1

Foundations

If we start from the notions of curve – of dimension 1, locally like the line $\mathbb{R}$ of real numbers, and surface – of dimension 2, locally like the plane $\mathbb{R}^2$, the general term is 'manifold'. We begin with perhaps the most elegant form of the definition, but will prove it equivalent to other versions.

We say that a function F defined on $\mathbb{R}^n$ (or on an open subset thereof) is *smooth* if it admits continuous partial derivatives of *all* orders. We use the term 'smooth' in this sense throughout.

In the opening section, we begin with the definition of smooth manifold, introduce the bump function, and proceed to the construction of partitions of unity. We then discuss connectedness.

Probably the most important property distinguishing smooth from topological manifolds is the existence of tangent vectors. Again we begin with a formal definition, then give alternative ways to view the concept. We introduce smooth maps, and discuss concepts of submanifold and embedding.

The tangent vectors to a smooth manifold form a vector bundle, so we next introduce the notions of Lie group and of fibre bundle, and establish the existence of a Riemannian structure on any smooth manifold.

An essential tool in the study of smooth manifolds is the integration of smooth vector fields. This becomes effective when combined with the use of partitions of unity to construct vector fields. We show how to reformulate the basic theorem asserting the existence solutions of ordinary differential equations in geometrical terms to yield flows on smooth manifolds.

Finally we extend the concept of smooth manifold to that of manifold with boundary, and establish the existence of a collar neighbourhood of the boundary.

1.1 Smooth manifolds

A *smooth m-manifold* is a Hausdorff topological space M^m with a family $\mathcal{F} = \mathcal{F}_M$ of continuous real-valued functions defined on M and satisfying the following conditions:

(M1) $\mathcal{F}$ is local. If $f : M \to \mathbb{R}$ is such that each point of M has a neighbourhood in which f agrees with a function of $\mathcal{F}$, then $f \in \mathcal{F}$.

(M2) $\mathcal{F}$ is differentiably closed. If $f_1, \ldots, f_k \in \mathcal{F}$, and F is a smooth function on $\mathbb{R}^n$, then $F(f_1, \ldots, f_k) \in \mathcal{F}$.

(M3) $(M, \mathcal{F})$ is locally Euclidean. For each point $P \in M$, there are m functions $f_1, \ldots, f_m \in \mathcal{F}$ such that $Q \mapsto (f_1(Q), \ldots, f_m(Q))$ gives a homeomorphism of a neighbourhood U of P in M onto an open subset V of $\mathbb{R}^m$. Every function $f \in \mathcal{F}$ coincides on U with $F(f_1, \ldots, f_m)$, where F is a smooth function on V.

(M4) M is a countable union of compact subsets.

We call functions $f \in \mathcal{F}$ *smooth functions* of M, and the mapping defined in (M3) (or, by abuse of language, the set U) a *coordinate neighbourhood* of P. It follows from (M2) that sums, products, and constant multiples of smooth functions are also smooth.

The integer m is called the *dimension* of the manifold M.

We now give some simple examples of smooth manifolds.

The empty set is a smooth m-manifold (the definition is vacuously satisfied).

Euclidean space $\mathbb{R}^m$, with smooth functions taken in the ordinary sense, is a smooth m-manifold. Condition (M1) is trivial; (M2) follows from the rule for differentiating a composite (a function of a function); for (M3), since the coordinate functions are smooth, we take the identity map; and $\mathbb{R}^m$ is the union of the compact subsets given by $\|x\| \leq n$.

The disjoint union of a finite or countable set of smooth m-manifolds is another. Define a function to be smooth if the induced function on each part is so; the conditions are then all trivial.

Let O be an open subset of $\mathbb{R}^m$. Write $\mathcal{G}_O$ for the restriction to O of functions of $\mathcal{F}_{\mathbb{R}^m}$; $\mathcal{F}_O$ for the set of functions locally agreeing with a function of $\mathcal{G}_O$. Then since O is open in $\mathbb{R}^m$, $(O, \mathcal{G}_O)$ satisfies conditions (M1), (M3); $(O, \mathcal{F}_O)$ satisfies them and also condition (M2).

For each positive integer i, consider the sets $D^m_x(\sqrt{m}/i)$[1] such that all the coordinates of ix are integers and which are contained in O. There are only countably many of these. For any $y \in O$, some $\mathring{D}^m_y(\delta) \subset O$. Choose $i > 2\sqrt{m}/\delta$. Then some x with $ix \in \mathbb{Z}^m$ is within a distance $\sqrt{m}/i$ of y, and

$$y \in D^m_x(\sqrt{m}/i) \subset D^m_y(2\sqrt{m}/i) \subset \mathring{D}^m_y(\delta) \subset O.$$

[1] For this notation and others, see the Index of Notations on p. 340.

Thus the chosen sets cover M, so (M4) also holds, and O is a smooth manifold.

More generally, let M be any smooth m-manifold and O be an open subset of M. Again write $\mathcal{G}_O$ for the restriction to O of functions of $\mathcal{F}_M$; $\mathcal{F}_O$ for the set of functions locally agreeing with a function of $\mathcal{G}_O$. We see as above that (M1)-(M3) hold. Now M is covered by coordinate charts, so any compact subset is covered by finitely many; hence M is covered by countably many charts U_α. Thus O is the union of countably many sets $O \cap U_\alpha$, each of which can be regarded as an open set in Euclidean space, so by the preceding paragraph is a countable union of compact sets. Thus (M4) also holds, and the structure of smooth m-manifold on M induces such a structure on O. We call O an *open submanifold* of M.

Let $M_1^{m_1}, M_2^{m_2}$ be smooth manifolds. Then the topological product $N^{m_1+m_2} = M_1^{m_1} \times M_2^{m_2}$ has a natural structure of smooth manifold. For let π_1, π_2 denote projections on the factors. Then for $f_1 \in \mathcal{F}_{M_1}$, $f_2 \in \mathcal{F}_{M_2}$, we define $f_1 \circ \pi_1$, $f_2 \circ \pi_2$ to belong to $\mathcal{F}_N$; any smooth functions of a finite set of these; and any function locally agreeing with one of these functions. This definition ensures that conditions (M1) and (M2) are satisfied. But so is (M3), for it now follows that if $\varphi_1 : U_1 \to \mathbb{R}^{m_1}$, $\varphi_2 : U_2 \to \mathbb{R}^{m_2}$ are coordinate neighbourhoods in M_1 and M_2, then $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^{m_1+m_2}$ can be taken as a coordinate neighbourhood in $M_1 \times M_2$. And (M4) follows since (see §A.2) the product of two compact sets is compact.

The first tool for working with our definition is a bump function. Define first a function B_1 on $\mathbb{R}$ by:

$$B_1(x) = \begin{cases} \exp\left(\frac{1}{(x(x-1))}\right) & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then B_1 is smooth, non-negative, and differs from zero when $0 < x < 1$. The *bump function* $Bp(x)$ is now given by

$$Bp(x) = \int_0^x B_1(t)dt \Bigg/ \int_0^1 B_1(t)dt.$$

Since $B_1(x)$ is smooth, so is $Bp(x)$. Also

$$\begin{aligned} Bp(x) &= 0 \quad \text{if} \quad x \le 0, \\ 0 < Bp(x) &< 1 \quad \text{if} \quad 0 < x < 1, \quad \text{and} \\ Bp(x) &= 1 \quad \text{if} \quad x \ge 1. \end{aligned}$$

The bump function is illustrated in Figure 1.1. Although we have given an explicit construction, the above are the essential properties of the bump

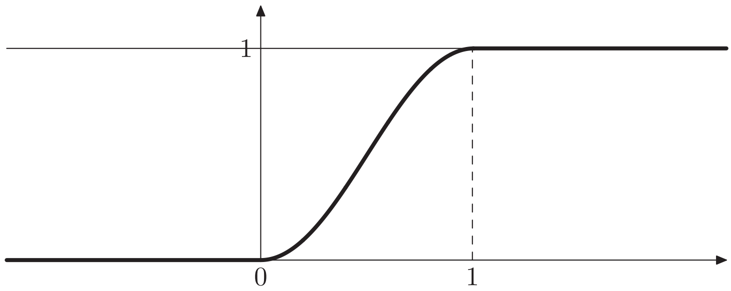

Figure 1.1 The bump function

function. We also note that since $B_1(1-x) = B_1(x)$ we have $Bp(1-x) = 1 - Bp(x)$; we also have $Bp'(x) > 0$ if $0 < x < 1$. We now have

Proposition 1.1.1 *Let $\varphi : U \to V$ be a coordinate neighbourhood of $P \in M$, and let F be a smooth function on V. Then there is a function $f \in \mathcal{F}$, agreeing with $F \circ \varphi$ in a neighbourhood of P, and zero outside U.*

Proof Without loss of generality, let $\varphi(P) = 0$. Since V is a neighbourhood of 0, we can find $r > 0$ with $\mathring{D}^m(3r) \subset V$. Define $\Phi(x) = Bp(2 - r^{-1}\|x\|)$. Then $\Phi(x) = 1$ for $\|x\| \le r$, $\Phi(x) = 0$ for $\|x\| \ge 2r$, and Φ is a smooth function on $\mathbb{R}^m$, hence also on V, since Bp is smooth, and $\|x\|$ is smooth except at 0. Then $F\Phi$ is also smooth on V, and $F(x)\Phi(x) = 0$ if $\|x\| \ge 2r$. We define a function f on M by:

$$f(P) = \begin{cases} F(\varphi(P))\Phi(\varphi(P)) & \text{if} \quad P \in M \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (M2), $f \in \mathcal{F}$, and f agrees with $f \circ \varphi$ in $\varphi^{-1}(D^m(r))$. □

Another commonly given version of the definition of manifold is as follows. For a Hausdorff topological space M, a *chart* is a homeomorphism of an open subset U of M onto an open subset V of $\mathbb{R}^m$. A collection of charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ is an *atlas* if the open sets U_α cover M. For any pair of charts, set $V_{\alpha,\beta} := \varphi_\alpha(U_\alpha \cap U_\beta)$; then there is a homeomorphism $\psi_{\alpha,\beta} : V_{\alpha,\beta} \to V_{\beta,\alpha}$ between open sets of Euclidean space induced by $\varphi_\beta \circ \varphi_\alpha^{-1}$. Then we say that the atlas is smooth if each $\psi_{\alpha,\beta}$ is smooth.

Lemma 1.1.2 *M is a (smooth) manifold if and only if it has a (smooth) atlas with countably many charts.*

Proof If $(M, \mathcal{F})$ satisfies (M1)–(M3), the coordinate neighbourhoods form a smooth atlas. Each compact subset of M is covered by open charts, hence by a finite subset; it follows from (M4) that a countable number of charts cover M.

Conversely, given a smooth atlas, we define $\mathcal{F}$ by letting $f \in \mathcal{F}$ if, for each α, $f \circ \varphi_\alpha^{-1}$ is a smooth function on V_α. It is immediate that this satisfies (M1) and (M2). As to (M3), for each $P \in M$, choose α with $P \in U_\alpha$; then the functions $x_i \circ \varphi_\alpha^{-1}$ are defined near P and, by the proposition, there are functions f_i, smooth on U_α, vanishing outside a neighbourhood of P, and agreeing with these on a smaller neighbourhood. Extend f_i to a function on M vanishing outside U_α. Then $f_i \in \mathcal{F}$, and $(f_1, \ldots, f_m)$ have the desired property. Now (M4) follows since each coordinate neighbourhood is a countable union of compact sets. □

Using the notion of atlas, we now give further important examples of smooth manifolds. If V is a vector space over $\mathbb{R}$, with O the origin, the projective space $P(V)$ is the quotient of $V \setminus \{O\}$ by the equivalence relation $\mathbf{v} \sim k\mathbf{v}$ for $0 \neq k \in \mathbb{R}$. For $(x_0, \ldots, x_n) \neq 0 \in \mathbb{R}^{n+1}$ we write $(x_0 : \ldots : x_n)$ for its image in $P^n(\mathbb{R}) := P(\mathbb{R}^{n+1})$ (since it is given by the ratios of the x_i). We define an atlas for $P^n(\mathbb{R})$ by taking open sets U_i given by $x_i \neq 0$ and defining $\varphi_i : U_i \to \mathbb{R}^n$ by $\varphi_i(x_0 : \ldots : x_n) := (x_0 x_i^{-1}, \ldots, x_n x_i^{-1})$ (with the ith term $x_i x_i^{-1}$ omitted). The coordinate transformations are multiplications by $x_i x_j^{-1}$ on each coordinate, so are smooth. We use the same notations with $\mathbb{C}$ in place of $\mathbb{R}$, giving the complex projective space $P^n(\mathbb{C})$.

For the next tools we will need condition (M4), and begin with a general result. First observe that any manifold is locally Euclidean, and hence locally compact.

Proposition 1.1.3 *Suppose that X is locally compact and a countable union of compact subsets. Then we can find compact subsets C_n and open subsets $B_{n+\frac{1}{2}}$ such that $X = \bigcup_n C_n$ and for all $n \geq 1$, $C_n \subset B_{n+\frac{1}{2}} \subset C_{n+1}$.*

Proof Suppose X the union of compact sets A_n. Define $C_1 := A_1$. Now suppose inductively C_n defined. Since X is locally compact, each $x \in (C_n \cup A_{n+1})$ has an open neighbourhood U_x with compact closure. These open sets cover the compact set $C_n \cup A_{n+1}$, so we can choose finitely many of them which together cover this set. Define $B_{n+\frac{1}{2}}$ to be the union of these open sets, and C_{n+1} to be its closure: this is a finite union of compact sets, so is compact. Finally since $X = \bigcup_n A_n$ and $A_n \subseteq C_n$, $X = \bigcup_n C_n$. □

Theorem 1.1.4 *For any manifold M^m, we can find a set of coordinate neighbourhoods $\varphi_\alpha : U_\alpha \to \mathring{D}^m(3)$ for M^m such that*

(i) The sets $\varphi_\alpha^{-1}(\mathring{D}^m)$ cover M.

(ii) Each $P \in M$ has a neighbourhood in M which meets only a finite number of sets U_α, i.e., the U_α are locally finite.

Moreover, the covering by the U_α may be chosen to refine any given covering of M.

Proof Choose subsets C_n and $B_{n+\frac{1}{2}}$ of X as in Proposition 1.1.3. Any $x \in X$ belongs to some $C_n \setminus C_{n-1}$, so the open set $B_{n+\frac{1}{2}} \setminus C_{n-1}$ is a neighbourhood of x: we may choose a coordinate neighbourhood $U_\alpha = \varphi_\alpha^{-1}(\mathring{D}^m(3))$ of x inside this, and also contained in one of the open sets of the given covering. The neighbourhoods $\varphi_\alpha^{-1}(\mathring{D}^m)$ cover the compact set $C_{n+1} \setminus B_{n-\frac{1}{2}}$, so we may choose a finite subcovering. The collection of these for all n covers M and refines the given open covering.

Now any $y \in X$ has $B_{m+\frac{1}{2}} \setminus C_{m-1}$ as a neighbourhood for some m, and this meets $B_{n+\frac{1}{2}} \setminus C_{n-1}$ only if $|m-n| \leq 1$, hence meets only finitely many of the U_α. □

The *support* of a continuous function ψ on M is the set $\{x \in M \mid \psi(x) \neq 0\}$. A set of non-negative continuous functions $\{\psi_\alpha\}$ on M is called a *partition of unity* if their supports U_α form a locally finite covering of M and $\sum_\alpha \psi_\alpha(P) = 1$. (Local finiteness is not strictly necessary, but ensures convergence and continuity of the infinite sum.) If we are given an open covering $\{U_\beta\}$ of M, the partition $\{\psi_\alpha\}$ of unity is said to be *subordinate* to the covering if the support of each ψ_α is contained in some set U_β of the covering.

If the ψ_α are smooth, we have a smooth partition of unity. These will be key to numerous constructions. It will be useful if we can say that if f_α is a smooth function defined on U_α, then the function equal to $f_\alpha \psi_\alpha$ on U_α and zero elsewhere is smooth on M. This holds if moreover the support of ψ_α is contained in a closed set in the interior of U_α. If this holds for each α, we will say that the partition $\{\psi_\alpha\}$ of unity is *strictly subordinate* to the covering.

Theorem 1.1.5 *For any open covering $\mathcal{V}$ of a smooth manifold M, there is a smooth partition of unity strictly subordinate to it.*

Proof By Theorem 1.1.4 there is a locally finite refinement of $\mathcal{V}$ by a set of coordinate neighbourhoods $\varphi_\alpha : U_\alpha \to \mathring{D}^m(3)$ such that the $\varphi_\alpha^{-1}(\mathring{D}^m)$ cover M. For each α, set

$$\Psi_\alpha(P) = \begin{cases} Bp(2 - \|x\|) & \text{if} \quad P \in U_\alpha, \phi_\alpha(P) = x, \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of Proposition 1.1.1, $\Psi_\alpha(P)$ is smooth. The above properties imply that for each $P \in M$, there is an α with $\Psi_\alpha(P) = 1$, and that each $P \in M$

has a neighbourhood on which all but a finite number of functions Ψ_α vanish. Hence the function $\Sigma(P) = \sum_\alpha \Psi_\alpha(P)$ can be defined, and is everywhere smooth. Thus the functions $\psi_\alpha(P) = \Psi_\alpha(P)/\Sigma(P)$ give a partition of unity.

The support of Ψ_α, hence also that of ψ_α, is $\varphi_\alpha^{-1}(D^m(2))$, which is in the interior of U_α, so the partition of unity is strictly subordinate to the given covering. □

The next result is of the same type, but needs more work. It will be needed for Theorem 4.6.1 and Lemma 4.6.2. A slight modification of the argument gives a corresponding result for subsets of Cartesian powers M^r with $r > 2$.

Lemma 1.1.6 *(i) There is a countable collection of pairs of disjoint compact sets (K_α, K'_α) in M such that for any $P, P' \in M$ with $P \neq P'$ there exists α with $P \in K_\alpha$ and $P' \in K'_\alpha$.*

(ii) Let U be an open neighbourhood of the diagonal $\Delta(M)$ in $M \times M$. Then we can find pairs of disjoint compact sets (L_β, L'_β) in M such that for any $(P, P') \in (M \times M \setminus U)$ there exists β with $P \in L_\beta$ and $P' \in L'_\beta$ and moreover such that $\{L_\beta, L'_\beta\}$ is locally finite.

Proof (i) Let δ_α be a partition of unity constructed as in Theorem 1.1.4. Then the closure K_α of the support of δ_α is compact. Given $P,\ P' \in M$, either (a) there exists α with $P, P' \in K_\alpha$ or (b) we can choose $P \in K_\alpha \setminus K_{\alpha'}$, $P' \in K_{\alpha'} \setminus K_\alpha$.

In case (a), the points P, P' lie in the same coordinate patch. Here we have a problem in Euclidean space, and the disjoint pairs of $\mathring{D}^n_x(\sqrt{m}/i)$ (where ix has integer coordinates) give what we want.

To deal with (b), define compact sets by $K_{\alpha,n} := \{P \in M \mid \delta_\alpha(P) \geq \frac{1}{n}\}$ for each α and $n \geq 2$. Then if $n > \delta_\alpha(P)^{-1}$ and similarly for n', P and P' lie in the disjoint sets $K_{\alpha,n}, K_{\alpha',n'}$.

(ii) First choose a locally finite cover $\{C_\gamma\}$ of M by compact sets – for example, the sets $C_{n+1} \setminus B_{n-\frac{1}{2}}$ of Proposition 1.1.3. For $(P, P') \in (M \times M \setminus U)$, choose sets of this cover with $P \in C_\gamma$, $P' \in C_{\gamma'}$. If $C_\gamma, C_{\gamma'}$ are disjoint, we can choose these as our pair (L_β, L'_β). If not, $K = C_\gamma \cup C_{\gamma'}$ is compact, hence so is $K \times K \setminus U$; thus it is covered by finitely many of the pairs $K_\alpha \times K'_\alpha$, and we choose these as our (L_β, L'_β).

Since each C_γ meets only finitely many other such sets, it meets only finitely many of the chosen L_β and L'_β. □

We return to partitions of unity, which are an essential tool in numerous proofs. As first applications, we can approximate continuous functions by smooth functions.

Proposition 1.1.7 *(i) Let f be a continuous positive function on M. Then we can find a smooth function g, with $0 < g(P) < f(P)$ for all $P \in M$.*

(ii) For any continuous function f on M and any $\varepsilon > 0$ there exists a smooth function h on M with $|h(P) - f(P)| < \varepsilon$ for every $P \in M$.

(iii) If $f : M \to \mathbb{R}$ is continuous, $\varepsilon > 0$, and F is a closed subset of M such that f is smooth on some open set $U \supset F$, we can find h such that also $h = f$ on a neighbourhood of F.

Proof (i) Let $\{\psi_\alpha\}$ be a smooth partition of unity, and choose $\delta_\alpha > 0$ less than the infimum of f on the support of ψ_α (since the support has compact closure, this infimum cannot be zero). Then $g := \sum_\alpha(\delta_\alpha\psi_\alpha)$ has $g(P) > 0$ since $\psi_\alpha(P) > 0$ for some α; on the other hand, for each α with $\psi_\alpha(P) > 0$ we have $\delta_\alpha < f(P)$, so $g(P) < \sum \psi_\alpha(P)f(P) = f(P)$.

(ii) The sets $\{P \in M \mid n < \frac{2}{\varepsilon}f(P) < n + 2\}$ form an open cover of M. By Theorem 1.1.5, we may choose a smooth partition $\{(U_\alpha, \psi_\alpha)\}$ of unity strictly subordinate to this cover. For each α choose $P_\alpha \in U_\alpha$. Now the function $h := \sum_\alpha f(P_\alpha)\psi_\alpha$ is well defined and smooth. Any $Q \in M$ belongs to U_α for a finite, non-empty set of α, and as each such U_α is contained in one of the sets of the original cover, $f(Q)$ and $f(P_\alpha)$ lie in the same interval of length ε. Thus we have $f(Q) - f(P_\alpha) < \varepsilon$, so

$$\begin{aligned} |h(Q) - f(Q)| &= \left|\sum_\alpha (f(P_\alpha) - f(Q))\psi_\alpha\right| \\ &\leq \sum_\alpha |(f(P_\alpha) - f(Q)|\psi_\alpha \\ &< \sum_\alpha \varepsilon\psi_\alpha = \varepsilon. \end{aligned}$$

(iii) As well as the open sets U_α, which we may take disjoint from F, we now choose open sets $U_\beta \subset U$ which cover U, and set $f_\beta := f$. Piecing together using a partition of unity now yields the result. □

This approximation technique is very useful. It is also flexible: we will show in Proposition 2.3.4 that the target can be any smooth manifold.

Our next topic is connectedness of smooth manifolds. A smooth map $\alpha : \mathbb{R} \to M$ is called a *path* in M. Two points P, Q in M are called *connected in M* if there is a path in M whose image contains P and Q.

Lemma 1.1.8 *Connectedness in M is an equivalence relation.*

Proof By definition, the relation is symmetric. It is reflexive, since a constant map is a path. To prove transitivity, first observe that if $h : I \to M$ is a smooth path, the normalised path $N(h) : \mathbb{R} \to M$ given by $N(h)(t) = h(Bp(t))$ is smooth. If now $h : (I, 0, 1) \to (M, P, Q)$ and $k : (I, 0, 1) \to (M, Q, R)$ are smooth paths, a smooth path joining P to R is given by setting $H(t) = N(h)(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $H(t) = N(k)(2t - 1)$ for $\frac{1}{2} \leq t \leq 1$. □

The equivalence classes are called the *components* of M.

Lemma 1.1.9 *(i) Each equivalence class is open and closed in M.*

(ii) A subset of M is open and closed if and only if it is a union of equivalence classes.

Proof (i) If $\varphi : U \to V$ is a coordinate neighbourhood of P such that V is convex, every point of U can be joined to P using the path corresponding to the straight line in V (suitably parametrised). Hence an equivalence class contains a neighbourhood of each of its points, so is open.

Since each equivalence class is the complement of the union of the other equivalence classes, it is closed in M.

(ii) Sufficiency follows by (i). For necessity, observe that since $\mathbb{R}$ is connected, any path which meets an open and closed subset is contained in it, so such a subset is saturated for the equivalence relation. □

It follows that M is *connected* in the usual sense if it only has one component. We also see that for smooth manifolds, connection and connection by smooth paths are equivalent. A component of M, being open, is an open submanifold; and M is the disjoint union of all its components. Thus to study M, it suffices to take the components separately; we shall frequently do this.

1.2 Smooth maps, tangent vectors, submanifolds

Let M^m, V^v be smooth manifolds. A mapping $\varphi : M \to V$ is called *smooth* if for each $f \in \mathcal{F}_V$, $f \circ \varphi \in \mathcal{F}_M$.

In view of (M3) this is equivalent to the requirement that each transformation of coordinates induced by φ between coordinate neighbourhoods in M and in V be smooth in the usual sense. The above definition is more convenient: for example, the following are immediate.

If $\varphi_1 : M_1 \to M_2$ and $\varphi_2 : M_2 \to M_3$ are smooth, then so is $\varphi_2 \circ \varphi_1 : M_1 \to M_3$.

If O is an open submanifold of M, $i : O \subset M$ is smooth.

A bijective correspondence $\varphi : M^m \to V^m$ between two smooth manifolds is a *diffeomorphism* if both φ and φ^{-1} are smooth. M^m and V^m are called *diffeomorphic*.

Thus a diffeomorphism is a bijective correspondence between the two manifolds under which smooth functions correspond: this is the equivalence relation which classifies manifolds.

A *tangent vector* at a point P of a smooth manifold M is a derivation on $\mathcal{F}$ to $\mathbb{R}$. In detail, a tangent vector at $P \in M$ is a mapping $\xi : \mathcal{F} \to \mathbb{R}$ which satisfies:

$$(i) \text{ if } a_1, a_2 \in \mathbb{R},\ f_1, f_2 \in \mathcal{F}, \text{ then } \xi(a_1 f_1 + a_2 f_2) = a_1 \xi(f_1) + a_2 \xi(f_2); \tag{1.2.1}$$

$$(ii) \text{ if } f_1, f_2 \in \mathcal{F}, \text{ then } \xi(f_1 f_2) = \xi(f_1) f_2(P) + f_1(P)\xi(f_2). \tag{1.2.2}$$

We next study the set of all tangent vectors to M. Since sums and real multiples of tangent vectors at P are also tangent vectors at P, the tangent vectors to M at P form a vector space: we call it the *tangent space* $T_P M$ to M at P.

If $p : U \to M$ is a smooth path (U open in $\mathbb{R}$), with $p(0) = P$, the expression $\xi(f) = \frac{d}{dt} p(f(t))|_{t=0}$ is defined, and the map $\xi : \mathcal{F} \to \mathbb{R}$ satisfies (i) and (ii), so $\xi \in T_P M$. We call ξ the tangent to the path. Thus tangent vectors correspond to displacement along the manifold.

Let $\varphi : U \to V \subset \mathbb{R}^m$ be a coordinate neighbourhood of P with $\varphi(P) = 0$. Let $x_1, \ldots, x_m$ be coordinates in $\mathbb{R}^m$. Then for each $f \in \mathcal{F}$, $F := f \circ \varphi^{-1}$ is a smooth function on V, so $\frac{\partial}{\partial x_i}(f) := \left.\frac{\partial F}{\partial x_i}\right|_0$ is well defined. Then $\frac{\partial}{\partial x_i}$ is a tangent vector at P: condition (i) is clear, and (ii) follows by the rule for differentiating a product. We will prove that the $\frac{\partial}{\partial x_i}$ form a basis for $T_P M$; first, however, we need a lemma, which will be used again.

Lemma 1.2.3 *Let f be a smooth function on an open convex subset V of $\mathbb{R}^m$ containing* 0, *and let* $f(0) = 0$. *Then there exist further smooth functions f_i $(1 \le i \le m)$ on V such that $f(x) = \sum_1^m x_i f_i(x)$. Moreover, if f is a smooth function of additional parameters a_j, we may suppose that f_i also are.*

Proof We may write

$$f(x) = f(x) - f(0) = \int_0^1 \frac{\partial f(tx)}{\partial t} dt.$$

But $\frac{\partial f(tx)}{\partial t} = \sum_1^m x_i \frac{\partial f}{\partial x_i}(tx)$. Substituting this gives $f(x) = \sum_1^m x_i f_i(x)$, where $f_i(x) := \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$. The last part also follows. □

Theorem 1.2.4 *The tangent vectors $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ form a basis for $T_P M$.*

Proof We first remark that a tangent vector is essentially local in nature: if $f = g$ in a neighbourhood U of P, and ξ is a tangent vector at P, then $\xi(f) = \xi(g)$. For by Proposition 1.1.1, we can find a function Φ on M, equal to 1 in a neighbourhood of P, and zero outside U. Then $\Phi f = \Phi g$, and so $f - g = (f - g)(1 - \Phi)$. Thus

$$\xi(f) - \xi(g) = \xi(f - g) = \xi(f - g)(1 - \Phi(P)) + (f(P) - g(P))\xi(1 - \Phi) = 0.$$

Hence it is sufficient to consider only functions defined and smooth in U, where $\varphi : U \to V$ is a coordinate neighbourhood of P with V convex; it will be simpler to speak directly of functions on V.

For any smooth function f on V, by Lemma 1.2.3, we can put

$$f(x) = f(0) + \sum x_i f_i(x).$$

For any tangent vector ξ at P, then,

$$\begin{aligned}\xi(f) &= \xi(f(0)) + \sum \xi(x_i f_i)\\ &= f(0)\xi(1) + \sum \xi(x_i) f_i(0) + \sum x_i(0)\xi(f_i).\end{aligned}$$

But $\xi(1) = \xi(1 \cdot 1) = 1 \cdot \xi(1) + \xi(1) \cdot 1 = 2\xi(1)$, and so $\xi(1) = 0$. Thus

$$\xi(f) = \sum \xi(x_i) f_i(0).$$

In particular

$$\frac{\partial}{\partial x_j}(f) = \sum \frac{\partial}{\partial x_j}(x_i) f_i(0) = \sum \delta_{ij} f_i(0) = f_j(0).$$

Thus $\xi(f) = \sum \xi(x_i)\frac{\partial f}{\partial x_i}$, and as this is true for all f, $\xi = \sum \xi(x_i)\frac{\partial}{\partial x_i}$. Hence the $\frac{\partial}{\partial x_i}$ span $T_P M$. Since $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, they are linearly independent. Hence they form a basis. □

For example, we may identify the tangent space to $\mathbb{R}^m$ at any point a with $\mathbb{R}^m$ itself, by identifying $\sum_i k_i \partial/\partial x_i$ with the vector $(k_1, \ldots, k_m)$. In particular, $T_a\mathbb{R}$ is identified with $\mathbb{R}$.

Now let $\varphi : M^m \to V^v$ be a smooth mapping, and let $\varphi(P) = Q$. The *differential* of φ at P, $d\varphi_P : T_P M \to T_Q V$ is defined by:

$$d\varphi_P(\xi)(f) = \xi(f \circ \varphi) \quad \text{for} \quad \xi \in T_P M, f \in \mathcal{F}_V.$$

Since f, φ are smooth, so is $f \circ \varphi$, so the right-hand side is defined. Then $d\varphi_P(\xi)$ is a derivation since ξ is. Clearly, $d\varphi_P$ is a linear mapping of $T_P M$ to $T_Q V$.

If $f \in \mathcal{F}_M$, then $f : M^m \to \mathbb{R}$ is a smooth mapping, so for any $P \in M$, we have $df_P : T_P M \to T_{f(P)}\mathbb{R} = \mathbb{R}$. Since df_P is linear, it is an element of the dual vector space $T_P^\vee M$ to $T_P M$. Now, if $x_1, \ldots, x_m$ are local coordinates at P, we have

$$dx_i(\partial/\partial x_j) = \partial x_i/\partial x_j = \delta_{ij}$$

so the dx_j form the basis of $T_P^\vee M$ dual to the basis $\partial/\partial x_i$ of $T_P M$.

Theorem 1.2.5 (Inverse Function Theorem) *Let $f_1, \ldots, f_n$ be smooth functions defined in a neighbourhood of $O \in \mathbb{R}^n$, and suppose $\left|\frac{\partial f_i}{\partial x_j}\right| \neq 0$ at O. Then $(f_1, \ldots, f_n)$ defines a diffeomorphism of some neighbourhood U of O on an open subset of $\mathbb{R}^n$.*

A proof can be found, for example, in [40, Theorem 10.2.1].

We can now give a simple test for coordinate neighbourhoods of a point.

Corollary 1.2.6 *Let M^n be a smooth manifold; $f_1, \ldots, f_n$ be smooth functions on M, $P \in M$. The f_i may be taken as coordinate functions for a coordinate neighbourhood of P if and only if the df_i form a basis for $T_P M^\vee$.*

Proof Let $\varphi : U \to \mathbb{R}^n$ be a coordinate neighbourhood of P. Then the $f_i \circ \varphi^{-1}$ are smooth functions on a neighbourhood of $\varphi(P) \in \mathbb{R}^n$; by the theorem, they define a diffeomorphism of some such neighbourhood if and only if the Jacobian determinant $|\frac{\partial(f_i \circ \varphi^{-1})}{\partial x_j}| \neq 0$ at $\varphi(P)$. But the elements of this matrix are just the coefficients in the df_i of basis elements dx_j of $T_P M^\vee$. $\square$

Theorem 1.2.7 (Implicit Function Theorem) *Let $f_1, \ldots, f_r$ be smooth functions defined in a neighbourhood of $O \in \mathbb{R}^{r+s}$ and suppose the determinant formed by their partial derivatives with respect to $x_1, \ldots, x_r$ is non-zero at O. Then there are r smooth functions $g_1, \ldots, g_r$ defined in a neighbourhood of $O \in \mathbb{R}^s$ such that within some neighbourhood of $O \in \mathbb{R}^{r+s}$, a point satisfies $f_i(P) = 0$ $(1 \leq i \leq r)$ if and only if it satisfies*

$$x_i = g_i(x_{r+1}, \ldots, x_{r+s}) \quad (1 \leq i \leq r).$$

Proof It follows from the hypothesis that the map defined by

$$(f_1, \ldots, f_r, x_{r+1}, \ldots, x_{r+s})$$

on a neighbourhood of O satisfies the hypothesis of the Inverse Function Theorem 1.2.5. Hence by that result, there is a smooth inverse map. We may write this map as $(h_1, \ldots, h_r, x_{r+1}, \ldots, x_{r+s})$. The result now follows on setting $g_i(x_{r+1}, \ldots, x_{r+s}) := h_i(0, \ldots, 0, x_{r+1}, \ldots, x_{r+s})$. $\square$

A subset M^m of a smooth manifold N^n is a *submanifold* (of dimension m and *codimension* $n - m$) if, for each point $P \in M$, there is a coordinate neighbourhood $\varphi : U \to \mathbb{R}^n$ of P in N such that $U \cap M = \varphi^{-1}(\mathbb{R}^m)$.

By Corollary 1.2.6, an equivalent requirement is that in a neighbourhood of each point of M, M is defined by the vanishing of $(n - m)$ functions with linearly independent differentials. For in the case above, M is defined by the vanishing of the last $(n - m)$ coordinate functions; while by that corollary, any set of functions with linearly independent differentials can be taken as functions of a coordinate neighbourhood. If M is a closed subset of N, we call it a *closed submanifold*.

A submanifold M^m of N^n has a natural induced structure of smooth m-manifold: the existence of coordinate neighbourhoods for M and the fact that overlaps are smooth follow immediately from the definition.

Lemma 1.2.8 *If M is a closed submanifold of N, $\mathcal{F}_M$ consists of the restrictions to M of the functions of $\mathcal{F}_N$.*

Proof We have an open covering of N consisting of charts U_α as in the definition of submanifold, and the subset $U_0 := N \setminus M$. By Theorem 1.1.5 we can pick a smooth partition of unity $(\{\delta_\alpha\}, \delta_0)$ strictly subordinate to this covering. For each $f \in \mathcal{F}_M$, the restriction $f \mid M \cap U_\alpha$ of f to U_α extends to a smooth function f_α on U_α using projection in the chart. Now $\sum \delta_\alpha f_\alpha$ is a smooth extension of f. □

If M is not closed, we can construct smooth functions on M that do not even extend to continuous functions on N: the simplest example is $N = \mathbb{R}$, $M = \{x \mid x > 0\}$ with $f(x) = x^{-1}$.

Many important examples of manifolds occur as submanifolds of Euclidean or projective space, often given (at least locally) by equations with linearly independent differentials: for example, we have the unit sphere $S^{n-1} \subset \mathbb{R}^n$ defined by $\|x\|^2 = 1$; in particular, the unit circle S^1.

There are plenty of examples of smooth manifolds.

Lemma 1.2.9 *Any finite simplicial complex X is homotopy equivalent to a smooth manifold.*

This result is proved by first embedding X in Euclidean space of high enough dimension, then taking a 'regular' neighbourhood N of X, which is a compact manifold with boundary, containing X in its interior, and having X as (strong) deformation retract, and then rounding the corner to make N a smooth manifold (for details see [71]). Characterising homotopy types of compact manifolds without boundary is much more delicate: we will turn to this in §7.8.

A map $f : V \to M$ between two smooth manifolds will be called a *smooth embedding* if $f(V)$ is a submanifold W of M, and f induces a diffeomorphism of V on W, where W has the induced structure. This is more stringent than the notion of (topological) embedding, where only a homeomorphism is required.

A map $f : V \to M$ between two smooth manifolds is called an *immersion* if f is smooth and, for each $P \in V$, $df_P : T_P V \to T_{f(Q)} M$ is injective. The following criterion uses the notion of proper map, which is defined and studied in §A.2.

Proposition 1.2.10 *(i) A map $f : V \to M$ is a smooth embedding if and only if it is both a (topological) embedding and an immersion.*

(ii) A map $f : V \to M$ is an embedding as a closed submanifold if and only if it is injective, proper, and an immersion.

Proof (i) It follows from the definition that if f is a smooth embedding, it is an embedding. To see that it is an immersion at P, choose a coordinate neighbourhood at $Q = f(P)$, with $x_1, \ldots, x_m$ the coordinate functions on M at Q, and such that $f(V)$ is given locally by $x_{\upsilon+1} = \ldots = x_m = 0$. By definition of the induced structure, $x_1 \circ f, \ldots, x_\upsilon \circ f$ define a coordinate neighbourhood of P in V say $y_i = x_i \circ f$. But then $df(\partial/\partial y_i) = \partial/\partial x_i$ and so df has rank υ at Q.

For the converse, let $f : V^\upsilon \to N^n$ be a smooth immersion and an embedding with image W. Let $P \in V$, $f(P) = Q$, and choose a coordinate neighbourhood $\varphi : U \to \mathbb{R}^m$ of Q in M such that $df^*(dx_1), \ldots, df^*(dx_\upsilon)$ form a basis for $T_P^\vee V$ - this is possible since f is an immersion. Write $y_i = x_i \circ f$: then since $dy_1, \ldots, dy_\upsilon$ form a basis for $T_P^\vee V$ by Corollary 1.2.6, $y_1, \ldots, y_\upsilon$ may be taken as coordinates in a neighbourhood of P. Since the other y_i are smooth functions, by the definition of smooth manifold we can write $y_i = g_i(y_1, \ldots, y_\upsilon)$ $(\upsilon < i \leq m)$ in a neighbourhood of P in V. Since f is an embedding, $x_i = g_i(x_1, \ldots, x_\upsilon)$ in a neighbourhood of Q in W. Thus W is locally defined by vanishing of the $n - \upsilon$ smooth functions $x_i - g_i(x_1, \ldots, x_\upsilon)$, which clearly have linearly independent differentials. So W is a submanifold, and it now follows that f defines a diffeomorphism of V on W.

(ii) follows since by Lemma A.2.3, a map is proper and injective if and only if it is an embedding as a closed subset. □

We have a hierarchy of conditions on a smooth map $f : V \to M$: proper embedding $\Rightarrow$ smooth embedding $\Rightarrow$ injective immersion $\Rightarrow$ immersion. None of the implications can be reversed: we now offer examples, which are illustrated in Figure 1.2.

The inclusion in $\mathbb{R}$ of $\{x \in \mathbb{R} \mid x > 0\}$ is a smooth embedding which is not proper; another example is a curve $(e^{-t} \cos t, e^{-t} \sin t)$ spiralling in to the origin in the plane.

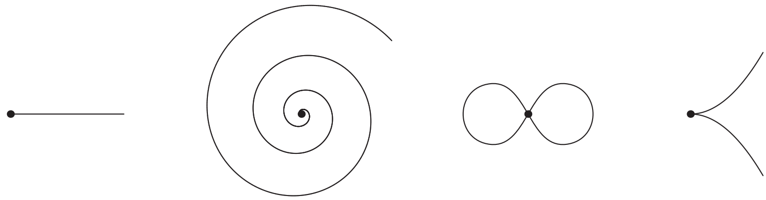

Figure 1.2 Examples which fail to give embeddings

The parametrisation $f(\theta) = (\sin(\frac{1}{2}\theta), \sin(\theta))$ defines a figure eight curve in the plane, with equation $y^2 = 4x^2(1 - x^2)$. As the differential df is nowhere zero, f is an immersion, but it is not injective. As θ runs from -2π to 2π, the point $f(\theta)$ starts at $(0, 0)$, describes a loop in $x < 0$ returning to the origin at $\theta = 0$, then describes a loop in $x > 0$.

However, if we take $t = \tan(\frac{1}{4}\theta)$ as parameter for the same curve, we have a map given by $g(t) = \left(\frac{2t}{1+t^2}, \frac{4t(1-t^2)}{(1+t^2)^2}\right)$. As t goes from $-\infty$ to $+\infty$, θ increases from -2π to 2π, so g is an injective immersion, but not an embedding.

The map $h : \mathbb{R} \to \mathbb{R}^2$ defined by $h(t) = (t^2, t^3)$ (a cusp) is a (topological) embedding which is not an immersion.

Theorem 1.2.11 *Any compact manifold M^m can be imbedded in a Euclidean space.*

Proof Let $\{\varphi_i : U_i \to \mathring{D}^m(3)\}$ be the coordinate neighbourhoods constructed in Theorem 1.1.4: since they are locally finite, and M compact, there are only a finite number. Also as in Theorem 1.1.5, let $\Phi_i(P) = Bp(2 - \|\varphi_i(P)\|)$ for P in the range of φ_i, 0 otherwise. Now define functions f_{ij} by

$$\begin{aligned} f_{i0}(P) &= \Phi_i(P) \\ f_{ij}(P) &= \Phi_i(P)x_j(\varphi_i(P)) \quad P \text{ in range of } \varphi_i \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then the f_{ij} are all smooth functions of P; if the range of i is $1 \leq i \leq N$, there are $(m + 1)N$ of them, so they define a smooth map $F : M^m \to \mathbb{R}^{(m+1)N}$. We assert that F is an embedding: since M is compact, it suffices by Proposition 1.2.10 to prove that F is injective and an immersion.

Since the $\varphi_i^{-1}(\mathring{D}^m(1))$ cover M, each $P \in M$ belongs to at least one of them. But in this set, $\Phi_i = 1$, $f_{ij}(P) = x_j(\varphi_i(P))$, and so the df_{ij} with $j > 0$ form a basis for $T_P^\vee M$. Thus $dF_P : T_P M \to T_{f(P)}\mathbb{R}^{(m+1)N}$ is injective, and so F is an immersion.

If $F(P) = F(Q)$, and $P \in \varphi_i^{-1}(\mathring{D}^m(1))$, then $1 = \Phi_i(P) = f_{i0}(P)$, and so $1 = f_{i0}(Q) = \Phi_i(Q)$, and $Q \in \varphi_i^{-1}(\mathring{D}^m(1))$ also. But in this set, we can take the $f_{ij}(= x_j)$ as coordinates. Since these have the same values for P and Q, we have $P = Q$. Thus F is also injective. □

Here we have presented a shortcut to the result: we will give a sharper statement in Theorem 4.2.2 and will see in Corollary 4.7.8 that embeddings are dense in the space of smooth maps between any manifolds M^m and V^v with $v > 2m$; if M is not compact we need V non-compact and must restrict to proper maps.

1.3 Fibre bundles

A map $\pi : T \to M$ is the projection of an *n-vector bundle* if M can be covered by open sets U_α such that

(i) There are homeomorphisms $\varphi_\alpha : U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$ such that, for all $x \in U_\alpha, y \in \mathbb{R}^n, \pi\varphi_\alpha(x, y) = x$.

(ii) For each pair (α, β) there is a continuous map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ such that, for all $x \in U_\alpha \cap U_\beta,\ y \in \mathbb{R}^n,\ \varphi_\beta(x, y) = \varphi_\alpha(x, g_{\alpha\beta}(x).y)$.

The space M is called the *base space* of the bundle, and T is its total space; $\mathbb{R}^n$ is the *fibre*; more precisely, the fibre over $m \in M$ is the preimage $\pi^{-1}(m)$.

If $\pi : T \to M$ is a vector bundle, and $V \subset T$ is such that $\pi \,|\, V$ is a vector bundle with $\pi^{-1}(x) \cap V$ a vector subspace of $\pi^{-1}(x)$ for each $x \in M$, then V is called a subbundle of T.

More generally, we can define fibre bundles. A *Lie group* is a smooth manifold G, which is also a group, such that the group operations $g \mapsto g^{-1}$, $(g, h) \mapsto gh$ are smooth maps $G \to G$, $G \times G \to G$. A *smooth action* of a Lie group G on a smooth manifold M is a smooth map $\phi : G \times M \to M$ which is a group action, i.e. which satisfies the identity $\phi(g_1, f(g_2, x)) = \phi(g_1 g_2, x)$. If the action is understood, it is frequently denoted by a dot: thus $\phi(g, x)$ becomes $g.x$. We will discuss Lie groups and smooth actions more fully in §3.

Given a smooth action of G on F, we define $\pi : T \to B$ to be the projection of a *smooth fibre bundle* with structure group G and fibre F if B can be covered by open sets U_α such that

(i) There are homeomorphisms $\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that, for all $x \in U_\alpha, y \in F, \pi\varphi_\alpha(x, y) = x$.

(ii) For each pair (α, β) there is a continuous map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that for $x \in U_\alpha \cap U_\beta,\ y \in F,\ \varphi_\beta(x, y) = \varphi_\alpha(x, g_{\alpha\beta}(x).y)$.

The simplest example is a product $T = M \times F$ given by a single chart: this is called a *trivial bundle*.

The structure of a bundle is determined by the maps $g_{\alpha\beta}$; two bundles with the same $g_{\alpha\beta}$ but different fibres are called *associated*. If the $g_{\alpha\beta}$ all have images in a subgroup G' of G, we say that the group of the bundle *reduces* to G', and we say that a bundle with group G', together with an isomorphism to the given bundle, defines a *reduction* of the structure group from G to G'. A map $\chi : M \to T$ is called a *cross-section* if $\pi \circ \chi = 1$.

A trivial vector bundle $\pi : T \to M$ is isomorphic to a product $M \times \mathbb{R}^n$. In this case each fibre of π is isomorphic to $\mathbb{R}^n$, each unit vector e_i of $\mathbb{R}^n$ defines a section $E_i : M \to M \times \mathbb{R}^n \cong T$, and for each $x \in M$ the vectors $E_i(x)$ give a basis of the vector space $\pi^{-1}(x)$ or, as one sometimes says, a *framing* of this vector space. Conversely, a set of sections of a vector bundle π defining a framing of each fibre gives an isomorphism $T \to M \times \mathbb{R}^n$, which may be called a framing or a trivialisation of the bundle; it is also a reduction of the structure group of π to the trivial group.

Given two vector bundles $\xi_1 = (\pi_1 : T_1 \to M)$ and $\xi_2 = (\pi_2 : T_2 \to M)$ over the same base space M we can construct a new vector bundle $\xi = \xi_1 \oplus \xi_2$ over M, called the direct sum or *Whitney sum* of the bundles ξ_1 and ξ_2: its fibre over any $m \in M$ is the direct sum of the fibres of ξ_1 and ξ_2 over m. In particular, the direct sum of ξ with a trivial line bundle is called the *suspension* of ξ. Two vector bundles ξ_1 and ξ_2 are said to be *stably isomorphic* if there exist a trivial bundle η and an isomorphism $\xi_1 \oplus \eta \cong \xi_2 \oplus \eta$.

If we have two fibre bundles $\pi_1 : T_1 \to M_1$, $\pi_2 : T_2 \to M_2$ with the same group G and fibre F, a *G-bundle map* is given by maps $f : T_1 \to T_2$, $b : M_1 \to M_2$ with $\pi_2 \circ f = b \circ \pi_1$ such that if $U_\alpha \subset M_1$ and $V_\beta \subset M_2$ are open sets as above, there exists a continuous map $g_{\alpha,\beta} : U \cap b^{-1}(V) \to G$ such that for $x \in (U \cap b^{-1}(V))$, $y \in F$ we have $\varphi_\beta(b(x), y) = f(\varphi_\alpha(x, g_{\alpha,\beta}(x).y))$.

The total space T of a smooth fibre bundle admits a natural structure as smooth manifold such that the maps φ_α are diffeomorphisms on open submanifolds. For if we use these to define coordinate neighbourhoods, then we have smooth transformations of coordinates on the intersections.

The reason for introducing these concepts at this point is that the set of all tangent vectors to a smooth manifold M has a natural structure of a vector bundle.

Write $\mathbb{T}(M) = \cup\{T_P M : P \in M\}$ for the set of all tangent vectors to M. Define $\pi : \mathbb{T}(M) \to M$ by $\pi(T_P M) = P$. Let $H_\alpha : U_\alpha \to V_\alpha$ be a set of local coordinate systems, with the U_α covering M, and for $P \in U_\alpha$, $v \in \mathbb{R}^m$, define $\varphi_\alpha(P, v)$ as the tangent vector at P determined by $\sum v_i \partial/\partial x_i$. Then for each α, the mapping $\varphi_\alpha : U_\alpha \times \mathbb{R}^m \to \pi^{-1}(U_\alpha)$ is bijective. On $U_\alpha \cap U_\beta$, denoting the two systems of coordinates by x^α, x^β; we have, by the usual transformation rule,

$$\partial/\partial x_i^\beta = \sum_\alpha (\partial x_j^\alpha/\partial x_i^\beta)(\partial/\partial x_j^\alpha),$$

so we define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_m(\mathbb{R})$ by

$$g_{\alpha\beta}(Q) = \left(\frac{\partial x_j^\alpha}{\partial x_i^\beta}\right)_Q .$$

Then $g_{\alpha\beta}$ is a smooth mapping, and satisfies the condition above. Now take the φ_α (or rather their inverses) as coordinate neighbourhoods, and thus define on $\mathbb{T}(M)$ the structure of smooth manifold, which in particular gives it a topology, with the φ_α homeomorphisms. Thus we have a smooth vector bundle.

We say that $\pi : \mathbb{T}(M) \to M$ is the *tangent bundle* to M. Write $\mathbb{T}^0(M)$ for the zero cross-section, i.e. the set of zero tangent vectors. In general, a smooth cross-section of $\mathbb{T}(M)$ is called a *vector field* on M. We can identify the set of smooth vector fields on M with the set of derivations from $\mathcal{F}_M$ to itself, for if ξ is such a derivation then for each $P \in M$, $f \mapsto \xi(f)(P)$ is a tangent vector at P.

Any bundle associated to $\mathbb{T}(M)$ via a linear representation of $GL_m(\mathbb{R})$ is called a *tensor bundle* (and a points of it are tensors, whose type is determined by the representation). The bundle $\mathbb{T}^\vee(M)$ given by the adjoint representation is the *bundle of differential 1-forms* on M^m; its fibre over P is the dual space $T_P^\vee M$ to $T_P M$.

The bundle whose fibre over P is the set of all positive definite quadratic forms on $T_P M$ is called the *Riemann bundle*, and any smooth cross-section of it a *Riemannian structure* on M. In local coordinates this takes the form $\sum_1^m g_{i,j}(x) dx_i dx_j$.

We now prove the fundamental.

Theorem 1.3.1 *Every smooth manifold M^m has a Riemannian structure.*

Proof Let $\{U_\alpha\}$ be an open covering such that we have charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ (see, for example, Theorem 1.1.4). Let Ψ_α be a partition of unity strictly subordinate to this cover. Now $\mathbb{R}^m$ has the standard Euclidean Riemannian structure: $\sum_{i=1}^m dx_i^2$. We write $ds^2 = \sum_\alpha \Psi_\alpha(\sum_{i=1}^m d(x_i \circ \varphi_\alpha)^2)$. Since the U_α are locally finite, the sum is defined; since the partition was strictly subordinate to the cover, the sum is smooth. Since a linear combination of positive definite quadratic forms is again positive definite, ds^2 is everywhere positive definite. Thus it defines a Riemannian structure on M^m. $\square$

Given a Riemannian structure on M^m, we can choose orthonormal bases in the fibres of $\mathbb{T}(M)$ by applying the Gram–Schmidt orthogonalisation process. This will modify the maps $\varphi_\alpha : U_\alpha \times \mathbb{R}^m \to \pi^{-1}(U_\alpha)$ so as to preserve the inner product on the fibres. Indeed, consider φ_α as a map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ depending on certain parameters, and set $\varphi'(e_i) = \sum_{j \le i} \lambda_{ij}\varphi(e_j)$, where the λ_{ij} with $j < i$ are chosen inductively to make the $\varphi'(e_i)$ orthogonal and the $\lambda_{ii} > 0$ so as to make the $\varphi'(e_i)$ unit vectors. Then the λ_{ij} are also smooth functions of the parameters.

A Riemannian structure on M^m determines a reduction of the group of the tangent bundle to the orthogonal group O_m; conversely, a reduction to O_m corresponds to a Riemannian structure. We also observe that the choice of an inner product on T_PM allows us to identify T_PM with $T_P^\vee M$. For a Riemannian manifold, we shall usually do this.

M^m is called *orientable* if the group of the tangent bundle is reducible to $GL_m^+(\mathbb{R})$, *oriented* if the group is so reduced. Since the coordinate transformations were given by the matrices $(\partial x_j^\alpha / \partial x_i^\beta)$, the condition is that all the Jacobian determinants are positive. The total space of the bundle associated to the tangent bundle with fibre $GL_m(\mathbb{R})/GL_m^+(\mathbb{R}) = \mathbb{Z}_2$ is a double covering $\tilde{M}$ of M, called the *orientation covering*. Its projection on M, together with coordinate neighbourhoods of M, can be taken as coordinate neighbourhoods, so $\tilde{M}$ is a smooth manifold. By the definition, all the Jacobians occurring are positive, so this manifold is orientable.

If M itself is orientable, $\tilde{M}$ consists of two copies of M; if M is connected and non-orientable, $\tilde{M}$ is connected. If M is non-orientable, we can find a closed chain of coordinate neighbourhoods, each overlapping the next, such that the number of negative Jacobians is odd.

We can specify an orientation of M at a point P by giving an isomorphism of $\mathbb{R}^m$ on T_PM, or equivalently, an ordered basis $(e_1, \ldots, e_m)$ of T_PM; another basis defines the same orientation if the determinant of the basis change is positive.

If M has a Riemannian structure, an orientation gives a reduction of the group of the tangent bundle from O_m to SO_m.

1.4 Integration of smooth vector fields

We have already seen that a smooth path in a manifold has a tangent vector at each of its points. We now show that, conversely, a tangent vector field can be integrated to give a deformation (family of paths) in the manifold. This is an essential technique for constructing deformations.

The key is Picard's existence theorem for differential equations.

Theorem 1.4.1 (Existence Theorem for Ordinary Differential Equations) *Let U be an open subset of $\mathbb{R}^n$, K a compact subset of U. Given a system of equations $\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x})$, where $\mathbf{X}$ is a smooth function on U to $\mathbb{R}^n$, then for some $\varepsilon > 0$ there exists a unique smooth function $\mathbf{x} = g(\mathbf{x}_0, t) = g_t(\mathbf{x}_0)$ on $K \times E$ to U, where E is the set $|t| < \varepsilon$, satisfying the equation, and such that $\mathbf{x}_0 = g_0(\mathbf{x}_0)$.*

A proof is given in [40, Theorem 10.4.5].

We next translate this from the language of analysis to that of geometry, and then see how to reformulate it. First write $\mathbf{X} = (X_1, \ldots, X_n)$ and define a vector field ξ on U by $\xi = \sum_i X_i \frac{\partial}{\partial x_i}$. Then the given equation becomes $\xi(x_i) = X_i$.

For any smooth function f on U, and $\mathbf{x} = \varphi_t(\mathbf{x}_0)$, we have

$$\frac{df(\mathbf{x})}{dt} = \sum_i \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \sum_i X_i \frac{\partial f}{\partial x_i} = \xi(f),$$

so this relation is not restricted to f being a coordinate function x_i.

We define a *flow* on a smooth manifold M^m as a map $\varphi : V \to M \times \mathbb{R}$ with V some neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$, where we write $\varphi_t(P)$ for $\varphi(P, t)$, such that

(i) $\varphi_0(P) = P$ for all $P \in M$,

(ii) $\varphi_s(\varphi_t(P)) = \varphi_{s+t}(P)$ whenever both are defined.

A flow gives rise to a vector field ξ on M as follows. For $f \in \mathcal{F}_M$, $P \in M$, we set

$$\xi_P(f) = \lim_{t\to 0} \frac{f(\varphi_t(P)) - f(P)}{t} = \left.\frac{d}{dt} f(\varphi_t(P))\right|_{t=0}.$$

It is clear that ξ_P is a tangent vector to M at P, and that ξ_P varies smoothly with P, so that ξ is a vector field. Substituting $P = \varphi_s(Q)$, and using (ii), it follows that

$$\xi_{\varphi_s(Q)}(f) = \frac{d}{dt} f(\varphi_{t+s}(Q))|_{t=0} = \frac{d}{dt} f(\varphi_t(Q))|_{t=s}.$$

We now show that any vector field defines a flow.

Theorem 1.4.2 *Let M^m be a smooth manifold, ξ a vector field on M. Then there is a flow $\varphi : U \to M \times \mathbb{R}$ giving rise to ξ, and any two such flows agree on some neighbourhood of $M \times \{0\}$.*

Proof Any $P \in M$ lies in a compact set K contained in the interior of some V, where $H : V \to U$ is a coordinate neighbourhood. In U, write ξ in local coordinates as $\sum_1^n X_i(x)\partial/\partial x_i$, and consider the system $\frac{dx_i}{dt} = X_i(x)$. Apply Theorem 1.4.1: we find $\varepsilon > 0$, and a smooth function $x = g(x_0, t)$ for $x_0 \in K$, $|t| < \varepsilon$, uniquely determined by the equation. We define φ_t in V by this relation in U.

The fact that the functions defined by different coordinate neighbourhoods agree on the intersection follows by the uniqueness, and the fact that the equations solved are simply derived from each other by change of variables.

The functions $\varphi_{s+t}(P) \to g(\mathbf{x}_0, s+t)$ satisfy the same equation, with initial value $g(\mathbf{x}_0, s)$. By the uniqueness, $g(\mathbf{x}_0, s+t) = g(g(\mathbf{x}_0, s), t)$, i.e., $\varphi_{s+t}(P_0) = \varphi_t\varphi_s(P_0)$, at least on some neighbourhood of $s = t = 0$ in $M \times \mathbb{R}^2$. □

We have seen that, for each point $P \in M$, $\varphi_t(P)$ is defined for $t = 0$, and that if it is defined for $t = t_0$, then it can be uniquely defined in some neighbourhood $|t - t_0| < \varepsilon$. There is thus a bound $B_P \leq \infty$ such that $\varphi_t(P)$ is defined for all $0 \leq t < B_P$, but no further.

Lemma 1.4.3 *Either $B_P = \infty$ or the map $[0, \infty) \to M$ given by $t \mapsto \varphi_t(P)$ is proper.*

Proof We need to show that if $B_P < \infty$ and K is a compact subset of M, then the set of t with $\varphi_t(P) \in K$ is compact, i.e. that it has an upper bound strictly less than B_P.

It follows from Theorem 1.4.2 and Corollary A.2.4 that there is a number $\varepsilon > 0$ such that $\varphi_t(Q)$ is defined for all $Q \in K$ and all t with $|t| < \varepsilon$. Suppose there exists $t > B_P - \varepsilon$ with $Q = \varphi_t(P) \in K$. Then it follows that the definition of $\varphi_t(P)$ extends beyond $t = B_P$, contradicting our hypothesis. □

One sometimes wishes to solve an equation of the form $\frac{dx}{dt} = X(x, t)$. This is not essentially different in nature: merely take t as an additional coordinate, with $\frac{dt}{dt} = 1$. In geometrical terms, we have a 'time-dependent vector field' $\xi(t)$ defined on M, and treat this as a vector field $\xi + \partial_t$ on $M \times \mathbb{R}$, i.e. a vector field on $M \times \mathbb{R}$ whose projection on $\mathbb{R}$ is equal to ∂_t. The corresponding flow $\varphi : V \to (M \times \mathbb{R}) \times \mathbb{R}$ then has the property that whenever $\varphi_s(P, t)$ is defined, its second component is equal to $s + t$.

If we have a flow on M defined on the whole of $M \times \mathbb{R}$ and satisfying $\varphi_s(\varphi_t(P)) = \varphi_{s+t}(P)$ everywhere, then each φ_t is a smooth map $M \to M$ and has an inverse map φ_{-t}, hence is a diffeomorphism. The map $\varphi : M \times \mathbb{R} \to M$ thus defines a differentiable group action of the additive group $\mathbb{R}$ on M, often called a 1-parameter group of diffeomorphisms of M. In general, a vector field on M is called *complete* if it generates a 1-parameter group of diffeomorphisms of M. We collect some simple sufficient conditions for completeness.

Proposition 1.4.4 *(i) If M^m is compact, each vector field on M is complete.*

(ii) The constant vector field $\partial/\partial t$ on $\mathbb{R}$ is complete.

(iii) If ξ is a complete vector field on V, and M is any manifold, $\xi \oplus 0$ is complete on $V \times M$.

(iv) If ξ is complete, and ξ' agrees with ξ outside a compact subset of M, then ξ' is also complete.

(v) If M has a complete metric, any bounded vector field is complete.

Proof (i) follows from Lemma 1.4.3, since there are no proper maps $[0, \infty) \to M$ if M is compact.

(ii) and (iii) are trivial.

(iv) $\varphi_t(P)$ is defined for all P and t, by hypothesis; since ξ and ξ' differ only on a compact set, there is an $\varepsilon > 0$ such that $\varphi'_t(P)$ is defined for all P and all $|t| < \varepsilon$. It follows that it is defined for all t.

(v) Since ξ is bounded, there is a uniform bound $\rho(\varphi_t(P), \varphi_s(P)) < A|s - t|$. Thus as t converges to any limit B, the points $\varphi_t(P)$ form a Cauchy sequence, so converge since the metric is complete. Thus a limit value B_P as in Lemma 1.4.3 cannot exist. □

1.5 Manifolds with boundary

We now extend the notion of manifold by considering manifolds with boundary. In the sequel these will play as much part as the manifolds already defined; we have merely deferred the definition till this point to help concentrate ideas.

N^n is a smooth *manifold with boundary*, or bounded manifold, if it satisfies all the defining conditions of a smooth manifold, with the exception that we allow coordinate neighbourhoods to map onto open sets in either $\mathbb{R}^n$ or $\mathbb{R}^n_+$, where $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$.

Since we will not always include the phrase 'with boundary', we also use the term *closed manifold* for a compact manifold without boundary (the phrase 'open manifold' is sometimes used for a non-compact manifold without boundary).

A point is a *boundary point* of N if its image by the chart lies on the boundary $\{x_1 = 0\}$ of $\mathbb{R}^n_+$: it is clear that this property is preserved on change of coordinate neighbourhood. The set of such points is the *boundary* of N, which we always denote by ∂N. The restrictions of coordinate charts give ∂N the structure of a smooth manifold of dimension $n - 1$. We write $\mathring{N} := N \setminus \partial N$, the 'interior' of N. This is a manifold, an open submanifold of N. A simple example of manifold with boundary is the unit disc D^n, with boundary $\partial D^n = S^{n-1}$.

The concept of smooth function on a manifold with boundary is clarified by the following.

Theorem 1.5.1 (Whitney's Extension Theorem) *Let f be a smooth function defined on the open set $x_1 > 0$ of $\mathbb{R}^n$, and suppose that f and all its partial* *derivatives extend to continuous functions on $\mathbb{R}^n_+$. Then there is a smooth function g on $\mathbb{R}^n$ which agrees with f in its range of definition.*

Whitney's proof, which establishes results of much greater generality, can be found in his paper [173].

A function on $\mathbb{R}^n_+$ is called smooth if it satisfies the equivalent conditions of the theorem. With this as the definition on a chart, we extend to a definition of smooth functions and maps on manifolds with boundary in general.

A diffeomorphism between manifolds with boundary is a smooth bijection whose inverse is smooth. Necessarily, the two boundaries correspond.

We also say N^n is a *manifold with corner* if it satisfies the defining conditions for a smooth manifold, except that coordinate neighbourhoods may map into open sets in any of $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, where $\mathbb{R}^n_{++}$ denotes the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \geq 0$, $x_2 \geq 0$. Topologically, as opposed to differentiably, N is a manifold with boundary; its boundary ∂N consists of points corresponding to $x_1 = 0$ (in $\mathbb{R}^n_+$) or to $x_1 x_2 = 0$ (in $\mathbb{R}^n_{++}$). Points corresponding to $x_1 = x_2 = 0$ in $\mathbb{R}^n_{++}$ form the *corner* $\angle N$, which is a smooth manifold of dimension $n - 2$.

If M_1, M_2 are manifolds with boundary, products of coordinate neighbourhoods of M_1 and M_2 give coordinate neighbourhoods in $M_1 \times M_2$ which (up to a permutation of coordinates) are appropriate for a manifold N with corner. We have $\partial(M_1 \times M_2) = (\partial M_1 \times M_2) \cup (M_1 \times \partial M_2)$ and $\angle(M_1 \times M_2) = \partial M_1 \times \partial M_2$. In this, as in most other important cases, $\angle N$ separates ∂N into two parts; of course this is always true locally.

The discussion of orientability and orientations for manifolds with boundaries (and perhaps corners) is essentially the same as before. However, at boundary points $P \in \partial N$, we must distinguish between *inward-* and *outward-pointing* tangent vectors: in terms of a coordinate neighbourhood of P, these are vectors $\Sigma \lambda_i \partial/\partial x_i$ with $\lambda_1 > 0$ resp. $\lambda_1 < 0$. If $\lambda_1 = 0$, we call the vector tangent to the boundary; indeed, if $i : \partial N \to N$ is the inclusion map, such vectors form the image of di, so do come from tangent vectors of ∂N. If $p : \mathbb{R}^+ \to N$ is a path with $p(0) = P$, we see by considering local coordinates that the tangent to p at P has $\lambda_1 \geq 0$. Thus the terminology is independent of the choice of local coordinates. Boundaries of manifolds and submanifolds are pictured in Figure 1.3.

In the presence of boundaries or corners, there are various corresponding extensions of the notion of submanifold. A subset M of a manifold N with boundary is a *submanifold* if it satisfies the same conditions as when N is not bounded, except that the coordinate neighbourhood φ may map U to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Thus in a neighbourhood of a point of M, the pair (N, M) is locally like

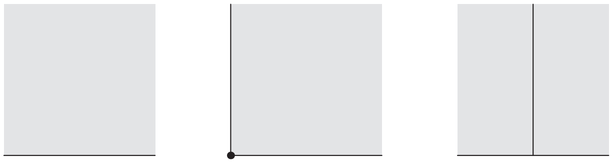

Figure 1.3 Boundaries of manifolds and submanifolds

$(\mathbb{R}^n, \mathbb{R}^m)$ or $(\mathbb{R}^n_+, \mathbb{R}^m_+)$. Geometrically, we can say that M meets ∂N transversely (for the general notion of transversality, see §4). M has an induced structure of manifold with boundary, just as above, and we have $\partial M = M \cap \partial N$. The definition includes the case when ∂M is empty, and M is disjoint from ∂N; then M is a submanifold of $\mathring{N}$. A result corresponding to Proposition 1.2.10 continues to hold.

As before, submanifolds which are not closed may have bad behaviour. We usually require the condition $\bar{M} \cap \partial N = M \cap \partial N$, which excludes such examples as $N = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, $M = \{(0, y) \in \mathbb{R}^2 \mid y > 0\}$.

We could go on to consider further cases where the pair (N, M) is modelled on *any* product of pairs $(\mathbb{R}, \mathbb{R})$, $(\mathbb{R}, \mathbb{R}^+)$, $(\mathbb{R}, 0)$, $(\mathbb{R}^+, \mathbb{R}^+)$, $(\mathbb{R}^+, 0)$, but restrict to the following.

If N^n is a manifold, perhaps with boundary, we define a closed subset M^m to be a *closed submanifold with boundary* of N^n if each point of M^m has a neighbourhood U in N and a smooth chart $\varphi : U \to \mathbb{R}^n$ with $\varphi(U)$ an open set in $\mathbb{R}^n$ or $\mathbb{R}^n_+$ and $\varphi(U \cap M)$ its intersection with $\mathbb{R}^m$ or with $\mathbb{R}^m \cap \{x \mid x_2 \geq 0\}$. Thus in the case when N has a boundary we allow M to have a corner, and $\angle M$ divides ∂M into $M \cap \partial N$ and the closure of $M \cap \mathring{N}$.

The results on vector fields and flows extend as follows to manifolds with boundary. First, the local existence theorem adapts as follows.

Lemma 1.5.2 *If U is open in $\mathbb{R}^n_+$, $K \subset U$ compact, and ξ a smooth vector field on U, inward pointing along $U \cap \partial\mathbb{R}^n_+$, then for some $\varepsilon > 0$ there is a map $\varphi : K \times [0, \varepsilon] \to U$ with $\partial\varphi(x, t)/\partial t = \xi$ and $\varphi(x, 0) = x$ for all $x \in K$.*

For the global case, if ξ is a vector field on M, inward pointing at all points of ∂M, it follows as for Theorem 1.4.2 that there is a flow $\varphi : V \to M \times \mathbb{R}_+$ for some neighbourhood V of $M \times \{0\}$ in $M \times \mathbb{R}^+$.

Now suppose more generally that along some components of M, whose union we denote by $\partial_- M$, ξ is inward pointing, and along the rest (forming $\partial_+ M$), it is outward pointing. Then for each $P \in M$ we have $\varphi(x, t)$ defined for t in some interval in $\mathbb{R}$ containing 0 and with end points A_P, B_P say, and we have

Lemma 1.5.3 *Either $B_P = \infty$ or $\varphi(P, B_P) \in \partial_+ M$ or the map $[0, \infty) \to M$ given by $t \mapsto \varphi_t(P)$ is proper.*

The following result is the first step towards the construction of diffeomorphisms.

Theorem 1.5.4 *Suppose M a compact manifold with boundary $\partial_- M \cup \partial_+ M$, ξ a vector field on M, pointing inward on $\partial_- M$ and outward on $\partial_+ M$, and $f : M \to \mathbb{R}$ such that $\xi(f) > 0$ on M. Then $M \cong \partial_- M \times I$.*

Proof Integrating ξ gives a flow φ which is defined on a neighbourhood of $\partial_- M \times \{0\}$ in $\partial_- M \times \mathbb{R}_+$. Now apply Lemma 1.5.3. Since M is compact, there is no proper map $[0, \infty) \to M$. Since $\xi(f) > 0$ on the compact manifold M, it has a positive lower bound c, so $f(\varphi_{t+T}(P)) \geq f(\varphi_t(P)) + cT$, and as f also must be bounded, the case $B_P = \infty$ is ruled out. Thus each orbit of the flow can terminate only on $\partial_+ M$ and likewise (as t decreases) on $\partial_- M$. There is thus a smooth positive function g on $\partial_- M$ such that for $P \in \partial_- M$, the flow is defined at (P, t) if and only if $0 \leq t \leq g(P)$. Now the map $(P, t) \mapsto \varphi(P, \frac{t}{g(P)})$ gives a diffeomorphism of $\partial_- M \times I$ on M. □

If N is a manifold with boundary, a *collar neighbourhood* of ∂N in N is an embedding $\psi : \partial N \times I \to N$ as submanifold with boundary, extending the projection of $\partial N \times 0$ on ∂N. The use of collars will often enable us, when discussing manifolds with boundary, to avoid special difficulties arising at the boundary. We now establish their existence.

Theorem 1.5.5 *For every manifold with boundary, the boundary has a collar neighbourhood.*

Proof Each point $P \in \partial N$ lies in the domain of a coordinate neighbourhood U_α with a map $\varphi_\alpha : U_\alpha \to \mathring{D}^n_+$. We may suppose these chosen so that the ∂U_α cover ∂N. Hence the U_α together with $U_0 := N \setminus \partial N$ form an open cover of N. By Theorem 1.1.5 we can pick a strictly subordinate locally finite smooth partition of unity δ_α, δ_0.

We next construct a vector field on N which is inward pointing along ∂N. The vector field $\partial/\partial x_1$ on U^+ corresponds under φ_α to a smooth vector field ξ_α on U_α, which is inward pointing. Then $\delta_\alpha \xi_\alpha$ gives a smooth vector field on N, vanishing outside U_α. Now consider the smooth vector field $\xi := \sum \delta_\alpha \xi_\alpha$ on N. Each point P of ∂N lies in the support of some δ_α, so in the chart φ_α, the coefficient of $\partial/\partial x_1$ in $\delta_\beta \xi_\beta$ at P is non-negative for every β and positive for α, hence ξ is inward pointing at P.

We can now integrate ξ on some neighbourhood of $N \times \{0\}$ in $N \times \mathbb{R}^+$ to give a map to N. We are only interested in the restriction ψ to a neighbourhood

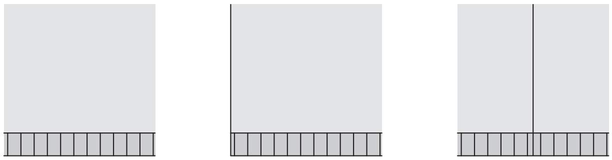

Figure 1.4 Collars of manifolds and submanifolds

W_0 of $\partial N \times \{0\}$ in $\partial N \times \mathbb{R}^+$. Along $\partial N \times \{0\}$ the map is the inclusion of ∂N in N, and by (iii) of the theorem, the derivative with respect to t is the vector field ξ. Since ξ is inward pointing, it follows from Theorem 1.2.5 that the map ψ is a local diffeomorphism. By Corollary A.2.6 there is a (smaller) neighbourhood W_1 of $\partial N \times \{0\}$ on which ψ is an embedding. By Proposition 1.1.7 (i) we can choose a smooth positive function g on ∂N such that W_1 contains $W_2 := \{(x, t) \in \partial N \times \mathbb{R} \mid x \in \partial N,\ 0 \le t \le g(x)\}$.

The map $(x, t) \to \psi(x, tg(x))$ now gives the desired collar neighbourhood. □

Extensions of the argument enable us to establish the existence of collars compatible with corners and submanifolds. It will be convenient to introduce the following terminology. For M a manifold with corner, a subset Q of ∂M is a *smooth part* if $\partial Q = Q \cap \angle M$. Thus the interior of Q is a union of connected components of $\partial M \setminus \angle M$.

Proposition 1.5.6 *(i) For N a manifold with corner and Q a smooth part of ∂N, there is a smooth embedding of $Q \times I$ in N giving a neighbourhood of Q in N.*

(ii) For N a manifold with boundary, M a submanifold, there is a collar neighbourhood of ∂N whose restriction to $\partial M \times I$ gives a collar neighbourhood for ∂M.

(iii) For N a manifold with boundary, M a submanifold with boundary, so $\angle M$ separates ∂M into $\partial_0 M := M \cap \partial N$ and $\partial_1 M$, there is a collar neighbourhood of ∂N whose restriction to $\partial_0 M \times I$ gives a collar neighbourhood as in (i).

Collars of manifolds and submanifolds are illustrated in Figure 1.4.

Proof Once we have constructed suitable vector fields in coordinate neighbourhoods, the piecing together using partitions of unity and integration of the vector field to give a local diffeomorphism proceeds in just the same way as above. But the local vector field can also be taken as $\partial/\partial x_1$ in each case.

For (i) it is sufficient to consider a chart $\varphi : U \to \mathbb{R}^n$ at $P \in \partial Q$ taking $U \cap (\partial N \setminus Q)$ to $x_2 = 0$ and $U \cap Q$ to $x_1 = 0$. Integrating $\partial/\partial x_1$ gives translation in the x_1 direction, which preserves $\partial N \setminus Q$.

For (ii) we need to consider points $P \in \partial M$. But here, by definition of submanifold, we have a chart $\varphi : U \to \mathbb{R}^n$ at P taking M to the subspace $\mathbb{R}^m$ where all but the first m coordinates vanish. Again this is preserved by the vector field $\partial/\partial x_1$.

For (iii), other points of $\partial_0 M$ are as in (ii), while at a point $P \in \angle M$, we have a chart $\varphi : U \to \mathbb{R}^n$ taking M to the subset of $\mathbb{R}^m$ with $x_1 \geq 0$ and $x_2 \geq 0$, and the same vector field remains suitable. □

1.6 Notes on Chapter 1

§1.1 The concept of manifold gradually evolved during the nineteenth century, beginning with the cases of curves and surfaces in Euclidean space, with successive steps taken by Riemann (who considered the n dimensional case) and Poincaré (who introduced charts). Manifolds not considered as subsets of Euclidean space first appeared in 1931 in the book [156] by Veblen and Whitehead; see also Weyl [172]. A decisive step was taken by Whitney [175] in 1936, who was the first to prove that any abstract manifold could be regarded as a manifold embedded in Euclidean space.

The use of atlases allows several variations of the definition giving related concepts: for example, instead of requiring the coordinate transformations $\psi_{\alpha,\beta}$ to be smooth, we could have required them merely to be continuous, giving topological manifolds; or to have all partial derivatives of order $\leq r$ defined and continuous, giving C^r-manifolds; or had charts as open subsets of $\mathbb{C}^r$ with holomorphic coordinate transformations, giving complex manifolds.

Any smooth atlas defines a smooth structure; conversely, the set of all smooth charts is a unique maximal atlas, and we could take 'maximal atlas' as the basic concept.

Alternatively, for each $P \in M$, we can write $\mathcal{F}_P$ for the ring of germs at P of elements of $\mathcal{F}$. The rings $\mathcal{F}_P$ fit to give a sheaf, and we can recover $\mathcal{F}$ from the sheaf of rings $\mathcal{F}_P$ as the ring of global sections. Axiom (M1) is part of the definition of sheaf; (M2)-(M3) easily translate into axioms on the sheaf.

Since our main interest is in compact manifolds (where the proofs are easier), the reader new to the subject can afford to ignore most of the references to topology, though of course the model example $\mathbb{R}^n$ is not compact.

It can be shown that in the presence of axioms (M1-M3), the following further conditions are equivalent for smooth manifolds which are connected (more generally if the set of components is (at most) countable):

M is a countable union of compact sets (the above condition (M4)),

Every open covering of M has a locally finite refinement (M is paracompact),

M has a countable base of open sets,

There is an embedding of M in Euclidean space,

The topology of M is metrisable.

We have seen in Theorem 1.1.4 that the first condition implies the second. The third follows since as M is covered by coordinate neighbourhoods, so any compact subset of M is contained in the union of finitely many, M is covered by countably many coordinate neighbourhoods. Since $\mathbb{R}^n$, and hence any open subset, has a countable base of open sets, the same follows for M. The other conditions follow from Theorems 1.2.11 and 2.1.1.

More general results of this kind are also known for topological spaces satisfying appropriate local conditions.

Examples satisfying (M1-M3) but not (M4) can be constructed, but such examples do not occur naturally. It is hard to obtain results of interest about such objects, and we do not consider them further.

§1.2 Lemma 1.2.3 is due to Marston Morse.

Proofs of Theorem 1.2.5 can be found in any good book on analysis, for example in [40].

§1.3 We give here merely the definitions necessary for the first two chapters of this book. Smooth group actions will be more fully treated in Chapter 3.

We refer the reader to Steenrod's book [144] for a systematic account of fibre bundles: this is the classic exposition. Many others have appeared since; another good reference is [77]. See also Appendix B.

§1.4 Proofs of Theorem 1.4.1 can also be found in any good book on analysis: in [40] both Theorems 1.2.5 and 1.4.1 are obtained as simple applications of the Contraction Mapping Theorem. Another reference is Hurewicz [75, 2.5]. The little book [83] gives slick treatments of all the topics up to this point, in a somewhat abstract framework.

§1.5 Manifolds with boundary were, I believe, first introduced by Poincaré.