Topics to be covered

- Milnor's book: degree theory
  $f: M^m \to N^n$

- Milnor's notes:
  Whitney Embedding Thm
  Transversality
  Vector bundles, Tubular Nbhds,
  Thom spaces
  Intersection theory

- Cobordism theory
  (Wall's Notes)

August 21

- $U \subset \mathbb{R}^k$ $\quad V \subset \mathbb{R}^\ell$ open
  $f: U \to V$ is smooth
  if C^1

- $X \subset \mathbb{R}^k$ $\quad Y \subset \mathbb{R}^\ell$
  $f: X \to Y$ is smooth

if $\forall x \in X$, $\exists$ $U \subset \mathbb{R}^k$ open
& $F: U \to Y \subset \mathbb{R}^\ell$
s.t. $F|_X = f$ & F is smooth

Def $f: X \to Y$ is diffeomorphism

Def $M \subset \mathbb{R}^k$ is smooth mfld

Ex $S^n \subset \mathbb{R}^{n+1}$

Tangent Spaces & derivatives

Def $df_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$

- Chain Rule
- $dI_x = id$

Inverse Function Thm

$x \in M \subset \mathbb{R}^k$ $\quad g: U \to M$

parametrization

$dg_x : \mathbb{R}^m \to \mathbb{R}^k$

$T_x M = \operatorname{Im}(dg_x)$

well-defined. $\dim T_x M = m$

· $f: M \to N$ def df_x

Regular values

Fundamental Thm of Alg.

Def'n of a smooth mfld

$$\begin{array}{ccc} \mathbb{R}^k & & \mathbb{R}^l \\ \cup & & \cup \\ U & \xrightarrow[f]{} & V \end{array}$$

f is <u>smooth</u> if

~~if~~, if is C^1 (eventually need C^∞)

$X \subset \mathbb{R}^k$ $\quad Y \subset \mathbb{R}^\ell$

subsets

$f : X \longrightarrow Y \subset \mathbb{R}^\ell$

f is smooth at x

if $\exists$ open $U \ni x$

& $F : U \longrightarrow \mathbb{R}^\ell$

s.t. $F|_X = f$.

<u>Def</u> $f : X \to Y$ is

a <u>diffeomorphism</u>

if it is homeomorphism

& f & f^{-1} are

both smooth

<u>Def</u> $M \subset \mathbb{R}^k$ is

smooth mfld. if $\forall x \in M$

$\exists$ open W s.t.

$W \cap M$ is diffeomorphic

to open subset U of $\mathbb{R}^m$

ψ coordinate chart

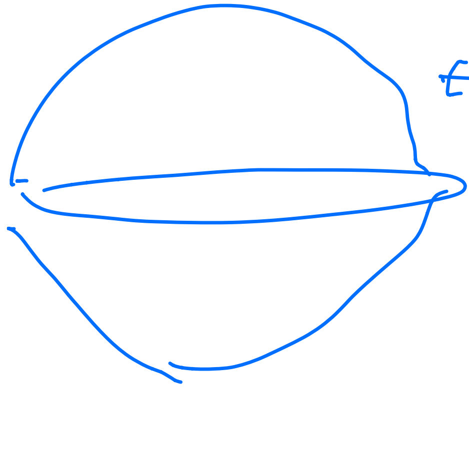
$$W \cap M \underset{g = \psi^{-1}}{\overset{\psi}{\rightleftarrows}} U$$

"parametrization"

<u>Ex</u> $S^n \subset \mathbb{R}^{n+1}$

$= \{x_1 \ldots x_{n+1} \mid \sum x_i^2 = 1\}$

$S^n \cap \overbrace{\{x \mid x_{n+1} > 0\}}^{W}$



$\leftarrow W \cap S^2$

Tangent Spaces & derivatives

$$U \xrightarrow{f} V$$

$$\cap \qquad\qquad \cap$$

$$\mathbb{R}^k \qquad \mathbb{R}^l$$

$$df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

Properties

i Chain Rule

$$\underset{u}{U} \xrightarrow{f} \underset{v}{V} \xrightarrow{g} \underset{w}{W}$$

$$d_u(g \circ f) = (d_v g)\, d_u f$$

$$I : U \to U \qquad \text{identity}$$

$$d_u I = id : \mathbb{R}^m \to \mathbb{R}^m$$

Claim $f: \underset{\mathbb{R}^k}{U} \longrightarrow \underset{\mathbb{R}^l}{V}$ (with $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$)

is diff, then

$k = l$ & df_x is iso.

$I_U = f^{-1} f \Rightarrow \underbrace{df^{-1} \circ df}_{\text{id}}$

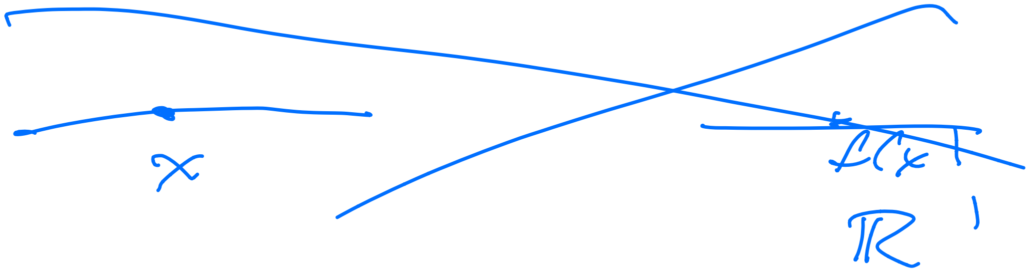
$I_V = f f^{-1}$

Inverse Function Thm

$x \in U \subset \mathbb{R}^k \quad f: U \to \mathbb{R}^l$

If df_x is nonsingular

Then f maps a nbhd of x diffeomorphically onto nbhd of $f(x)$



$$M^m \subset \mathbb{R}^k$$

<u>Def of Tangent Space</u>

$$\begin{array}{ccc} U & \xrightarrow{g} & g(U) \subset M \ \left(\cancel{= M \cap W}\right) \\ \cap & & \cup \\ \mathbb{R}^m & & \\ u & \longrightarrow & x \end{array}$$

Def

$$T_x M = \operatorname{Im}(dg_u) \subset \mathbb{R}^k$$

linear subspace of

$$T_x \mathbb{R}^k = \mathbb{R}^k$$

Well-defined

$$\begin{tikzcd} & \mathbb{R}^k & x \in M \\ U \arrow[ur, "g"] \arrow[rr, "h^{-1}\circ g"'] & & V \subset \mathbb{R}^n \arrow[ul, "h"'] \end{tikzcd}$$

diffeo

$\Longrightarrow$ $d(h^{-1}\circ g)$ is iso

$\qquad$ " $dh^{-1}\circ dg$

$\operatorname{Im} dg = \operatorname{Im} dh$

$T_x M$ is well-defined

$\dim T_x M = m$

$g^{-1} : g(U) \longrightarrow U \subset \mathbb{R}^n$

$\exists$ open $W \subset \mathbb{R}^k$

if g^{-1} extends to F on $\mathbb{R}$

$$\begin{tikzcd} & W \arrow[dr, "F"] & \\ U_0 \arrow[ur, "g"] \arrow[rr, "\text{inclusion}"'] & & \mathbb{R}^m \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dr, "dF_x"] & \\ \mathbb{R}^m \arrow[ur, "dg_u"] \arrow[rr, "id"'] & & \mathbb{R}^n \end{tikzcd}$$

$\operatorname{rank} dg = m$

$f: M^{\subset \mathbb{R}^k} \longrightarrow N^{\subset \mathbb{R}^\ell}$ smooth

Then define df_x

$df_x : T_x M \longrightarrow T_{f(x)} N$

f

$$\begin{tikzcd} x \in M \arrow[r] & N \ni f(x) \\ U \arrow[u, "g"] \arrow[r, "h^{-1}fg"] & V \arrow[u, "h"'] \end{tikzcd}$$

$h^{-1}fg : U \longrightarrow V$ smooth

$$\begin{tikzcd} T_xM \arrow[r, "df_x"] & T_{f(x)}V \\ \mathbb{R}^m \arrow[u, "dg", "\cong"'] \arrow[r, "d(h^{-1}fg)"'] & \mathbb{R}^n \arrow[u, "dh \cong"'] \end{tikzcd}$$

<u>Def</u> $f : M^m \longrightarrow N^n$

$x \in M^m$ is regular pt. if $df_x : T_xM \to T_{f(x)}N$ is iso.

$y \in N$ is a

regular value
if each $x \in f^{-1}(y)$
is a regular pt

Obs ~~M is cpt~~

y regular $\Rightarrow$ $f^{-1}(y)$ is discr

~~$\exists$~~ $x \in f^{-1}(y)$

$\exists$ open U_x which
maps diffeo ont
nhd of y.

M is compact
Then $\# f^{-1}(y)$ is finite
is ~~too~~

August 24

<u>Inverse Function Thm</u> $U' \subset \mathbb{R}^k$

$f: U' \to \mathbb{R}^k$ $\quad df_x$ on iso.

Then f maps nbhd U of x diffeo onto $f(U) \subset \mathbb{R}^k$.

<u>Cor</u> $f: M^m \to N^n$ then

$df_x : T_x M \to T_{f(x)} N$

is well-defined

<u>Def</u> of regular value
$m \geq n$.

x is a regular pt
if $df_x : TM_x \to T_{f(x)} N$
is onto. $y \in N$ in

regular value, if $f^{-1}(y)$ is only regular pts

Assume $m = n$. Then df_x an iso $\Rightarrow \exists$ nbhd U of x s.t. $f: U \to f(U) = V$ is diff.

Assume $\boxed{M \text{ is compact}}$ y reg. value. Then $\# f^{-1}(y) < \infty$

(Pf: $f^{-1}(y)$ is closed & discrete.)

Fact: $\# f^{-1}(y)$ y reg value is locally constant on $N - \{\text{crit pts}\}$

Pf y = reg value

$f^{-1}(y) = \{x_1 \dots x_k\}$

$\exists$ open nbhd U_i of x_i

$f: U_i \xrightarrow{\cong} V_i$

$V = V_1 \cap \dots \cap V_k - f(M - (U_1 \cup \dots \cup U_k))$

open nbhd of y

~~$U = f^{-1}(V)$~~ $\qquad \# f^{-1}(y) = k$

$\forall y \in V$.

Fund. Thm of Algebra

$P: \mathbb{C} \to \mathbb{C}$ a nonconstant poly. Then P has a root.

Pf ① Extend P to smooth map $f: S^2 \to S^2$

② z is critical pt

$\iff P'(z) = 0$ since

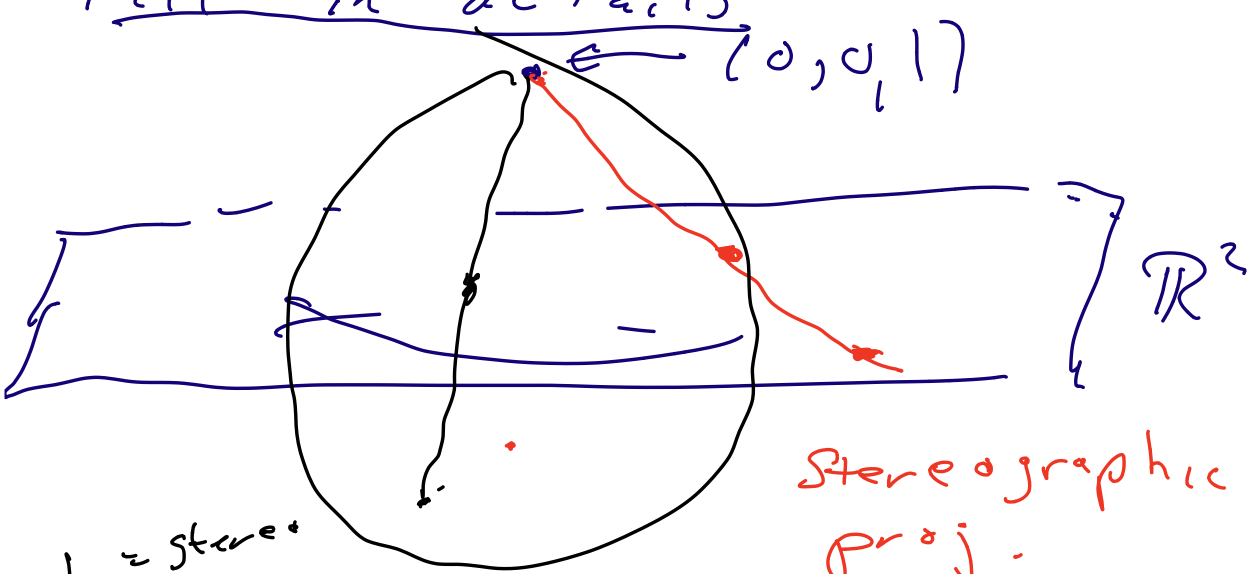
$P(z)$ not constant $P'(z)$ has only finitely many 0's $z_1, \ldots, z_k$

③ $S^2 - \{z_1 \ldots z^k\}$ is connected

so $\# f^{-1}(y)$ is constant & $\neq 0$

$\therefore f^{-1}(0) \neq \emptyset$.

Fill in details



$$S^2 - \{(0,0,1)\} \xleftarrow{h_+^{-1}} \mathbb{C} = \mathbb{R}^2 : h_+$$

S^2

$f(z)$ h_x P $h_x^{-1}(z)$ $\in S^2$

$f(0,0,1) = (x,0,1)$

Sard's Thm $f: U^m \to \mathbb{R}^n$ $\subset \mathbb{R}^m$

smooth (C^∞?) Let

$C = \{x \in U \mid df_x < n\}$

Then $f(C) \subset \mathbb{R}^n$ has measure 0.

Assume $m \geq n$

Cor Set of regular values of $f: M^m \to N^n$ is everywhere dense

Lemma 1 $f: M^m \to N^n$

$y \in N$ a regular value

Then $f^{-1}(y)$ is a smooth submanifold of dim $m-n$.

Lemma 2 $M' = f^{-1}(y)$.

$\mathcal{N}$ = null space of

$T_x M \to T_y(N)$

Then $\mathcal{N} = T_x M'$.

Def of mfld with ∂

H^m = upper half space

Each x has nbhd iso

to open subset $\mathcal{U} \subset H^m$

<u>Lemma 3</u> $g: M \to \mathbb{R}$

smooth map 0 a reg value

Then $X = \{x \in M \mid g(x) \geq 0\}$

is a manifold with ∂.

$\partial X = g^{-1}(0)$

<u>Lemma 4</u> $f: X^m \to N$, $m > n$

y a regular value for

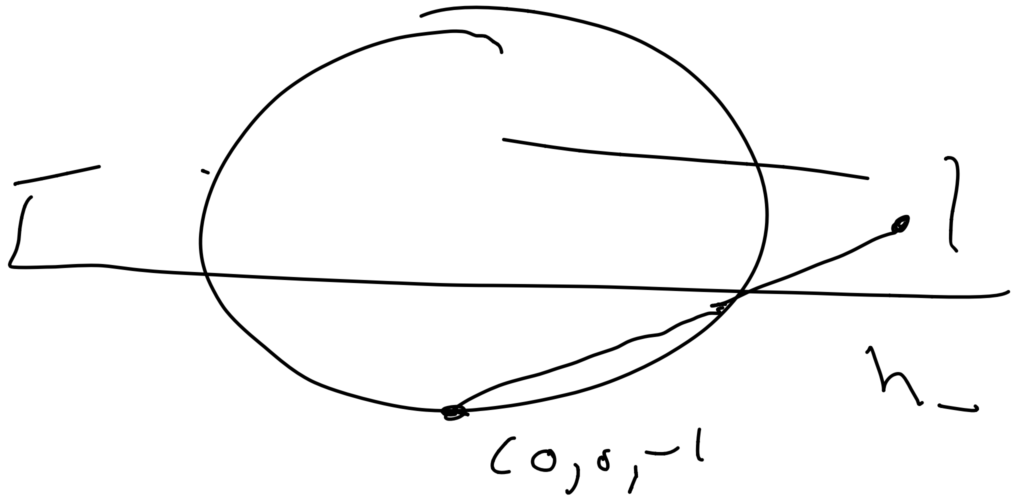
both $f|\partial X$ & f.

Then $f^{-1}(y) =$ smooth mfld with ∂.

Moreover, $\partial f^{-1}(y) = f^{-1}(y) \cap \partial X$

Proof of

Fund ~~Gen~~ Thm of Algebra Continued



$$h_- : \mathbb{C} \to S^2 - \{0,0,1)$$

$$Q(z) = h_- f \; h_-^{-1}$$

$$h_+ h_-^{-1}(z) = \frac{1}{\bar{z}}$$

$$P(z) = a_0 z^n + \ldots + a_n$$

$$Q(z) = h_- h_+^{-1} P h_+ h_-^{-1}(z)$$

$$= \frac{1}{a_0\left(\frac{1}{\bar{z}}\right)^n + \qquad + a_n}$$

$-(z)$

$$= \frac{z^n}{a_0 + \cdots + a_n z^n}$$

$P'(z)$ has finitely many 0's

So f has finitely many crit. $\# f^{-1}(y) = $ constant

so $\neq 0$

$M \subset \mathbb{R}^k$

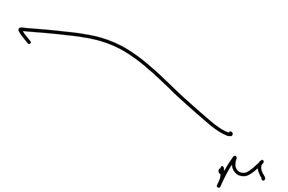
Pf of Lemma 1

$\mathcal{N} =$ null space of $df_x : T_x M \to T_y N$

Choose linear subspace

~~$L \subset \mathbb{R}^k$~~

$L \perp \mathcal{N}$ ~~is~~ no



$b \quad - m - n$

$L : \mathbb{R}^{m} \longrightarrow \mathbb{R}^{m-n}$

$L|_{M}$ is non-singular

$F : M \longrightarrow N \times \mathbb{R}^{m-n}$

$F(x) = (f(x), L(x))$

$dF_x = (df_x ; L)$

is non singular

In. Fun Thm $\Rightarrow \exists$

nbhd V of $(y, L(x))$

nbhd U of x $U \cong V$.

$\cancel{(y \times \mathbb{R}^{m-n})}$

F takes

$F : f^{-1}(y) \times U \longrightarrow$

$(y \times \mathbb{R}^{m-n}) \cap V$.

Lemma 2. $\mathcal{N} =$
$T_x(f^{-1}(y))$

Pf

$~~M^m \xrightarrow{f} N^n~~$

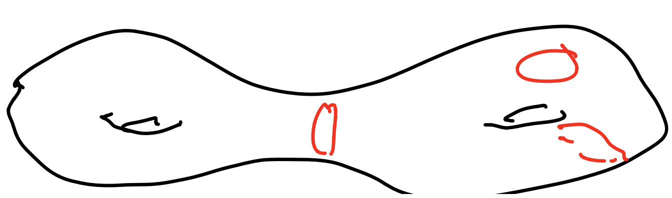
$$\begin{tikzcd} M' \arrow[r, "i"] \arrow[d, "i"] & M \arrow[d, "f"] \\ y \arrow[r, hook] & N \end{tikzcd}$$

$M' = f^{-1}(y)$

Lemma 1 (Implicit function Thm)

$f : M^m \to N^n$ $\quad y =$ reg value

$f^{-1}(y)$ is smooth submanifold of dim $m-n$



$\xrightarrow{f} S^1$

Aug 28

1. Def'n of manifold with ∂

2. Pf of Lemma 3
   Ex D^n

3. Lemma 4 (= Lemma 1 for mfld with ∂)

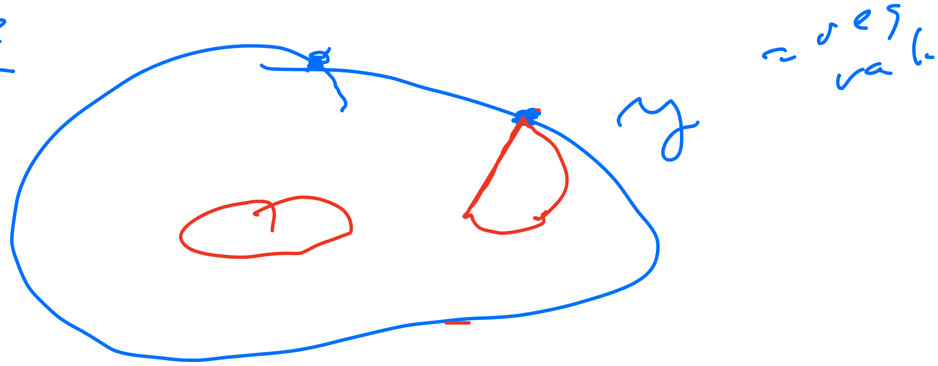
4. Statement of Sard's Thm

Brouwer Fixed Pt Thm

Lemma 5 : $\nexists$ a smooth map $f : X^m \longrightarrow \partial X$ s.t. $f|\partial X = id_{\partial X}$.

X cpt.

Picture



Lemma C (Smooth Brouwer's Fixed Pt Thm) $g : D^n \to D^n$ smooth. Then g has a fixed pt, i.e., $\exists x \in D^n$ s.t

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$H^m = \{(x_1, \dots x_n) \mid x_m \geq 0\}$

"upper half-space"

$X \subset \mathbb{R}^k$ is m-mfld with ∂. $\forall x \in X$

$\exists$ nbhd U s.t

$U \cap X \underset{\text{differ}}{\cong}$ open subset of H^m.

Follows that

$\partial X = \{ x \mid x$ corresponds to pt in $\partial H^m \}$

$= (m-1)$ dim submfld

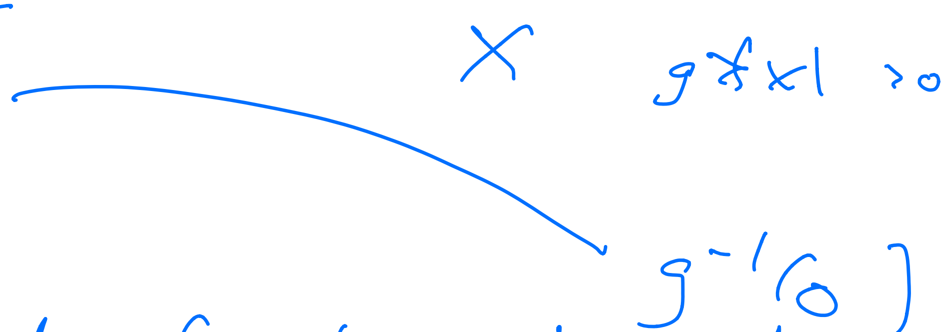
<u>Lemma</u> $g : M^m \to \mathbb{R}$

$0 \in \mathbb{R}$ regular value

$X = \{ x \in M \mid g(x) \geq 0 \}$

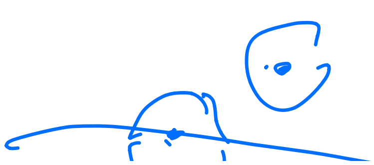
is a mfld with ∂.

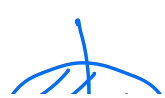
<u>Pf</u>

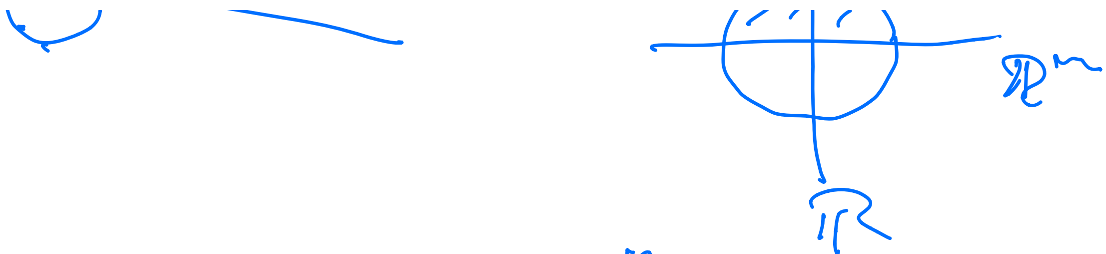


Add function $L : \mathbb{R}^n \to \mathbb{R}$

$M \to \mathbb{R} \times \mathbb{R}^{m-1}$







Ex $X = D^n =$ $\quad g : \mathbb{R}^n \to \mathbb{R}$

$g(x) = (1 - \|x\|^2)$

$g(x) \geq 0$ defines D^n

Pf of Lemma 4 $\quad f^{-1}(y)$

$=$ smfld with ∂.

Can assume $M = H^m$

Case 1 $\quad x \in f^{-1}(y) \cap (X - \partial X)$

$\qquad\qquad = H^m - \partial H^m \cong \mathbb{R}^m$

$\Rightarrow f^{-1}(y)$ is mfld near x

Case 2 $\quad \bar{x}$ is ∂ pt

$\bar{x} \in H^m \subset \mathbb{R}^m$

open nbhd U in $\mathbb{R}^m$

f extends to

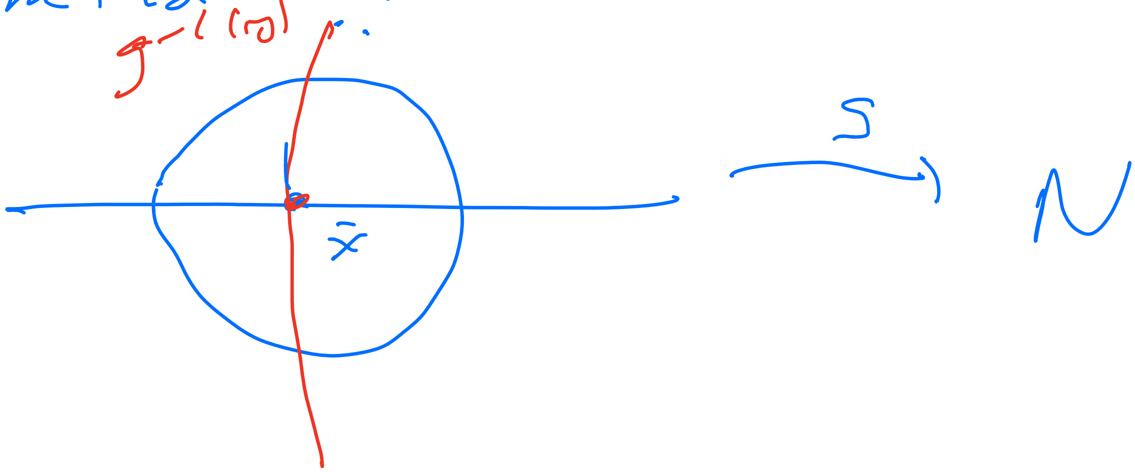
$g : \cancel{\mathbb{R}^m}\ U \longrightarrow N$

$df_{\bar{x}}$ is onto

So U by smaller nbh

dg is onto

$\therefore\ g^{-1}(y)$ is smooth mfld : $m-1$



$\pi : g^{-1}(y) \longrightarrow \mathbb{R}$

$\pi : x \longrightarrow x_m$

0 is regular value

$T_{\bar{x}}\, g^{-1}(y) = n_{\bar{x}}$

$= \ker df_x = \ker dg_x$

$f|\partial H$ has a reg pt at $\bar{x}$

So $\eta_x \notin \mathbb{R}^{m-1} = \partial H^m$
$= T_x \partial H^m$

$\therefore g^{-1}(y) \cap H^m = f^{-1}(y) \cap U$
$g(x) \geq 0$
$=$ ~~mfld with ∂~~.

by previous Lemma ?

is mfld with ∂

<u>Sard's Thm</u>

$U \subset \mathbb{R}^m \quad f : U \to \mathbb{R}^n$

which smooth (C^∞).

$C = \{x \in U \mid \operatorname{rank} df_x < n\}$ (ker above rank)

$f(C)$ has measure 0.

Cor If $m \geq n$. Then

~~{crit. values} is nowhere dense~~

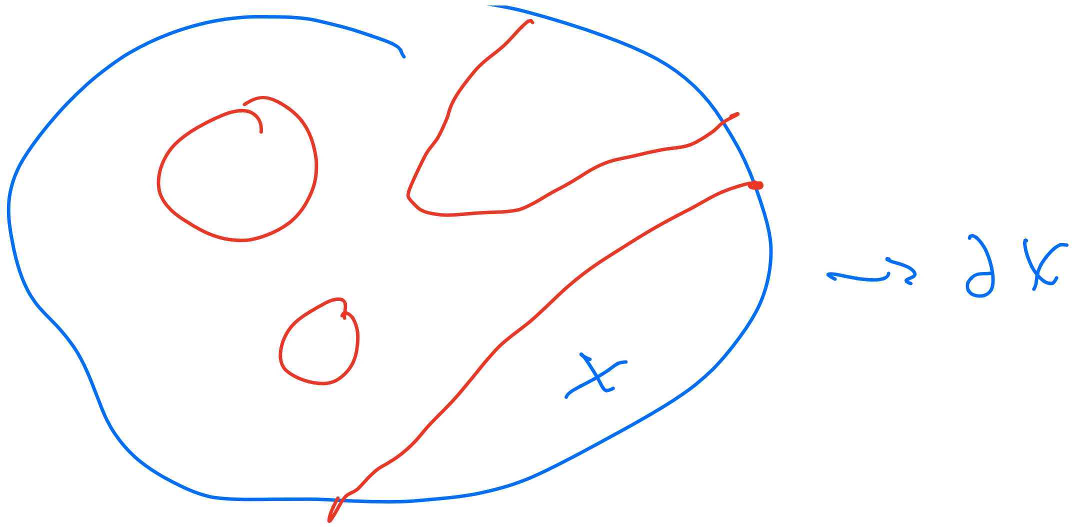
{reg values} is everywhere dense.

Pf of Lemma 5,

$f^{-1}(y)$ is smooth cpt 1-mfld with ∂

$\partial f^{-1}(y) = \{y\}$

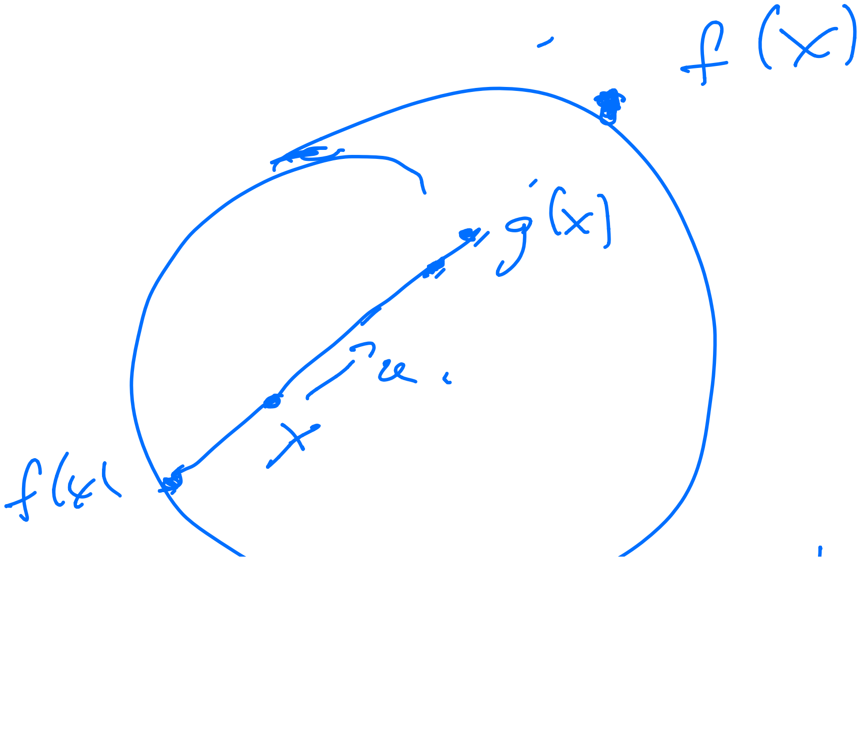
must have even number of pt



Lemma (Smooth Brouwer Fixed pt Thm)

Any $g: D^n \to D^n$ has a fixed pt ($\exists x \ \ g(x) = x$)

Pf



$f(x)$ is closest pt to ~~to $g(x)$~~ on $S^{n-1} = \partial D$ than to $g(x)$

Pf That f is smooth

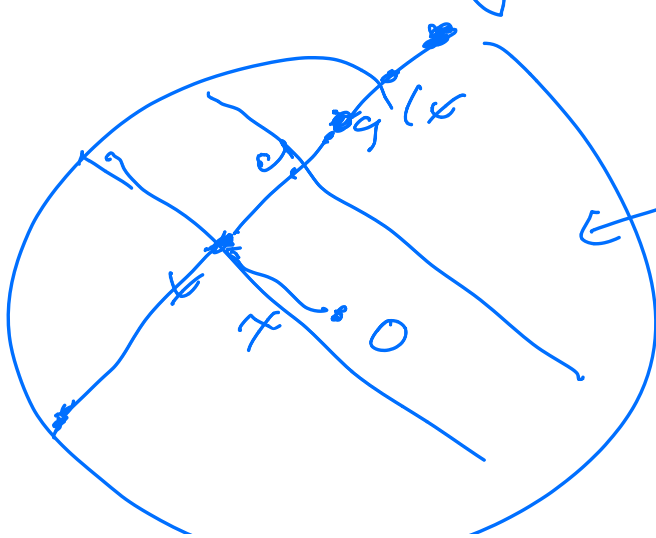
$$f(x) = x + tu$$

$$u = \frac{x - g(x)}{|x - g(x)|}$$

$$(x+tu)\cdot(x+tu) = 1$$

$$x\cdot x + 2t(x\cdot u) + t^2 = 1$$

$$t = (-x\cdot u) + \sqrt{1 - (x\cdot x) + (x\cdot u)^2}$$



$(u\cdot x)^2 \geq 0$

$x \cdot u \leq 0$
under square root
is positive

$$\boxed{\text{Brouwer Fixed Pt Thm for continuous } G : D^n \to D^n}$$

<u>Pf.</u> Weierstrass $\Rightarrow$ $\exists$ poly P

$P : D^n \longrightarrow \mathbb{R}^n$ s.t.

$|G(x) - P(x)| < \varepsilon$. Replace

P by $P_1(x) = P(x) / 1+\varepsilon$.

Suppose $G(x) \neq x$ $\forall x$

Then $|G(x) - x|$ has min. value

$\mu > 0$. Choose $\varepsilon < \mu$

Then $|P_1(x) - G(x)| < \mu$ so
$P_1(x) \neq x$.

August 30

<u>Sard's Thm</u> $f : U^{\subset \mathbb{R}^n} \to \mathbb{R}^p$ smooth ($= C^\infty$)

$C = \{x \mid \operatorname{rk} df_x < p\}$

Then $f(C)$ has measure 0

Instead of C^∞ can assume C^k where $k+1 > n/p$

Proof by induction on n

$C_i = \{x \in U \mid$ all partials of f of order $\leq i$ vanish$\}$

$C \supset C_1 \supset C_2 \supset \ldots$

decreasing chain of closed sets. 3 steps to proof

① $\mu(f(C - C_1)) = 0$

② $\mu(f(C_i - C_{i+1})) = 0$

③ large enough k $\left[k > \frac{n}{p} - 1\right]$

$\mu(f(C_k)) = 0$

Facts from measure theory

- Def'n of measure 0
- Countable union of measure 0 sets is measure 0
- Fubini's Thm. Assum $A \subset \mathbb{R}^p$ is measurable

Then $\mu_p(A) = 0 \Longleftarrow \forall$ hyperplane $t \times \mathbb{R}^{p-1}$, $\mu_{p-1}(A \cap (t \times \mathbb{R}^{p-1})) = 0$.

$p \geq 2$

Pf of Step 1 $\bar{x} \in C - C_1$

So $\frac{\partial f_i}{\partial x_j} \neq 0$ at $\bar{x}$ so

∂x_j

$$\frac{\partial f_1}{\partial x_1}(\bar{x}) \neq 0$$

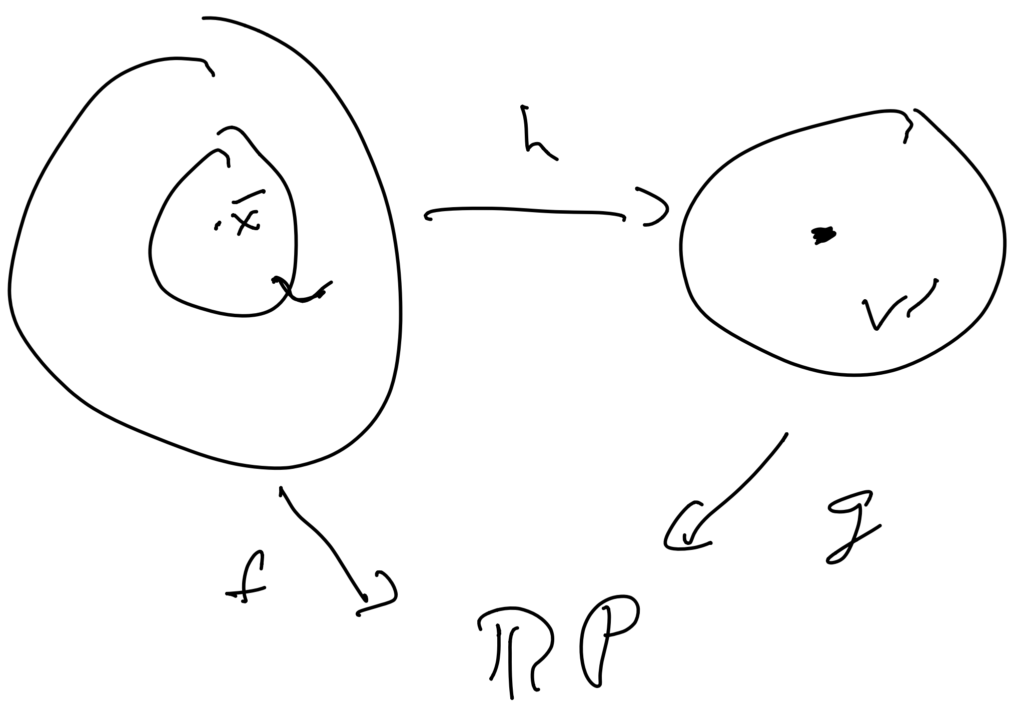
$h : U \to \mathbb{R}^n$

$$h(x) = (f_1(x), x_2, \dots x_n)$$

$dh_{\bar{x}}$ is iso.

So h maps nbhd V of $\bar{x}$ $\underbrace{\text{diffeo}}_{\text{to}}$ V'.

$$g = f h^{-1} : V' \to \mathbb{R}^p$$



$C' =$ crit pts of g

$C =$ $h(V \cap C)$

$g(C') = f(V \cap C)$

claim

g takes ~~to~~ hyperplane to hyperpl

$(t, x_2, \ldots x_n) \in V'$

$g(t, x_2, \ldots x_n) \subset t \times \mathbb{R}^{p-1} \subset \mathbb{R}^p$

$t = f_1(x)$

Let $g' =$ restriction g to $t \times \mathbb{R}^{p-1}$

$\mathrm{Crit}(g') = \mathrm{Crit}(g)$

$$\frac{\partial g_i}{\partial x_j} = \begin{pmatrix} 1 & 0 & 0 \\ * & \partial g_i' / \partial x_j & \\ * & & \\ * & & \end{pmatrix}$$

By induction ~~Fubini~~

$\mu_{p-1}\left(\text{Crit } \overset{\text{values}}{\cancel{pts}}\ g\right) = 0$

$f_x(V \cap C)$

Fubini $\Longrightarrow$ $\mu_p(g(C')) = 0$

($g(C')$ is measurable)

<u>Pf</u> <u>Step 2</u> $f(C_k - C_{k+1})$

$\bar{x} \in C_k - C_{k+1}$

$$\frac{\partial^{k+1} f_r}{\partial x_{p_1} \cdots \partial x_{p_{k+1}}}(\bar{x}) \neq 0$$

Assume $x_{p_1} = x_1$

$$w(x) = \frac{\partial^k f_r}{\partial x_{p_2} \cdots \partial x_{p_k}} = 0 \text{ on } \bar{x}$$

$h(x) = (w(x), x_2 \ldots x_n)$

as before, local diff

$$\begin{array}{ccc} V & \xrightarrow[\cong]{h} & V' \\ \cup & & \\ x & & \end{array}$$

$g = f \circ h^{-1} : V' \to \mathbb{R}^p$

$\bar{g}$ = restriction of g to $(0 \times \mathbb{R}^{n-1}) \cap V'$

Induction $\Rightarrow$ ~~crit~~ crit values of $\bar{g}$ have measure 0

each pt $h(C_k \cap V)$ is crit. pt

$\bar{g}\, h(C_k \cap V) = f(C_k \cap V)$ has measure 0

Pf of Step 3 Show for large k

$$\mu(f(C_k)) = 0$$

$$I^n \subset \bigcup_{x \in C_k} \text{a cube}$$

of edge length δ

Suffices to show

$$\mu(f(C_k \cap I^n)) = 0$$

Taylor's Thm $\Rightarrow$

$$f(x+h) = f(x) + R(x,h)$$

$$|R(x,h)| \leq c\,|h|^{k+1}$$

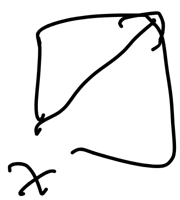
Subdivide I^n into r^n cubes of edge length δ/r

$I_1 \ni x \in C_k \cap I$

" such " "

any pt in I_1

has form $x + h$



$|h| \le \sqrt{n}\; \delta / r$

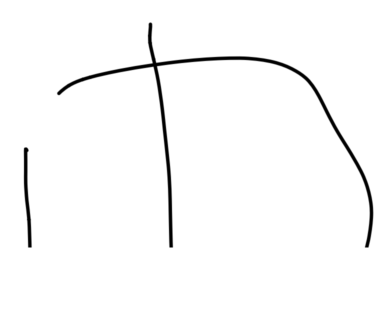
$f(I_1)$ lies in cube of edge length

$\frac{a}{r^{k+1}}$ $\quad 2C\left(\sqrt{n}\,\frac{\delta}{r}\right)^{k+1}$

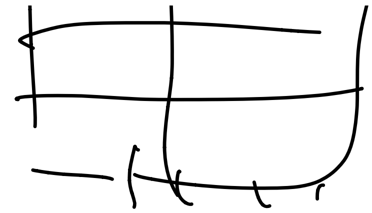
$a = 2C\left(\sqrt{n}\,\delta\right)^{k+1}$

Vol. of cube in $\mathbb{R}^p$

$\left(\frac{a}{r^{k+1}}\right)^p$



Total volum of cubes in I



r^n

$$= r^n \left(\frac{a}{n^{k+1}}\right)^p$$

$$= a^p \cancel{r}\, n^{n - p(k+1)}$$

exponent negative

$\Rightarrow$ total volume $< \varepsilon$

$$n < p(k+1)$$

$$\frac{n}{p} < k+1$$

$a \quad p \sim 1$