6

Immersions and embeddings

We saw in Chapter 4 that a map $V^v \to M^m$ in general position is already an embedding if $m > 2v$. If this condition fails, we still have effective techniques for constructing embeddings, and will describe some of the main results in this chapter.

For immersions, the results give a complete reduction of the problem to a problem in homotopy theory. The proof of this major result is somewhat technical, and the details will not be required elsewhere.

We will now need to assume rather more familiarity with homotopy theory than in earlier chapters, and refer to Appendix B for a summary of the relevant definitions and results.

The theory of embeddings begins with a technique introduced by Whitney for removing pairs of self-intersections of a smooth n-manifolds in a $2n$-manifold (if $n \geq 3$). We describe this in some detail in §6.3: it was used as a key tool in §5.5. We then apply it to discuss embeddings of S^n in $2n$-manifolds.

The essential idea of this technique was generalised by Haefliger to maps $V^v \to M^m$ whenever $2m \geq 3(v+1)$ – a condition we call the *metastable* range. There are several related results giving homotopy theoretic criteria for deforming a map to an immersion, or to an embedding, or for finding a regular homotopy of an immersion to an embedding; each one also has a simplified form when the target is Euclidean space, and also a companion criterion for finding a diffeotopy of the constructed embeddings. We describe these results, but confine ourselves to an outline of the rather involved proof.

6.1 Fibration theorems

A map $f : E \to B$ is said to be a *fibration* if given a space K, a map $a : K \to E$ and a homotopy $b : K \times I \to B$ such that $b \,|\, (K \times 0) = p \circ a$, there exists a homotopy $c : K \times I \to X$ such that $a = c \,|\, (K \times 0)$ and $b = p \circ c$. We also say that f has the covering homotopy property (CHP). If this holds for K a finite CW-complex, it follows for any CW-complex; it also follows if (K, L) is a CW-pair that c can be chosen to extend a lift already given on $L \times I$. It suffices to require this condition for pairs $(K, L) = (D^n, S^{n-1})$.

If $f : E \to B$ is a fibration, $*$ a point of B, and $F = f^{-1}(*)$ (called the *fibre*), there is an exact homotopy sequence $\dots \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \dots$.

The term 'fibration' recalls the fact that (see Lemma B.1.1) the projection map of a fibre bundle has the CHP.

If $p : E \to B$ and $p' : E' \to B'$ are fibrations, a map $f : E \to E'$ is called a *fibre map* if $p(e_1) = p(e_2)$ implies $p'(f(e_1)) = p'(f(e_2))$, so that there is a map $g : B \to B'$ with $g \circ p = p' \circ f$.

In this section I give fibration theorems for spaces of cross-sections and of (smooth) embeddings to prepare the way for the next section.

Theorem 6.1.1 *Let M be a smooth manifold, $V \subset M$ a compact submanifold. Then the map Diff$(M) \to$ Emb(V, M) is a fibration.*

This is an upgrading of the Diffeotopy Extension Theorem 2.4.2, and the same proof goes through with minor changes.

Proof We may suppose given a space P, a map $f : P \to \mathrm{Diff}(M)$. and a homotopy $g : P \times I \to \mathrm{Emb}(V, M)$ of the restriction of f, and need to lift g to a homotopy of f. Denote by $i : V \to M$ the inclusion, $f' : P \times M \to M$ and $g' : P \times I \times V \to M$ the maps associated to f and g (thus $f'(p, x) = f(p)(x)$). Thus for $p \in P, x \in V$ we have $g'(p, 0, x) = f'(p, x)$.

For each $(p, x) \in P \times V$ we have a path $g'(p, t, x)$ in M; denote the tangent vector to this path at $g'(p, t, x)$ by $\xi(p, t, x)$. We need to construct a tangent vector field $\eta(p, t, y)$ to M for each $(p, t) \in P \times I$, depending smoothly on $y \in M$ and continuously on p and t, and extending ξ.

The argument of Theorem 2.4.2 now goes through, but (i) allowing the additional parameter $p \in P$ and (ii) not insisting on smoothness as a function of the variables p and t: these do not significantly affect the argument. □

As for Theorem 2.4.2, compactness of V is essential to the argument. In Cerf [36] and Palais [118] we find a more precise result: the fibration is locally trivial, where the spaces of sections have the C^∞ topology.

Lemma 6.1.2 *Let $f : E \to B$ be a fibration; let $K \subset L \subset B$ be CW-complexes. Write $\Gamma(K)$ for the space of cross-sections of f over K. Then restriction defines a fibration $\Gamma(L) \to \Gamma(K)$.*

Proof We may suppose given a map $P \to \Gamma(L)$ and a homotopy of the composed map to $\Gamma(K)$; i.e. $f : P \times L \to E$ and $g : P \times K \times I \to E$ with $g(p, a, 0) = f(p, a)$ if $a \in K$. We seek to construct a homotopy $P \times L \times I \to E$ covering the projection on B and also extending g. But this exists since f has the CHP for the pair $(P \times L, P \times K)$. □

In case $K \subset L$ are smooth manifolds, there is a corresponding result for spaces of smooth sections.

A map $f : X \to Y$ is said to be a *weak homotopy equivalence* if, for any CW-pair (K, L) and maps $a : L \to X$ and $b : K \to Y$ with $b \mid L = f \circ a$ there exists $c : K \to X$ with $c \mid L = a$ and $f \circ c$ homotopic to b keeping L fixed. For this it suffices to consider pairs $S^{k-1} \subset D^k$ instead of $L \subset K$; thus for X connected it suffices if f induces isomorphisms $f_* : \pi_r(X) \to \pi_r(Y)$ of homotopy groups.

Lemma 6.1.3 *Suppose given a commutative diagram*

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

with p and p′ fibrations and g a weak homotopy equivalence. Then if h is a weak homotopy equivalence, so its restriction to each fibre of p.

Conversely, if the fibre map h induces a weak homotopy equivalence on each fibre, g is a weak homotopy equivalence.

This result is an easy deduction from the homotopy exact sequences of the fibrations and the five lemma.

6.2 Geometry of immersions

If $f : V \to M$ is an immersion, at each $P \in V$ the map $df_P : T_PV \to T_{f(P)}M$ is injective. When we were discussing submanifolds, we remarked that the restriction of $\mathbb{T}(M)$ to V had $\mathbb{T}(V)$ as a sub-bundle, and described the quotient as $\mathbb{N}(M/V)$. If f is an immersion, instead of the restriction of $\mathbb{T}(M)$ we have its pullback $f^*\mathbb{T}(M)$ by f, and an embedding of $\mathbb{T}(V)$ as a sub-bundle of $f^*\mathbb{T}(M)$. The main result about immersions is a converse to this statement.

A homotopy $g_t : V \to M$ is called a *regular homotopy* if g_t is an immersion for each t. We also seek to classify immersions up to regular homotopy. In fact, not only is the main result stated in more precise terms, but the result is a special case of a principle, formulated by Gromov [59] and called by him the h-principle, which holds for a variety of other geometric structures as well as immersions.

Again let $f : V \to M$ be an immersion, then $j^1 f : V \to J^1(V, M)$ avoids the set $\Sigma^*(V, M) := \bigcup_{i>0} \Sigma^i(V, M)$ of singular jets, and so defines a section g to the projection map $\pi_1^* : J^1(V, M) \setminus \Sigma^*(V, M) \to V$, and this section carries the information about bundles. Not every section of $\pi_1 : J^1(V, M) \to V$ has the form $j^1 f$ for a map $f : V \to M$: apart from the requirement of differentiability, these sections satisfy the additional equations given in local coordinates as $u^i_j = \partial y_j / \partial x_i$. Nevertheless, we will see that in many situations any section of π_1^* can be deformed to one of the form $j^1 f$, hence arising from an immersion f.

In fact the proof gives a stronger result, and this strengthening is key to the proof. Instead of considering a single map, we consider spaces of maps. Taking 1-jets defines a map J from the space $\mathrm{Imm}(V, M)$ of immersions to the space $\Gamma(V, M)$ of sections of π_1^*. We now state the main theorem.

Theorem 6.2.1 *Provided that either $v < m$ or V^v is open, the map*

$$J : Imm(V, M) \to \Gamma(V, M)$$

is a weak homotopy equivalence.

This will usually only be applied in the following form.

Corollary 6.2.2 *Any section of $\Gamma(V, M)$ is homotopic to one induced by an immersion, and two immersions are regularly homotopic if and only if the corresponding sections of $\Gamma(V, M)$ are homotopic.*

The necessity of the condition in the theorem is clear: for example, there is certainly no immersion $S^1 \to \mathbb{R}^1$, since any map has bounded image, while the image of an immersion would be open.

We can state the result in a more concrete way. Recall that $\Gamma(V, M)$ is the space of sections of $\pi_1 : J^1(V, M) \to V$, and that a 1-jet with source P and target Q is determined by these points and a linear map $T_P V \to T_Q M$. Thus a section σ of $\Gamma(V, M)$ assigns to each point $P \in V$ a point $f(P) = Q \in M$ and a linear map $g(P) : T_P V \to T_Q M$. The component f is a smooth map $V \to M$, and g gives a map $\mathbb{T}(V) \to f^*\mathbb{T}(M)$, which we require to be injective on each fibre.

We pause to introduce the Stiefel manifold $V_{m,v}$, defined as the set of isometric embeddings $\mathbb{R}^v \to \mathbb{R}^m$, and hence diffeomorphic to O_m/O_v. This is a deformation retract of the space of linear embeddings $\mathbb{R}^v \to \mathbb{R}^m$, which we denote $V'(m, v)$ and call the weak Stiefel manifold.

The above bundle map g is injective on each fibre if and only if it is a section of a bundle ξ'_f over V with fibre $V'_{m,\upsilon}$. Thus the existence part of the criterion can be formulated as follows.

Proposition 6.2.3 *Provided that either $\upsilon < m$ or V^υ is open, a map $f : V^\upsilon \to M^m$ is homotopic to an immersion if and only if the above bundle ξ'_f over V admits a section.*

A corresponding formulation for the uniqueness part concerns a homotopy between maps $f_0,\ f_1$ and a homotopy of sections g_i lying over this.

Note also that homotopy classes of sections of ξ'_f correspond to classes of sections of a bundle ξ_f over V with fibre $V_{m,\upsilon}$.

In the case $M = \mathbb{R}^m$, the tangent bundle $\mathbb{T}(M)$ is trivial and ξ_f is associated to the normal bundle $\mathbb{N}(M/V)$ of V. An easy application is to the case when $\mathbb{T}(V)$ is trivial.

Corollary 6.2.4 *If V^υ has trivial tangent bundle, there is an immersion of V in $\mathbb{R}^{\upsilon+1}$; if V is open, it immerses in $\mathbb{R}^\upsilon$.*

The idea of the proof of the theorem is to build V up as a union of stages V^i and to show, by induction on i, that the result holds at each stage. At each stage we attach a k-handle for some k, and need to show that the property remains true. This step is established by induction on k.

We recall the notation $D^k(a)$ for the disc $\{x \in \mathbb{R}^k \mid \|x\| \leq a\}$; we now also write $D^k(a, b) := \{x \in \mathbb{R}^k \mid a \leq \|x\| \leq b\}$.

The theorem will be deduced from three lemmas.

Lemma 6.2.5 *The theorem holds if $V = D^\upsilon$ is a disc.*

Lemma 6.2.6 *Suppose V^+ obtained from V by introducing a corner or attaching a collar. Then the restriction maps $Imm(V^+, M) \to Imm(V, M)$ and $\Gamma(V^+, M) \to \Gamma(V, M)$ are weak homotopy equivalences.*

Lemma 6.2.7 *The restriction map $Imm(D^k(2) \times D^{\upsilon-k}, M) \to Imm(D^k(1, 2) \times D^{\upsilon-k}, M)$ is a fibration.*

Proof of Theorem 6.2.1 By Corollary 5.1.7 (if V is compact) and Lemma 5.1.8 (if not), V has a handle decomposition. If $\upsilon < m$, this can have no m-handles; if $\upsilon = m$ and V is open we may suppose by Proposition 5.4.1 that there are none. First suppose V compact.

We will prove by induction on k that the result holds for any V' which has only j-handles for $j < k$. Lemma 6.2.5 provides the start of the induction. We also induct on the number of handles of V: write V_i for the manifold with i handles, and suppose V_{i+1} obtained by attaching a k-handle. By the description in §5.1, V_{i+1} is obtained from V_i by first introducing a corner to obtain V_i^+, say, then attaching a copy of $D^k \times D^{v-k}$, which we take as $D^k(2) \times D^{v-k}$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Imm}(D^k(2) \times D^{v-k}, M) & \xrightarrow{J} & \Gamma(D^k(2) \times D^{v-k}, M) \\ \downarrow & & \downarrow \\ \mathrm{Imm}(D^k(1,2) \times D^{v-k}, M) & \xrightarrow{J} & \Gamma(D^k(1,2) \times D^{v-k}, M) \end{array}. \quad (6.2.8)$$

By Lemma 6.2.7, the left-hand downward map is a fibration; the right-hand one is by Lemma 6.1.2. By Lemma 6.2.5 together with Lemma 6.2.6, the upper map J is a weak homotopy equivalence. By inductive hypothesis, so is the lower map J. Hence by Lemma 6.1.3 J induces a weak homotopy equivalence on each fibre. Now the diagram

$$\begin{array}{ccc} \mathrm{Imm}(V_{i+1}, M) & \xrightarrow{J} & \Gamma(V_{i+1}, M) \\ \downarrow & & \downarrow \\ \mathrm{Imm}(V_i^+, M) & \xrightarrow{J} & \Gamma(V_i^+, M) \end{array} \quad (6.2.9)$$

maps by restriction to diagram (6.2.8). The vertical maps in (6.2.9) are the pull-backs of the vertical maps in (6.2.8) which are fibrations; hence they too are fibrations. The restriction of J to each fibre in (6.2.9) is a weak homotopy equivalence. Since the lower map J is a weak homotopy equivalence by Lemma 6.2.6, it follows that the upper also is.

For the case when V is not compact, so we have an infinite number of handles, we note that $\mathrm{Imm}(V, M)$ is the inverse limit of the $\mathrm{Imm}(V_i, M)$, $\Gamma(V, M)$ is the inverse limit of the $\Gamma(V_i, M)$, and apply Lemma B.1.3. □

Proof of Lemma 6.2.5 Since the disc is contractible, the space $\Gamma(D^v, M)$ of sections of $\pi_1^* : J^1(D^v, M) \setminus \Sigma^*(D^v, M) \to D^v$ is homotopy equivalent to the space of sections over the origin, which is the space W of injective linear maps from $\mathbb{R}^v$ to the tangent space $T_Q M$.

We need to look at a map from D^k to $\Gamma(D^v, M)$ and a lift to $\mathrm{Imm}(D^v, M)$ of its restriction to S^{k-1}. So for each $x \in D^k$ we have an injective linear map from $\mathbb{R}^v$ to some $T_Q M$, i.e. a 1-jet j_x^1 at 0 of a map $f_x : D^v \to M$. To see that we can choose the f_x to depend continuously on x, take a closed embedding $h : M \to \mathbb{R}^K$ (which exists by Corollary 4.7.8), a tubular neighbourhood of its image with image N, and hence a smooth retraction $\rho : N \to M$, the projection of the tube. Now j_x^1, composed with the inclusion h gives a 1-jet of map $\mathbb{R}^v \to \mathbb{R}^K$ which has polynomial (in fact, linear) representative g_x. Composing with ρ gives $f_x = \rho \circ g_x$, defined on a neighbourhood of 0, and depending continuously on x. Moreover since D^k is compact we can choose the same neighbourhood for all $x \in D^k$.

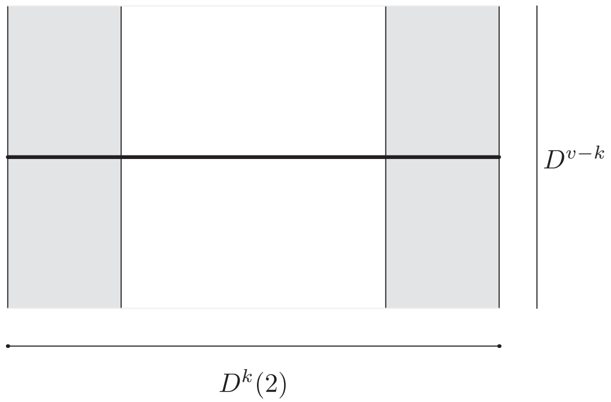

Figure 6.1 The core of $D^k(2) \times D^{v-k}$

We have constructed a lift, but on a smaller disc $D^v(\varepsilon)$. Composing with a diffeotopy J_t of D^v, equal to the identity near 0, and compressing D^v inside $D^v(\varepsilon)$ by a diffeotopy, we obtain a lift of maps of the larger disc.

This does not yet agree on the boundary S^{k-1} with the given lift. For $x \in \partial D^k$ we have the given immersion $g_x : D^v \to M$ and the map f_x just constructed, both with the same 1-jet at 0. Working again in $\mathbb{R}^N$, we consider the linear interpolation $\lambda g_x(y) + (1-\lambda) f_x(y)$ and compose with ρ to obtain a homotopy in M. Since the 1-jets at $0 \in D^v$ are constant, this restricts to a regular homotopy on a smaller disc $D^v(\varepsilon')$. Using J_t again, we obtain a regular homotopy on D^v. □

The proof of Lemma 6.2.6 is simple: the second result holds since $V \subset V^+$ is a homotopy equivalence; as to the first, we have embeddings $V \to V^+ \to V$ with composite diffeotopic to the identity.

Proof of Lemma 6.2.7 The proof that

$$\mathrm{Imm}(D^k(2) \times D^{v-k}, M) \to \mathrm{Imm}(D^k(1,2) \times D^{v-k}, M)$$

is a fibration is the key to the whole result. Define the *core* (of $D^k(2) \times D^{v-k}$) to be $C := (D^k(2) \times \{0\}) \cup (D^k(1,2) \times D^{v-k})$: this is pictured in Figure 6.1.

The parameter space P plays very little part below (we just use the fact that P is compact). Nor does M: we have to make sure that each map into M is an immersion, but can use the fact that immersions form an open set. Let us call a map $\phi : A \times B \to M$, with A a submanifold of $D^k(2) \times D^{v-k}$, *admissible* if, for each $b \in B$, the induced map $a \to \phi(a, b)$ is an immersion.

We are thus given a continuous P-parameter family of immersions giving an admissible map $g : D^k(2) \times D^{\upsilon-k} \times \{0\} \times P \to M$ and a homotopy of its restriction $f : D^k(1,2) \times D^{\upsilon-k} \times I \times P \to M$, and seek to extend this to a homotopy (through admissible maps) of g. The first step is to extend the maps f and g to a map $D^k(2) \times D^{\upsilon-k} \times I \times P \to M$. This is not in general admissible, but by openness of immersions, the restriction f' of this map to $D^k(\alpha, 2) \times D^{\upsilon-k} \times I \times P \to M$ is admissible for some $\alpha < 1$.

Next define $K : D^k(\alpha, 2) \times D^{\upsilon-k} \times I \times P \times I \to M$ by setting

$$\begin{aligned} K(x, y, t, z, T) &= f'(x, y, t, z) && \text{if } T = t \text{ or if } \|x\| > (\alpha + 2)/3 \\ &= f'(x, y, T, z) && \text{if } |\|x\| - \alpha| < (1 - \alpha)/3 \end{aligned}$$

and extending smoothly to other values. This defines an admissible map on some neighbourhood of $T = t$, hence for some $\varepsilon > 0$ whenever $|t - T| < \varepsilon$. We can thus find $0 = t_0 < t_1 < \ldots < t_s = 1$ such that $k_i(x, y, t, z) := K(x, y, t, z, t_i)$ is admissible for $t_i \leq t \leq t_{i+1}$.

We will now inductively construct an admissible extension g_n for $0 \leq t \leq t_n$ which is equal to f' on $\|x\| \geq a_n$, where $\alpha = a_0 < \cdots a_{n-1} < a_n < 1$. We start the induction by setting

$$\begin{aligned} g_0(x, y, p, t) &= g(x, y, p, t) && \text{if } \|x\| \leq \alpha, \\ &= k_0(x, y, p, t) && \text{if } \|x\| \geq \alpha. \end{aligned}$$

The key step is now a diffeotopy $h_n : D^k(2) \times D^{\upsilon-k} \times [0, t_n] \to D^k(2) \times D^{\upsilon-k}$ such that

(i) $h_n(x, y, t) = (x, y)$ on a neighbourhood of $\|x\| = a_n,\ y = 0$, and outside a neighbourhood of $a_{n-1} \leq \|x\| \leq a_{n+1}$,

(ii) $h_n(x, y, t) = (x, y)$ if $t \leq t_{n-1}$,

(iii) $h_n(*, *, t_n)$ maps $S^{k-1}(a_{n+1}) \times \{0\}$ onto $S^{k-1}(a) \times \{0\}$.

To construct this it is essential that $k < \upsilon$, so that dim $D^{\upsilon-k} > 0$ and there is enough space within $D^{\upsilon-k}$ to move one point past another. The effect of using this diffeotopy is to introduce folds in the immersion, thus giving extra space to move.

We define g_{n+1} by

$$\begin{aligned} g_{n+1}(x, y, t, z) &= g_n(x, y, t, z) && \text{if } 0 \leq t \leq t_n,\ \|x\| \leq a_n \\ &= f'(h_n(x, y, t), t, z) && \text{if } 0 \leq t \leq t_n,\ a_n \leq \|x\| \leq 2 \\ &= k_n(h_n(x, y, t_n), t, z) && \text{if } t_n \leq t \leq t_{n+1},\ a_{n+1} \leq \|x\| \leq 2 \\ &= g_{n+1}(x, y, t_n, z) && \text{if } t_n \leq t \leq t_{n+1},\ \|x\| \leq a_{n+1} \end{aligned}$$

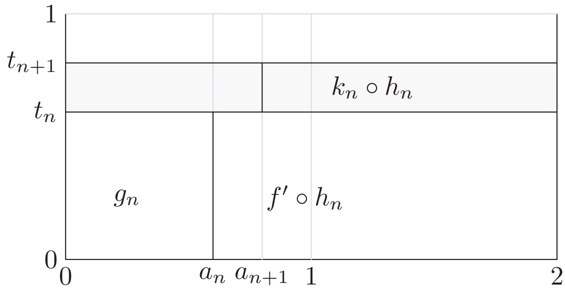

Figure 6.2 Piecing together the construction of g_{n+1}

We offer Figure 6.2 to help follow this 'piecing together' construction of g_{n+1}: note also that g_{n+1} has already been defined for $t = t_n$ by the lines above, and in the upper left rectangle, the value is independent of t.

We now check that, at least for (x, y) in some neighbourhood C^* of C, these formulae agree on the intersections of the different domains of definition.

First consider $0 \le t \le t_n,\ \|x\| = a_n$. Since $h_t(x, y)$ is constant near $\|x\| = a_n,\ y = 0$, the second formula here reduces to f', as does the first.

Along $t = t_n$, we have
(i) $g_n(x, y, t_n, z)$ ($\|x\| \le a_n$),
(ii) $f'(h_{t_n}(x, y), t_n, z)$ ($a_n \le \|x\| \le 2$),
(iii) $k_n(h_{t_n}(x, y), t_n, z) = g'(h_{t_n}(x, y), t_n, z, t_n) = f'(h_{t_n}(x, y), t_n, z)$ ($a_{n+1} \le \|x\| \le 2$), while the final formula agrees by definition, so indeed all match up.

Finally consider $t_n \le t \le t_{n+1},\ \|x\| = a_{n+1}$. Again we need only check in a neighbourhood of $y = 0$, and use (iii) above. The fourth formula is independent of t in this range, and we have already checked agreement at $t = t_n$. Now $k_n(h_{t_n}(x, y), t, z) = g'(h_{t_n}(x, y), t, z, t_n)$, and indeed this is independent of t if t_n is near to a.

Choose a diffeotopy H_t of $D^k(2) \times D^{\upsilon-k}$ into itself which is the identity on a neighbourhood of C and has $H_1(D^k(2) \times D^{\upsilon-k}) \subset C^*$. We use this to re-parametrise our maps to obtain the desired extension: the map $g_{n+1}(H_1(x, y), p, t)$ is admissible, and allows us to continue the induction. $\square$

Since the proof proceeded by attaching successively to V handles of dimension less than υ, it follows without further argument that if we have already constructed an immersion on a closed submanifold W of V of the same dimension, this can be extended over the rest of V, provided $\dim V < \dim M$ or no component of $W \setminus V$ has compact closure. Applying this to the case when W is a collar neighbourhood of ∂V, we deduce that the theorem extends to the case of immersions $(V, \partial V) \to (M, \partial M)$: given a map covered by an injective map of tangent spaces, then if $\upsilon < m$ we can construct an immersion.

A similar argument yields a much more general result. Let $E(M)$ be any bundle over M naturally associated to the differential structure, in the sense that any diffeomorphism $M \to N$ induces an isomorphism $E(M) \to E(N)$ and for any open set $U \subset M$, the restriction of $E(M)$ to U is naturally isomorphic to $E(U)$: for example, for any manifold P we can take $E(M) = J^k(M, P)$. Write $E^r(M)$ for the bundle of r-jets of sections of $E(M)$. Let $E_0^r(M)$ be an open subbundle of $E^r(M)$ with the same invariance property.

Define $\Gamma E_0^r(M)$ to be the space of sections of $E_0^r(M)$ and $\Gamma_0 E(M)$ to be the space of sections σ of $E(M)$ whose r-jet is a section of $E_0^r(M)$. We assign these spaces the C^∞ topologies. The following result is called by Gromov the *h-principle*.

Theorem 6.2.10 *[59] If M is open, the map $j^r : \Gamma_0(M) \to \Gamma E_0^r(M)$ is a weak homotopy equivalence.*

For example, if we have an immersion $V^\upsilon \to \mathbb{R}^k$ with a continuous section to the bundle of unit normal vectors, we have an immersion $V^\upsilon \to \mathbb{R}^{k-1}$. We will use this to show in Theorem 6.3.6 that any manifold immerses in $\mathbb{R}^{2\upsilon-1}$.

A smooth map $f : V \to M$ is called a k-mersion if, at each point $P \in V$, the map $df_P : T_P V \to T_P M$ has rank $\geq k$. The h-principle applies to k-mersions: given a map $f : V \to M$ covered by a map $\mathbb{T}(V) \to f^*\mathbb{T}(M)$ having rank $\geq k$ at each point then f is homotopic to a k-mersion.

6.3 The Whitney trick

We have seen from general position arguments that any manifold M^m embeds in $\mathbb{R}^{2m+1}$ and immerses in $\mathbb{R}^{2m}$. It was shown by Whitney [177] that in fact M^m embeds in $\mathbb{R}^{2m}$ and, again by Whitney, in [178] that M^m immerses in $\mathbb{R}^{2m-1}$.

In this section we explain the construction used by Whitney to establish these results. It has further applications, which will be frequently used in Chapter 7.

By Corollary 4.5.8 to the transversality theorem, if we have two embeddings of compact manifolds $f : V^\upsilon \to M^m$ and $f' : V'^{\upsilon'} \to M^m$ with $m = \upsilon + \upsilon'$ we may suppose, up to a (small) diffeotopy of f, that the images are distinct except that there are finitely many pairs $P_i \in V$ and $P_i' \in V'$ with $f(P_i) = f'(P_i') = R_i$ and the two intersections are transverse.

Similarly, by Theorem 4.7.7, if $m = 2v$, for a dense set of maps $g : V^v \to M^m$, g is an immersion, and fails to be injective only insofar as there are finitely many pairs of distinct points (P_i, Q_i) in V with $g(P_i) = g(Q_i) = R_i$, but the two branches of V are transverse.

The so-called Whitney trick is a procedure which, under some conditions, perturbs the embeddings f and f' to be disjoint, or perturbs the map g to obtain an embedding of V in M. Given orientations, each intersection is assigned (at least locally) a sign. The construction will enable us to cancel a pair of intersections of opposite signs. Moreover this is achieved by a diffeotopy of f or a regular homotopy of g.

A second construction, also due to Whitney, allows us to introduce a single self-intersection (of either sign) of g by taking connected sum with a standard map inside a coordinate neighbourhood. Combining the two constructions gives a further chance to modify g to an embedding.

Suppose given orientations of V, V' and M. At a point R where V and V' intersect transversely, we have $T_RM = T_RV \oplus T_RV'$. Choose bases $(e_1, \ldots, e_v)$ of T_RV and $(e'_1, \ldots, e'_{v'})$ of T_RV' defining the given orientations. Then the local intersection number of V and V' at R is defined to be $+1$ if the basis $(e_1, \ldots, e_v, e'_1, \ldots, e'_{v'})$ of T_RM defines the given orientation of M and -1 if it does not.

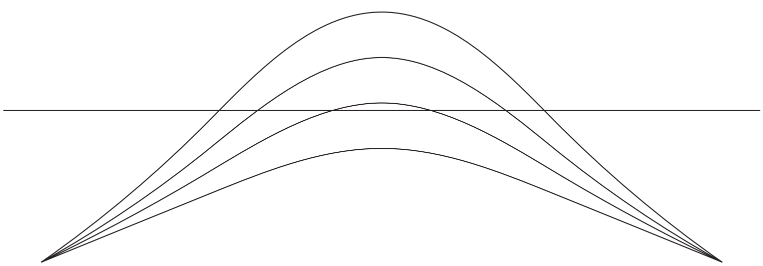

Figure 6.3 Model of the deformation

The model picture, which is illustrated in Figure 6.3, is to take the line C : $y = 0$ in the plane and the curve $C' : y = x^2 - 1$ intersecting it at the two points $A_1 = (1, 0)$, $A_2 = (-1, 0)$ and deform the curve C' vertically to C_t given by $y = x^2 - 1 + t$: for $t > 1$ the intersections have disappeared. More precisely, we choose a deformation $y(x, t)$ for $|x| \leq 1 + 2\varepsilon$ and $t \in I$ such that for $|x| \leq 1$ we have $y(x, t) = x^2 - 1 + t$ and for $|x| \geq 1 + \varepsilon$ we have $y(x, t) = x^2 - 1$. Write D^* for the region spanned by the two arcs, and D^+ for a neighbourhood of D^* in $\mathbb{R}^2$.

Suppose given connected manifolds V, V' of dimensions υ, υ', and embeddings $\phi : V \to M^m$, $\phi' : V' \to M^m$ with $m = \upsilon + \upsilon'$ which intersect transversely. The key idea is to embed the above model in the manifold.

Take two points of intersection, say $\phi(P_i) = \phi'(P_i') = R_i$ $(i = 1, 2)$. Choose paths $f : (I, 0, 1) \to (V, P_1, P_2)$ and $f' : (I, 0, 1) \to (V', P_1', P_2')$. Since υ and υ' are each ≥ 3, general position shows that we may take each of f, f' to be a smooth embedding. Taking x as parameter on C and C' allows us to regard the paths as maps $f : (C, A_1, A_2) \to (V, P_1, P_2)$ and $f' : (C', A_1, A_2) \to (V', P_1', P_2')$. Then $\phi \circ f$ and $\phi' \circ f'$ together define a loop $F : C \cup C' \to M$.

Proposition 6.3.1 *Suppose also that V, V' and M are all orientable, that the intersections at R_1 and R_2 have opposite signs, and that either*

(i) $\upsilon, \upsilon' \geq 3$ and F defines a nullhomotopic loop in M or

(ii) $\upsilon > 2 = \upsilon'$ and F defines a nullhomotopic loop in $M \setminus V$.

Then there is an embedding $\phi : D^+ \times \mathbb{R}^{\upsilon-1} \times \mathbb{R}^{\upsilon'-1} \to M$ such that $\phi^{-1}(V) = (D^+ \cap C) \times \mathbb{R}^{\upsilon-1} \times \{0\}$ and $\phi^{-1}(V') = (D^+ \cap C') \times \{0\} \times \mathbb{R}^{\upsilon'-1}$.

Proof Since F is nullhomotopic in M, the map of the two arcs extends to a map of the disc D^*, and hence to a map g of a neighbourhood D^+, which we can take as smooth. We next put the map g in general position. Since $m \geq 5$, we may suppose that g is an embedding.

In case (i) as $\upsilon, \upsilon' > 2$ we may suppose using general position that the only intersections of $g(D^+)$ with V and V' occur along the images of C and C'. In case (ii) we can avoid V' by general position, and the fact that the extension g, outside a neighbourhood of C, can be taken to avoid V holds by our hypothesis.

For short, write C for $g(C \cap D^+)$ and C' for $g(C' \cap D^+)$. Write ζ for the tangent vector $\zeta = d(\phi \circ f)(\partial/\partial t)$ along C. Let η_1 be the vector field along C, normal to C, and inwards tangent to $g(D^*)$. Similarly write ζ' for the tangent along C' and ξ_1 for the normal pointing inwards along $g(D^*)$. Observe that we have $\xi_1 = \zeta$ and $\eta_1 = \zeta'$ at R_1, and $\xi_1 = -\zeta$, $\eta_1 = -\zeta'$ at R_2. These steps are illustrated in Figure 6.4.

We next construct smooth vector fields ξ_i $(2 \leq i \leq \upsilon)$ and η_j $(2 \leq j \leq \upsilon')$ on $g(D^+)$ such that

(i) at each point, they form a base for the normal space to $g(D^+)$,

(ii) along C the ξ_i $(2 \leq i \leq \upsilon)$ are tangent to V, and

(iii) along C' the η_j $(2 \leq j \leq \upsilon')$ are tangent to V'.

First choose vectors $\xi_2, \ldots, \xi_\upsilon$ tangent to V at R_1 such that $(\zeta, \xi_2, \ldots, \xi_\upsilon)$ is a base defining the orientation of V. Since C is contractible, we can extend this to give a base of $T_P V$ at all $P \in C$. We can also extend $\xi_1, \xi_2, \ldots, \xi_\upsilon$ to give a base of the normal space $N_P(M/V')$ at all $P \in C'$, but now need compatibility at R_2. Since the intersections at R_1 and R_2 have opposite signs, the two orientations

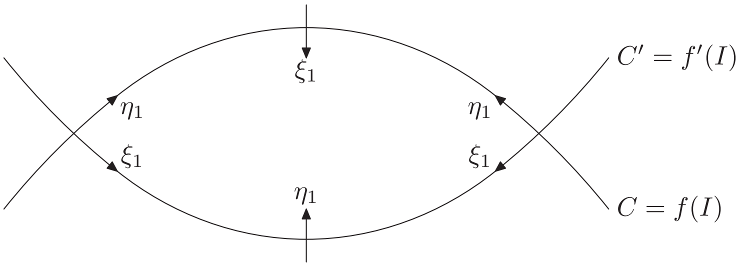

Figure 6.4 Using the model to construct a deformation

of $T_{R_2}V$ differ. But as $\zeta = -\xi_1$ at R_2, the two bases $\xi_2, \ldots, \xi_v$ at R_2 have the same orientation, so by choosing a different extension above, we may suppose that they coincide, so these vectors are defined over $C \cup C'$.

The bundle over D^+ of orthonormal $(v-1)$-frames orthogonal to D^+ is a trivial bundle whose fibre is the Stiefel manifold $V_{v+v'-2,v-1}$. We have constructed a section over a circle; since $v' \geq 3$, $V_{v+v'-2,v-1}$ is simply connected (see §B.3(xv)), so we can extend the section over D^+.

Since D^+ is contractible, all bundles over D^+ are trivial. We may thus extend $\eta_2, \ldots, \eta_{v'}$ to a base for the bundle of vectors normal to D^+ and to the ξ_i. It follows from our choice of these that these satisfy (iii) above. We have thus constructed normal vector fields $(\xi_2, \ldots, \xi_v, \eta_2, \ldots, \eta_{v'})$ along D^+ such that $(\xi_2, \ldots, \xi_v)$ are tangent to V at all points of V $(\eta_2, \ldots, \eta_{v'})$ are tangent to V' at all points of V'.

By Theorem 2.5.5, a neighbourhood of D^+ is diffeomorphic to a disc bundle over D^+, which must be trivial, hence diffeomorphic to $D^+ \times \mathbb{R}^{v+v'-2}$; corresponding statements hold for C and C'.

By Proposition 2.5.10 there exists a tubular neighbourhood of C in M whose restriction gives a tubular neighbourhood of C in V; by the remark following that result, we may suppose this neighbourhood compatible with D^+. We use this to define ϕ on a neighbourhood of C.

We argue similarly for C'; moreover, these neighbourhoods are constructed by glueing together pieces, so if we begin with charts at P and P', we can ensure that these maps agree, thus defining ϕ on a neighbourhood of $C \cup C'$.

As above, we can use general position to extend ϕ over D^+. Finally, the above maps may be regarded as defining a tubular neighbourhood for D^+ on a neighbourhood of $C \cup C'$, and the proof of Proposition 2.5.10 shows how to extend this over D^+: note that our construction of bases for the normal spaces shows that these partial tubular neighbourhoods do indeed define an embedding of our model. □

Theorem 6.3.2 *Suppose we have connected orientable manifolds V^{υ}, $V'^{\upsilon'}$ and embeddings $\phi : V \to M^{\upsilon+\upsilon'}$, $\phi' : V' \to M^{\upsilon+\upsilon'}$ which intersect transversely, with two intersection points $\phi(P_i) = \phi'(P_i') = R_i$ $(i = 1, 2)$ of opposite signs. Choose paths giving smooth embeddings $f : (C, A_1, A_2) \to (V, P_1, P_2)$ and $f' : (C', A_1, A_2) \to (V', P_1', P_2')$ defining a loop $F : C \cup C' \to M$. Suppose also that either*

(i) $\upsilon,\ \upsilon' \geq 3$ and F defines a nullhomotopic loop in M or

(ii) $\upsilon > 2 = \upsilon'$ and F defines a nullhomotopic loop in $M \setminus V$.

Then there is a diffeotopy of $\phi : V \to M$, supported on $\phi(I)$, such that $h_1(V) \cap V'$ agrees with $V \cap V'$ less the points $R_1,\ R_2$.

Proof It suffices to construct a diffeotopy in the model $D^+ \times \mathbb{R}^{\upsilon-1} \times \mathbb{R}^{\upsilon'-1}$ which is constant near the boundary, then transport it into M by the embedding ϕ.

Begin with the diffeotopy $\phi_t : C \times I \to D^+$, modified using a bump function to be the identity outside a neighbourhood of D^*. Now define

$$\Phi : C \times \mathbb{R}^{\upsilon-1} \times I \to D^+ \times \mathbb{R}^{\upsilon-1} \times \mathbb{R}^{\upsilon'-1}$$

by $\Phi_t(x, y) = (\phi_{t\alpha(y)}(x), y, 0)$, where $\alpha(y)$ is equal to 1 when $y = 0$ and to 0 for $\|y\| \geq \varepsilon$. □

Although the details involve local orientations, the hypothesis of global orientability is not needed for this argument. If V, for example, is non-orientable, by replacing f by its composite with an orientation reversing loop we can change the local orientation, so in this case we do not need to assume the two intersections of opposite signs. However the condition that (i) or (ii) holds is essential.

The same construction is used, taking $V = V'$, to eliminate self-intersections of an n-manifold in a $2n$-manifold. Here we can do somewhat more. First suppose n odd. Then the sign of the intersection number is changed if we reverse the order of the two branches V, V' at R. Thus (provided $n > 2$ and M is simply connected) we can eliminate any pair of transverse self-intersections by a regular homotopy, as we can start by joining P_1 to P_2' and P_1' to P_2 instead.

We also have

Proposition 6.3.3 *There is a self-transverse immersion $\Xi : S^n \to S^{2n}$ with a single transverse self-intersection, and with normal bundle isomorphic to the tangent bundle of S^n.*

Proof We begin with the immersion $f : S^1 \to \mathbb{R}^2$ given by $f(\theta) = (\sin\theta, \frac{1}{2}\sin 2\theta)$, where θ denotes the angular coordinate on $S^1 =$

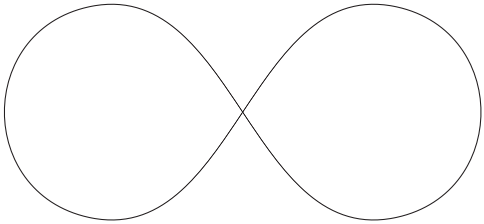

Figure 6.5 An immersed sphere

$\{(\sin\theta, \cos\theta)\}$. (This is essentially the same as an example already used in §1.2.) The curve passes through the origin when $\theta = 0$ and when $\theta = \pi$, and the two branches are transverse.

Rotating this, define $F : S^{n-1} \times [0, \pi] \to \mathbb{R}^{2n}$ by $F(\xi, \theta) = (\xi \sin\theta, \frac{1}{2}\xi \sin 2\theta)$, where we regard S^{n-1} as a subset of $\mathbb{R}^n$ and identify $\mathbb{R}^{2n}$ with $\mathbb{R}^n \times \mathbb{R}^n$. We observe that $F(\xi, \theta) = F(-\xi, -\theta)$. All the points with $\theta = 0$ or π are again mapped to the origin. Hence F factors as $F = G \circ p$, where $p : S^{n-1} \times [0, \pi] \to S^n$ is defined by $p(\xi, \theta) = (\xi \sin\theta, \cos\theta)$. I claim that $G : S^n \to \mathbb{R}^{2n}$ is a smooth immersion.

Near the point $(0, \dots, 0, 1)$ on S^n we can write $x = \xi sin\,\theta$: then $\|x\| = sin\,\theta$, so $y = cos\,\theta = \sqrt{1 - \|x\|^2}$ and $G(x, y) = (x, x\sqrt{1 - \|x\|^2})$. Here we can take x as giving local coordinates, and $\sqrt{1 - \|x\|^2}$ is a smooth function of x near $x = 0$. Thus the map is smooth at this point, and the image has tangent the diagonal $\{(x, x)\}$. A similar calculation deals with the point $(0, \dots, 0, -1)$ (here $\theta = \pi$). The image (for $n = 1$) is pictured in Figure 6.5.

For the immersion f, both tangent and normal bundle are trivial. We can define an isomorphism between them by rotating each tangent vector through an angle $+\frac{\pi}{2}$ in the plane. A corresponding rotation can be made in $\mathbb{R}^{2n}$, using the matrix (in block form) $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and again this gives an isomorphism of the tangent space to $G(S^n)$ on its normal space. That this also works for each branch at the origin follows from the above calculation of the tangent space there.

Composing G with an embedding $\mathbb{R}^{2n} \subset S^{2n}$ given the desired map Ξ. □

Given any immersion $V^n \to M^{2n}$ we can take a connected sum with J, at a smooth point of each submanifold, and this produces (up to diffeomorphism) another immersion $V^n \to M^{2n}$, but now with an additional point of self-intersection. Moreover, changing the orientation, this point can be supposed to have intersection number of either sign.

Theorem 6.3.4 *Given compact smooth manifolds with V^n connected, M^{2n} simply-connected, and $n \geq 3$, any map $f : V^n \to M^{2n}$ is homotopic to a smooth embedding.*

Proof We may suppose by general position that f is an immersion, and is self-transverse, so that there are finitely many points P_i of self-crossing.

First suppose V orientable. Choose orientations of V and M, and hence a sign ± 1 attached to each P_i. As just observed, we may introduce a further self-intersection point of either sign. Introduce such points until there are equal numbers of both signs.

Given two points of opposite sign, we apply Theorem 6.3.2 to construct a diffeotopy of the neighbourhood of an arc in V joining the points, hence inducing a regular homotopy of V, which removes these intersection points and introduces no new ones.

If V is non-orientable, we first introduce a self-intersection point, if necessary, to make the number of such points even. Now we can cancel any pair of P_i by the same construction: we just need to choose the arc in V of the desired parity. □

The hypotheses are necessary. If V has two components, they may have non-zero intersection number in M: for example, consider $(S^n \times *) \cup (* \times S^n)$ in $S^n \times S^n$. If M is not simply-connected, counting self-intersections more carefully gives an obstruction lying in the group ring $\mathbb{Z}[\pi_1(M)]$: see [167, §5]. For a counterexample, we can take the above map Ξ with M a neighbourhood of its image. If $n = 2$, the whole Whitney trick fails.

For any manifold M^{2n}, we know by general position that any map $S^n \to M^{2n}$ is homotopic to an immersion, and from Theorem 6.2.1 that immersions in a given homotopy class are classified up to regular homotopy by $\pi_n(E)$, where $E \to M$ is the bundle associated to $\mathbb{T}M$ and with fibre the Stiefel manifold $V_{2n,n}$. We have an exact sequence

$$\pi_n(V_{2n,n}) \to \pi_n(E) \to \pi_n(M) \to \{1\};$$

by §B.3(xvi), the first term is cyclic of order ∞ or 2 according as n is even or odd.

If n is even, the immersion ϕ determines a homology class $[\phi]$ with self-intersection number $[\phi].[\phi]$. It also has a normal bundle, with Euler class giving a number $e(\phi)$. We may suppose ϕ has transverse self-intersections; summing the intersection numbers at these points gives a further integer $I(\phi)$.

Lemma 6.3.5 *We have $[\phi].[\phi] = e(\phi) + 2I(\phi)$.*

Proof The homological self-intersection is the intersection number of the image of ϕ with a small perturbation of it; this is the sum of the contribution $e(\phi)$ of the self-intersection in the normal bundle and a contribution of 2 from each point of self-intersection of ϕ. □

Taking the connected sum with the example of Proposition 6.3.3 has the effect of changing the regular homotopy class by the action of a generator of $\pi_n(V_{2n,n})$, and of adding 2 to $e(\phi)$ and subtracting 1 from $I(\phi)$.

If n is odd, there are two regular homotopy classes of immersions in each homotopy class of maps $S^n \to M^{2n}$, one in which the number of self-intersections is even, and one with this number odd: since the parity is invariant under regular homotopy, the map $\pi_n(V_{n,n}) \to \pi_n(E)$ is injective. If also $n \geq 3$ and M is simply-connected, just the former regular homotopy class contains embeddings. The two classes have different normal bundles if $\mathbb{T}(S^n)$ is non-trivial, i.e. if $n \neq 3, 7$.

Similar conclusions to these apply with any V^n in place of S^n.

Now consider immersions of N^n in Euclidean spaces. By Proposition 6.2.3, a map $f : N^n \to M^m$ with $n < m$ is homotopic to an immersion if and only if a certain bundle η over N with fibre $V_{m,n}$ admits a section. Obstruction theory tells us that obstructions to the existence of sections lie in $H^i(N; \pi_{i-1}(V_{m,n}))$, and by §B.3(xv) $V_{m,n}$ is $(m-n-1)$-connected. So the primary obstruction is in $H^{m-n+1}(N; \pi_{m-n}(V_{m,n}))$; and by (xvi), $\pi_{m-n}(V_{m,n})$ is isomorphic to $\mathbb{Z}$ if $(m-n)$ is even and to $\mathbb{Z}_2$ if $(m-n)$ is odd. This obstruction is denoted $W_{m-n+1}(\eta)$; its image in $H^{m-n+1}(N; \mathbb{Z}_2)$ is the Stiefel–Whitney class $w_{m-n+1}(\eta)$ (see §8.6).

First take $m = 2n$: since $V_{2n,n}$ is $(n-1)$-connected, there is (as expected) no obstruction.

Theorem 6.3.6 *For $n \geq 2$, any smooth n-manifold immerses in $\mathbb{R}^{2n-1}$.*

Proof The result is due to Whitney [178]; we follow the account of Hirsch [70].

We set $m = 2n-1$ in the above. Since $V_{2n-1,n}$ is $(n-2)$-connected, the only obstruction is the primary obstruction, which lies in $H^n(N : \pi_{n-1}(V_{2n-1,n}))$.

If N is non-compact, or has boundary, then the obstruction lies in a zero group, so vanishes, and immersions exist. Otherwise the obstruction lies in the group $H^n(N; \mathbb{Z})$, where the coefficients are twisted if N is non-orientable, hence the group is isomorphic to $\mathbb{Z}$ in both cases.

Now proceed indirectly, and start with an immersion in $\phi : N^n \to \mathbb{R}^{2n}$. If we find a non-zero normal vector field to this immersion, this implies the existence of a section to the bundle with fibre $V_{2n-1,n}$ and hence of an immersion in $\mathbb{R}^{2n-1}$. Such a normal vector field exists if and only if the normal Euler class $e(\phi)$

vanishes. (We refer to [103, VIII] for background on this class.) The class $e(\phi)$ also lies in $H^n(N; \mathbb{Z})$ and can be identified with the above obstruction.

If n is even, we recall the equation $[\phi].[\phi] = e(\phi) + 2I(\phi)$. As intersection numbers in $\mathbb{R}^{2n}$ vanish, it follows that $e(\phi)$ is even. Taking the connected sum with a suitable numbers of copies of the example of Proposition 6.3.3 gives an immersion ψ with $e(\psi) = 0$. We can thus find a non-vanishing normal vector field to $\psi(N)$, and hence obtain an immersion in $\mathbb{R}^{2n-1}$.

If n is odd, the Euler class satisfies $2e(\phi) = 0$. Since it lies in a group isomorphic to $\mathbb{Z}$, it vanishes, so a normal vector field exists. □

Some results can be obtained for the problem of immersibility of N^n in $\mathbb{R}^{2n-2}$. It was shown in [70] that if $n \equiv 1 \pmod 4$, an immersion exists if and only if $W_{n-1}(\eta) = 0$.

Among the more interesting problems is to determine the lowest dimensions into which one can immerse or embed the real projective spaces $P^n(\mathbb{R})$. By studying the conditions on bundles, Atiyah [13] proved that, if we define $\sigma(n)$ to be the greatest integer s such that $2^{s-1}\binom{n+s}{s}$ is *not* divisible by $2^{\phi(n)}$ (where ϕ denotes Euler's phi function) then $P^n(\mathbb{R})$ cannot be immersed in $\mathbb{R}^{n+\sigma(n)-1}$ or embedded in $\mathbb{R}^{n+\sigma(n)}$.

6.4 Embeddings and immersions in the metastable range

Given an embedding $f : V^v \to \mathbb{R}^m$, as in §4.2 we associate to any pair of distinct points P, Q of V the non-zero vector $\Delta_f(P, Q) := f(Q) - f(P) \in \mathbb{R}^m$, and hence the unit vector $\delta_f(P, Q) := \mathbf{u}(\Delta_f(P, Q)) \in S^{m-1}$. Recalling the notation $V^{(2)}$ for the set of pairs of distinct points of V, we have a map $\delta_f : V^{(2)} \to S^{m-1}$ with the property $\delta_f(Q, P) = -\delta_f(P, Q)$. If f is an immersion, the same formula defines a map on $U \setminus \Delta(V)$ for some neighbourhood U of $\Delta(V)$ in $V \times V$.

Since we will have a number of similar conditions in this section, let us agree that we have standard actions of the group $\mathbb{Z}_2$ of order 2, given on $V \times V$ or on $V^{(2)}$ by interchange of the factors, and on a vector bundle by the map which is minus the identity on each fibre. Thus when we say a map is 'equivariant' we mean with respect to this action. We also say that an equivariant map $\alpha : A \to B$ is *isovariant* if the preimage of the fixed set (of the action) in B is the fixed set in A.

For any map $f : V \to M$ the product $f \times f : V \times V \to M \times M$ is equivariant. It is isovariant if and only if $f(x) = f(y)$ implies $x = y$, i.e. if f is injective. In this case, $f \times f$ restricts to an isovariant map $f^{(2)} : V^{(2)} \to M^{(2)}$.

If f is an immersion rather than an embedding, we know at least that it is locally injective, so there is a neighbourhood U of the diagonal $\Delta(V)$ in $V \times V$ such that the restriction of $f \times f$ to U is isovariant and the map $f^{(2)}$ is defined on $U \setminus \Delta(V)$. We will thus consider isovariant maps $g : U \to M \times M$ defined on an unspecified neighbourhood U of $\Delta(V)$ in $V \times V$; for such g denote by $g_\Delta : V \to M$ the map such that $g(x, x) = (g_\Delta(x), g_\Delta(x))$ for $x \in V$. If g, g' are isovariant maps defined on U, U' we say that they are *isovariant germ homotopic* if there is an isovariant homotopy of their restrictions to some unspecified neighbourhood of $\Delta(V)$ in $V \times V$. We will also refer to equivariant germ homotopy classes for their restrictions to sets $U \setminus \Delta(V)$.

In this section we will describe results showing that conversely, the existence of a suitable isovariant map implies existence and uniqueness up to diffeotopy or regular homotopy of an embedding or immersion giving rise to a map homotopic to the given map. These results hold under the condition that $2m > 3v + \epsilon$, where ϵ is a small number (0, 1 or 2) whose exact value depends on precisely which result is in question. We refer to a dimensional condition of this type as the *metastable range*, in contrast to the stable range $m > 2v + \epsilon$ in which any map is homotopic to an embedding (or immersion), unique up to diffeotopy (or regular homotopy).

Such results will imply simplified statements for the special case when the target is Euclidean space as follows.

Lemma 6.4.1 *There is a natural bijection between isovariant homotopy classes of isovariant maps $g : V \times V \to \mathbb{R}^m \times \mathbb{R}^m$ and equivariant homotopy classes of equivariant maps $F : V^{(2)} \to S^{m-1}$.*

There is a natural bijection between isovariant germ homotopy classes of isovariant maps defined on some neighbourhood U of $\Delta(V)$ in $V \times V$ and equivariant germ homotopy classes of equivariant maps defined on $U \setminus \Delta(V)$ for some neighbourhood U of $\Delta(V)$ in $V \times V$.

Proof Given an isovariant map g, we define $r(g) : V^{(2)} \to S^{m-1}$ by $r(g)(P, Q) := \mathbf{u}(s(g(P, Q)))$, where $s : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ denotes the subtraction map $s(x, y) = x - y$. Conversely, given an equivariant map $F : V^{(2)} \to S^{m-1}$, define an isovariant map by

$$l(F)(P, Q) := \rho(P, Q)(F(P, Q), -F(P, Q)),$$

where ρ denotes the distance in some Riemannian metric. For any F, we have $r(l(F)) = F$. For any g, $s(l(r(g)))$ is a non-zero multiple of $s(g)$, hence $l(r(g))$ is isovariantly homotopic to g.

The second assertion follows from the same argument, by restricting the maps to appropriate neighbourhoods of $\Delta(V)$. □

We first treat immersions, and start from the above Corollary 6.2.2, which we can restate as:

Any injective bundle map $\mathbb{T}(V) \to \mathbb{T}(M)$ is homotopic to one induced by an immersion, and two immersions are regularly homotopic if and only if the corresponding bundle maps are homotopic through injective maps.

The first step is to choose metrics on V and M, and observe that (by an easy homotopy argument) the space of injective bundle maps is homotopy equivalent to the space of bundle maps which preserve distances in the fibres. Thus in each fibre we have an isometric embedding $\mathbb{R}^v \to \mathbb{R}^m$ and hence a map $f : \mathbb{R}^v \to \mathbb{R}^m$ with $f^{-1}(0) = 0$ and having the (equivariance) property that $f(-x) = -f(x)$. We call fibre maps with this property *skew maps*, and homotopies preserving this condition *skew homotopies*.

The next step, which will be accomplished in Proposition 6.4.3, is to replace the space of isometric bundle maps by the space of skew maps.

Now $V'_{m,v}$ is the space of injective linear maps $\mathbb{R}^v \to \mathbb{R}^m$; denote by $W_{m,v}$ the space of skew maps $\mathbb{R}^v \to \mathbb{R}^m$, and by $\rho_{m,v} : V'_{m,v} \to W_{m,v}$ the inclusion.

Lemma 6.4.2 *The map $\rho_{m,v}$ is $(2m - 2v - 1)$-connected.*

Proof The Stiefel manifold $V_{m,v}$ is a deformation retract of $V'_{m,v}$; similarly the subspace $Y_{m,v} \subset W_{m,v}$ of radial skew maps that preserve length along each radius is a deformation retract: the retraction is given by taking the skew map $f : \mathbb{R}^v \to \mathbb{R}^m$ to g, where for $t > 0$, $||x|| = 1$, we have $g(tx) = tf(x)/||f(x)||$. A deformation is given by

$$h(u, tx) = \left(\frac{t}{||f(x)||}\right)^{1-u} f(t^u x).$$

We can identify $Y_{m,v}$ with the space of maps $S^{v-1} \to S^{m-1}$ that commute with the antipodal map; $V_{m,v}$ is the subspace of isometric embeddings.

We prove the result by induction on v; for $v = 1$, we have $X_{1,m} = Y_{1,m} = S^{m-1}$. We have the diagram

$$\begin{array}{ccc} V_{m,v} & \longrightarrow & Y_{m,v} \\ p_X \downarrow & & p_Y \downarrow \\ V_{v-1,m} & \longrightarrow & Y_{v-1,m} \end{array},$$

where the vertical arrows are induced by restriction to S^{v-2}. The map p_X is the projection of a fibre bundle; p_Y is a fibration (compare Lemma 6.1.2: the covering homotopy property for p_Y follows from the homotopy extension property for $S^{v-2} \subset S^{v-1}$). Let $x \in V_{v-1,m}$ have image $y \in Y_{v-1,m}$: then the result will follow from the homotopy exact sequences of the fibrations if we can show that $p_X^{-1}(x) \to p_Y^{-1}(y)$ is $(2m - 2v - 1)$-connected.

Now $p_X^{-1}(x)$ is homeomorphic to S^{m-v}. The space $p_Y^{-1}(y)$ consists of equivariant maps $S^{v-1} \to S^{m-1}$ agreeing with y on S^{v-2}, hence determined by the extension of y to one of the hemispheres bounded by S^{v-2}. The space of such extensions has the same homotopy type as when the map $S^{v-2} \to S^{m-1}$ is constant, and hence is homotopy equivalent to the iterated loop space $\Omega^{v-1}(S^{m-1})$. Thus we need to show that $S^{m-v} \to \Omega^{v-1}(S^{m-1})$ is $(2m-2v-1)$-connected, or equivalently that $\pi_r(S^{m-v}) \to \pi_{r+v-1}(S^{m-1})$ is surjective for $r \le 2m-2v-1$ and bijective for $r \le 2m-2v-2$. But this is the standard stability property of the suspension map (see §B.3(vi)). □

Now let $\pi : E \to B$ and $\pi' : E' \to B'$ be vector bundles over CW-complexes B, B' with respective fibres $\mathbb{R}^v$ and $\mathbb{R}^m$.

Proposition 6.4.3 *(i) If* $\dim(B) \le 2m-2v-1$ *and* $\phi : E \to E'$ *is a skew map, there is a bundle map* $\psi : E \to E'$*, with* $\overline{\phi} = \overline{\psi}$*, skew homotopic to* ϕ*.*

(ii) If $\dim(B) < 2m-2v-1$ *and* $\phi_0, \phi_1 : E \to E'$ *are skew homotopic bundle maps, there is a bundle homotopy of* ϕ_0 *to* ϕ_1*, covering the given homotopy of maps* $B \to B'$*.*

Proof The skew maps $\phi : E \to E'$ that cover $\overline{\phi}$ are in bijective correspondence with the cross-sections of the bundle $\mathcal{W}$ over B whose fibre over $x \in B$ is the space of skew maps of E_x to E'_x, which can be identified with $W_{m,v}$; correspondingly for the bundle $\mathcal{L}$ of fibrewise injective bundle maps, with $V'_{m,v}$. By Lemma 6.4.2, we have $\pi_r(W_{m,v}, V'_{m,v}) = 0$ for $r < 2m-2v-1$. Since the obstructions to deforming a cross-section of $\mathcal{W}$ into $\mathcal{L}$ lie in these groups, the results follow. □

Now choose a complete metric on V. By Proposition 2.2.6, the map $e_V : \mathbb{T}(V) \to V \times V$ given by $e_V(\xi) = (\exp(\xi), \exp(-\xi))$ is a local diffeomorphism along $\Delta(V)$ and there exist neighbourhoods A_V of $\mathbb{T}^0(V)$ in $\mathbb{T}(V)$ and O_V of $\Delta(V)$ in $V \times V$ such that e_V gives a diffeomorphism of A_V on O_V; make corresponding choices for M.

Proposition 6.4.4 *There is a natural bijection between isovariant germ homotopy classes of isovariant maps* $F : U \to M \times M$ *defined on some neighbourhood* U *of* $\Delta(V)$ *in* $V \times V$ *and skew homotopy classes of skew bundle maps* $\mathbb{T}(V) \to \mathbb{T}(M)$*.*

Proof Let $F : U \to M \times M$ be an isovariant map, with U a neighbourhood of $\Delta(V)$ in $V \times V$. The composite $F_1 := e_M^{-1} \circ F \circ e_V$ is an isovariant map $\mathbb{T}(V) \to \mathbb{T}(M)$ defined on a neighbourhood of $\mathbb{T}^0(V)$. The restriction of F to the diagonal defines a map $F_0 : V \to M$ agreeing with the restriction of F_1 to zero vectors.

We next deform F_1 to a fibre map over F_0. If we had a trivialisation $\mathbb{T}(M) \cong M \times \mathbb{R}^m$ we could separate base and fibre components, write $F_1(X) = (F_b(X), F_f(X))$ and simply define $F_2 := (F_0, F_f)$. In general we define a natural metric on $\mathbb{T}(M)$: at a point given by a tangent vector v at $x \in M$ we can identify $T_v(\mathbb{T}(M)$ with $T_x(M) \oplus T_x(M)$ and use the Riemann metric on M twice. For $X \in \mathbb{T}(M)$ and $y \in M$ consider the point $\sigma(X, y) \in T_y(M)$ closest in $\mathbb{T}(M)$ to X. It follows from Theorem 2.3.2 that we have a smooth map $\sigma : \mathbb{T}(M) \times M \rightsquigarrow \mathbb{T}(M)$ defined on a neighbourhood of the diagonal. Moreover σ is a submersion along (and hence near) the set of points (X, x) with $X \in T_x(M)$, so the preimage of the zero cross-section is smooth, hence coincides (near the diagonal) with $T^0(M) \times M$.

Now define $F_2(X) := \sigma(F_1(X), F_0(\pi(X)))$. It follows that if X is a non-zero vector, so is $F_2(X)$. Thus F_2 is isovariant and, in some neighbourhood of $\mathbb{T}^0(V)$, is homotopic to F_1 through isovariant maps.

Using a partition of unity, we construct a positive continuous function $\varepsilon_V(X)$ on $\mathbb{T}(V)$ such that $\varepsilon_V(X) = 1$ for X in a neighbourhood of $\mathbb{T}^0(V)$, $\varepsilon_V(X)X \in U \cap A_V$ for all $X \in \mathbb{T}(V)$, and $\varepsilon_V(-X) = \varepsilon_V(X)$ for all X.

Now define $F_3 : \mathbb{T}(V) \to \mathbb{T}(M)$ by $F_3(X) = \varepsilon_V(X)^{-1}(F_2(\varepsilon_V(X)X))$. This is still isovariant, and is defined on all of $\mathbb{T}(V)$.

The converse procedure is more straightforward: if $G : \mathbb{T}(V) \to \mathbb{T}(M)$ is a skew map, $e_M \circ G \circ e_V^{-1}$ already gives an isovariant map on some neighbourhood of the diagonal. □

Putting these results together, we have

Theorem 6.4.5 *(i) If $2m > 3v$, U_V is a neighbourhood of $\Delta(V)$ in $V \times V$ and $F : U_V \to M \times M$ is isovariant, F_Δ can be approximated by immersions $f : V \to M$ such that F and $f^{(2)}$ are isovariantly germ homotopic.*

(ii) If $2m > 3v + 1$, two immersions $f, g : V \to M$ are regularly homotopic if and only if $f^{(2)}$ and $g^{(2)}$ are isovariantly germ homotopic.

Proof By Theorem 6.2.1, regularly homotopy classes of immersions $V \to M$ correspond bijectively to homotopy classes of fibrewise injective linear maps $\mathbb{T}(V) \to \mathbb{T}(M)$. It follows from Proposition 6.4.3 that if $\dim(V) < 2m - 2v - 1$ these correspond bijectively to skew homotopy classes of skew bundle maps $\mathbb{T}(V) \to \mathbb{T}(M)$. Finally by Proposition 6.4.4 there is a natural bijection between these and isovariant germ homotopy classes of isovariant maps $F : U \to M \times M$. A corresponding argument yields (i). □

Corollary 6.4.6 *If $2m > 3v + 1$ the classification of immersions $V \to M$ depends only on the topology of V and M (not on the differential structure).*

In the case when M is Euclidean space, the statement can be simplified using Lemma 6.4.1. Recall that an immersion $f : V \to \mathbb{R}^m$ induces a map $f^{(2)} : V^{(2)} \to (\mathbb{R}^m)^{(2)}$ whose restriction to some neighbourhood U of $\Delta(V)$ is isovariant, hence induces an equivariant map, $f_\delta : U \setminus \Delta(V) \to S^{m-1}$.

Corollary 6.4.7 *(i) If $2m > 3v$, U_V is a neighbourhood of $\Delta(V)$ in $V \times V$ and $F : U_V \setminus \Delta(V) \to S^{m-1}$ is equivariant, there is an immersion $f : V \to \mathbb{R}^m$ such that F and f_δ are equivariantly germ homotopic.*

(ii) If $2m > 3v + 1$, two immersions $f, g : V \to \mathbb{R}^m$ are regularly homotopic if and only if f_δ and g_δ are equivariantly germ homotopic.

We come to embeddings. As promised above, the main result is

Theorem 6.4.8 *Let V^v, M^m be manifolds with the former compact. Then*

(i) If $2m \geq 3(v + 1)$, a continuous map $f : V \to M$ is homotopic to a smooth embedding if and only if $f \times f$ is equivariantly homotopic to an isovariant map.

(ii) If $2m > 3(v + 1)$, two smooth embeddings f_0, $f_1 : V \to M$ are diffeotopic if and only if $f_0 \times f_0$ and $f_1 \times f_1$ are isovariantly homotopic.

In view of Theorem 6.4.5, this will be an immediate consequence of

Theorem 6.4.9 *Let V^v, M^m be manifolds with the former compact. Then*

(i) If $2m \geq 3(v + 1)$, an immersion $f : V \to M$ is regularly homotopic to a smooth embedding if and only if there is an equivariant homotopy H of $f \times f$ to an isovariant map such that $\Delta(V)$ is open in $H_t^{-1}(\Delta(M))$ for each t.

(ii) If $2m > 3(v + 1)$, a regular homotopy f_t between two smooth embeddings f_0, $f_1 : V \to M$ is regularly homotopic to a diffeotopy if and only if there is a map $H : V \times V \times I \times I \to M \times M$ {write $H_{t,u}(v, w)$ for $H(v, w, t, u)$} such that $H_{t,0} = f_t \times f_t$, $H_{0,u} = f_0 \times f_0$, $H_{1,u} = f_1 \times f_1$, $H_{t,1}$ is isovariant and $\Delta(V)$ is open in $H_{t,u}^{-1}(\Delta(M))$ for each (t, u).

Proof The proof of this result follows the same lines as that of the Whitney trick. We need to construct a model for the deformation, then show how to embed the model in M. As the details are somewhat involved, we confine ourselves here to an outline of the key points of the proof, and refer to the original paper [63] for a careful account. We deal only with (i), in the case when V has no boundary, and try to keep our notation close to that of §6.3.

The core of the model is a smooth manifold C together with an involution σ of C and a σ-invariant function $\lambda : C \to \mathring{D}^1$. The double point set will be $C_0 := \lambda^{-1}(0)$. The core C is smoothly embedded in the source manifold V, so a tubular neighbourhood is determined by a vector bundle L over C, which we may take to be an orthogonal bundle, and consists of the set L_ε of vectors in L of length $\leq \varepsilon$.

We will slide the image of C across the space D defined as the quotient of $C \times \mathring{D}^1$ by identifying, for $x \in C$ and $-1 < t < 1$, $(x, t) \sim (\sigma(x), -t)$: write $[x, t]$ for the image of (x, t). The map $\phi : C \to D$ is given by $\phi(x) = [x, \lambda(x)]$ (thus C_0 is indeed the double point set) and the deformation by $\phi(x, t) = [x, \lambda(x) - t\mu(x)]$ for a suitable μ. Since the first component remains at x, this is a regular homotopy; provided μ is σ-invariant, the double point set is given by $\lambda(x) - t\mu(x) = 0$, so ϕ_1 is an embedding provided $\mu(x) > \lambda(x)$ for all x; we also need $\mu(x) < \lambda(x) + 1$ for the map to be defined.

As before, we expect the normal bundle of D in M to be locally the direct sum of two bundles, one restricting on C to the isomorphic image of L and the other to the normal bundle along C of the image of V in M; but the roles of the summands at C_0 interchange. Explicitly, define $L \oplus_\sigma L$ to be the pull-back of the external direct sum $L \times L$ over $C \times C$ by the antidiagonal map $x \mapsto (x, \sigma(x))$, then let W be the bundle over D given as the quotient of the bundle $(L \oplus_\sigma L) \times \mathring{D}^1$ over $C \times \mathring{D}^1$ by the identification $(e_x, e_{\sigma(x)}, t) \sim (-e_{\sigma(x)}, -e_x, -t)$; again use square brackets to denote a point in the quotient.

The regular homotopy ϕ now extends to the map $\Phi : L_\varepsilon \times I \to W_\varepsilon$ given by $\Phi_t(e_x) = [e_x, 0, \lambda(x) - t\alpha(\|e_x\|)\mu(x)]$: here we require α to be a bump function with $\alpha(0) = 1$ and $\alpha(y) = 0$ for $|y| \geq \varepsilon$: for example, we can take $\alpha(y) = Bp(1 - (|y|/\varepsilon)^2)$.

This concludes the construction of the model; now we need to embed it in V and M. We extend the given homotopy H to a map $V \times V \times [-1, 1] \to M \times M$; away from $\Delta(V) \times [-1, 1]$, we may suppose by transversality that H is transverse to $\Delta(M)$, so that $X := H^{-1}(\Delta(M)) \setminus \Delta(V)$ is a closed submanifold of $V \times V \times [-1, 1]$, of dimension $2v + 1 - m$. The first projection defines a map $p_1 : X \subset V \times V \times [-1, 1] \to V$. Since $v > 2(2v + 1 - m)$, we may suppose by general position that p_1 is an embedding. Since H is equivariant, the second projection p_2 is also an embedding, with the same image.

Define C to be $p_1(X)$, $\sigma : C \to C$ to be $p_2 \circ p_1^{-1}$ and $\lambda : C \to \mathring{D}^1$ to be $p_3 \circ p_1^{-1}$. Since $H_0 = f \times f$, the double point set C_0 of the model is indeed the double point set of f. Taking L to be the normal bundle of C in V, the choice of a tubular neighbourhood of C gives an embedding $L \to V$. This completes the constructions in V.

We begin the construction of a map $\psi : D \to M$ by defining

$$\psi[x, t] := p_1(H(x, \sigma(x), \lambda(x) - t)) \;\text{ whenever }\; 0 \leq t \leq \lambda(x).$$

Since the subset $0 \le t \le \lambda(x)$ is a deformation retract of D, this can be extended to a continuous map ψ of D. Recalling that $2m \ge 3(v+1)$ and that $\dim D = 2v - m + 1$, we may suppose in turn

(i) that ψ is smooth,

(ii) that along the image of C, ψ is an embedding not tangent to V (it is 'transverse' in the sense that the intersection of the tangent spaces to the images of D and of V is that of C) this we can ensure using general position since $\dim D < m - v$,

(iii) that ψ is an embedding, by general position, since $m > 2 \dim D$,

(iv) that the image of ψ meets V only along C, again by general position, since $m > \dim D + \dim V$.

Now define $\xi : C \to M$ by $\xi(x) = \psi[x, 0]$.

We now need to identify W with the normal bundle of $\psi(D)$ in M. Since p_1 and p_2 are embeddings, the normal bundle $\mathbb{N}(V \times V \times [-1, 1]/X)$ splits into components, leading to an isomorphism of the normal bundle $\mathbb{N}(V \times V \times [-1, 1]/X)$ onto $\mathbb{N}(V/C) \oplus_\sigma \mathbb{N}(V/C) \oplus \mathbb{T}(C) \oplus E$, where E is a trivial line bundle. On the other hand, since X is the transverse preimage of $\Delta(M)$ it follows by Lemma 4.5.1 that this normal bundle is the pullback of $\mathbb{N}((M \times M)/M) \cong \mathbb{T}(M)$. We thus have an isomorphism

$$\Xi : \mathbb{N}(V/C) \oplus_\sigma \mathbb{N}(V/C) \oplus \mathbb{T}(C) \oplus E \to \mathbb{T}(M).$$

We can identify L with $\mathbb{N}(V/C)$ and D as the quotient of $C \times \mathring{D}^1$ by $\mathbb{Z}_2$. We would now like to identify the summands $\mathbb{T}(C) \oplus E$ of $\mathbb{T}(M)$ with $\mathbb{T}(D)$ and (hence) the normal bundle $\mathbb{N}(M/D)$ with $\mathbb{N}(V/C) \oplus_\sigma \mathbb{N}(V/C) \cong L \oplus_\sigma L$ and hence with W.

A number of details need attention. The restriction of Ξ to C_0 is equal to $df \oplus_\sigma (-df) \oplus 0$ on $\mathbb{N}(V/C) \oplus_\sigma \mathbb{N}(V/C) \oplus 0$. It is now not difficult to identify the bundle maps over C_0.

If σ denotes the involution of $\mathbb{N}(V/C) \oplus_\sigma \mathbb{N}(V/C) \oplus \mathbb{T}(C) \oplus E$ given by $\sigma(e_x, e_{\sigma(x)}, f_x, r) = (e_{\sigma(x)}, e_x, f_{\sigma(x)}, r)$, it follows from equivariance of H that $\Xi \circ \sigma = -\Xi$. Using again the dimension condition, we can extend to an embedding η of $\mathbb{N}(V/C) \oplus_\sigma \mathbb{N}(V/C)$ in $\mathbb{T}(M)$, covering ξ, with $\eta(e_x, e_{\sigma(x)}) = -\eta(e_{\sigma(x)}, e_x)$ and such that, for $x \in C_0$, $\eta(e_x, e_{\sigma(x)}) = df(e_x) - df(e_{\sigma(x)})$.

To construct the desired isomorphism $\chi : L \oplus_\sigma L \to W$ over ψ and agreeing with η on $(L \oplus_\sigma L) \times [0, 1]$, we first restrict to $(L \oplus 0) \times [0, 1]$, and define $\chi(e_x, 0, \lambda(x)) = df(e_x)$ if $\lambda(x) \ge 0$, and check that the obstructions to extending over $C \times I$ lie in zero groups. Then extend using $\chi(0, -e_x, -\lambda(x)) = df(e_x)$ for $\lambda(x) \le 0$.

As in the proof of Proposition 6.3.1, we can now use Proposition 2.5.10 to construct embeddings of a neighbourhood of C in L into V and of a neighbourhood of D in W into M compatible with the given maps. We have already shown how to construct a regular homotopy to an embedding. A final calculation is necessary to check compatibility of this embedding with the given isovariant map. $\square$

In view of Lemma 6.4.1, we now have

Corollary 6.4.10 *Let V^v be a compact manifold.*

(i) If $2m \geq 3(v+1)$, there is a smooth embedding $f : V \to \mathbb{R}^m$ if and only if there is an equivariant map $(V \times V) \setminus \Delta(V) \to S^{m-1}$.

(ii) If $2m > 3(v+1)$, two smooth embeddings $f_0, f_1 : V \to \mathbb{R}^m$ are diffeotopic if and only if $(f_0)_\delta$ and $(f_1)_\delta$ are equivariantly homotopic.

One can also formulate a Euclidean version of Theorem 6.4.9.

The above are not the only important results about embedding in the metastable range. The following result is also due to Haefliger, and was originally proved using the normal forms for singularities obtained in Theorem 4.8.5.

Theorem 6.4.11 *Let V^v be a compact connected manifold (without boundary), M^m a manifold and $f : V \to M$ a $(k+1)$-connected map.*

(a) If $m \geq 2v - k$ and $2m \geq 3(v+1)$, f is homotopic to an embedding.

(b) If $m > 2v - k$ and $2m > 3(v+1)$, any two embeddings homotopic to f are diffeotopic.

This is deduced in [62] from Theorem 6.4.8 and the following

Proposition 6.4.12 *If $f : V \to M$ is $(2v - m + 1)$-connected, V is closed and $m > v$, then $f \times f$ is equivariantly homotopic to an isovariant map.*

This Proposition is proved by an obstruction theory argument. Some applications of Theorem 6.4.11 were given in §5.6: we now give others following [62]. Taking M to be Euclidean space, we deduce

Corollary 6.4.13 *Let V^v be a compact k-connected manifold (without boundary).*

(a) If $m \geq 2v - k$ and $2m \geq 3(v+1)$, V embeds in $\mathbb{R}^m$.

(b) If $m > 2v - k$ and $2m > 3(v+1)$, any two embeddings of V in $\mathbb{R}^m$ are diffeotopic.

Corollary 6.4.14 *If $2m > 3(v+1)$, any two embeddings of S^v in $\mathbb{R}^m$ are diffeotopic.*

For the equivariant homotopy class of $(S^v \times S^v) \setminus \Delta(S^v) \to S^{m-1}$ is unique. We will see in §8.8 that this result is best possible.

Corollary 6.4.15 *Provided* $2m > 3\max(p, q) + 1$, *the isotopy classes of embeddings of* $S^p \cup S^q$ *in* $\mathbb{R}^m$ *correspond bijectively to* $\pi_{p+q}(S^{m-1})$.

For, using the preceding result, we just need isovariant homotopy classes of $(S^p \times S^q) \cup (S^q \times S^p) \to S^{m-1}$, i.e. homotopy classes $[S^p \times S^q : S^{m-1}] = \pi_{p+q}(S^{m-1})$.

In some situations, the results of Corollary 6.4.13 can be sharpened: we refer the interested reader to [72]. In particular ([72, Theorem 8]).

Proposition 6.4.16 *Let* V^v *be a compact k-connected manifold and* $N \geq \max(2v - 2k - 1, \frac{1}{2}(3v - k), v + 2)$. *Then* V *embeds in* $\mathbb{R}^N$ *if and only if* $W_{N-v+1} = 0$.

6.5 Notes on Chapter 6

§6.1 The main result in the following section is best stated at the level of function spaces. We have collected here some fundamental definitions and results, so as not to interrupt the exposition in the next section.

§6.2 The breakthrough in obtaining a general theory of immersions was made by Steve Smale – his lecture at the International Congress in 1958 was one I found particularly exciting. His work appeared in [137], and was quickly generalised by Moe Hirsch [70]. This theory is often referred to as Smale–Hirsch theory.

The next major step was taken by Misha Gromov [58], who created a general theory. The account given above follows closely the version in lectures by André Haefliger [65]. Another account is given in [2].

§6.3 Whitney had used general position arguments to show in [175] that any m-manifold embeds in $\mathbb{R}^{2m+1}$ and immerses in $\mathbb{R}^{2m}$. He introduced the 'Whitney trick' in [177] to show that any m-manifold embeds in $\mathbb{R}^{2m}$. In the same paper he gave a construction of an m-sphere immersed in $\mathbb{R}^{2m}$ with a single self-intersection. He went on in [178] to show that any m-manifold immerses in $\mathbb{R}^{2m-1}$.

The Whitney trick fails if $m = 2$: here finding the embedding of the 2-disc required for Proposition 6.3.1, which is given by general position if $m \geq 3$, is a problem of the same type as the theorem it seeks to establish. Not only the proof but the result fails: see Section 5.7 for more details. As a result of this failure, the study of 4-manifold topology has a completely different nature to that in higher dimensions. In studying the Whitney trick, Casson was led to introduce an infinite sequence of such problems, leading to the notion of 'grope', and to new results in topological topology. Smooth manifolds behave differently and require new techniques, for which we refer to [46].

Theorem 6.3.4 becomes trivial if $n = 1$: here V must be I or S^1 and $M = S^2$. I do not know what happens if $n = 2$.

§6.4 The first major result on embeddings in the metastable range was obtained by Haefliger [60]. In this impressive paper Haefliger, following the idea of the proof of the Whitney trick, uses the description in Theorem 4.7.3 of singularities of maps in the metastable range to construct a model for a deformation of a map to an embedding.

All the results in this section are due to Haefliger, some in collaboration with Hirsch. For this account we have followed [66] to construct immersions, and [63] for embeddings. A different approach is used in [67]. These theorems are so powerful that much of the subsequent literature is devoted to calculations required for applications. We will return to embeddings of spheres in Euclidean space in the final section §8.8.