has S^n as a zero element (up to orientation preserving diffeomorphism).

For convenience we will consider only compact connected oriented n-manifolds without boundary. Let $\mathcal{M}_n$ be the set of classes of these manifolds under orientation preserving diffeomorphism. Then $\mathcal{M}_n$, with the composition operation $\#$, is a semi-group.

$\mathcal{M}_1 = 0$.
$\mathcal{M}_2$ is a free commutative semi-group on one generator, the torus.
$\mathcal{M}_3$ is a free commutative semi-group on $\aleph_0$ generators.

(This result is essentially due to H. Kneser. See J. Milnor, *A unique decomposition theorem for 3-manifolds*, to appear.[1])

The structure of $\mathcal{M}_4$ is not known. However, $\mathcal{M}_4$ does not satisfy the cancellation law, as shown by an example of Hirzebruch, and hence cannot be free commutative. Let $M^4 = \mathbf{CP}^2$ be complex projective 2-space. Let $\overline{M}^4 = M^4$ with the opposite orientation. Then it can be shown that

$$M^4 \# M^4 \# \overline{M}^4 \approx M^4 \# (S^2 \times S^2)$$

but

$$M^4 \# \overline{M}^4 \not\approx S^2 \times S^2.$$

For higher values of n it may happen that $M \# N \approx S^n$ but $M \not\approx S^n$. This definitely occurs for $n = 7$.

THEOREM 3.1. *If $M \# N \approx S^n$ then M is homeomorphic to S^n.*

Proof. We form the infinite sum $M \# N \# M \# N \# \cdots$. Since $M \# N \approx N \# M \approx S^n$ we have

$$\begin{aligned} M \# N \# M \# N \# \cdots &\approx (M \# N) \# (M \# N) \# \cdots \\ &\approx S^n \# S^n \# S^n \# \cdots \\ &\approx \mathbf{R}^n. \\ M \# N \# M \# N \# \cdots &\approx M \# (N \# M) \# (N \# M) \# \cdots \\ &\approx M \# S^n \# S^n \# \cdots \\ &\approx M \# \mathbf{R}^n. \end{aligned}$$

So $\mathbf{R}^n \approx M \# \mathbf{R}^n$. But since $M \approx M \# S^n$ we have M-point $\approx M \# \mathbf{R}^n \approx \mathbf{R}^n$. Therefore M is homeomorphic to S^n.

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[1] *A unique decomposition theorem for 3-manifolds*, Amer. J. Math. (1962), 1–7; or Collected Papers II: The Fundamental Group. Publish or Perish (1995).

4. Differentiable Structures on Spheres.

This section will describe constructions by which one can obtain exotic differentiable structures on spheres. However the proof that some of the resulting manifolds are not diffeomorphic to the usual sphere will only be given later.

Definition. Let $f : S^{n-1} \to S^{n-1}$ be a diffeomorphism. We wish to define a manifold by taking the disjoint union of two hemispheres and identifying their boundary $(n-1)$-spheres by f. As a matter of convenience, for defining the differentiable structure on this new manifold, we adopt the following procedure: Take $\mathbf{R}^n + \mathbf{R}^n$ and identify $t\vec{u}$ in the first $\mathbf{R}^n$ with $t^{-1}f(\vec{u})$ in the second, for $\vec{u} \in S^{n-1}$ and $0 < t$. The resulting smooth manifold is denoted $M(f)$ and is called a *twisted sphere with twist* f. Clearly $M(f)$ is topologically a sphere.

As an example, if i denotes the identity map of S^{n-1}, we have $M(i) \approx S^n$.

Remark. It can be shown that a closed manifold M is a twisted sphere if and only if there exists a smooth real valued function on M with exactly two critical points, both being non-degenerate. Smale has recently proved that, at least for $n \neq 3, 4$, M is a twisted sphere if and only if M has the homotopy type of a sphere.

LEMMA 4.1. *The correspondence $f \to M(f)$ defines a homomorphism from the group $\pi_0 \operatorname{Aut} S^{n-1}$ into the semi-group $\mathcal{M}_n$. That is,*

(1) *if $f \approx g$ then $M(f)$ is orientation preserving diffeomorphic to $M(g)$,*

(2) *$M(g \circ f)$ is orientation preserving diffeomorphic to $M(f) \# M(g)$.*

Proof. If $f \approx g$ we have the diagram of figure 7, where $f : S_1^{n-1} \to S_3^{n-1}$ and $g : S_2^{n-1} \to S_4^{n-1}$; and there is an extension H from the annular region A between S_1^{n-1} and S_2^{n-1} to the annular region B between S_3^{n-1} and S_4^{n-1} (since $f \approx g$). Let $M = (S_+^n \cup A) + (S_-^n \cup B)$ with x identified with $H(x)$ for $x \in A$.

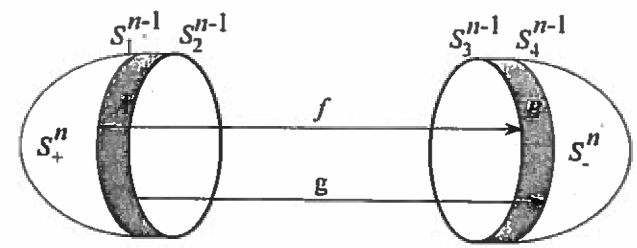


FIGURE 7.

It is clear that $M = M(f)$ and also $M = M(g)$.

To show $M(g \circ f) = M(f) \# M(g)$ consider the diagram shown in figure 8:
To form $M(f) \# M(g)$ we may remove cell A from $M(f)$ and cell B from $M(g)$ and identify their boundary $n-1$ spheres by the identity map. This is clearly the same

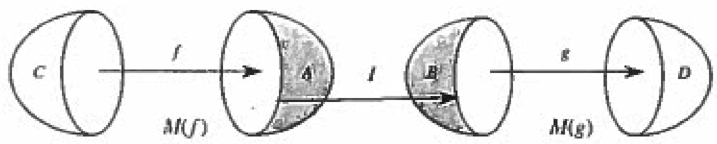


FIGURE 8.

as identifying the hemispheres C and D along their boundary $n-1$ spheres by $g \circ f$. □

Definition. The image group of $\pi_0 \operatorname{Aut} S^{n-1}$ under this homomorphism is denoted by Γ_n. We may describe Γ_n as the *group of twisted n-spheres.*

Next we describe a procedure for constructing non-trivial diffeomorphisms of S^{n-1}. Let $n-1 = p+q$. Start with two homotopy classes

$$(f) \in \pi_p(SO_q) \text{ and } (g) \in \pi_q(SO_p)$$

where SO_q denotes the group of rotations of $\mathbf{R}^q$. These homotopy classes are represented by smooth maps

$$f : \mathbf{R}^p \to SO_q, \quad \text{and } g : \mathbf{R}^q \to SO_p$$

where $f(x) = e$ for $\|x\| \geq 1$, $g(y) = e$ for $\|y\| \geq 1$. Now define automorphisms F and G of $\mathbf{R}^{p+q} = \mathbf{R}^p \times \mathbf{R}^q$ by

$$\begin{aligned} F(x, y) &= (x, f(x) \cdot y) \\ G(x, y) &= (g(y) \cdot x, y). \end{aligned}$$

Then $FGF^{-1}G^{-1}$ is an automorphism of $\mathbf{R}^{p+q}$ which leaves everything outside of $D^p \times D^q$ fixed. Hence $FGF^{-1}G^{-1}$ extends to an automorphism of the smooth manifold S^{p+q}.

LEMMA 4.2. *This construction gives rise to a bilinear pairing*

$$\pi_p(SO_q) \otimes \pi_q(SO_p) \to \pi_0 \operatorname{Aut} S^{p+q}.$$

Proof. If f_0 and f_1 represent the same element of $\pi_p(SO_q)$ then one can choose a smooth homotopy f_t between them, where $f_t(x) = e$ for $\|x\| \geq 1$. Now the formula $F_tGF_t^{-1}G^{-1}$ describes a diffeotopy between the corresponding automorphisms of S^{p+q}.

Next consider the sum of two homotopy classes (f_1) and (f_2) in $\pi_p(SO_q)$. We may assume that

$$f_1(x) = e \quad \text{for } \|x\| \geq \frac{1}{2}$$

and that $f_2(x) = e$ except for x lying in a ball contained in the region $1/2 \leq \|x\| \leq 1$. Then

$$f_3(x) = f_1(x) \cdot f_2(x)$$

represents the homotopy class $(f_1) + (f_2)$. The identity

$$(F_1GF_1^{-1}G^{-1})(F_2GF_2^{-1}G^{-1}) = F_3GF_3^{-1}G^{-1}$$

now shows that our construction is bilinear.

As an example, suppose that $p = q = 3$. Then the rotation group SO_3 has the sphere S^3 as 2-fold covering space. Hence

$$\pi_3(SO_3) = \pi_3(S^3) = \mathbf{Z};$$

and we have homomorphisms

$$\mathbf{Z} = \mathbf{Z} \otimes \mathbf{Z} \xrightarrow{\alpha} \pi_0 \operatorname{Aut} S^6 \xrightarrow{\beta} \Gamma_7.$$

It will be shown later that $\beta \circ \alpha$ is non trivial. In fact it is known that Γ_7 is a cyclic group of order 28 generated by $\beta\alpha(1)$. Almost nothing is known about π_0 Aut S^6 or image α or kernel β.

5. Tubular Neighborhoods

Definition. An *ℓ-disk bundle* is a fiber bundle with the ℓ-disk D^ℓ as fiber and the orthogonal group $O(\ell)$ as a structure group. An *ℓ-dimensional vector bundle* is a fiber bundle with $\mathbf{R}^\ell$ as fiber and the general linear group $GL(\ell, \mathbf{R})$ as structure group. (For the definition of a fiber bundle see N. Steenrod, *The Topology of Fibre Bundles.*) A fiber bundle is *smooth* if all spaces and maps appearing in the definition are smooth.

Definition. Let M and V be smooth manifolds without boundary, of dimensions k and $n = k+\ell$, respectively. Suppose that M is compact and that $i : M \hookrightarrow V$ is an embedding. A *tubular neighborhood of M in V* is a smooth ℓ-disk bundle with base space M and total space a closed neighborhood N of $i(M)$, such that the embedding $i : M \hookrightarrow N$ agrees with the canonical cross-section of the bundle obtained by selecting the center of D^ℓ at each point. The projection map π of the bundle is therefore a retraction of N onto M.

We will first show that tubular neighborhoods always exist. For the case $V = \mathbf{R}^n$ we will give a detailed proof; we will then indicate how the proof may be modified for the general case.

If M is a smooth compact k-dimensional manifold without boundary and $i : M \hookrightarrow \mathbf{R}^n$ is an embedding, where $n = k + \ell$, define

$$E = \{(x, v) \in M \times \mathbf{R}^n : v \text{ is perpendicular to } i_*(M_x)\}.$$

If $(x, v) \in E$ we can choose a coordinate system $u^1, \ldots, u^n$ for a neighborhood U of $i(x)$ in $\mathbf{R}^n$ so that

$$i(M) \cap U = \{y \in \mathbf{R}^n \cap U : u^{k+1}(y) = \ldots = u^n(y) = 0\}.$$

Let $g_{ij}(y) =$ inner product of $\frac{\partial}{\partial u^i}\Big|_y$ and $\frac{\partial}{\partial u^j}\Big|_y$. Then a point $(y, a^1, \ldots, a^n)$ of $M \times \mathbf{R}^n$ is in E if and only if

$$\sum_{j=1}^{n} g_{ij}(y)a^j = 0 \quad \text{for } i = 1, \ldots, k.$$

Since the $k \times n$ matrix $g_{ij}(y)$ has rank k this shows that E is an n-dimensional smooth submanifold of $M \times \mathbf{R}^n$.

Let $\exp : E \to \mathbf{R}^n$ be the map (the exponential map) defined by $\exp(x, v) = i(x) + v$; this map is easily seen to be smooth. Let

$$E_\epsilon = \{(x, v) \in E : \|v\| < \epsilon\}$$

and

$$N_\epsilon = \{y \in \mathbf{R}^n : d(y, i(M)) < \epsilon\}.$$

THEOREM 5.1. *For sufficiently small ϵ, the map $\exp$ is a diffeomorphism from E_ϵ onto N_ϵ.*

It is then clear that $\overline{N}_{\epsilon/2}$ is a tubular neighborhood of M.

Proof. Let $U \subset E$ be the set of non-critical points of exp. It is easy to see that any pair $(x, 0) \in E$ belongs to U; and that $\exp|_U$ is a local diffeomorphism. Choose ϵ_1 so that $U \supset U_1 = \{(x, v) \in E : \|v\| \leq \epsilon_1\}$.

Then U_1 is compact and we may use the following lemma.

LEMMA 5.2. *Let A_0 be a closed subset of a compact metric space A. Let $f : A \to B$ be a local homeomorphism such that $f|_{A_0}$ is one-to-one. Then there is a neighborhood U of A_0 such that $f|_U$ is one-to-one.*

Proof. Let $C \subset A \times A$ be the set of all pairs (x, y) with $x \neq y$ but $f(x) = f(y)$. Then C is closed: let (x_n, y_n) be a sequence in C with $x_n \to x$ and $y_n \to y$; since $f(x_n) = f(y_n)$ for every n we have $f(x) = f(y)$; moreover $x \neq y$ since f is locally one-to-one.

Let $g : C \to \mathbf{R}$ be defined by $g(x, y) = d(x, A_0) + d(y, A_0)$. Then g is everywhere positive. Since C is compact there is an $\epsilon > 0$ such that $g \geq 2\epsilon$ on C. Then f is one-to-one on the ϵ-neighborhood of A_0.

We may apply the lemma to the map exp which is a local homeomorphism on the compact metric space U_1 and one-to-one on $\{(x, 0) \in E\}$. It follows that for sufficiently small ϵ the map exp is a diffeomorphism of E_ϵ into N_ϵ.

If $p \in N_\epsilon$ there is a point x of M which is nearest to p. Then the vector $\overrightarrow{i(x), p}$ is perpendicular to $i_*(M_x)$ and has length $< \epsilon$. Therefore $p = \exp(x, \overrightarrow{i(x), p})$ so that exp takes E_ϵ diffeomorphically onto N_ϵ.

In the general case (when V is not $\mathbf{R}^n$) choose some Riemannian metric g for V. Define

$$\begin{aligned} E &= \{(x, v) : x \in M \text{ and } v \text{ is a tangent vector of } V_{i(x)} \text{ with } g(v, i_*(M_x)) = 0\} \\ E_\epsilon &= \{(x, v) \in E : \sqrt{g(v, v)} < \epsilon\} \\ N_\epsilon &= \{y \in V : \rho(y, i(M)) < \epsilon\} \end{aligned}$$

where ρ is the (topological) metric on V induced by g.

For small enough ϵ the map exp may be defined as follows: given a tangent vector v at $i(x)$ let w be the geodesic with $w(0) = i(x)$ and $w'(0) = v$. If ϵ is sufficiently small w may be defined on $[0, 1]$. The point $w(1)$ is defined as $\exp(x, v)$.

Theorem 5.1 remains true; the proof is the same.

Next we will prove a theorem which asserts that tubular neighborhoods are essentially unique. Let $i : M \to V$ be an embedding of the smooth compact

manifold M into the smooth manifold V. Suppose that we are given two tubular neighborhoods of M with projection maps.

$$\pi : N \to M, \quad \pi' : N' \to M$$

respectively.

THEOREM 5.3. *There exists a strong diffeotopy F_t of V leaving $i(M)$ pointwise fixed so that*

(1) F_0 = identity map of V, and

(2) $F_1|_N$ is a bundle map of N onto N'.

Thus F_1 is an automorphism of V which throws each ℓ-disk $\pi^{-1}(x)$ linearly onto the disk $\pi'^{-1}(x)$.

The proof begins as follows. We may clearly assume that $N \subset E \subset V$ [or $N' \subset E' \subset V$] where E [or E'] has the structure of a smooth ℓ-dimensional vector bundle over M which extends the structure of the given k-disk bundle.

LEMMA 5.4. *There is a bundle map $f : E \to E'$ such that $f : E \to V$ is weakly diffeotopic to the inclusion $E \subset V$.*

Proof. **Case 1:** Suppose that $E \subset E'$, and let $j : E \to E'$ be the inclusion map. Define

$$j_t(e) = t^{-1} j(te) \text{ for } t \neq 0,\ e \in E;$$

where multiplication by t [or t^{-1}] is the multiplication within the fiber over $\pi(e)$ [or $\pi' j(te)$].

Choosing local coordinates (x, y) for E where $x = (x^1, \ldots, x^k)$ gives the point in the base space M and y the point in the fiber $\mathbf{R}^\ell$; and choosing similar coordinates (x', y') for E' we have

$$j_t(x, y) = (\alpha(x, ty), t^{-1}\beta(x, ty))$$

for certain smooth maps α and β, where

$$\alpha(x, 0) = x \quad \beta(x, 0) = 0$$

(since j leaves M fixed). The first coordinate $\alpha(x, ty)$ is clearly a smooth function of x, y and t even when $t \to 0$.

We can write $\beta(x, y) = \sum y^i g_i(x, y)$ by Lemma 2.1. (The extra variables x do not affect the proof of Lemma 2.1.) Thus

$$j_t(x, y) = (\alpha(x, ty), \sum y^i g_i(x, ty))$$

which makes sense, and is smooth, even for $t = 0$. In fact

$$j_0(x, y) = (x, \sum y^i g_i(x, 0)).$$

Thus the second coordinate of j_0 is a linear function L of y, for fixed x. In order to complete the proof of Case 1 it is sufficient to show that L in non-singular.

Since $g_i(x, 0) = \frac{\partial \beta^j}{\partial y^i}(x, 0)$ the matrix of L is

$$\left(\frac{\partial \beta^j}{\partial y^i}(x, 0)\right).$$

The determinant of this matrix is the same as the determinant of the matrix of j_1, so it is non-zero. Thus j_0 is a bundle isomorphism from E to E'.

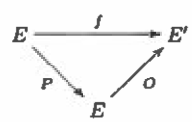
Case 2 (the General Case): The set $E \cap E'$ is a manifold containing M. There is an ℓ-dimensional vector bundle $E'' \subset E \cap E'$ (since there is a tubular neighborhood of M in $E \cap E'$). Then Case 1 may be applied to E and E'' and to E' and E''. This completes the proof of Lemma 5.4. □

We now wish to prove that F_1 can be made a bundle map when the group is reduced to the orthogonal group $O(\ell)$. Let $L : W \to W'$ be a non-singular linear transformation between two real vector spaces with a Euclidean inner product. We will need the following facts:

(1) Any non-singular matrix A can be expressed uniquely as $A = OP$ where O is orthogonal and P is symmetric positive definite. (See C. Chevalley, *Theory of Lie Groups I*, p. 14.) Furthermore O and P depend differentiability on A.

(2) The set of symmetric positive definite matrices is convex.

Fact **(1)** implies that L can be factored uniquely $L = OP$ (where $P : W \to W$ and $O : W \to W'$) with P positive definite and O an isometry. We may apply this factorization to the bundle map f constructed in Lemma 5.4 fiber by fiber to obtain a factorization of f,



$f = OP$ where O and P are smooth fiber maps, P is symmetric positive definite on each fiber, and O is an isometry on each fiber. Since the symmetric positive definite matrices are convex the family of bundle maps

$$f_t = 0 \cdot (tI + (1-t)P)$$

is a diffeotopy between f and a bundle map $E \to E'$ which is an isometry on each fiber. Therefore we have proved:

LEMMA 5.5. *There is a bundle map $g : E \to E'$ which is an isometry on each fiber such that $g : E \to V$ is weakly diffeotopic to the inclusion $E \subset V$.*

Proof of Theorem 5.3. Clearly $g|_N$ gives a bundle map from N to N', which is weakly diffeotopic, in V, to the inclusion $N \subset V$. Since N is compact we can apply Theorem 1.1 and conclude that $g|_N$ is strongly diffeotopic to the inclusion. This completes the proof. □

Smooth Manifolds with Boundary

John Milnor

BASED ON LECTURES AT THE UNIVERSITY OF MEXICO IN 1960.
NOTES BY H. CÁRDENAS, E. LLUIS, F. RECILLAS AND R. VÁZQUEZ
AS TRANSLATED BY I. E. COLON-ROLDAN AND S. SIMANCA

Contents:

1. Basic Constructions: Smooth Manifolds and Smooth Maps.

Let $\mathbf{H}^n$ be the Euclidean half-space consisting of all points $x = (x_1, \ldots, x_n)$ of the real Euclidean space $\mathbf{R}^n$ such that $x_n \geq 0$. Define the *boundary* $\partial\mathbf{H}^n \cong \mathbf{R}^{n-1}$ of $\mathbf{H}^n$ as the set of all points in $\mathbf{H}^n$ with $x_n = 0$. (For the case $n = 0$, we define $\mathbf{H}^0 = \mathbf{R}^0$ to be a single point with vacuous boundary.)

The concept of smoothness (or more precisely C^∞-smoothness[1]) will first be defined for maps between open subsets of Euclidean spaces or half-spaces and then for more general smooth manifolds. If V is an open subset of $\mathbf{R}^n$ or of $\mathbf{H}^n$, and if V' is an open subset of $\mathbf{R}^m$ or $\mathbf{H}^m$, then a continuous map $f : V \to V'$ will be called *smooth* if the partial derivatives of all orders $\frac{\partial^r f_i}{\partial x_{j_1} \cdots \partial x_{j_r}}$ are defined and continuous as mappings from V to the real numbers. Note that the identity map

[1]For the purposes of differential topology, one could equally well work with C^r-smooth manifolds and C^r-smooth maps for any integer $r \geq 1$. In fact any C^r-smoothness structure can be upgraded to a C^∞-smoothness structure and any C^r-diffeomorphism can be approximated by a C^∞-diffeomorphism, provided that $r \geq 1$. However, C^∞ structures are particularly convenient to work with.

of V is smooth, and that the composition of smooth maps between open subsets of Euclidean spaces or half-spaces is again smooth.

Smooth functions to the real numbers will play a special role. We will make use of the fact that a smooth real valued function on an open subset of $\mathbf{H}^n$ can always be extended locally, near any point, to a smooth real valued function on an open subset of $\mathbf{R}^n$. (Compare the Appendix.[2])

Let V be an open subset of $\mathbf{H}^n$ or $\mathbf{R}^n$. By a *germ* of a function at a point $x \in V$ is meant an equivalence class of functions, each defined on some neighborhood of x in V, where two functions f_1, f_2 defined in neighborhoods V_1, V_2 of x are said to be equivalent if there is a neighborhood V_3 of x, with $V_3 \subset V_1 \cap V_2$, such that f_1 and f_2 coincide on V_3.

Fixing some point $x \in V$, let $S_x(V)$ be the ring consisting of all germs of smooth real valued functions defined on open neighborhoods of the point x in V. (Note that $S_x(V) \cong S_x(\mathbf{H}^n)$ if V is open in $\mathbf{H}^n$, and that $S_x(V) \cong S_x(\mathbf{R}^n)$ if V is open in $\mathbf{R}^n$.) The union $S(V) = \bigcup_{x \in V} S_x(V)$ consists of all germs of smooth real valued functions at the points of V. Note that a function $f : V \to \mathbf{R}$ is smooth if and only if its germ at each point $x \in V$ belongs to this ring $S_x(V)$. Furthermore, a function $x \mapsto f(x) = (f_1(x), \dots, f_m(x))$ from V to $V' \subset \mathbf{R}^m$ or $V' \subset \mathbf{H}^m$ is smooth if and only if each component function $f_i : V \to \mathbf{R}$ satisfies this condition. Still another equivalent condition can be given as follows.

LEMMA 1.1. A smoothness criterion in $\mathbf{R}^n$. *Again let V and V' be open subsets of Euclidean spaces or half-spaces. A function $f : V \to V'$ is smooth (that is, has continuous partial derivatives of all orders) if and only if, for each $x \in V$ and for each germ $g \in S_{f(x)}(V')$ of a smooth real valued function at the point $y = f(x)$, the germ of the composition $g \circ f$ at x belongs to $S_x(V)$.*

The proof is completely straightforward. These constructions should help to motivate the following.

Definition 1.2. A smooth manifold is a pair $(M, S(M))$ consisting of a Hausdorff space M together with a set $S(M)$ of germs of real valued functions on open subsets of M, satisfying the following condition:

> For each point $p \in M$ there is a neighborhood U of p, an open subset V of $\mathbf{H}^n$ or of $\mathbf{R}^n$ for some $n \geq 0$, and a homeomorphism $h : U \to V$, such that a germ of real valued functions in V belongs to $S(V)$ if and only if the corresponding germ in U belongs to $S(M)$.

The set $S(M)$ will be called a *smoothness structure* on M. The triple (U, V, h) is called a *coordinate chart* for $(M, S(M))$, and U is called a *coordinate neighborhood* of the point p.

(Occasionally, when the smoothness structure $S(M)$ is completely clear, we may simply refer to M as a smooth manifold.)

[2]The proof requires some work. For an easier presentation, one could just define a smooth function on an open subset of $\mathbf{H}^n$ to be one which extends locally to a smooth function on $\mathbf{R}^n$.

As examples, the half-space $(\mathbf{H}^n, S(\mathbf{H}^n))$ and the full Euclidean space $(\mathbf{R}^n, S(\mathbf{R}^n))$ are certainly examples of smooth manifolds. Similarly, any open subset $V \subset \mathbf{H}^n$ or $V \subset \mathbf{R}^n$ has a natural smoothness structure $S(V)$, defined as above. Other examples will be described presently.

Remarks.

(1) The dimension n is a continuous integer valued function on M, and is therefore constant on each connected component of M. In most applications, this dimension function takes the same value everywhere, and one speaks of an n-dimensional manifold, or briefly an n-manifold. However, occasionally it can be convenient to allow the more general case, where different connected components may have different dimensions.

(2) If the charts (U, V, h) all have the property that each V is an open subset of $\mathbf{R}^n$ (or of the open half-space $x_n > 0$ in $\mathbf{H}^n$), then we speak of a manifold *without boundary*.

(3) It is often convenient to describe a smooth manifold structure by means of its collections of coordinate charts. We will see in Lemma 1.4 that this form of the definition is completely equivalent to the one given above.

The concept of a smooth map between arbitrary smooth manifolds can now be defined as follows.

Definition 1.3. Let $(M, S(M))$, $(N, S(N))$ be smooth manifolds. A continuous map $f : M \to N$ is *smooth* if for each $x \in M$ and each $\varphi \in S_{f(x)}(N)$, the germ of $\varphi \circ f$ at x belongs to $S_x(M)$. If the inverse map $f^{-1} : N \to M$ is also defined and smooth, then f is called a *diffeomorphism* (or more precisely a C^∞-*diffeomorphism*) between $(M, S(M))$ and $(N, S(N))$.

Evidently the identity map of M is smooth, as is the composition of any two smooth maps. In the special case where M and N are open subsets of Euclidean spaces or half-spaces, it follows from Lemma 1.1 that f is smooth in this new sense if and only if it is smooth in the original sense (continuous partial derivatives of all orders).

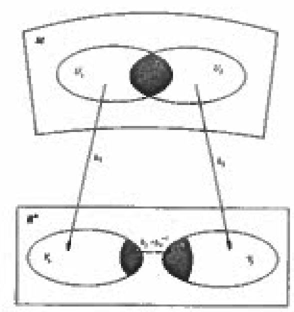


FIG. 1. Overlapping coordinate charts.

LEMMA 1.4. Overlapping coordinate charts. *Let (U_i, V_i, h_i) and (U_j, V_j, h_j) be two coordinate charts for the smooth manifold $(M, S(M))$, and suppose that $U_i \cap U_j \neq \emptyset$. Then the composition*

$$h_j \circ h_i^{-1} : h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$$

is a smooth map between open subsets of $\mathbf{R}^n$ or $\mathbf{H}^n$. Conversely, let M be a Hausdorff space and $\{(U_i, V_i, h_i)\}$ a collection of triples where the U_i are open subsets with union equal to M, where each V_i is an open subset of some $\mathbf{R}^{n_i}$ or $\mathbf{H}^{n_i}$, and each h_i is a homeomorphism of U_i onto V_i. If each transition map

$$h_j \circ h_i^{-1} : h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$$

has continuous partial derivatives of all orders, then there is one and only one smoothness structure $S(M)$ so that $(M, S(M))$ is a smooth manifold and so that each h_i is a diffeomorphism.

Thus the definition of smooth manifold as given above, in terms of germs of smooth real valued functions, is completely equivalent to the more customary definition in terms of overlapping coordinate charts. The proof is straightforward. □

1.1. Operations with smooth manifolds. To conclude this section, we describe seven basic constructions involving smooth manifolds.

(1) Submanifolds. Let $(M, S(M))$ be a smooth manifold and let N be any subset of M. Define the restriction of the smoothness structure $S(M)$ to N as follows: A germ g of a real valued function at the point $x \in N$ belongs to this restriction $S(N) = S(M)|_N$ if and only if there exists a neighborhood U of x in M and a smooth real valued function on U whose restriction to $U \cap N$ represents the given germ g. If this $S(N)$ is a smoothness structure, or in other words if the pair $(N, S(N))$ is a smooth manifold, then $(N, S(N))$ is called a *smooth submanifold* of $(M, S(M))$. We may also say that $(N, S(N))$ is *smoothly embedded* in $(M, S(M))$. Note that this will always be the case if N is an open subset of M.

(2) Boundary. Recall that the boundary of the half-space $\mathbf{H}^n = \{(x_1, \ldots, x_n) ; x_n \geq 0\}$ is defined to be the subset $\partial\mathbf{H}^n = \{(x_1, \ldots, x_n) ; x_n = 0\}$. Similarly, for any open subset $V \subset \mathbf{H}^n$, the boundary is defined to be the set $\partial V = V \cap \partial\mathbf{H}^n$. It is not difficult to check that any diffeomorphism between open subsets of $\mathbf{H}^n$ must carry boundary to boundary.

For any smooth manifold $(M, S(M))$, the *boundary* $(\partial M, S(\partial M))$ is a possibly vacuous smooth submanifold. We first define the subset $\partial M \subset M$. If (U, V, h) is a coordinate chart for $(M, S(M))$, then we set $\partial U = h^{-1}(\partial V)$, and define ∂M to be the union $\bigcup \partial U$ over all such coordinate charts. The closed subset $\partial M \subset M$ defined in this way is always a smooth submanifold. In other words, the restriction $S(\partial M) = S(M)|_{\partial M}$ is necessarily a smoothness structure on ∂M. In fact it is not difficult to check that $\partial U = U \cap \partial M$, and that the collection of triples $\{(\partial U_i, \partial V_i, h_i|_{\partial U_i})\}$ constitutes a family of coordinate charts with $\bigcup \partial U_i$ equal to

∂M. The complement $\mathring{M} = M \setminus \partial M$ is called the *interior* of M. If M is a manifold without boundary (i.e., with vacuous boundary), then every point of M is an interior point.

Since ∂V is an open subset of $\partial\mathbf{H}^n \cong \mathbf{R}^{n-1}$, it follows that ∂V is an $(n-1)$-dimensional manifold without boundary: $\partial(\partial V) = \emptyset$. For any M it follows that ∂M is a manifold without boundary. If M has constant dimension n (for example if M is connected), note that ∂M has constant dimension $n - 1$.

(3) Tangent bundle. If p is an interior point of a smooth manifold M, then a *tangent vector* $\mathbf{v}$ at p can be defined for example as an equivalence class $[\phi]$ of smooth curves $\phi : (-\epsilon, \epsilon) \to M$, with $\phi(0) = p$, where $(-\epsilon, \epsilon)$ is some neighborhood of zero in $\mathbf{R}$. By definition, two such curves ϕ and ψ are *equivalent* if, for some coordinate chart (U, V, h) with $p \in U$, the derivatives $dh_i(\phi(t))/dt$ at $t = 0$ are equal to the corresponding derivatives $dh_i(\psi(t))/dt$ at $t = 0$. This condition does not depend on the particular choice of coordinate chart. The set of all such tangent vectors at p form a vector space TM_p. In fact if V has dimension n and if $x(t) = h(\phi(t))$, then the correspondence

$$[\phi] \mapsto \left.\frac{dx(t)}{dt}\right|_{t=0} = \left.\left(\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right)\right|_{t=0}$$

maps the tangent vector space TM_p isomorphically onto $\mathbf{R}^n$. Note that any smooth map $f : M \to N$ induces a linear map $\mathcal{D}f_p : TM_p \to TN_{f(p)}$, called the *derivative* of f at p, by the formula $\mathcal{D}f_p[\phi] = [f \circ \phi]$.

If p is a boundary point of M, then we must modify this definition by allowing curves $\phi : [0, \epsilon) \to M$ or $\phi : (-\epsilon, 0] \to M$ which are defined only on a one-sided neighborhood of zero. The corresponding tangent vectors $[\phi]$ are said to point *inward* in the first case and *outward* in the second. *Thus the tangent space TM_p at a boundary point is the union of a half-space of inward vectors together with a half-space of outward vectors.* The intersection of these two half-spaces is the tangent space $T(\partial M)_p$ of the boundary manifold at p.

The *tangent manifold* TM of an n-dimensional smooth manifold M is a $2n$-dimensional smooth manifold which can be expressed as the union of disjoint subsets TM_p, where p ranges over M. This union TM is topologized and given a smoothness structure so that the following two conditions are satisfied:

(a) If V is an open subset of $\mathbf{R}^n$ or $\mathbf{H}^n$, then TV is diffeomorphic to $V \times \mathbf{R}^n \subset \mathbf{R}^{2n}$ under the correspondence $[\phi] \leftrightarrow (\phi(0), \frac{d\phi}{dt}(0))$.

(b) If (U, V, h) is a coordinate chart for M, then $(TU, TV, \mathcal{D}h)$ is a coordinate chart for TM. In particular, the tangent manifold TU is an open subset of TM, and the derivative $\mathcal{D}h$ maps TU diffeomorphically onto TV, which can be identified with $V \times \mathbf{R}^n$ as above.

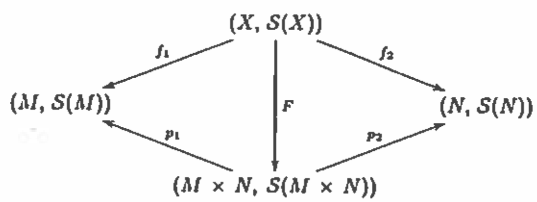
It follows easily that every smooth map $f : M \to N$ gives rise to an induced smooth map $\mathcal{D}f : TM \to TN$ which carries each subset TM_p to the subset $TN_{f(p)}$ by the linear map $\mathcal{D}f_p$. Further details will be left to the reader.

(4) Orientation. By an *orientation* for a finite dimensional vector space W is meant an equivalence class of ordered bases, where two bases $\{e_j\}$ and $\{e'_i = \sum_j a_{ij} e_j\}$ are equivalent (or define the same orientation) if and only if the matrix $[a_{ij}]$ has positive determinant. An orientation for a smooth manifold is a choice of orientation for every tangent vector space TM_p, where these orientations are required to depend continuously on the point p, in the following sense. Each $p_0 \in M$ should have a neighborhood U and a choice of basis $\{p \mapsto e_i(p) \in TM_p\}$ depending continuously on $p \in U$, so that $\{e_i(p)\}$ determines the specified orientation for each $p \in U$. The manifold is said to be *orientable* if it possesses such an orientation. If a connected manifold is orientable, then evidently it can be given exactly two possible orientations.

(5) Disjoint sum. Let $\{(M_i, \mathcal{S}(M_i))\}_{i \in I}$ be an arbitrary family of smooth manifolds, and let $\biguplus M_i$ be the topological sum (that is the disjoint union of the M_i, topologized so that each M_i is embedded as an open subset). We make this into a smooth manifold by setting $\mathcal{S}(\biguplus M_i)$ equal to the union of the $\mathcal{S}(M_i)$. Evidently each $(M_i, \mathcal{S}(M_i))$ is embedded as a smooth open submanifold of this union $(\biguplus M_i, \mathcal{S}(\biguplus M_i))$.

(6) Product. If $(M, \mathcal{S}(M))$, $(N, \mathcal{S}(N))$ are smooth manifolds and if at least one of the two has vacuous boundary, then we define $\mathcal{S}(M \times N)$ as the unique smoothness structure on the cartesian product $M \times N$ for which the projections p_1 and p_2 are smooth and which satisfies the following universal property:

Given any smooth manifold $(X, \mathcal{S}(X))$ and smooth maps f_1, f_2, as in the diagram below, the product function $F = (f_1, f_2)$ is also smooth.



This definition can easily be extended to any finite number of factors, provided that at most one of these factors has non-vacuous boundary.

(7) Quotient manifolds. Let $(M, \mathcal{S}(M))$ be a smooth manifold and let $\sim$ be an equivalence relation on M. Then we can form the quotient space $M/\sim$, together with a canonical projection $j : M \to M/\sim$. If this quotient space is sufficiently well behaved, then it can be given the structure of a smooth manifold, with smoothness structure $\mathcal{S}(M/\sim)$ defined as follows. A germ of a real valued function φ at the point $y \in M/\sim$ belongs to the ring $\mathcal{S}_y(M/\sim)$ if and only if the germ of the composition $\varphi \circ j$ belongs to $\mathcal{S}_x(M)$ for every x in the equivalence class $j^{-1}(y)$. If $M/\sim$ is a Hausdorff topological manifold, and if $\mathcal{S}_y(M/\sim)$ is a smoothness structure on this manifold, then $(M/\sim,\ \mathcal{S}(M/\sim))$ is called the quotient smooth manifold. As an example, if $M = \mathbf{R}^{n+1} \setminus \{0\}$ and if the equivalence class $j^{-1}(j(x))$ is the set $(\mathbf{R} \setminus \{0\})x$ consisting of all non-zero real multiples, then the quotient space $\mathbf{R}^{n+1}/\sim$ is called the *real projective space* $\mathbf{RP}^n$. If we carry out the same construction using complex numbers in place of real numbers, then we obtain the *complex projective space* $\mathbf{CP}^n$, which is a $2n$-dimensional smooth manifold.

Still another important construction, pasting together the boundaries of two smooth manifolds via a boundary diffeomorphism, will be described at the end of the following section.

2. Manifolds are Homogeneous.

This section will first prove the following.

THEOREM 2.1. Equivalence of disk-embeddings. *Let $f : D^n \to M$ and $g : D^n \to M$ be two smooth embeddings of the closed disk D^n as submanifolds of the interior of a smooth connected n-dimensional manifold M. If M is orientable, we assume also that these two embeddings have compatible orientations. Then there exists a diffeomorphism $h : M \to M$ such that $h \circ f = g$, where h is equal to the identity outside a compact subset of the interior of M.*

The general strategy of the proof can be described as follows. (Compare Palais [1960].) We first compose f with a diffeomorphism h_1 so that $h_1 \circ f(0) = g(0)$. Then we compose with a further diffeomorphism h_2 so that the embedding $h_2 \circ h_1 \circ f$ is actually tangent to g at 0. After composing with still another diffeomorphism, we may suppose that the two maps coincide throughout some neighborhood of 0. Finally, by deforming both f and g down into this small neighborhood we can complete the argument.

This proof will depend on a number of lemmas, starting with the following global form of the inverse function theorem.

LEMMA 2.2. Diffeomorphism criterion. *A smooth map $f : M \to N$ is a diffeomorphism if and only the induced map $\mathcal{D}f : TM \to TN$ is bijective (ie., one-to-one and onto).*

In other words, f is a diffeomorphism if and only if (a) f itself is bijective, and (b) for each $p \in M$ the linear map $\mathcal{D}f_p : TM_p \to TN_{f(p)}$ between tangent spaces is also bijective.

Proof. First suppose that $f : V \to V'$ is a map between open subsets of $\mathbf{R}^n$ or $\mathbf{H}^n$. Then the proof proceeds just as for the standard inverse function theorem. It is convenient to identify TV with $V \times \mathbf{R}^n$, and to use the slightly modified notation

$$\mathcal{D}f(x, v) = \big(f(x),\ \mathcal{D}f_x(v)\big)$$

for the induced map of tangent bundles in this case, where $\mathcal{D}f_x$ can be identified with the $n \times n$ matrix of first derivatives at x. Write the Taylor series at a point as

$$f(x + \Delta x) = f(x) + \mathcal{D}f_x(\Delta x) + \mathcal{R}\,,$$

where the remainder term satisfies $\|\mathcal{R}\|/\|\Delta x\| \to 0$ as $\Delta x \to 0$. If $y = f(x)$ and $y + \Delta y = f(x + \Delta x)$, then we have

$$f^{-1}(y + \Delta y) = x + \Delta x = f^{-1}(y) + (Df_x)^{-1}(\Delta y) + \mathcal{R}'\,,$$

where $\mathcal{R}' = -(\mathcal{D}f_x)^{-1}(\mathcal{R})$. It follows easily that $\|\mathcal{R}'\|/\|\Delta y\| \to 0$ as $\|\Delta y\| \to 0$, so that f^{-1} is differentiable at y, with a derivative $\mathcal{D}(f^{-1})_y = (\mathcal{D}f_x)^{-1}$ which depends continuously on y. Thus f^{-1} is at least C^1-smooth throughout the set $f(V) = V'$.

Now let us apply this same argument to the smooth map $F = \mathcal{D}f : TV \to TV'$. A brief computation shows that the $(2n) \times (2n)$ matrix $\mathcal{D}F_{(x,v)}$ has the form

$$\begin{pmatrix} \mathcal{D}f_x & 0 \\ * & \mathcal{D}f_x \end{pmatrix}$$

where $*$ stands for the matrix of second derivatives of f at x. Thus $\mathcal{D}F$ is also bijective, hence F^{-1} is at least C^1-smooth. Since the matrix of first derivatives of F^{-1} contains the matrix of second derivatives of f^{-1}, this implies that f^{-1} is at least C^2-smooth. Now applying this same argument to F, it follows that F^{-1} is at least C^2-smooth, hence f is at least C^3-smooth. Continuing inductively, it follows that f^{-1} is C^r-smooth for every positive integer r.

This proves Lemma 2.2 for open subsets of $\mathbf{R}^n$ or $\mathbf{H}^n$. The transition to arbitrary smooth manifolds is straightforward, using local coordinate charts. □

Using Lemma 2.2, we will prove that smooth maps of $\mathbf{R}^n$ which are sufficiently close to the identity must necessarily be diffeomorphisms.

LEMMA 2.3. A diffeomorphism criterion in $\mathbf{R}^n$. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map such that*

$$\left|\frac{\partial f_i}{\partial x_j} - \delta_{ij}\right| \;\le\; \frac{1}{2n}$$

for all points in $\mathbf{R}^n$ and all i, j, where $[\delta_{ij}]$ is the identity matrix. Then f is a diffeomorphism.

Proof. Let us write $f(x) = x + a(x)$, with components $f_i(x) = x_i + a_i(x)$. These functions $a_i(x)$ evidently satisfy, $|\partial a_i/\partial x_j| \le 1/2n$, for all i and j. It will be convenient for this proof to use the *max norm* on Euclidean space, defined by the formula $\|x\|_{\max} = \max\{|x_1|, \ldots, |x_n|\}$. If we follow a broken path from x to y in $\mathbf{R}^n$, changing just one coordinate at a time, and apply the mean value theorem to each segment of this path, then we see that

$$|a_i(x) - a_i(y)| \;\le\; \sum_{j=1}^{n} \tfrac{1}{2n}|x_j - y_j| \;\le\; \sum_{j=1}^{n} \tfrac{1}{2n}\|x - y\|_{\max} \;=\; \tfrac{1}{2}\|x - y\|_{\max}$$

hence

$$\|a(x) - a(y)\|_{\max} \;\le\; \tfrac{1}{2}\|x - y\|_{\max}$$

for all $x, y \in \mathbf{R}^n$. Thus the mapping $x \mapsto a(x)$ contracts distances uniformly.

It follows that f is bijective. For if we choose any base point $z \in \mathbf{R}^n$ and set

$$g_z(x) \;=\; z - a(x)\,,$$

then clearly the map g_z from $\mathbf{R}^n$ to $\mathbf{R}^n$ also contracts distances uniformly,

$$\|g_z(x) - g_z(y)\| \;\le\; \tfrac{1}{2}\|x - y\|.$$

Hence, by a standard argument, g_z has one and only one fixed point $x = g_z(x)$. In fact, if we start with an arbitrary point $y \in \mathbf{R}^n$ and apply the map g_z over and over again, then the resulting orbit $y \mapsto g_z(y) \mapsto g_z(g_z(y)) \mapsto \ldots$ will necessarily converge to this unique fixed point x. But clearly x is a solution to the equation $f(x) = x + a(x) = z$ if and only if $g_z(x) = z - a(x)$ is equal to x. Thus the equation $f(x) = z$ has one and only one solution x.

For any $x \in \mathbf{R}^n$, the linear map $\mathcal{D}f_x : T\mathbf{R}^n_x \to T\mathbf{R}^n_{f(x)}$ also satisfies the hypothesis of Lemma 2.3. Hence the argument above also shows that each $\mathcal{D}f_x$ is bijective. Hence, by Lemma 2.2, f is a diffeomorphism. □

LEMMA 2.4. Moving a point in $\mathbf{R}^n$. *If V is a neighborhood of the point $p \in \mathbf{R}^n$, then for any $q \in \mathbf{R}^n$ which is sufficiently close to p there exists a diffeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ with $f(p) = q$ such that f is the identity outside some compact subset of V.*

Without loss of generality, we may assume that $p = 0$ and that V contains the closed unit disk. Let $\sigma : \mathbf{R} \to \mathbf{R}$ be a smooth function such that

$$\sigma(u) = \begin{cases} 1, & \text{if } \; u \le 1/2\,, \\ 0, & \text{if } \; u \ge 1\,. \end{cases}$$

(Compare Lemma A.1 in the appendix.) Let q be a point in $\mathbf{R}^n$ and $f_i(x) = x_i + \sigma(\|x\|^2)\,q_i$, where $\|x\| = \sqrt{\sum x_i^2}$ is the Euclidean norm. Then

$$\partial f_i/\partial x_j = \delta_{ij} + 2x_j\sigma'(\|x\|^2)\,q_i\,,$$

from which it follows that

$$|\partial f_i/\partial x_j - \delta_{ij}| = |2x_j\sigma'(\|x\|^2)\,q_i|\,.$$

Therefore, it suffices to take $\|q\|$ less than $1/\big(4n\max(|\sigma'(u)|)\big)$ to meet the conditions of Lemma 2.3 and conclude that f is a diffeomorphism which maps the origin to q and equals the identity outside the compact subset $\|x\| \le 1$ of V. This proves Lemma 2.4. □

LEMMA 2.5. Moving a point in a manifold. *Let M be a connected smooth manifold with interior $\mathring{M}$. Given any two points p and q in $\mathring{M}$, there exists a diffeomorphism $f : M \to M$ such that $f(p) = q$. In fact we can always choose f to be a diffeomorphism with compact support contained in $\mathring{M}$.*

That is, we can choose f so that $f(x) = x$ for x outside of some compact subset of the interior of M.

Proof of Lemma 2.5. Let p be a point of M and (U, V, h) a coordinate chart, with $p \in U$ and $h : U \xrightarrow{\cong} V \subset \mathbf{R}^n$. By Lemma 2.4, there is a neighborhood V' of $h(p)$ in V so that we can move $h(p)$ to any point of V' by a diffeomorphism ϕ which is the identity outside of a compact subset of V. Hence we can move p to any point of $h^{-1}(V')$ by a diffeomorphism $h^{-1} \circ \phi \circ h$ which is the identity outside a compact subset of U. Evidently $h^{-1} \circ \phi \circ h$ extends to a diffeomorphism of the whole manifold M.

Now define an equivalence relation on $\mathring{M}$ by setting p equivalent to q whenever there exists a diffeomorphism of M onto itself with compact support in $\mathring{M}$ that maps p to q. By the above arguments, the equivalence classes are open, and therefore closed. Since $\mathring{M}$ is connected, there can be only one equivalence class. □

LEMMA 2.6. Realizing a prescribed first derivative map at a point. *Given an isomorphism (orientation preserving if the manifold is orientable) of the tangent space TM_p of M at p onto the tangent space TM_q, the diffeomorphism of Lemma 2.5 can be chosen such that the derivative $\mathcal{D}f_p : TM_p \to TM_q$ is equal to the given isomorphism.*

Proof. First consider the case $p = q = 0 \in \mathbf{R}^n$. Let

$$f(x) = x + \sigma(\|x\|^2)L(x)$$

where $L(x)$ is a linear transformation of $\mathbf{R}^n$ into itself, and where

$$\sigma(u) = \begin{cases} 1, & \text{if } u \le 1/2, \\ 0, & \text{if } u \ge 1, \end{cases}$$

as before. Then, $\partial f_i/\partial x_j = \delta_{ij} + 2x_j\sigma'(\|x\|^2)L_i(x) + \sigma(\|x\|^2)\partial L_i/\partial x_j$. If the $\partial L_i/\partial x_j$ are sufficiently small, or in other words if the linear transformation L is sufficiently close to the zero map, then, $|\partial f_i/\partial x_j - \delta_{ij}| \le 1/2n$. Therefore f is a diffeomorphism, with $f(x) = x$ for $\|x\| \ge 1$, and

$$f(x) = x + L(x) \quad \text{for} \quad \|x\|^2 \le 1/2 .$$

Thus $f(0) = 0$, and the derivative $\mathcal{D}f_0$ is equal to $I + L$, where I is the identity map and L can be any linear map sufficiently close to zero.

This implies that the subgroup consisting of all $I + L \in \mathrm{GL}(n, \mathbf{R})$ for which there exists a diffeomorphism f of $\mathbf{R}^n$ with compact support, with $f(0) = 0$, and with $\mathcal{D}f_0 = I + L$, is open. But an open subgroup is necessarily also closed, since its complement is the union of cosets which are open. Hence this subgroup must contain the connected component of the identity.

When the manifold is orientable, Lemma 2.6 follows immediately. If the manifold is not orientable, an argument similar to the proof of Lemma 2.5 shows that there must exist a diffeomorphism with compact support that maps p onto p and reverses the local orientation. The conclusion then follows easily. □

LEMMA 2.7. Extending a given germ to a diffeomorphism. *Given a point $p \in \mathring{M}$, and a diffeomorphism f of some neighborhood of p into $\mathring{M}$ which preserves the orientation, there exists a diffeomorphism h of M onto itself which coincides with f in a smaller neighborhood of p.*

Proof. After composing f with a diffeomorphism of M, we may assume by Lemma 2.6 that $f(p) = p$ and that the derivative $\mathcal{D}f_p$ is the identity map of the tangent space T_pM. Thus, choosing a coordinate chart, it suffices to consider the case where M is a neighborhood of the origin $0 \in \mathbf{R}^n$, with $p = f(p) = 0$ and with $(\partial f_i/\partial x_j)(0) = \delta_{ij}$. Let $f(x) = x + a(x)$, with $a(0) = 0$ and $(\partial a_i/\partial x_j)(0) = 0$. Now choose $r > 0$ sufficiently small so that f is defined throughout the disk $\|x\| < 2r$, and set,

$$h(x) = x + a(x)\sigma(\|x\|^2/r^2) ,$$

with σ as above. Then h is a smooth map which extends throughout $\mathbf{R}^n$ with

$$h(x) = \begin{cases} f(x), & \text{if } \|x\| \le r/\sqrt{2}, \\ x, & \text{if } \|x\| \ge r. \end{cases}$$

Furthermore, h is a diffeomorphism if r is sufficiently small. In fact,

$$\partial h_i/\partial x_j = \delta_{ij} + 2x_j\,a(x)\,\sigma'(\|x\|^2/r^2)/r^2 + (\partial a_i/\partial x_j)\,\sigma ,$$

and as $r \to 0$ we have, $\|a\|/r \to 0$ and $\partial a_i/\partial x_j \to 0$ for all $\|x\| \le r$, with σ, σ' and $|x_j|/r$ bounded. The conclusion follows, using Lemma 2.3. □

LEMMA 2.8. Extending disk embeddings. *Any smooth embedding $f : D^n \to \mathring{M}$ of the closed unit n-disk into the interior of a smooth n-manifold can be extended to an embedding $\hat{f} : D^n(1+\varepsilon) \to M$ of a slightly larger disk.*

Proof. Identifying the tangent space TD^n as usual with the product $D^n \times \mathbf{R}^n$, consider the outward unit vector field $x \mapsto (x, x/\|x\|)$ for $x \in D^n \setminus \{0\}$. Note that the associated differential equation $dx/dt = x/\|x\|$ has a unique integral curve

$$x(t) = tu \quad \text{for} \quad 0 < t \le 1$$

which meets the boundary $\partial D^n = S^{n-1}$ at a specified unit vector u. Using the diffeomorphism $\mathcal{D}f : TD^n \to T(f(D^n))$, we obtain a corresponding vector field $y \mapsto w(y) \in TM_y$ for y belonging to the image disk $f(D^n) \subset M$. Thus the differential equation $dy/dt = w(y)$ has general solution $t \mapsto y = f(tu)$ for $0 < t \le 1$, where u can be any point in the sphere S^{n-1}. Using a partition of unity, it is not difficult to extend the vector field w throughout some neighborhood of $f(D^n)$. (Compare Lemma 2.14 of the Munkres Lecture Notes on page 162.) The solution curves then extend also, so as to be defined say for $0 < t < 1 + \varepsilon'$. There is then a unique extension $\hat{f}$ of f to the open $(1+\varepsilon')$-disk so that these extended solution curves are given by $t \mapsto y = \hat{f}(tu)$. Using Lemma 2.2, it follows that $\hat{f}$ is a diffeomorphism on some sufficiently small neighborhood of D^n, say the neighborhood $D^n(1+\varepsilon)$ of radius $1 + \varepsilon$. □

Proof of Theorem 2.1. Recall that f and g are embeddings of the closed disk D^n into the interior of M. Choose extensions $\hat{f}$ and $\hat{g}$ as in Lemma 2.8. By Lemma 2.7, after composing one of these two embeddings with a diffeomorphism of M, we may assume that $\hat{f}(x) = \hat{g}(x)$ for $\|x\| \le \varepsilon$, provided that ε is small enough. Now let $\sigma_\varepsilon : \mathbf{R} \to \mathbf{R}$ be a monotone smooth function such that

$$\sigma_\varepsilon(r) = \begin{cases} \varepsilon, & \text{if } r \le 1, \\ 1, & \text{if } r \ge 1 + \varepsilon/2, \end{cases}$$

and consider the diffeomorphism $S_\varepsilon : \mathbf{R}^n \to \mathbf{R}^n$ defined by $S_\varepsilon(x) = x\sigma_\varepsilon(\|x\|)$. Making use of S_ε, we can define an auxiliary diffeomorphism $F : M \to M$ by the formula

$$F(p) = \begin{cases} p, & \text{if } p \notin \hat{f}(D^n(1+\varepsilon/2)), \\ \hat{f} \circ S_\varepsilon \circ \hat{f}^{-1}(p), & \text{if } p \in \hat{f}(D^n(1+\varepsilon)). \end{cases}$$

Then, $F \circ f(x) = \hat{f} \circ S_\varepsilon(x) = f(\varepsilon x) = g(\varepsilon x)$, for all $x \in D^n$. Similarly define a diffeomorphism G of M so that $G \circ g(x) = g(\varepsilon x)$ for $x \in D^n$. Then applying G^{-1} to both sides of the equation $F \circ f = G \circ g$, we see that $h = G^{-1} \circ F$ is the required diffeomorphism of M, with $h \circ f = g$. □

To conclude this section, we discuss a way of constructing new manifolds out of old by pasting together diffeomorphic boundary components. Let $(M, S(M))$ be a smooth manifold with non-vacuous boundary.

Definition. An open neighborhood $U \subset M$ of ∂M is called a *collar neighborhood* of ∂M if $(U, S(U))$ is diffeomorphic to the product $(\partial M \times [0,1),\ S(\partial M \times [0,1)))$. Whenever this is so, the diffeomorphism can be chosen in such a way that each point p in $\partial M \subset U$ is mapped to the pair $(p, 0) \in \partial M \times [0,1)$.

THEOREM 2.9. Collar neighborhoods. *If the boundary ∂M is compact, then ∂M has a collar neighborhood within the manifold M.*

A proof can be outlined as follows. (Compare the discussion of tubular neighborhoods in §3 of the Munkres notes, page 168, or in §5 of the Lectures on Differentiable Structures, page 187.) Using a smooth partition of unity, one can construct a smooth Riemannian metric on M. Then any point $p \in M$ which is sufficiently close to ∂M can be joined to ∂M by a unique geodesic of shortest length. (This shortest path is necessarily smooth, and meets the boundary at right angles.) Let $f(p) \in \partial M$ be its boundary endpoint, and let $\ell(p)$ be the length of this shortest path, or in other words the Riemannian distance between p and $f(p)$. Then the correspondence $p \mapsto (f(p), \ell(p)/\epsilon)$ maps the ϵ-neighborhood of ∂M diffeomorphically onto $\partial M \times [0,1)$, provided that ϵ is sufficiently small. □

Now let $(M, S(M))$, $(N, S(N))$ be smooth manifolds, and let M_1 and N_1 be connected components of the boundaries ∂M and ∂N, respectively. We suppose that these two boundary components are compact, and that they are diffeomorphic to each other under a diffeomorphism $h : M_1 \to N_1$. Let $P = M \cup_h N$ be the identification space in which M_1 and N_1 are pasted together under h. In other words, let

$$P = M \cup_h N = (M \uplus N)/r$$

where the equivalence relation r is such that each $p \in M_1$ is equivalent to $h(p) \in N_1$, but otherwise points are equivalent only to themselves.

COROLLARY 2.10. Pasting together boundaries. *Under these conditions, there exists a smoothness structure $S(P)$ for P such that $(P, S(P))$ is a smooth manifold and contains both $(M, S(M))$ and $(N, S(N))$ as smooth submanifolds.*

Proof. We need only specify the smoothness structure $S_x(P)$ at the points of the identified boundary. Choose collar neighborhoods $U \cong M_1 \times [0,1)$ and $U' \cong N_1 \times (-1, 0]$ of M_1 and N_1 respectively. Then the union $U \cup_h U'$ can clearly be given a smoothness structure so that it is diffeomorphic to $M_1 \times (-1, 1)$. Further details are straightforward. □

Remark. If we are given explicit collar neighborhoods with explicit diffeomorphisms to $M_1 \times [0,1)$ and $N_1 \times (-1, 0]$, then this smoothness structure is unique. In fact, with a little more work, one show that collar neighborhoods with prescribed product structure are unique up to diffeomorphism of the ambient manifold. Hence the resulting smooth manifold $(P, S(P))$ is also unique up to diffeomorphism.

3. Connected sums and the group $\mathcal{A}_n$ of invertible exotic spheres.

Let M and N be smooth n-dimensional manifolds which are connected and oriented. Then the connected sum $M \# N$ is a new smooth manifold, well defined up to orientation preserving diffeomorphism. Intuitively, it can be constructed by cutting a small disk out of each manifold and then pasting together the resulting boundaries. However, to keep control of orientation and smoothness structure, we proceed more carefully as follows. Let $f : D^n \to \mathring{M}$, $g : D^n \to \mathring{N}$ be smooth embeddings such that f preserves the orientation and g reverses it. Define an orientation preserving diffeomorphism $h : f(\mathring{D}^n \setminus 0) \to g(\mathring{D}^n \setminus 0)$ by

$$h\big(f(ru)\big) = g\big((1-r)u\big)$$

for every unit vector $u \in S^{n-1} = \partial D^n$ and for every $0 < r < 1$. Then, by definition, the *connected sum*

$$M \# N = \Big(M \setminus f(0)\Big) \cup_h \Big(N \setminus g(0)\Big)$$

is obtained by removing one point each from M and N and then pasting together the neighborhoods $f(\mathring{D}^n \setminus 0)$ and $g(\mathring{D}^n \setminus 0)$ of these points under h.

Evidently $M \# N$ is also a smooth oriented n-dimensional manifold, compact if both M and N are compact, and connected provided that $n \geq 2$. It follows easily from Theorem 2.1 that this sum is well-defined up to orientation preserving diffeomorphism.

LEMMA 3.1. Properties of connected sums. *If M, N, P are connected, oriented, smooth manifolds of dimension $n \geq 2$, then:*

(1) $M \# S^n \cong M$,
(2) $M \# N \cong N \# M$,
(3) $M \# (N \# P) \cong (M \# N) \# P$,
(4) $M \# \mathbf{R}^n \cong M \setminus (point)$, *and similarly*
(5) $M \# D^n \cong M \setminus f(\mathring{D}^n)$, *where $f(D^n)$ is a smoothly embedded disk. Furthermore:*
(6) *The connected sum of a surface of genus g and a surface of genus h is diffeomorphic to a surface of genus $g + h$.*

Here $\cong$ stands for the relation of orientation preserving diffeomorphism, and S^n stands for the unit sphere in $\mathbf{R}^{n+1}$ with its standard smoothness structure.

Proof. The proof is not difficult, and will be left to the reader. □

It follows that the set of all oriented diffeomorphism classes of smooth oriented n-manifolds forms a *commutative monoid*, that is, a commutative associative semigroup with 2-sided identity element. We will be particularly interested in the submonoid made up out of compact manifolds without boundary:

Definition. Let $\mathfrak{M}_n$ be the set of equivalence classes, under orientation preserving diffeomorphism, of connected, orientable smooth n-manifolds which are compact and without boundary. Evidently the connected sum operation makes

$\mathfrak{M}_n$ into a commutative monoid with the equivalence class of S^n as identity element, for $n \geq 1$. (For compact manifolds without boundary, even the case $n = 1$ makes sense, although it is not very interesting.) As examples:

$\mathfrak{M}_1$ has only the identity element.

$\mathfrak{M}_2$ is a free monoid on one generator, namely the surface $S^1 \times S^1$ of genus one.

$\mathfrak{M}_3$ is a free commutative monoid with infinitely many generators. (See KNESER [1929], MILNOR [1962]. These papers consider only piecewise-linear manifolds. However, in dimensions ≤ 3, it follows from MUNKRES [1960a, 1964] and HIRSCH [1963], together with SMALE [1959b], that there is a one-to-one correspondence between smooth manifolds up to diffeomorphism and piecewise-linear manifolds up to piecewise-linear homeomorphism. Furthermore, MOISE [1952] showed that topological manifolds have an essentially unique piecewise-linear structure, in dimensions ≤ 3.)

However, $\mathfrak{M}_4$ is not free. Indeed, let $\mathbf{CP}^2$ be the oriented complex projective plane and let $\overline{\mathbf{CP}}^2$ be the complex projective plane with opposite orientation. Then

$$(\mathbf{CP}^2\#\overline{\mathbf{CP}}^2)\#\overline{\mathbf{CP}}^2 \cong (S^2 \times S^2)\#\overline{\mathbf{CP}}^2 , \quad \text{although} \quad \mathbf{CP}^2\#\overline{\mathbf{CP}}^2 \not\cong S^2 \times S^2 .$$

Thus the cancellation law does not hold in $\mathfrak{M}_4$, so this monoid is certainly not free.

Here is an outline proof of these statements. If M is any complex manifold of complex dimension 2, then the connected sum $M\#\overline{\mathbf{CP}}^2$ can be identified with the manifold obtained by "blowing up" a point in M. (Compare MCDUFF AND SALAMON [1995 p. 216].) On the other hand, identifying S^2 with $\mathbf{CP}^1$, the birational correspondence, $(1:z) \times (1:w) \leftrightarrow (1:z:w)$, between $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2$ gives rise to an analytic isomorphism between $\mathbf{CP}^1 \times \mathbf{CP}^1$ with one point $(0:1)\times(0:1)$ blown up and $\mathbf{CP}^2$ with two points $(0:1:0)$ and $(0:0:1)$ blown up. (Compare GRIFFITHS AND HARRIS [1978, pp. 478–480].) However, the two manifolds $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$ and $S^2 \times S^2$ are not diffeomorphic; in fact their cohomology rings are different.

We will be particularly interested in elements of $\mathfrak{M}_n$ which admit an inverse under the connected sum operation.

Definition. Let $\mathcal{A}_n$ be the subgroup of $\mathfrak{M}_n$ consisting of all invertible elements. As examples, it follows from the discussion above that $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ contain only the identity element. On the other hand, we will see that $\mathcal{A}_7$ is non-trivial.

THEOREM 3.2. (Mazur). #-Invertible manifolds are topological spheres. *If M and N are smooth connected n-manifolds, and if $M\#N \cong S^n$, then M as a topological space is homeomorphic to S^n.*

COROLLARY 3.3. *If the group $\mathcal{A}_n$ is non-trivial, then S^n admits two or more essentially distinct differentiable structures.*

In order to prove Theorem 3.2 we introduce the concept of an infinite connected sum.

Definition 3.4. Let $M_1, M_2, \ldots$ be smooth connected n-manifolds and let

$$f_1 : D^n \longrightarrow M_1 ,$$
$$f_i, g_i : D^n \longrightarrow M_i , \qquad i = 2, 3, \ldots$$

be smooth embeddings such that the f_i preserve orientation and the g_i reverse it, with $f_i(D^n) \cap g_i(D^n) = \emptyset$. Consider the disjoint union

$$(1) \qquad (M_1 \setminus f_1(0)) \uplus \biguplus_{i\geq 2} (M_i \setminus \{f_i(0), g_i(0)\})$$

and the equivalence relation,

$$f_i(ru) \sim g_{i+1}((1-r)u) , \quad \text{for} \quad \|u\| = 1,\ 0 < r < 1,\ i = 1, 2, \ldots$$

The quotient of the disjoint union of Equation (1) with respect to the equivalence relation $\sim$ is by definition the infinite connected sum, $M_1 \# M_2 \# M_3 \# \cdots$.

One then shows, as for finite connected sums, that the infinite sum is well-defined and associative.

Example 3.5. It is easy to see that, $S^n \# S^n \# \cdots \cong \mathbf{R}^n$.

We can now give a proof of the theorem.

Proof of Theorem 3.2. If $M\#N \cong S^n$ then we have,

$$\mathbf{R}^n \cong (M\#N)\#(M\#N)\#\cdots \cong M\#(N\#M)\#\cdots \cong M\#\mathbf{R}^n \cong M \setminus \text{point},$$

which implies that M is homeomorphic to the n-sphere. □

The preceding theorem is contained in the following.

THEOREM 3.6. Characterization of #-invertible manifolds. *If M is an oriented, compact and connected smooth manifold without boundary, then the following conditions are equivalent:*

(1) *$M\#N \cong S^n$ for some N.*
(2) *$M \setminus \{point\} \cong \mathbf{R}^n$.*
(3) *$M = U_1 \cup U_2$ with U_1, U_2 open in M and both diffeomorphic to $\mathbf{R}^n$.*
(4) *If $f(D^n) \subset M$ is an embedded disk, then the complement $M \setminus f(D^n)$ can be smoothly embedded in S^n.*

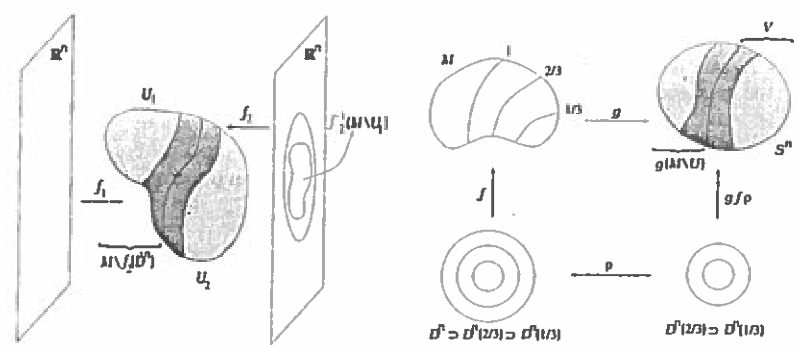


FIG. 2. *On the right: Proof that (3) ⇒ (4). On the left: Proof that (4) ⇒ (1).*

Proof. The proof of Mazur's theorem shows that (1) implies (2). To see that (2) implies (3) it is enough to take $U_i = M \setminus p_i$, with distinct points p_1 and p_2. To show that (3) implies (4), first note that $M \setminus U_1$ is compact and is contained in U_2. Let $f_i : \mathbf{R}^n \xrightarrow{\cong} U_i \subset M$. Then $f_2^{-1}(M \setminus U_1)$ is a compact subset of $\mathbf{R}^n$, so it is contained in some disk $D^n(r)$ of large radius. Then $M \setminus f_2(D^n(r)) \subset U_1 \cong \mathbf{R}^n$. Consequently, $M \setminus f_2(D^n(r))$ can be embedded in $\mathbf{R}^n \subset S^n$, as required. (It is enough to prove this for just one smooth embedding of a closed n-disk, since any two such embeddings are equivalent by Theorem 2.1.)

The proof that (4) implies (1) can be outlined as follows. Intuitively, the statement that $M\#N \cong S^n$ means that there exists a smooth embedding $\phi : S^{n-1} \to S^n$, cutting the n-sphere into two regions $\hat{M}$ and $\hat{N}$ with $\phi(S^{n-1})$ as common boundary, so that if we fill the hole in $\hat{M}$ with an n-disk we obtain the manifold M, while if we fill the hole in $\hat{N}$ with an n-disk we obtain N. But condition (4) implies that the manifold $\hat{M}$ (that is, M with an open disk removed) can indeed be embedded in S^n, so that we can construct the required manifold N. To make this argument more precise, let f be a smooth embedding of the disk D^n in M. By (4), there exists a diffeomorphism g from the open set $M \setminus f(D^n(1/3))$ onto an open set $U \subset S^n$. Let $A = \mathring{D}^n(2/3) \setminus D^n(1/3)$ be the n-dimensional 'annulus' consisting of vectors $x \in D^n$ with $1/3 < \|x\| < 2/3$. We construct a new manifold N from the disjoint sum of the disk $\mathring{D}^n(2/3)$ and the open set $V = S^n \setminus g(M \setminus f(\mathring{D}^n(2/3)))$ by pasting $A \subset \mathring{D}^n(2/3)$ onto $g(f(A)) \subset V$ by the diffeomorphism $g \circ f \circ \rho$, where $\rho(ru) = (1-r)u$. Then we claim that the resulting manifold $N = V \cup_{gf\rho} \mathring{D}^n(2/3)$ has the required property that $M\#N \cong S^n$. In fact, let $i : \mathring{D}^n(2/3) \to N$ be the inclusion map. Then $M\#N$ can be obtained from the disjoint sum of $M \setminus f(D^n(1/3)) \cong U$ and $N \setminus i(D^n(1/3)) \cong V$ by pasting together the subsets $f(A)$ and $i(A)$ under $i \circ \rho \circ f^{-1}$ But g on the first summand and the inclusion map $V \to S^n$ on the second induce a diffeomorphism of this identification space with S^n. □

Remarks. Except in dimension 4, we can make the sharper statement that a smooth n-dimensional manifold is homeomorphic to S^n if and only if it is invertible under $\#$. For $n \le 3$ this is true trivially, since there is only one differentiable structure on the n-sphere, up to diffeomorphism, in these dimensions. (Compare the discussion of $\mathfrak{M}_3$ above.) For the discussion of dimensions $n \ge 5$, see the next section.

4. The group $\Gamma_n \subset \mathcal{A}_n$ of twisted n-spheres, and the homomorphisms $\pi_0\mathrm{Diff}^+(S^{n-1}) \to \Gamma_n$ and $\pi_m(\mathrm{SO}_n) \otimes \pi_n(\mathrm{SO}_m) \to \pi_0\mathrm{Diff}^+(S^{m+n})$.

One particularly easy way of constructing an exotic n-sphere is to take two copies of the closed disk D^n and paste their boundaries together by some diffeomorphism

$$f : S^{n-1} \to S^{n-1} .$$

More precisely, in order to keep track of smoothness structures, let us work with the full Euclidean space $\mathbf{R}^n$ in place of the unit disk.

Definition 4.1. Let $\mathrm{Diff}^+(S^{n-1})$ be the group consisting of all orientation preserving diffeomorphisms of the $(n-1)$-sphere. Given any $f \in \mathrm{Diff}^+(S^{n-1})$ we construct a new smooth manifold

$$\Sigma(f) = \mathbf{R}^n \cup_F \mathbf{R}^n ,$$

homeomorphic to the n-sphere, by pasting together the subsets $\mathbf{R}^n \setminus \{0\}$ under the diffeomorphism

$$F(tu) = f(u)/t \quad \text{for} \quad u \in S^{n-1} \quad \text{and} \quad 0 < t < \infty .$$

This manifold $\Sigma(f)$, with the orientation coming from the first copy of $\mathbf{R}^n$, will be called a *twisted n-sphere*, with *twist f*.

Since $\Sigma(f)$ is homeomorphic to the first copy of $\mathbf{R}^n$, together with a single point at infinity, it is certainly homeomorphic to the standard n-sphere.

Definition 4.2. Two diffeomorphisms $f, g : M \to N$ are said to be *smoothly isotopic* if there is a family of diffeomorphisms $h_t : M \to N$, where t ranges over the real numbers, so that

$$h_t = f \text{ for } t \le 0$$
$$h_t = g \text{ for } t \ge 1 ,$$

and so that the correspondence

$$(t, x) \mapsto h_t(x)$$

defines a smooth mapping from $\mathbf{R} \times M$ to N.

It is easy to check that this is an equivalence relation between diffeomorphisms. In the special case where $M = N$, the set of all smooth isotopy classes of diffeomorphisms from M to itself will be denoted by $\pi_0\mathrm{Diff}(M)$. The composition of diffeomorphisms defines a group structure in the set $\mathrm{Diff}(M)$ of all diffeomorphisms, and hence in $\pi_0\mathrm{Diff}(M)$. This is a non-abelian group, in general. (As an example, $\pi_0\mathrm{Diff}(S^1 \times S^1)$ can be identified with the group $\mathrm{GL}(2, \mathbf{Z})$ of 2×2 invertible matrices.) However:

LEMMA 4.3. Commutativity. *The group $\pi_0\mathrm{Diff}^+(S^n)$ of isotopy classes of orientation preserving diffeomorphisms of the n-sphere is abelian.*

Proof. We will make use of Theorem 2.1 in a slightly sharper form: Any two smooth orientation preserving embeddings of the n-disk in a connected n-manifold are equivalent under a diffeomorphism of the manifold which is isotopic to the identity. In fact the proof in §2 can easily be modified to provide this sharper statement. Now let f_1, f_2 be any two elements of $\mathrm{Diff}^+(S^n)$. Denote by D^n_+ and D^n_- the upper and lower hemispheres of S^n, respectively. There exists h_1 and h_2, both isotopic to the identity, such that $h_1 \circ f_1 |_{D^n_-}$ is the identity map of D^n_- and $h_2 \circ f_2 |_{D^n_+}$ is the identity map of D^n_+. Then clearly $h_1 \circ f_1$ commutes with $h_2 \circ f_2$. Therefore $f_1 \circ f_2$ is isotopic to $f_2 \circ f_1$. □

LEMMA 4.4. Twisted spheres. *The correspondence* $(f) \mapsto \Sigma(f)$ *gives rise to a homomorphism*

$$\pi_0\mathrm{Diff}^+(S^{n-1}) \;\to\; \mathcal{A}_n$$

from the commutative group of isotopy classes of orientation preserving diffeomorphisms of S^{n-1} *into the commutative group* $\mathcal{A}_n$ *of invertible exotic n-spheres.*

By definition, the image of this homomorphism will be called the group $\Gamma_n \subset \mathcal{A}_n$ of *twisted n-spheres.*

Proof of Lemma 4.4. We must first show that if f is isotopic to g, or in other words if $g^{-1}\circ f$ is isotopic to the identity, then $\Sigma(f)$ is diffeomorphic to $\Sigma(g)$. Let $\{h_t\}$ be a smooth family of diffeomorphisms $h_t : S^{n-1} \to S^{n-1}$, where we may assume that h_t is the identity map for $t \le 1$ and that $h_t = g^{-1}\circ f$ for $t \ge 2$. Then the required diffeomorphism from $\Sigma(f) = \mathbf{R}^n \cup_F \mathbf{R}^n$ to $\Sigma(g) = \mathbf{R}^n \cup_G \mathbf{R}^n$ is given by the formula

$$tu \mapsto t\,h_t(u)$$

on the first copy of $\mathbf{R}^n$, and by

$$f(u)/t \;\mapsto\; g\circ h_t(u)/t$$

on the second. It is easy to check that this expression is compatible with the identifications, and defines the required diffeomorphism.

Now we must prove that

$$\Sigma(f)\#\Sigma(g) \;\cong\; \Sigma(g\circ f)\,.$$

In fact if $i_1, i_2 : \mathbf{R}^n \to \Sigma(f)$ are the embeddings of the two copies of $\mathbf{R}^n$ into $\Sigma(f)$, and if $j_1, j_2 : \mathbf{R}^n \to \Sigma(g)$ are the corresponding embeddings for $\Sigma(g)$, then we can then form a manifold diffeomorphic to the connected sum from the disjoint union

$$(\Sigma(f) \smallsetminus \{i_2(0)\}) \;\uplus\; (\Sigma(g) \smallsetminus \{j_1(0)\})$$

by identifying the subsets $i_2(\mathbf{R}^n \smallsetminus \{0\})$ and $j_1(\mathbf{R}^n \smallsetminus \{0\})$ under the correspondence $tu \leftrightarrow u/t$. (This is a slight variation on our construction of the connected sum, but is easily seen to be equivalent to it. Compare Theorem 2.9 and its Corollary.) In other words, we form the disjoint sum $i_1(\mathbf{R}^n) \uplus j_2(\mathbf{R}^n)$ and then identify the complements of the origin under the composition

$$tu \;\mapsto\; f(u)/t \;\mapsto t\,f(u) \;\mapsto g(f(u))/t\,.$$

But this is just the required manifold $\Sigma(g\circ f)$. Thus our correspondence

$$\pi_0\mathrm{Diff}^+(S^{n-1}) \;\to\; \mathcal{A}_n$$

is a well defined group homomorphism. Evidently the image is just the group of oriented diffeomorphism classes of twisted spheres. □

Remarks. There are several other important characterizations of twisted spheres. An argument due to REEB [1952] shows that a closed n-manifold is a twisted sphere if and only if it admits a Morse function with only two critical points. According to THOM [1959], the manifold M^n is a twisted sphere if and only

if a C^1-triangulation $K \to M^n$ yields a simplicial complex K which is piecewise-linearly homeomorphic to the standard piecewise-linear sphere.

In dimensions $n \ne 4$, every smooth manifold homeomorphic to S^n is a twisted sphere, so that $\Gamma_n = \mathcal{A}_n$. In fact, for $n \le 3$, as noted in the previous section, we have $\Gamma_n = \mathcal{A}_n = 0$. In dimensions ≥ 7, STALLINGS [1960] proved that any compact combinatorial manifold without boundary having the homotopy type of an n-sphere is the union of two open subsets piecewise-linearly homeomorphic to $\mathbf{R}^n$, and hence is invertible and homeomorphic to S^n. (Compare Theorem 3.6.) SMALE [1960, 1961, 1962] proved an even sharper version of this statement, showing that every smooth homotopy n-sphere is a twisted sphere if $n \ge 5$. Thus only dimension four remains open. In fact, CERF [1968] showed that $\Gamma_4 = 0$, and FREEDMAN [1982] showed that every homotopy 4-sphere is a topological 4-sphere. However, the possibility that $\mathcal{A}_4 \ne 0$, and also the possibility that there exist smoothness structures on the 4-sphere which are not #-invertible, remain open, as far as I know.

For these constructions to be useful, we must have some way of constructing non-standard diffeomorphisms of spheres. One easy construction can be described as a bilinear pairing

$$\beta : \pi_m(\mathrm{SO}_n) \otimes \pi_n(\mathrm{SO}_m) \;\to\; \pi_0\mathrm{Diff}^+(S^{m+n})\,.$$

Here SO_n is the rotation group, consisting of $n\times n$ orthogonal matrices of determinant +1, which acts linearly on the Euclidean space $\mathbf{R}^n$. (As a general reference for this material, see STEENROD [1951, p. 26].) Every element of the homotopy group $\pi_m(\mathrm{SO}_n)$ can be represented by a smooth map

$$\phi : \mathbf{R}^m \;\to\; \mathrm{SO}_n$$

with compact support. That is, we assume that $\phi(x)$ is the identity matrix I_n for $\|x\|$ sufficiently large. The set of all smooth homotopy classes (with uniform compact support) of such maps ϕ forms a group, using the composition operation $\phi_1\cdot\phi_2$ which comes from the product in SO_n:

$$(\phi_1\cdot\phi_2)(x) \;=\; \phi_1(x)\cdot\phi_2(x)\,.$$

(This is completely equivalent to the more standard definition in terms of continuous maps from the m-sphere and continuous homotopies.) The resulting group $\pi_m(\mathrm{SO}_n)$ is commutative, since we can always deform one of the two maps ϕ_i by a homotopy, so that the two have disjoint support.

Given $\phi : \mathbf{R}^m \to \mathrm{SO}_n$ and $\psi : \mathbf{R}^n \to \mathrm{SO}_m$, both smooth with compact support, we use the action of the rotation group on the corresponding Euclidean space to construct diffeomorphisms

$$\Phi,\,\Psi \;\in\; \mathrm{Diff}^+(\mathbf{R}^m\times\mathbf{R}^n)$$

as follows. For each $x\in\mathbf{R}^m$, $y\in\mathbf{R}^n$, set

$$\Phi(x,y) \;=\; (x,\,\phi(x)y)\,,\qquad \Psi(x,y) \;=\; (\psi(y)\,x,\,y)\,.$$

The commutator $\Phi\circ\Psi\circ\Phi^{-1}\circ\Psi^{-1}$ is then a diffeomorphism of $\mathbf{R}^m\times\mathbf{R}^n$. It is easy to check that this commutator has compact support, in the sense that

$$\Phi\circ\Psi\circ\Phi^{-1}\circ\Psi^{-1}(x,y) \;=\; (x,y)$$

whenever either $\|x\|$ or $\|y\|$ is sufficiently large.

Now identify $\mathbf{R}^m\times\mathbf{R}^n\cong\mathbf{R}^{m+n}$ with the complement of a point $p_0\in S^{m+n}$, under stereographic projection. Then this commutator extends uniquely to a diffeomorphism of S^{m+n} which is the identity in some neighborhood of p_0. Let

$$\beta(\phi,\psi)\;\in\;\pi_0\mathrm{Diff}^+(S^{m+n})$$

be the isotopy class of this extended commutator.

LEMMA 4.5. The pairing β. *This construction gives rise to a well defined bilinear pairing from $\pi_m(\mathrm{SO}_n)\times\pi_n(\mathrm{SO}_m)$ to the group $\pi_0\mathrm{Diff}^+(S^{m+n})$.*

Proof. Clearly the isotopy class $\beta(\phi,\psi)$ depends only on the smooth homotopy class of ϕ and ψ. To prove that

$$\beta(\phi_1\cdot\phi_2,\psi) \;=\; \beta(\phi_1,\psi)+\beta(\phi_2,\psi)$$

we simply deform ϕ_1 so that its support is contained in the disk $\{x\in\mathbf{R}^m\,;\,\|x\|<1\}$ and deform ϕ_2 so that its support is contained in $\{x\in\mathbf{R}^m\,;\,1<\|x\|<2\}$. After this modification, the two corresponding diffeomorphisms $\Phi_1\circ\Psi\circ\Phi_1^{-1}\circ\Psi^{-1}$ and $\Phi_2\circ\Psi\circ\Phi_2^{-1}\circ\Psi^{-1}$ will have disjoint support. The conclusion then follows easily. □

The final two sections will outline a proof that, for suitable choice of m and n, the composition

$$\pi_m(\mathrm{SO}_n)\otimes\pi_n(\mathrm{SO}_m)\;\to\;\pi_0\mathrm{Diff}^+(S^{m+n})\;\to\;\Gamma_{m+n+1}$$

is non-trivial.

5. The invariant $\lambda(M^{4k-1})\in\mathbf{Q}/\mathbf{Z}$.

This section will first describe the Hirzebruch Signature Theorem. For all details of proof the reader is referred to HIRZEBRUCH [1966], or to MILNOR AND STASHEFF [1974]. We then use this theorem to construct a diffeomorphism invariant λ which can be used to distinguish between different smoothness structures on the sphere S^{4k-1} for suitable k.

Let B be a commutative ring and $x_1,x_2,\ldots$ indeterminates, where by definition the indeterminate x_d is assigned degree d. Consider the ring of polynomials $\mathfrak{P}=B[x_1,x_2,\ldots]$. We define the *degree* of a monomial $x_{i_1}x_{i_2}\cdots x_{i_r}$ as $i_1+i_2+\cdots+i_r$ and we denote by $\mathfrak{P}_k$ the finitely generated B-module consisting of polynomials in $\mathfrak{P}$ which are homogeneous of degree k. Thus $\mathfrak{P}=\bigoplus\mathfrak{P}_k$ is a graded B-algebra, with $\mathfrak{P}_0=B$ and with $\mathfrak{P}_r\mathfrak{P}_s\subset\mathfrak{P}_{r+s}$.

Let $K_0, K_1, K_2,\ldots$ be a sequence of polynomials

$$K_j \;=\; K_j(x_1,\ldots,x_j)\;\in\;\mathfrak{P}_j\,,$$

with $K_0=1$. Then to every formal power series of the form

$$f(z)=1+p_1z+p_2z^2+\cdots$$

we can assign a new formal power series

$$K(f(z)) \;=\; 1+K_1(p_1)z+K_2(p_1,p_2)z^2+K_3(p_1,p_2,p_3)z^3+\cdots,$$

or in other words

$$K\Big(1+\sum_{j=1}^{\infty}p_jz^j\Big)=\sum_{j=0}^{\infty}K_j(p_1,\ldots,p_j)z^j\,.$$

We say that $\{K_j\}$ is a *multiplicative sequence* if this correspondence $f(z)\mapsto Kf(z)$ is a multiplicative homomorphism. Equivalently, setting

$$1+\sum_{i=1}^{\infty}x_i''z^i \;=\; \Big(1+\sum_{i=1}^{\infty}x_iz^i\Big)\Big(1+\sum_{i=1}^{\infty}x_i'z^i\Big)$$

where z, x_i, x_i' are indeterminates, the sequence $\{K_j\}$ is multiplicative if

$$\sum_{j=0}^{\infty}K_j(x_1'',\ldots,x_j'')z^j=\Big(\sum_{j=0}^{\infty}K_j(x_1,\ldots,x_j)z^j\Big)\Big(\sum_{j=0}^{\infty}K_j(x_1',\ldots,x_j')z^j\Big)\,.$$

LEMMA 5.1. Classification of multiplicative sequences. *Given a formal power series $\sum_{j=0}^{\infty}b_jz^j$ with coefficients in B, where $b_0=1$, there exists one and only one multiplicative sequence $\{K_j\}$ which satisfies the condition that the coefficient of x_1^j in $K_j(x_1,\ldots,x_j)$ is equal to b_j for each j, or in other words the condition that K maps the formal power series $f(z)=1+z+0+0+\cdots$ to*

$$K(1+z) \;=\; 1+b_1z+b_2z^2+\cdots\,.$$

By definition, $\sum_{j=0}^{\infty}b_jz^j$ is the *characteristic series* for the multiplicative sequence $\{K_j\}$.

Proof. The proof of this lemma is based on the identity

$$K(1+t_1z)\cdots K(1+t_nz) \;=\; K(1+\sigma_1z+\sigma_2z^2+\cdots+\sigma_nz^n)\,,$$

where the t_i are indeterminates of degree one, and where the σ_j are the elementary symmetric functions of the t_i. Details will be omitted. □

Now let us specialize to polynomials with coefficients in the field $\mathbf{Q}$ of rational numbers. To every smooth manifold M^n there are associated the Pontrjagin classes $p_j\in H^{4j}(M^n;\mathbf{Q})$ of its tangent bundle. Thus to any multiplicative sequence $\{K_j\}$ with rational coefficients we can associate cohomology classes

$$K_j(p_1,\ldots,p_j)\;\in\;H^{4j}(M^n;\mathbf{Q})\,.$$

In particular, if M^n is compact and oriented, without boundary, of dimension $n=4k$, then we can integrate the cohomology class $K_k(p_1,\ldots p_k)$ over the manifold (or in other words evaluate it on the fundamental homology class $[M^{4k}]\in H^{4k}(M^{4k};\mathbf{Z})$) to obtain a *characteristic number* $K_k(p_1,\ldots,p_j)[M^{4k}]$, or briefly

$$K[M^{4k}]\;\in\;\mathbf{Q}\,.$$

We set $K[M^n] = 0$ if n is not of the form $4k$. The multiplicative property of the sequence $\{K_j\}$ yields the identity

$$K[M^m \times N^n] = K[M^m] \cdot K[N^n] .$$

Now recall the theory of *cobordism* as defined by Thom [1954]. Two smooth closed oriented manifolds M^n and N^n are *oriented cobordant* if there is a smooth compact oriented $(n+1)$-dimensional manifold whose boundary consists of M^n with its given orientation, together with N^n with the opposite of its given orientation. Thom showed that the collection Ω_* consisting of all cobordism classes of closed oriented manifolds forms a graded ring which, up to torsion, is a polynomial ring

$$\Omega_* \otimes \mathbf{Q} \cong \mathbf{Q}[\mathbf{CP}^2, \mathbf{CP}^4, \mathbf{CP}^6, \ldots] ,$$

generated by the complex projective spaces $\mathbf{CP}^{2k}$ of real dimension $4k$. Furthermore, he showed that a manifold M^{4k} represents the zero element of $\Omega_{4k} \otimes \mathbf{Q}$ if and only if all of its Pontrjagin numbers $p_{i_1} \cdots p_{i_r}[M^{4k}]$ are zero.

One important topological invariant of a closed oriented $4k$-manifold is the *signature*. (The term 'index' is also used in the literature.) The definition follows. Given any basis $a_1, \ldots, a_\beta$ for the middle dimensional cohomology $H^{2k}(M^{4k}; \mathbf{Q})$, the cup products $a_i a_j \in H^{4k}(M^{4k}; \mathbf{Q})$ give rise to a symmetric matrix

$$a_i\, a_j\, [M^{4k}]$$

of rational numbers. Choosing the basis so that this matrix is diagonal, the sum of the signs of the diagonal entries $a_i^2[M^{4k}]$ is defined to be the *signature* $\mathrm{sgn}[M^{4k}] \in \mathbf{Z}$. (This choice of notation is supposed to suggest that the signature has properties quite similar to those of a characteristic number.) If the dimension n is not of the form $4k$, then we define $\mathrm{sgn}[M^n]$ to be zero. Thom showed that this signature is a cobordism invariant, with

$$\mathrm{sgn}[M^m \times N^n] = \mathrm{sgn}[M^m] \cdot \mathrm{sgn}[N^n] .$$

He then concluded that the signature can be expressed as a rational linear combination of Pontrjagin numbers. More precisely, following Hirzebruch [1966]:

THEOREM 5.2. The signature formula. *There is one and only one multiplicative sequence $\{L_j\}$ with rational coefficients which satisfies the identity*

$$L[\mathbf{CP}^{2k}] = 1$$

for every complex projective space $\mathbf{CP}^{2k}$. It follows that

$$L[M^n] = \mathrm{sgn}[M^n]$$

for every smooth closed oriented manifold. The associated characteristic series is given by the formula

$$L(1+z) = \frac{\sqrt{z}}{\tanh \sqrt{z}} = 1 + \frac{z}{3} - \frac{z^2}{45} + \frac{2z}{945} - + \cdots = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k z^k ,$$

where $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42, \ldots$ are Bernoulli numbers.

For example

$$L_1(p_1) = \frac{p_1}{3}, \quad L_2(p_1, p_2) = \frac{7p_2 - p_1^2}{45}, \quad L_3(p_1, p_2, p_3) = \frac{62p_3 - 13p_1p_2 + 2p_1^3}{945}, \ldots .$$

The proof will be omitted.

Since the polynomial L_k is homogeneous of degree k, we can express it as a sum

$$L_k(p_1, \ldots, p_k) = L_k(p_1, \ldots, p_{k-1}, 0) + s_k p_k ,$$

where the coefficients

$$s_1 = 1/3, \quad s_2 = 7/45, \quad s_3 = 62/945, \ \ldots, \ s_n = 2^{2n}(2^{2n-1} - 1)B_n/(2n)!$$

are certain non-zero rational numbers. From this we obtain the following.

COROLLARY 5.3. *The expression*

$$\frac{\mathrm{sgn}[M] - L_k(p_1, \ldots, p_{k-1}, 0)[M]}{s_k}$$

is an integer, equal to the Pontrjagin number $p_k[M]$.

Now let W be a compact, connected smooth manifold of dimension $4k$ with boundary $\partial W = M$. The cohomology sequence of the pair (W, M) takes the form

$$\cdots \longrightarrow H^{i-1}(M) \xrightarrow{\delta} H^i(W, M) \xrightarrow{j} H^i(W) \xrightarrow{r} H^i(M) \xrightarrow{\delta} \cdots ,$$

where rational coefficients are to be understood. If M has the rational cohomology of a sphere, that is if $H^i(M) \cong H^i(S^{4k-1})$, then the homomorphism j is an isomorphism for $0 < i < 4k$, and δ is an isomorphism for $i = 4k$. For $0 < i < k$, it follows that the Pontrjagin class $p_i \in H^{4i}(W)$ pulls back to a well defined class $j^{-1}p_i \in H^{4i}(W, M)$.

The orientation of W determines a homomorphism $[W] : H^{4k}(W, M) \to \mathbf{Q}$, still using rational coefficients. We define $\mathrm{sgn}[W]$ to be the signature of the bilinear form determined by the composition

$$H^{2k}(W, M) \otimes H^{2k}(W, M) \xrightarrow{\cup} H^{4k}(W, M) \xrightarrow{[W]} \mathbf{Q} .$$

Then we can define a rational number $\Lambda(W, M)$ by the formula

$$\Lambda(W, M) = \frac{\mathrm{sgn}[W] - L_k(j^{-1}p_1, \ldots, j^{-1}p_{k-1}, 0)[W]}{s_k} \in \mathbf{Q} .$$

As examples, for $k = 1, 2, 3$ (simplifying the notation slightly) we obtain the expressions

$$\Lambda = 3 \cdot \mathrm{sgn}[W^4], \qquad \frac{45 \cdot \mathrm{sgn} + p_1^2}{7}[W^8], \qquad \frac{945 \cdot \mathrm{sgn} - 2p_1^3 + 13p_1p_2}{62}[W^{12}]$$

respectively.

LEMMA 5.4. The invariant λ. *The residue class of this rational number $\Lambda(W, M)$ modulo $\mathbf{Z}$ depends only of the smooth oriented manifold M, and not on the particular choice of oriented manifold W with boundary M. In other words, if $\partial W_1 = \partial W_2 = M$ then*

$$\Lambda(W_1, M) \equiv \Lambda(W_2, M) \pmod{\mathbf{Z}} .$$

We now define the invariant $\lambda(M) \in \mathbf{Q}/\mathbf{Z}$ to be this common residue class $\Lambda(W, M)$ mod $\mathbf{Z}$. As an example, if M is diffeomorphic to the standard sphere under a diffeomorphism $h : M \to S^{4k-1}$, then we can attach a standard $4k$-disk to W. The resulting union $U = W \cup_h D^{4k}$ can be given a compatible smoothness structure by the Corollary to Theorem 2.9. It then follows easily from the discussion above that $\Lambda(W, M) = p_k[U] \in \mathbf{Z}$. Hence the invariant $\lambda(M) \in \mathbf{Q}/\mathbf{Z}$ is zero.

The proof that $\lambda(M)$ is independent of the choice of W proceeds as follows. Suppose that $M = \partial W_1 = \partial W_2$. Form a closed oriented manifold $U = W_1 \cup_\iota W_2$ from the disjoint sum $W_1 \uplus W_2$ by pasting together boundaries under the identity map ι of M. Here we give U the orientation from W_1, so that the embedding $W_1 \to U$ is orientation preserving but the embedding $W_2 \to U$ is orientation reversing. As above, this union U has a compatible smoothness structure.

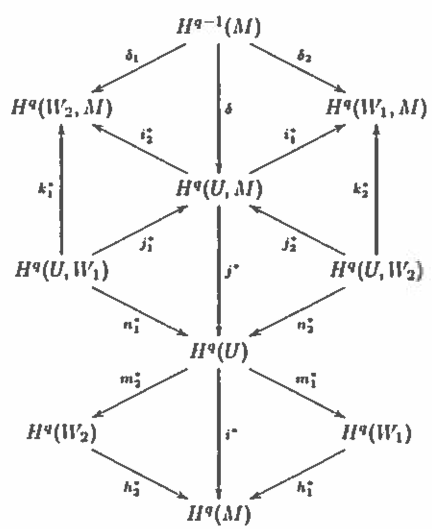
We next note that the relative cohomology ring of the pair (U, M) splits naturally as a direct sum

$$H^*(U, M) \cong H^*(W_1, M) \oplus H^*(W_2, M) .$$

In fact we have the following diagram, where the inclusion maps

$$k_1 : (W_2, M) \hookrightarrow (U, W_1), \qquad k_2 : (W_1, M) \hookrightarrow (U, W_2),$$

induce isomorphism of cohomology rings. (This is the Eilenberg-Steenrod 'excision' property.)



Here each triangle is commutative and each straight line corresponds to an exact sequence, so that

$$\begin{array}{llll} \operatorname{Im} \delta = \ker j^*, & \operatorname{Im} j^* = \ker i^*, & \\ \operatorname{Im} j_\alpha^* = \ker i_\alpha^*, & \operatorname{Im} n_\alpha^* = \ker m_\alpha^*, & \text{for } \alpha = 1, 2. \end{array}$$

It follows easily from this diagram that the homomorphisms j_α^* and i_α^* induce isomorphisms

$$H^q(U, W_1) \oplus H^q(U, W_2) \xrightarrow{\cong} H^q(U, M) \xrightarrow{\cong} H^q(W_1, M) \oplus H^q(W_2, M) ,$$

as required. Furthermore, these isomorphisms commute with cup products.

Next note that we have a commutative diagram of homomorphisms

$$\begin{array}{ccc} H^{4k}(U) \longrightarrow H^{4k}(U, M) & \xrightarrow{\cong} & H^{4k}(W_1, M) \oplus H^{4k}(W_2, M) \\ \downarrow [U] & & \downarrow [W_1] - [W_2] \\ \mathbf{Q} & \xrightarrow{=} & \mathbf{Q} \end{array}$$

where the minus sign on the right is used since W_2 and U have opposite orientations. Since

$$H^{2k}(U) \cong H^{2k}(U, M) \cong H^{2k}(W_1, M) \oplus H^{2k}(W_2, M) ,$$

it follows easily that

$$\operatorname{sgn}[U] = \operatorname{sgn}[W_1] - \operatorname{sgn}[W_2] .$$

On the other hand, for every $0 < i < k$ it follows from naturality properties of the Pontrjagin class p_i that the isomorphism

$$H^{4i}(U) \xrightarrow{\cong} H^{4i}(W_1) \oplus H^{4i}(W_2)$$

carries $p_i(U)$ to $p_i(W_1) \oplus p_i(W_2)$. From this, it follows easily that the Pontrjagin numbers of the union U are given by the formula

$$p_{i_1} \cdots p_{i_r}[U] = p_{i_1} \cdots p_{i_r}[W_1] - p_{i_1} \cdots p_{i_r}[W_2] ,$$

provided that $i_1, \ldots, i_r < k$. Combining these statements, we see that the difference $\Lambda(W_1, M) - \Lambda(W_2, M)$ is equal to

$$\frac{\operatorname{sgn}[U] - L_k(p_1, \ldots, p_{k-1}, 0)[U]}{s_k} = p_k[U] .$$

Hence this difference is an integer, which completes the proof of Lemma 5.4. □

This invariant $\lambda(M^{4k-1}) \in \mathbf{Q}/\mathbf{Z}$ is defined for any smooth oriented $(4k-1)$-dimensional manifold which has the rational cohomology $H^*(M^{4k-1}; \mathbf{Q})$ of a sphere, and which represents the zero element of the cobordism group Ω_{4k-1}. In fact we can clarify these conditions by noting the following result.

Assertion 5.5. *If a closed oriented manifold M^n has the mod 2 cohomology of a sphere, then it is a boundary, that is it represents the zero element of Ω_n.*

Proof. In fact, such a manifold M^n must also be a rational cohomology sphere. It then follows from "On the cobordism ring Ω^* and a complex analog, I" (pages 257–273 of this volume), or from AVERBUH [1959] or NOVIKOV [1960, 1962], together with THOM [1954], that M^n can only represent a 2-torsion element in the cobordism group Ω_n; and according to WALL [1960] it must actually represent the zero element. □ In particular, the invariant $\lambda(M^{4k-1})$ is defined for any smooth

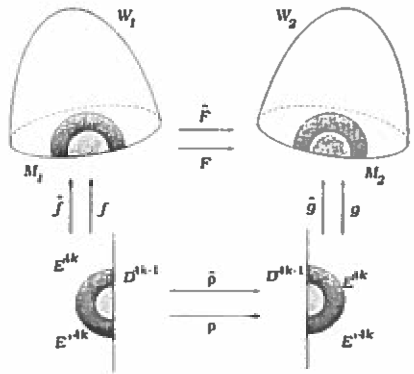


FIG. 3. *The connected sum of boundaries.*

$(4k-1)$-manifold which is a topological sphere.

LEMMA 5.6. The homomorphism λ. *This construction yields a well defined homomorphism from the group $\mathcal{A}_{4k-1}$ of #-invertible manifolds into $\mathbf{Q}/\mathbf{Z}$.*

Proof. We must show that

$$\lambda(M_1 \# M_2) \;=\; \lambda(M_1) + \lambda(M_2).$$

Consider the manifolds $M_1 = \partial W_1$, $M_2 = \partial W_2$. Let f, g be proper embeddings of D^{4k-1} in M_1 and M_2 respectively, f preserving the orientation and g reversing it; let $\rho : (\dot{D}^{4k-1} \smallsetminus 0) \to (\dot{D}^{4k-1} \smallsetminus 0)$ be defined by $\rho(ru) = (1-r)u$ and $F = g \circ \rho \circ f^{-1}$. Then with F we obtain $M_1 \# M_2$. We consider D^{4k-1} as contained in the boundary of H^{4k}. Let E^{4k}, E'^{4k} be the half-disks or radius 1 and 1/2 respectively (see Figure 3). Then ρ, f, g, F can be extended to diffeomorphisms $\hat{\rho}$, $\hat{f}$, $\hat{g}$, $\hat{F}$ respectively:

$$\begin{aligned}
\hat{\rho} :\; (\dot{E}^{4k} \smallsetminus 0) &\to (\dot{E}^{4k} \smallsetminus 0),\\
\hat{f} :\; E^{4k} &\to W_1,\\
\hat{g} :\; E^{4k} &\to W_2,\\
\hat{F} = \hat{g} \circ \hat{\rho} \circ \hat{f}^{-1} : \hat{f}(\dot{E}^{4k} \smallsetminus 0) &\to \hat{g}(\dot{E}^{4k} \smallsetminus 0).
\end{aligned}$$

Let us set

$$W = (W_1 \smallsetminus f(0)) \cup_{\hat{F}} (W_2 \smallsetminus g(0)).$$

We have then $\partial W = M_1 \# M_2$.

Let $W_1' = W_1 - \hat{f}(\dot{E}'^{4k})$, $W_2' = W_2 - \hat{g}(\dot{E}'^{4k})$. Then W_i' is diffeomorphic to W_i, and $W = W_1' \cup_{\hat{F}'} W_2'$, where $\hat{F}'$ is the restriction of $\hat{F}$ to the image under $\hat{f}$ of the boundary of E'^{4k} in H^{4k}. So we conclude that $W_1' \cap W_2'$ is homeomorphic to D^{4k-1}. Therefore it is contractible and when forming the Mayer-Vietoris sequence of the proper triad $(W; W_1', W_2')$ we obtain the isomorphisms

$$H^q(W) \xrightarrow{\;\varphi\;} H^q(W_1') \oplus H^q(W_2'), \qquad q > 0.$$

In this way we obtain the relation $\sigma(W) = \sigma(W_1) + \sigma(W_2)$, and also the analogous relation for the Pontrjagin numbers. From this it follows that $\Lambda(W) = \Lambda(W_1) + \Lambda(W_2)$, and the lemma is proved. □

6. Existence of Exotic Spheres (added in 2006).

In §4 we showed how to construct examples of twisted spheres, and in §5 we described an invariant $\lambda(M) \in \mathbf{Q}/\mathbf{Z}$ which can distinguish between different twisted spheres. However, to evaluate $\lambda(M)$ we must present this twisted sphere M as the boundary of a smooth compact manifold. This final section will tie these two chapters together by carrying out this construction. (Compare the discussion in the paper "Differentiable Structures on Spheres", on pages 35–45 of this volume. This will be referred to briefly as [DSS].)

In fact we will describe certain smooth manifolds $M(f_1, f_2)$ of the form

$$M(f_1, f_2) = \partial W(f_1, f_2),$$

where $W(f_1, f_2)$ is a smooth compact manifold of dimension $p+q$ having the homotopy type of the union $S^p \vee S^q$ of two spheres intersecting in a single point. Here, for the moment,

$$f_1 : S^{p-1} \to \mathrm{SO}_q \qquad \text{and} \qquad f_2 : S^{q-1} \to \mathrm{SO}_p$$

can be arbitrary smooth maps. To begin the construction, consider the union

$$(D_1^p \times D_1^q) \cup (D^p \times D^q) \cup (D_2^p \times D_2^q)$$

of three disjoint copies of the product $D^p \times D^q$. (This is illustrated in Figure 4 for the case $p = q = 1$. Here each dotted line is supposed to be identified with the opposite dotted line.) Now paste the two partial boundaries $S^{p-1} \times D^q$ and $S_1^{p-1} \times D_1^q$ together by the diffeomorphism

$$F_1(x, y) = (x,\; f_1(x) \cdot y),$$

where $\|x\| = 1$. Then the union $(D^p \times D^q) \cup_{F_1} (D_1^p \times D_1^q)$ can be described as the D^q-bundle over the p-sphere associated with the homotopy class $(f_1) \in \pi_{p-1}(\mathrm{SO}_q)$. (Compare STEENROD [1951].) I will use the notation $\xi(f_1)$ for this disk bundle.

We will be particularly interested in maps which satisfy any one of the following three equivalent conditions.

(1) The disk bundle $\xi(f_1)$ admits a nowhere zero cross-section.

(2) The associated vector bundle splits as the Whitney sum of an $\mathbf{R}^{q-1}$-bundle and a trivial line bundle.

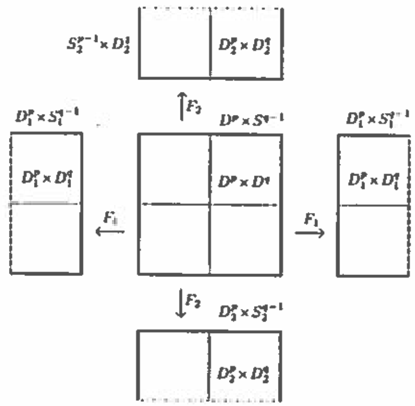


FIG. 4. *Construction of* $W(f_1, f_2)$.

(3) The map f_1 factors as a composition $S^{p-1} \to SO_{q-1} \subset SO_q$, up to homotopy.

Similarly, we can paste the two partial boundaries $D^p \times S^{q-1}$ and $D_2^p \times S_2^{q-1}$ together by the diffeomorphism

$$F_2(x, y) = \left(f_2(y) \cdot x,\ y\right),$$

where $\|y\| = 1$. Then $(D^p \times D^q) \cup_{F_2} (D_2^p \times D_2^q)$ will be a D^p-bundle $\xi(f_2)$ over the q-sphere. The union

$$(D_1^p \times D_1^q) \cup_{F_1^{-1}} (D^p \times D^q) \cup_{F_2} (D_2^p \times D_2^q)$$

will be the required manifold $W(f_1, f_2)$. Evidently the zero-sections of these two disk bundles are spheres of dimension p and q which intersect transversally at the midpoint of $D^p \times D^q$.

There is of course a technical difficulty here, in that $W(f_1, f_2)$, as described, would have a corner along the submanifold $S^{p-1} \times S^{q-1}$ of its boundary. We will assume that the differentiable structure has been modified near this corner, so as to smooth it out. (Compare the discussion in §8 of "Differentiable Manifolds which are Homotopy Spheres", on pages 65–88 of this volume.)

THEOREM 6.1. *If at least one of the two disk-bundles* $\xi(f_1)$ *and* $\xi(f_2)$ *has a nowhere zero cross-section, then the boundary*

$$M(f_1, f_2) = \partial W(f_1, f_2)$$

is a twisted sphere. If both of these bundles have nowhere zero sections, then this twisted sphere can be identified with the image of $\beta(\phi_1, \phi_2)$ *under the homomorphism*

$$\beta : \pi_{p-1}(SO_{q-1}) \otimes \pi_{q-1}(SO_{p-1}) \to \pi_0 Diff^+(S^{p+q-2}) \to \Gamma_{p+q-1}$$

of Lemma 4.5, where Γ_{p+q-1} *is the group of all twisted* $(p+q-1)$*-spheres. Here* ϕ_1 *and* ϕ_2 *are to be homotopy classes which map to* (f_1) *and* (f_2) *under the natural homomorphisms*

$$\pi_{p-1}(SO_{q-1}) \to \pi_{p-1}(SO_q) \quad \text{and} \quad \pi_{q-1}(SO_{p-1}) \to \pi_{q-1}(SO_p).$$

Proof. The first statement is an immediate consequence of [DSS, Lemma 1, p. 36]. To prove the second statement, note that the boundary $M(f_1, f_2) = \partial W(f_1, f_2)$ can be obtained from the disjoint union $(D_1^p \times S_1^{q-1}) \cup (S^{p-1} \times D^q)$ by gluing boundaries together under the diffeomorphism

$$F_2 \circ F_1^{-1} : S_1^{p-1} \times S_1^{q-1} \to S_2^{p-1} \times S_2^{q-1}.$$

However, letting F_1 act as a diffeomorphism of $S_2^{p-1} \times D_2^q$, we see that the gluing map

$$F_1 \circ F_2 \circ F_1^{-1} : S_1^{p-1} \times S_1^{q-1} \to S_2^{p-1} \times S_2^{q-1}$$

will give rise to a diffeomorphic manifold. Similarly, we can compose on the right with the diffeomorphism F_2^{-1} of $D_1^p \times S_1^{q-1}$, so as to obtain the commutator $F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1}$ as gluing map.

Now let us use the hypothesis that the image $f_1(S^{p-1})$ lies in the subgroup $SO_{q-1} \subset SO_q$, and that $f_2(S^{q-1}) \subset SO_{p-1} \subset SO_p$. Intuitively we can think of each $f_1(x)$ as a rotation which fixes the north and south poles of S^{q-1}, and each $f_2(y)$ as a rotation which fixes the poles of S^{p-1}. Furthermore, after deforming f_1 and f_2 by homotopies, we may assume that $f_1(x)$ is the identity rotation, except for x in a small disk N_1 around the north pole, and similarly that $f_2(y)$ is the identity unless $y \in N_2$. It then follows easily that

$$F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1}(x, y) = (x, y) \quad \text{unless} \quad (x, y) \in N_1 \times N_2.$$

In other words, a manifold diffeomorphic to $M(f_1, f_2)$ can be obtained from the standard $(p+q-1)$-sphere of radius $\sqrt{2}$ by cutting it open along the set $N_1 \times N_2 \subset S^{p-1} \times S^{q-1}$ and then pasting the two resulting copies of $N_1 \times N_2$ together by $F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1}$. The construction of Lemma 4.5 is almost identical, except that we cut along a bi-disk in the equator of S^{p+q-1}. Since it is easy to deform on set onto the other by an isotopy of S^{p+q-1}, this completes the proof. □

Now let us compute that invariant

$$\lambda\left(M(f_1, f_2)\right) \in \mathbf{Q}/\mathbf{Z}$$

of Lemma 5.4. It is easy to see that this invariant is zero unless both p and q are divisible by 4, so let us assume that $p = 4m$ and $q = 4n$. We will prove the following.

LEMMA 6.2. *Let* $\widehat{p}_m(f_1) \in \mathbf{Z}$ *be the integer obtained by evaluating the Pontrjagin class* $p_m(\xi(f_1))$ *on the fundamental homology class of the base space* S^{4m}*, and define* $\widehat{p}_n(f_2)$ *similarly. Suppose that at least one of the bundles* $\xi(f_1)$ *and* $\xi(f_2)$ *has a non-zero section, so that* $M(f_1, f_2)$ *is a twisted sphere. Then*

$$\lambda\left(M(f_1, f_2)\right) \equiv \pm \widehat{p}_m(f_1)\, \widehat{p}_n(f_2) s_m s_n / s_{m+n} \pmod{\mathbf{Z}},$$

where the sign depends on orientation choices.

Proof. (Compare [DSS, §3].) It is not hard to see that the signature of $W = W(f_1, f_2)$ is zero, and that the only relevant Pontrjagin number is $p_m p_n[W]$. Thus the only problem is to compute the coefficient of $p_m p_n$ in the Hirzebruch polynomial $L_{m+n}(p_1, p_2, \ldots, p_{m+n})$. To do this, consider a closed manifold N^{4n} whose only non-zero Pontrjagin number is $p_n[N^{4n}]$, so that the signature formula reduces to

$$\operatorname{sgn}(N^{4n}) = s_n\, p_n[N^{4n}].$$

If M^{4m} is an analogous manifold of dimension $4m$, then using the identity

$$\operatorname{sgn}(M \times N) = \operatorname{sgn}(M) \cdot \operatorname{sgn}(N),$$

a brief computation shows that the coefficient of $p_m p_n$ in the polynomial L_{m+n} is equal to

$$\begin{array}{lll} s_m s_n - s_{m+n} & \text{if} & m \neq n, \\ (s_m s_n - s_{m+n})/2 & \text{if} & m = n, \end{array}$$

where s_{m+n} is the coefficient of p_{m+n}. On the other hand, $p_m p_n[W]$ is equal to either $\widehat{p}_m(f_1)\,\widehat{p}_n(f_2)$ or to twice this number according as $m \neq n$ or $m = n$. The conclusion follows. □

According to Hirzebruch [1966, p. 12], the coefficient s_n is given by the formula

$$s_n = 2^{2n}(2^{2n-1} - 1)B_n/(2n)!,$$

where the B_n are Bernoulli numbers. (For relevant information about Bernoulli numbers, compare Milnor and Stasheff [1974, Appendix B].)

Here are some numerical values, expressed as quotients of products of primes.

n	1	2	3	4	5	6	7	8
B_n	$\frac{1}{2\cdot3}$	$\frac{1}{2\cdot3\cdot5}$	$\frac{1}{2\cdot3\cdot7}$	$\frac{1}{2\cdot3\cdot5}$	$\frac{5}{2\cdot3\cdot11}$	$\frac{691}{2\cdot3\cdot5\cdot7\cdot13}$	$\frac{7}{2\cdot3}$	$\frac{3617}{2\cdot3\cdot5\cdot17}$
s_n	$\frac{1}{3}$	$\frac{7}{3^2\cdot5}$	$\frac{2\cdot31}{3^3\cdot5\cdot7}$	$\frac{127}{3^3\cdot5^2\cdot7}$	$\frac{2\cdot73}{3^5\cdot5\cdot11}$	$\frac{2\cdot23\cdot89\cdot691}{3^6\cdot5^3\cdot7^2\cdot11\cdot13}$	$\frac{2^2\cdot8191}{3^6\cdot5^2\cdot7\cdot11\cdot13}$	$\frac{31\cdot151\cdot3617}{3^7\cdot5^4\cdot7\cdot11\cdot13\cdot17}$

However, in order to make use of Lemma 6.2 we need to know what values of $\widehat{p}_m(f_1)$ are possible.

LEMMA 6.3. *If $q \leq 2m$ then the homomorphism*

$$\widehat{p}_m : \pi_{4m-1}(\mathrm{SO}_q) \to \mathbf{Z}$$

is zero; but if $q > 2m$ then its image is a non-zero additive group. The generator of this group is a multiple of $(2m-1)!$, and its prime divisors are all $\leq 2m$.

In fact, if $q \geq 4m$ then Bott showed that that this image is generated by either $(2m-1)!$ or $2(2m-1)!$ according as m is even or odd. (Compare Bott and Milnor on page 229–231 of this volume.) The general case follows by using results of Serre on stable homotopy groups of spheres. For details, see [DSS, Lemma 5] on page 43. □

Below is a table showing the denominators of the quotients $s_m s_n/s_{m+n}$, expressed as fractions in lowest terms, for all m and n with $m \leq n < 2m$ and

$m + n \leq 8$. Those prime factors which are less than $2n$ are shown in parenthesis, since by Lemma 6.3 they will cancel against factors of $\widehat{p}_n(f_2)$. (In fact, all of the useful factors seem to come from the numerator of s_{m+n}.)

m	n	dimension	denom$(s_m s_n/s_{m+n})$
1	1	7	7
2	2	15	$(3)\cdot127$
2	3	19	$(5)\cdot73$
3	3	23	$23\cdot89\cdot691$
3	4	27	$(2\cdot5\cdot7)\cdot8191$
4	4	31	$(7)\cdot31\cdot151\cdot3617$
3	5	31	$(3)\cdot31\cdot151\cdot3617$

As an example, using either of the last two lines together with Lemmas 6.2 and 6.3, we see that there exist at least $31\cdot151\cdot3617 = 16931177$ distinct differentiable structures on the 31-dimensional sphere.

Appendix: Construction and Extension of Smooth Real Valued Functions.

The first two lemmas will construct certain smooth functions of one real variable.

LEMMA A.1. *There exists a C^∞-function $\psi : \mathbf{R} \to [0,1]$ such that $\psi(t) = 0$ for $|t| \geq 1$, with $\psi(0) = 1$, and with n-th derivative $\mathcal{D}^n\psi(0) = 0$ for all $n \geq 1$.*

Proof. Start with

$$\phi(t) = \begin{cases} e^{-1/t}, & \text{for} \quad t > 0 \\ 0, & \text{for} \quad t \leq 0, \end{cases}$$

the standard example of a function which is C^∞ but not real analytic. Evidently $\phi(t) > 0$ if and only if $t > 0$. It follows that $\phi(t) + \phi(1-t) > 0$ everywhere. The ratio

$$\eta(t) = \phi(1-t)/\big(\phi(t) + \phi(1-t)\big)$$

is then a C^∞-function satisfying $0 \leq \eta(t) \leq 1$, with $\eta(t) = 0$ for $t \geq 1$ and $\eta(t) = 1$ for $t \leq 0$. Setting $\psi(t) = \eta(t^2)$, we obtain a function with the required properties. □

LEMMA A.2. *Given a completely arbitrary sequence of real numbers $a_0, a_1, a_2, \ldots$ there exists a C^∞ map $f : \mathbf{R} \to \mathbf{R}$ whose n-th derivative at the origin is equal to a_n for each $n \geq 0$.*

Proof. Choose numbers $b_k \geq |a_k| + 1$, and set

$$f(t) = \sum_{k \geq 0} a_k\, \psi(b_k t)\, t^k/k!\,, \qquad (2)$$

with ψ as above. If the k-th term of this series is non-zero, then we must have $|b_k t| < 1$ hence

$$\big|\, a_k\, \psi(b_k t)\, t^k/k! \,\big| \;<\; b_k \cdot 1 \cdot (1/b_k)^k/k! \;\leq\; 1/k!$$

for $k \geq 1$. Thus the series of Equation (2) converges uniformly, hence f is a well defined continuous function. Note also that $f(t) = 0$ for $|t| \geq 1$, since $b_k \geq 1$ for all k.

It follows by induction on n that the n-th derivative of the k-th term in the series can be expressed as a sum

$$(3) \qquad \mathcal{D}^n\big(a_k\,\psi(b_k t)\,t^k/k!\big) = \sum_{j=0}^{\min(k,n)} \binom{n}{j} a_k b_k^{n-j} \psi^{(n-j)} t^{k-j}/(k-j)!\,,$$

where $\psi^{(n-j)}$ stands for the derivative $\mathcal{D}^{n-j}\psi$ evaluated at the point $b_k t$, where again we may assume that $|b_k t| \leq 1$. From this, we easily obtain an upper bound of the form

$$\big|\mathcal{D}^n\big(a_k\,\psi(b_k t)\,t^k/k!\big)\big| \leq C_n\, b_k^{1+n-k}/(k-n)!$$

for $k \geq n$, where C_n is a constant which depends only on n. Now sum over $k \geq n$ for fixed n. Since the resulting series converges uniformly for each n, it follows that f has continuous derivatives of all orders.

Finally, for the special case $t = 0$, note that the j-th term of the summation of Equation (3) is equal to a_n if $j = k = n$, and is zero otherwise. It follows that $\mathcal{D}^n f(0) = a_n$, as required. □

We now apply this lemma to study a local smooth function on the closed half-space $\mathbf{H}^n \subset \mathbf{R}^n$.

LEMMA A.3. *Let V be a neighborhood of the origin in $\mathbf{R}^n$, and suppose that $f : V \cap \mathbf{H}^n \to \mathbf{R}$ has continuous partial derivatives of all orders. Then there exists a smaller neighborhood V' of the origin in $\mathbf{R}^n$ and a smooth function $g : V' \to \mathbf{R}$ which coincides with f throughout the intersection $V' \cap \mathbf{H}^n$.*

Proof. It is convenient to identify points of $\mathbf{H}^n$ with pairs $(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R}$ where $t \geq 0$. Let $a_k(x)$ be the k-th partial derivative of $f(x, t)$ with respect to t at $t = 0$. Let D_ϵ be the closed ϵ-disk centered at the origin of $\mathbf{R}^{n-1} \times \mathbf{R}$, where ϵ is small enough so that $D_\epsilon \subset V$, and choose real numbers b_k so that $b_k \geq 1 + |a_k(x)|$ whenever $\|x\| \leq \epsilon$. For every (x, t) in D_ϵ set

$$g(x,t) = \begin{cases} f(x,t), & \text{if} \quad t \geq 0 \\ \sum_{k\geq 0} a_k(x)\,\psi(b_k t)\,t^k/k!, & \text{if} \quad t \leq 0 \end{cases}$$

Then it is not difficult to check that g is smooth throughout the interior of D_ϵ. Since g clearly coincides with f whenever both functions are defined, the conclusion follows. □

For a much more general statement about smooth extensions of real valued functions defined on subsets of Euclidean space, the reader is referred to WHITNEY [1936].

PART 3. RELATIONS WITH ALGEBRAIC TOPOLOGY

This section will consist of the following four papers:

On the parallelizability of the spheres (with R. Bott), Bulletin American Mathematical Society 64 (1958) 87–89.

Some consequences of a theorem of Bott, Annals of Mathematics 68 (1958) 444–449.

On the Whitehead homomorphism J, Bulletin American Mathematical Society 64 (1958) 79–82.

Bernoulli numbers, homotopy groups, and a theorem of Rohlin (with M. Kervaire), in "Proceedings International Congress of Mathematics 1958", Cambridge Univ. Press (1960) 454–458.

FIG. 1. Raoul Bott at the Bonn Arbeitstagung in 1969