Appendix B

Homotopy theory

I do not know any book on homotopy theory which covers all the material to which I need to refer, but one useful introduction is May's book [89].

B.1 Definitions and basic properties

A continuous map $X \times I \to Y$ is said to be a *homotopy* between the maps $X \to Y$ given by its restrictions to $X \times \{0\}$ and $X \times \{1\}$. The relation of homotopy between maps is an equivalence relation. A major concern of homotopy theory is the set of homotopy equivalence classes of maps $X \to Y$, which in this appendix we denote by $[X : Y]$. Unless otherwise stated we fix base points in X and Y and require maps and homotopies to respect the base point. The base point is usually denoted $*$, but is often suppressed from the notation. A map $X \to Y$ homotopic to the constant map $X \to *$ is said to be nullhomotopic. We write X^+ for the disjoint union of X and a point, taken as base point.

An important type of homotopy occurs when $B \subset A$, $h : A \times I \to A$ satisfies $h(x, 0) = x$ for all $x \in A$, $h(x, t) = x$ for all $x \in B$, $t \in I$ and $h(A \times \{1\}) = B$: B is then called a *deformation retract* of A and h is a *deformation retraction*. A simple example is when A is a square and B the union of three sides.

Two spaces X, X' are said to be *homotopy equivalent* if there are maps $f : X \to X'$ and $f' : X' \to X$ such that each composite $f \circ f'$, $f' \circ f$ is homotopic to the identity map.

If $f : S^{n-1} \to X$ is a continuous map, we define a space $X \cup_f e^n$: as a set, we have the disjoint union of X and $\mathring{D}^n$; the map $g : D^n \to X \cup_f e^n$ is given by the identity on $\mathring{D}^n$ and by f on S^{n-1}; and we declare a subset to be open if its preimages by both g and the inclusion of X are open. This process is called attaching an n-cell to X. We can allow $n = 0$: S^{-1} is the empty set, so $X \cup_f e^0 = X^+$ is the disjoint union of X and a point.

A space obtained by attaching a finite number of cells to the empty set is a cell complex. A *CW-complex* is obtained by a (possibly infinite) sequence of attachments of cells to $\emptyset$, subject to the condition that each attaching map has image in a finite subcomplex, and that the topology is given by declaring a set to be open if its intersection with each finite subcomplex is. A *CW-pair* (K, L) consists of a CW-complex L and a CW-complex K obtained from L by attaching cells. We are mainly interested in finite CW-complexes and pairs, or at worst those with a finite number of cells of each dimension.

Given a CW-complex (or pair) we can change the attaching maps by homotopies (and K by a homotopy equivalence) to ensure that cells are attached in order of increasing dimension: the argument parallels that of §5.2, which is modelled on the CW case. The space obtained at the intermediate stage when all cells of dimension $\leq n$ have been attached, is called the *n-skeleton* of K and denoted $K^{(n)}$.

In general, we use the term 'space' for a topological space homotopy equivalent to a CW-complex. This class of objects is closed under various natural constructions, including fibrations and formation of function spaces (with the compact-open topology).

For any space X and $n \geq 1$, the set $[S^n : X]$ has the structure of a group and is denoted $\pi_n(X)$. The group is abelian if $n \geq 2$; if X is connected, it is independent of the base point. The group $\pi_1(X)$ is called the fundamental group of X.

Given a space Y and subspace X, we can similarly define $\pi_n(Y, X)$ using maps $f : D^n \to Y$ with $f(S^{n-1}) \subset X$; more generally given any map $j : X \to Y$ we define $\pi_n(j)$. There is an exact sequence

$$\ldots \pi_n(X) \xrightarrow{j_*} \pi_n(Y) \to \pi_n(j) \to \pi_{n-1}(X) \ldots.$$

Going one further, given a commutative diagram

$$\begin{array}{ccc} & A & \xrightarrow{p} & B \\ \Phi : & q \downarrow & & r \downarrow, \\ & C & \xrightarrow{s} & D \end{array}$$

we can define $\pi_n(\Phi)$ by homotopy classes of commutative diagrams of maps of an n-sphere, the upper and lower hemispheres of its boundary, and the equator into Φ: this is a group for $n \geq 3$. There are exact sequences

$$\ldots \pi_n(p) \to \pi_n(s) \to \pi_n(\Phi) \to \pi_{n-1}(p),$$
$$\ldots \pi_n(q) \to \pi_n(r) \to \pi_n(\Phi) \to \pi_{n-1}(q).$$

A space X is *contractible* if it is homotopy equivalent to a point. It is *weakly contractible* if any map $K \to X$, with K a finite CW-complex, is homotopic to a constant map. It is sufficient to check this for K a sphere, i.e. that $\pi_n(X)$ is trivial for all $i \geq 0$.

If we merely suppose that every map $K \to X$, with K a finite CW-complex of dimension $\leq n$, is homotopic to a constant map, X is called *n-connected*. For this, it is sufficient that $\pi_n(X)$ is trivial for all $0 \leq i \leq n$.

Recall that a map $f : X \to Y$ is said to be a *weak homotopy equivalence* if, for any CW-pair (K, L) and maps $a : L \to X$ and $b : K \to Y$ with $b \,|\, L = f \circ a$ there exists $c : K \to X$ with $c \,|\, L = a$ and $f \circ c$ homotopic to b keeping L fixed.

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d, "i"] & X \arrow[d, "f"] \\ K \arrow[r, "b"] \arrow[ur, "c"] & Y \end{tikzcd}$$

For this it suffices to consider pairs $S^{k-1} \subset D^k$ instead of $L \subset K$; thus for X connected it suffices if f induces isomorphisms $f_* : \pi_r(X) \to \pi_r(Y)$ of homotopy groups.

The map $f : X \to Y$ is said to be n-connected if this condition holds for all (K, L) with K of dimension $\leq n$. If f is the inclusion of a subset, we say that the pair (Y, X) is *n-connected*. For this it is sufficient that $\pi_n(Y, X)$ is trivial for all $0 \leq i \leq n$: equivalently (if $n \geq 2$) that X and Y are connected, the map $f_* : \pi_r(X) \to \pi_r(Y)$ is an isomorphism for $r < n$ and surjective for $r = n$.

For any K, we define the *cylinder* on K to be the product $K \times I$, the *cone* CK on K to be obtained from $K \times I$ by identifying the subspace $K \times \{0\}$ to a point (so there is an inclusion $K \to CK$ with $x \mapsto (x, 1)$), and the *suspension* SK to be obtained by further identifying $(* \times I) \cup (K \times \{1\})$ to a point. More generally, for any map $f : K \to L$ we define the *mapping cone* $L \cup_f CK$ to be obtained from the disjoint union $L \cup CK$ by identifying, for each $x \in K$, the point $(x, 1) \in CK$ with $f(x) \in L$: this generalises the procedure of attaching a cell to L using a map $f : S^{n-1} \to L$. We also define the *mapping cylinder* $Cyl(f) := L \cup_f (K \times I)$ to be obtained from the disjoint union $L \cup (K \times I)$ by identifying, for each $x \in K$, the point $(x, 1) \in (K \times I)$ with $f(x) \in L$: this contains $K \times \{0\}$ as a subspace, and has L as a deformation retract.

The *join* of two spaces K and L is the space $K * L$ obtained from $K \times L \times I$ by identifying each $\{k\} \times L \times \{0\}$ to $k \in K$ and each $K \times \{l\} \times \{1\}$ to $l \in L$. The *smash product* of spaces K and L is defined to be

$$K \wedge L := (K \times L)/(K \times \{*\} \cup \{*\} \times L).$$

In particular, the suspension $SK = S^1 \wedge K$.

A map $i : K \to L$ is said to have the *homotopy extension property* (HEP) if given any map $f : L \to Y$ and homotopy $g : K \times I \to Y$ such that $g(x, 0) = f(i(x))$ for each $x \in K$ there is a homotopy $h : L \times I \to Y$ such that $h(i(x, t)) = g(x, t)$ for each $(x, t) \in K \times I$ and $h \circ (i \times 1_I) = g$. This is a typical property of inclusion maps: the inclusion of a subcomplex L in a CW-pair (K, L) has the HEP. Any map $f : K \to L$ is homotopy equivalent to the inclusion $K \to Cyl(f) = L \cup_f (K \times I)$, which has the HEP. If $i : K \to L$ has the HEP, identifying CK to a point gives a homotopy equivalence $L \cup_i CK \to L/K$: to obtain a homotopy inverse, extend the homotopy of CK which shrinks the cone to its vertex to a homotopy of the identity map of $L \cup_i CK$: at the end of the homotopy is a map sending CK to a point, hence factoring through L/K.

For any $f : K \to L$ and any X, the sequence

$$[K : X] \leftarrow [L : X] \leftarrow [L \cup_f CK : X]$$

is exact, for a map $L \to X$ extends to $L \cup_f CK$ if and only if its restriction to K is nullhomotopic. For any $f : K \to L$, denote by Af the inclusion $L \to L \cup_f CK$. Since Af has the HEP, $(L \cup_f CK) \cup_g CL$ is homotopy equivalent to $CL/(L \cup_f CK) = SK$, so up to homotopy $A^2 f$ is a map $L \cup_f CK \to SK$. Iterating once more gives a map $A^3 f : SK \to SL$ which differs from the suspension Sf by reversing orientation in I. Thus the sequence $A^r f$ of maps induces, for any X, an exact sequence

$$[K : X] \leftarrow [L : X] \leftarrow [L \cup_f K : X] \leftarrow [SK : X] \leftarrow [SL : X] \dots$$

Each set $[SK : X]$ admits a natural group structure, and $[S^2K : X]$ is abelian.

A map $p : X \to Y$ is said to be have the *covering homotopy property (CHP)* if given a space K, a map $a : K \to X$ and a homotopy $b : K \times I \to Y$ such that $b \,|\, (K \times 0) = p \circ a$, there exists a homotopy $c : K \times I \to X$ such that $a = c \,|\, (K \times 0)$ and $b = p \circ c$.

$$\begin{tikzcd} K \times 0 \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ K \times I \arrow[r, "b"] \arrow[ur, "c"] & Y \end{tikzcd}$$

If this holds for K a finite CW-complex, it follows for any CW-complex; it also follows if (K, L) is a CW-pair that c can be chosen to extend a lift already given on $L \times I$. It suffices to require this condition for pairs $(K, L) = (D^n, S^{n-1})$. We may regard the CHP as a sort of dual notion to the HEP.

We recall from §1.3 that if G is a Lie group acting on a smooth manifold F, a map $\pi : E \to B$ is the projection of a fibre bundle (with base space B, total space E, and fibre F) if B can be covered by open sets U_α such that

(i) There are homeomorphisms $\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that for all $m \in U_\alpha, x \in F, \pi\varphi_\alpha(m, x) = m$.

(ii) For each pair (α, β) there is a continuous map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that for $m \in U_\alpha \cap U_\beta, x \in F, \varphi_\beta(m, x) = \varphi_\alpha(m, g_{\alpha\beta}(m).x)$.

Lemma B.1.1 *The projection map $\pi : E \to B$ of a fibre bundle has the CHP.*

This is trivial if π is the projection of a product $B \times F \to B$, thus we can lift a homotopy whose image is contained in some U_α; and now the result is proved by subdividing $K \times I$ into small pieces.

This result motivates the definition that a map $\pi : E \to B$ is a *fibration* if it has the CHP. Given a fibration, write F for the fibre $F := \pi^{-1}(*)$. Then for any space X, the sequence $[X : F] \to [X : E] \to [X : B]$ is exact, for given a map $f : X \to B$ with $\pi \circ f$ homotopic to the map to $*$, we can lift the homotopy to give a homotopy of f to a map into F.

Now let X be a connected space and consider the space EX of continuous maps $\alpha : I \to X$. There are two projections $p_0,\ p_1 : EX \to X$ given by $p_0(\alpha) = \alpha(0)$ and $p_1(\alpha) = \alpha(1)$: each has the CHP. The map p_0 is a homotopy equivalence: a homotopy inverse is given by constant maps $c : X \to EX$ with $c(x)(t) = x$; the map $h : EX \times I \to EX$ given by $h(\alpha, t) = \alpha_t$ with $\alpha_t(u) = \alpha(min(t, u))$ is a homotopy of $c \circ p_0$ to the identity. Thus $PX := p_0^{-1}(*)$ is contractible. The restriction $q_1 := p_1 \,|\, PX$ also has the CHP, and $\Omega X := q_1^{-1}(*)$ is called the *loop space* of X.

For any map $f : K \to L$ we form the pullback

$$X := \{(k, \alpha) \in K \times EL \,|\, f(k) = \alpha(0)\};$$

write $i = (i_1, i_2)$ for the inclusion of X in $K \times EL$. Since p_0 is a homotopy equivalence, so is the projection $i_1 : X \to K$. The composite $f \circ i_1 = p_0 \circ i_2 : X \to L$ is homotopic to the map $\pi : p_1 \circ i_2$.

Lemma B.1.2 *The projection $\pi : X \to L$ defined above has the CHP.*

Proof Given $g : Y \to X$ and a homotopy $G : Y \times I \to L$ such that $G \,|\, Y \times \{0\} = \pi \circ g$ we need to construct $h : Y \times I \to X$ with $h \,|\, Y \times \{0\} = g$ and $\pi \circ h = g$. To this end, write $i \circ g = (g_1, g_2)$, $i \circ h = (h_1, h_2)$; use t as parameter for paths belonging to EL and s as the homotopy parameter in I; thus write h_2 as $h_2(y, t, s) \in L$.

Then the conditions that (g_1, g_2) and (h_1, h_2) factor through X are

$$f(g_1(y)) = g_2(y, 0),\ f(h_1(y, s)) = h_2(y, 0, s);$$

that h extends g is

$$h_1(y, 0) = g_1(y), \quad h_2(y, t, 0) = g_2(y, t),$$

and that h lifts G is

$$h_2(y, 1, s) = G(y, s).$$

We take $h_1(y, s) = g_1(y)$, and then the equations define $h_2(y, t, s)$ if either $t = 0$, $s = 0$ or $t = 1$: moreover the two values for $h_2(y, 0, 0)$ agree since $f(h_1(y, 0)) = f(g_1(y)) = g_2(y, 0)$ and those for $h_2(y, 1, 0)$ do since $G(y) =$

$\pi(g(y)) = g_2(y, 1)$. Since the union of 3 sides of the square $I \times I$ is a retract of the whole square, we can extend these values to define h_2 for all values. □

The fibre of π is called the *mapping fibre* of f; we may denote it by M_f. Thus $M_f := \{(k, \alpha) \in K \times EL \mid f(k) = \alpha(0),\ \alpha(1) = *\}$. We have seen that if f has the HEP, $L \cup_f K \simeq L/K$. Dually, if $f : K \to L$ has the CHP, with fibre F, then F is homotopy equivalent to M_f. Let us write Bf for the map $M_f \to K$: up to homotopy, if f has the CHP, this agrees with the inclusion $F \subset K$. As π has the CHP, so does $M_f \to K$, and this has fibre ΩL, so $B^2 f : \Omega L \to M_f$. Analogously to the above discussion of Af, up to homotopy we can identify $B^3 f$ with $\Omega f : \Omega K \to \Omega L$. It follows that for any space X, there is an exact sequence

$$\ldots [X : \Omega K] \to [X : \Omega L] \to [X : M_f] \to [X : K] \to [X : L].$$

Composition of loops induces a group structure on the set $[X : \Omega K]$, and there is a natural bijection of this set on $[SX : K]$. In particular, $\pi_r(\Omega X) \cong \pi_{r+1}(X)$. Taking X a sphere in the exact sequence gives

$$\ldots \pi_n(K) \to \pi_n(L) \to \pi_{n-1}(M_f) \to \pi_{n-1}(K) \to \pi_{n-1}(L).$$

Here we may identify $\pi_{n-1}(M_f)$ with the group $\pi_n(f)$ and the sequence with the exact homotopy sequence described above. If also $f : K \to L$ has the CHP, with fibre F, then M_f is homotopy equivalent to F.

Lemma B.1.3 *Given a sequence $A_{i+1} \xrightarrow{\alpha_i} A_i$ where the maps α_i are fibrations, there are natural isomorphisms $q_n : \pi_n(\varprojlim A_i) \cong \varprojlim \pi_n(A_i)$.*

Given a sequence of maps $f_i : A_i \to B_i$ between two sequences of fibrations, with each f_i a weak homotopy equivalence and $f_i \circ \alpha_i = \beta_i \circ f_{i+1}$ for each i, the induced map $\varprojlim A_i \to \varprojlim B_i$ is a weak homotopy equivalence.

For a map $S^n \to \varprojlim A_i$ defines a sequence of maps $S^n \to A_i$, so we have a natural map q_n. Since α_i is a fibration, if the homotopy class of a map $S^n \to A_i$ lifts to that of a map to A_{i+1}, so does the map itself. It follows that q_n is surjective; injectivity follows similarly.

The second assertion now follows.

Many of the definitions and results in this section have a formal nature. A set of axioms for homotopy theory, with a development along these lines, was given by Quillen [126].

B.2 Groups and homogeneous spaces

We observed in §3.1 that for any Lie group G and Lie subgroup H, we have a fibre bundle with projection $G \to G/H$ and fibre H; and that if we have two

Lie subgroups $H_1 \subset H_2 \subset G$, the projection $G/H_2 \to G/H_1$ is that of a fibre bundle, with fibre H_2/H_1, so has the CHP.

The group $GL_n(\mathbb{R})$ acts transitively on the space P of positive definite quadratic forms on $\mathbb{R}^n$, and O_n is the isotropy group of the usual inner product, so we have an induced diffeomorphism of $GL_n(\mathbb{R})/O_n$ on P, and hence a fibre bundle $O_n \to GL_n(\mathbb{R}) \to P$. Since P is a convex subset of a Euclidean space, it is contractible. Thus $GL_n(\mathbb{R})$ is homotopy equivalent to O_n. It is usually more convenient to work with the compact group O_n.

Similarly, any Lie group G has maximal compact subgroups K, any two are conjugate, and G/K is contractible. Thus for homotopy purposes, we may replace G by K. In particular, we may replace $GL_n(\mathbb{C})$ by U_n.

Since O_n acts transitively on the Grassmann manifold $Gr_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$, and the subgroup leaving $\mathbb{R}^k \oplus \{0\}$ can be identified with $O_k \times O_{n-k}$, we can identify $Gr_{n,k}$ with the coset space $O_n/(O_k \times O_{n-k})$. This is a smooth manifold, and there is a natural vector bundle $\gamma_{n,k}$ over $Gr_{n,k}$ whose fibre is the k dimensional linear subspace.

The space $V'_{n,k}$ of injective linear maps $\mathbb{R}^k \to \mathbb{R}^n$ is homotopy equivalent to the space of isometric linear embeddings $\mathbb{R}^k \to \mathbb{R}^n$. The latter is called the *Stiefel manifold*, and denoted $V_{n,k}$ (we call $V'_{n,k}$ the *weak Stiefel manifold*). It can be identified with O_n/O_{n-k}, hence with SO_n/SO_{n-k}. For any n-vector bundle $\xi : E \to B$ with group O_n there is an associated bundle with fibre $V_{n,k}$: a point in its total space can be interpreted as an isometry of $\mathbb{R}^k$ into some fibre of ξ.

For any Lie group G, there is a contractible space $E(G)$ admitting a free action of G. Write $B(G) := E(G)/G$ and $\pi_G : E(G) \to B(G)$ for the projection. Then this is a principal G-bundle, and for any principal G-bundle ξ over any space X there is a map $f : X \to B(G)$, unique up to homotopy, such that ξ is equivalent to $f^*\pi_G$. The bundle $\pi_G : E(G) \to B(G)$ is determined uniquely up to homotopy by this condition.

The space $B(G)$ is called a *classifying space* for G. Since $E(G)$ is contractible, it follows that G is homotopy equivalent to the loop space $\Omega B(G)$.

The classical construction of a classifying space is based on the Grassmann manifolds. The natural inclusion $Gr_{n,k} \subset Gr_{n+1,k}$, is $(n-k)$-connected, and the union $\bigcup_m Gr_{n,k}$ can be taken as a classifying space $B(O_k)$ for bundles with group O_k. This construction may be adapted for other Lie groups.

There is an alternative construction, due to Milnor [91], using the sequence of iterated joins $G * G * \ldots * G$ (on which G acts freely), and taking $E(G)$ as the union.

Yet another approach is axiomatic. The set $\mathcal{E}_G(X)$ of equivalence classes of bundles over X with a given structure group G is a contravariant functor of

X, and it is not difficult to verify the hypotheses of Brown's representability theorem [33]. This shows again that there exists a space $B(G)$ and a bundle ξ_G over it with structure group G such that taking a map $f : X \to B(G)$ to the bundle $f^*\xi_G$ induces a bijection of $[X : B(G)]$ on $\mathcal{E}_G(X)$.

In some sense, we can regard any space X as a classifying space for ΩX, which plays the part of the group, since we have a fibration $\Omega X \to PX \to X$ with PX contractible.

An $(n-1)$-spherical fibration consists of a fibration $\pi : F \to E \to X$ together with a homotopy equivalence $S^{n-1} \to F$. It follows from the axiomatic approach that there is a classifying space $B(G_n)$ for the set $\mathcal{E}^n_S(X)$ of homotopy equivalence classes of $(n-1)$-spherical fibrations over X and a fibration $\nu_n : S^{n-1} \to S(G_n) \to B(G_n)$, such that $f \mapsto f^*\nu_n$ gives a bijection $[X : B(G_n)] \to \mathcal{E}^n_S(X)$.

This notation goes with writing G_n for the set of maps of S^{n-1} to itself of degree ± 1, with the multiplication given by composition of maps. Although this is not a group, it can be treated as one for the purposes of homotopy theory. In particular we have a homotopy equivalence $G_n \to \Omega B(G_n)$. Restricting to maps of degree $+1$, or to fibrations with a fixed orientation of the fibre, gives a monoid SG_n and a classifying space $B(SG_n)$. The inclusion $O_n \subset G_n$ gives rise to a natural map $B(O_n) \to B(G_n)$.

We write $F_n \subset G_{n+1}$ for the set of base-point preserving maps $S^n \to S^n$ of degree ± 1, and SF_n for those of degree $+1$. The suspension of a self-map of S^{n-1} is a self-map of the same degree of S^n which fixes a base point; thus we also have an inclusion $G_n \subset F_n$. Since all components of $\Omega^n S^n$, including SF_n, are homotopy equivalent, we have $\pi_r(F_n) \cong \pi_{r+n}(S^n)$. We have a fibration $SF_{n-1} \to SG_n \to S^{n-1}$, and hence an exact sequence

$$\ldots \to \pi_{r+n-1}(S^{n-1}) \to \pi_r(G_n) \to \pi_r(S^{n-1}). \tag{B.2.1}$$

The classifying spaces $B(G)$ are infinite dimensional, and not homotopy equivalent to finite dimensional spaces. They may, however, be approximated by smooth manifolds. Since the map $Gr_{m,k} \to B(O_k)$ is m-connected, for a manifold M of dimension at most m, the set of homotopy classes of maps $M \to Gr_{m,k}$ maps bijectively to that of maps $M \to B(O_k)$. In general, we first replace the original $B(G)$, or indeed any space X, by the $(N+1)$-skeleton X_1 of its singular complex. Next, provided the homotopy groups of X are countable, we can replace X_1 by a countable $(N+1)$-simplicial complex X_2; then by a locally finite complex X_3, and finally imbed X_3 properly in Euclidean $(2N+3)$-space and take an open neighbourhood X_4 of which it is a deformation retract.

In the construction of classifying spaces we have emphasised principal bundles. However, for any G-space L we can study bundles with group G and fibre L, and the classification is the same as for the associated principal bundles: they are induced from the universal bundle $E(G) \times_G L$. For example, using the action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$, we obtain a universal vector bundle over $B(GL_n(\mathbb{R}))$.

Likewise we have a universal orthogonal vector bundle γ_k over $B(O_k)$, whose total space contains the associated unit disc bundle $A(O_k)$. Writing $S(O_k)$ for its boundary sphere bundle, we have the Thom space $T(O_k) = A(O_k)/S(O_k)$. Thus for any group G with a given homomorphism $G \to O_k$ we have induced bundles $S(G) \subset A(G)$ and $T(G)$ is obtained from $A(G)$ by identifying $S(G)$ to a point.

More generally, since each sphere bundle is a spherical fibration, we have an inclusion $O_n \subset G_n$ and maps $B(O_n) \to B(G_n)$, $S(O_n) \to S(G_n)$. Here the role of $A(G_n)$ is played by the mapping cylinder $Cyl(\pi)$, where $\pi : S(G_n) \to B(G_n)$ denotes the projection, and we define $T(G_n)$ to be its mapping cone. Again, any map $X \to B(G_n)$ induces a spherical fibration ξ over X and we have a Thom space. In this situation there is still a natural isomorphism, called the Gysin isomorphism

$$H^r(X) \to H^{k+r}(A_\xi, S_\xi) \cong \tilde{H}^{k+r}(T(\xi)).$$

A summary of calculations of cohomology of classifying spaces is in §8.6.

In general, if $x \in H^n(B(G); A)$ is a cohomology class, and $\pi : E \to X$ is a G-bundle, π is induced by a map $f : X \to B(G)$, so we have a class $f^*x \in H^n(X; A)$. Such a class is called a *characteristic class* of the bundle π, and denoted $x(f)$. For example, we have $H^*(B(O_n) : \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \dots, w_n]$, so any polynomial in $w_1, \dots, w_n$ defines a characteristic class for vector bundles of fibre dimension n.

If M is a smooth manifold, its tangent bundle $\mathbb{T}(M)$ is classified by a map $\phi : M \to B(O)$, so a class $x \in H^n(B(O); A)$ induces a characteristic class $x(M) := \phi^*(x) \in H^n(M; A)$. If $\mathbf{J}$ is a stable group and M has a J structure, we may replace O by $\mathbf{J}$ here.

If moreover M has the same dimension n, we have $\phi^*(x)[M] \in A$: this is called a *characteristic number* of M (if $A = \mathbb{Z}$ we do just have a number). If W is a cobordism of M to M', $x \in H^n(B(G) : A)$, and $\psi : W \to B(G)$ classifies a G-structure on W, then $\phi^*(x)[M] = \phi'^*(x)[M']$, since ψ restricts to ϕ and ϕ', and $\langle \psi^*(x), [M] - [M'] \rangle = 0$ since $[M] - [M'] = 0$ in homology, as the boundary of W. Thus characteristic numbers are cobordism invariants.

The same argument applies with any non-classical homology theory; for example, with KO-theory.

B.3 Homotopy calculations

In this section we summarise the results of a large number of homotopy calculations. We have included text intended to make the summary less unreadable, but make no attempt to give proofs. The results may be found in texts on homotopy theory, but the author has not discovered a convenient single reference for these results.

(i) There are natural maps $\pi_n(X) \to H_n(X; \mathbb{Z})$ and $\pi_n(X, Y) \to H_n(X, Y; \mathbb{Z})$. The *Hurewicz Isomorphism Theorem* states that if X is $(n-1)$-connected (and $n \geq 2$), the natural map $\pi_n(X) \to H_n(X; \mathbb{Z})$ is an isomorphism.

It follows that $\pi_r(S^n)$ is zero for $r < n$ and isomorphic to $\mathbb{Z}$ for $r = n$. We write ι_n for the class in $\pi_n(S^n)$ of the identity map.

The Hurewicz theorem has a relative version: if (K, L) is $(n-1)$-connected (and K, L are simply-connected), the natural map $\pi_n(K, L) \to H_n(K, L; \mathbb{Z})$ is an isomorphism. If we define the homology groups of a map $f : A \to B$ as those of the pair $(Cyl(f), A)$ we can write this as: if f is $(n-1)$-connected, $\pi_k(f) \to H_k(f; \mathbb{Z})$ is an isomorphism for $k \leq n$.

(ii) The group SU_2 is homeomorphic to the sphere S^3, and its action on $P^1(\mathbb{C}) \simeq S^2$ gives a fibre bundle map $\eta_2 : S^3 \to S^2$ called the *Hopf map*; similarly using quaternions or Cayley numbers gives maps $\eta_4 : S^7 \to S^4$ and $\eta_8 : S^{15} \to S^8$: using the real numbers gives $\eta_1 : S^1 \to S^1$ of degree 2, so homotopic to $2\iota_1$.

(iii) There is a natural homomorphism $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$, called the *Hopf invariant*. Given $f : S^{2n-1} \to S^n$, form $X_f := S^n \cup_f e^{2n}$, then $H^n(X_f)$ and $H^{2n}(X_f)$ are infinite cyclic with preferred generators u, v, say, and we set $u^2 = H(f)v$. This invariant vanishes for n odd (the cup product is skew-symmetric here), and takes the value 1 for each of η_2, η_4, η_8.

One generalisation of H is defined as follows. The map $\pi_r(j_n)$ induced by the inclusion $j_n : S^n \vee S^n \to S^n \times S^n$ has a right inverse given by adding the maps induced by the two projections of $S^n \times S^n$. Then H is the composite

$$\pi_r(S^n) \to \pi_r(S^n \vee S^n) \to \pi_{r+1}(S^n \times S^n, S^n \vee S^n) \to \pi_{r+1}(S^{2n}),$$

where the first map is induced by collapsing the equator to a point, the second by the splitting in the exact homotopy sequence of $(S^n \times S^n, S^n \vee S^n)$ and the third by collapsing $S^n \vee S^n$ to a point.

(iv) Let $f : (D^m, S^{m-1}) \to (X, *)$ represent $\alpha \in \pi_m(X)$ and $g : (D^n, S^{n-1}) \to (X, *)$ represent $\beta \in \pi_n(X)$: then the *Whitehead product* $[\alpha, \beta] \in \pi_{m+n-1}(X)$ is the homotopy class of the map $F : \partial(D^m \times D^n) \to X$ given by $F(x, y) = f(x)$ if $y \in \partial D^n$ and $= g(y)$ if $x \in \partial D^m$.

We have $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$, and $H([\iota_n, \iota_n])$ is 0 if n is odd, and 2 if n is even.

(v) The ‘Hopf invariant 1’ problem, the question whether $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is surjective, was solved by Adams [3]: it is surjective only if n is 2, 4, or 8.

This is analogous to the Kervaire invariant problem.

(vi) A further relative version of the Hurewicz theorem is the *Blakers–Massey Theorem* [18]. Given a commutative square

$$\begin{array}{cccc} & A & \xrightarrow{p} & B \\ \Phi : & q\downarrow & & r\downarrow, \\ & C & \xrightarrow{s} & D \end{array}$$

of simply-connected spaces, we can define $H_*(\Phi, \mathbb{Z})$ so that there are exact sequences $H_*(q; \mathbb{Z}) \to H_*(r; \mathbb{Z}) \to H_*(\Phi, \mathbb{Z}) \to H_{*-1}(q; \mathbb{Z})$. Then if p is $(r-1)$-connected, q is $(s-1)$-connected, and $H_*(\Phi, \mathbb{Z}) = 0$, $\pi_n(\Phi)$ vanishes for $n < r+s-1$ and $\pi_{r+s-1}(\Phi) \cong H_r(p; \mathbb{Z}) \otimes H_s(q; \mathbb{Z})$.

(vii) We can apply (vi) to the square given by the inclusions of S^n in the two hemispheres E_-^{n+1} and E_+^{n+1} of S^{n+1} (these inclusions are n-connected), and theirs in S^{n+1}. This gives $\pi_r(\Phi) = 0$ for $r \leq 2n$ and $\pi_{2n+1}(\Phi) \cong \mathbb{Z}$. Since the hemispheres are contractible, the sequence $\pi_r(E_-^{n+1}, S^n) \to \pi_r(S^{n+1}, E_+^{n+1}) \to \pi_r(\Phi)$ becomes $\pi_{r-1}(S^n) \to \pi_r(S^{n+1}) \to \pi_r(\Phi)$.

The map $\pi_{r-1}(S^n) \to \pi_r(S^{n+1})$ is called the *suspension map*. It is thus an isomorphism for $r \leq 2n-1$, so the groups $\pi_{n+k}(S^n)$ for $n \geq k+2$ are all isomorphic; the limit value is denoted π_k^S. Also we have an exact sequence

$$\pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1}) \to \mathbb{Z} \to \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1}) \to 0.$$

Here the second map is the Hopf invariant, and $1 \in \mathbb{Z}$ maps to $[\iota_n, \iota_n]$. It follows from the above that if n is even, the second map is zero so we have an exact sequence $0 \to \mathbb{Z} \to \pi_{2n-1}(S^n) \to \pi_{n-1}^S \to 0$; if $n \neq 1, 3, 7$ is odd we must replace $\mathbb{Z}$ by $\mathbb{Z}_2$ here.

(viii) For the groups $\pi_r(S^n)$ we have a range given by $r < n$ where the groups vanish, and a range $n \leq r < 2n-1$ where they are stable. We get information in the next ‘metastable’ range $2n-1 \leq r < 3n-2$ as follows.

We use the isomorphism of $\pi_r(\Omega S^{n+1})$ on $\pi_{r+1}(S^{n+1})$. Up to homotopy, ΩS^{n+1} has a cell structure with one kn-cell for each $k \in \mathbb{N}$. Hence $(\Omega S^{n+1}, S^n)$ is $(2n-1)$-connected and, by the relative Hurewicz theorem, $\pi_{2n}(\Omega S^{n+1}, S^n) \cong \mathbb{Z}$. Now applying (vi) to the square

$$\begin{array}{ccc} S^n & \longrightarrow & \Omega S^{n+1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Omega S^{n+1}/S^n \simeq S^{2n} \cup e^{3n} \ldots \end{array},$$

we find that $\pi_r(\Omega S^{n+1}, S^n) \to \pi_r(S^{2n})$ is an isomorphism for $r < 3n-1$. This yields the so-called *EHP sequence*

$$\pi_{n+k}(S^n) \xrightarrow{E} \pi_{n+k+1}(S^{n+1}) \xrightarrow{H} \pi_{n+k+2}(S^{2n+2}) \xrightarrow{P} \pi_{n+k-1}(S^n) \xrightarrow{E} \ldots, \quad \text{(B.3.1)}$$

generalising the sequence (vii), and valid for a range $k < 2n-1$. Here the map P agrees (up to suspension) with the Whitehead product with ι_n: $\pi_k(S^n) \to \pi_{n+k-1}(S^n)$.

A more general version can be obtained using the fibration $S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$ (after localisation at 2) constructed by James [1] and Toda [6].

(ix) The homotopy group $\pi_r(S^n)$ is finite for $r > n$ except if n is even and $r = 2n-1$ when it is the direct sum of $\mathbb{Z}$ and a finite group.

(x) The calculation of the homotopy groups $\pi_r(S^n)$ is a massive enterprise: see [129] for the state of the art. The stable groups form a ring under composition; the first few, with generators (here we use the same notation η_2 for the class of the suspension in π_1^S of η_2), are given by

$$\pi_1^S \cong \mathbb{Z}_2[\eta_2],\ \pi_2^S \cong \mathbb{Z}_2[\eta_2^2],\ \pi_3^S \cong \mathbb{Z}_{24}[\eta_4],\ \pi_4^S = 0,\ \pi_5^S = 0.$$

We have $\eta_2^3 = 0 \in \pi_3^S$.

(xi) The group SO_n acts transitively on the unit sphere S^{n-1} in $\mathbb{R}^n$, and the stabiliser of the unit point on the x_n-axis is the subgroup SO_{n-1}. Thus there is a fibre bundle $SO_{n-1} \to SO_n \to S^{n-1}$, with an exact homotopy sequence. Since $\pi_i(S^n)$ vanishes for $i < n$, we have isomorphisms $\pi_r(SO_{n-1}) \to \pi_r(SO_n)$ for $r \leq n-3$. More generally, if X has dimension $\leq r$, the suspension map $[X : BSO_n] \to [X : BSO_{n+1}]$ is bijective for $n \geq r+1$, so stably isomorphic vector bundles over X of fibre dimension $\geq r+1$ must be isomorphic.

Also all groups $\pi_r(SO_N)$ for $N \geq r+2$ are isomorphic; the common value is denoted $\pi_r(SO)$.

(xii) It was proved by Bott [21] that $\pi_r(SO)$ is infinite cyclic if $r \equiv 3$ (mod 4), isomorphic to $\mathbb{Z}_2$ if $r \equiv 0$ or $r \equiv 1$ (mod 8), and zero otherwise. A good account of Bott's proof is given in [98].

(xiii) The exact sequence of the fibre bundle $SO_{n-1} \to SO_n \to S^{n-1}$ includes

$$\to \pi_{n-1}(SO_{n-1}) \xrightarrow{i_*} \pi_{n-1}(SO_n) \xrightarrow{\pi_*} \mathbb{Z} \xrightarrow{\partial} \pi_{n-2}(SO_{n-1}) \xrightarrow{i_*} \pi_{n-2}(SO) \to 0. \tag{B.3.2}$$

If $x \in \pi_{n-1}(SO_n)$ classifies a bundle ξ, then $\pi_* x$ can be identified with the Euler number of ξ. If $x = \partial\iota_n$, then ξ is the tangent bundle of S^n, so $\pi_*\partial\iota_n$ is 2 for n even, and 0 for n odd. The image of π_* is 0 for n odd, $\mathbb{Z}$ for $n = 2, 4, 8$ and $2\mathbb{Z}$ for n even otherwise.

(xiv) Using (ix) and (xi), we see inductively that each group $\pi_r(SO_k)$ is finitely generated; the rank is 0 except if

(a) $k = 2s+1$, $r = 4i-1$, $1 \leq i \leq s$, or

(b) $k = 2s+2$, either $r = 4i-1$ with $1 \leq i \leq s$ or $r = 2s+1$.

In these cases the rank is 1 except if $k = 4s$ and $r = 4s-1$ when the rank is 2.

(xv) The Stiefel manifolds $V_{n,k} = SO_n/SO_{n-k}$ occur in fibre bundles $SO_{n-k} \to SO_n \to V_{n,k}$ $(1 < k < n)$ and $V_{n-k,l-k} \to V_{n,l} \to V_{n,k}$ $(k < l < n)$, which give further exact homotopy sequences. It now follows that for $r \leq n-k-1$ we have $\pi_r(V_{n,k}) = 0$, i.e. $V_{n,k}$ is $(n-k-1)$-connected.

(xvi) The calculation (xiii), that the kernel of $\pi_{n-2}(SO_{n-1}) \to \pi_{n-2}(SO)$ is isomorphic to $\mathbb{Z}$ for n odd, and to $\mathbb{Z}_2$ for n even, now implies that the first non-vanishing homotopy group $\pi_{n-k}(V_{n,k})$ is isomorphic to $\mathbb{Z}$ if $(n-k)$ is even and to $\mathbb{Z}_2$ if $(n-k)$ is odd.

(xvii) There is a homomorphism $J : \pi_k(SO_n) \to \pi_{n+k}(S^n)$, called the *J-homomorphism*, defined as follows. An element $\phi \in \pi_k(SO_n)$ is represented by a map $f : S^k \times D^n \to D^n$. Write $c : D^n \to S^n$ for a map which collapses ∂D^n to $*$. Write S^{n+k} as the union of $S^k \times D^n$ and $D^{k+1} \times S^{n-1}$, and define $g : S^{n+k} \to S^n$ to map the first part by $c \circ f$ and the second to $*$. Then $J(\phi)$ is the class of g in $\pi_{n+k}(S^n)$. An equivalent definition in the language of cobordism is given in §8.8.

For $x \in \pi_k(SO_n)$, we have $H(J(x)) = S^n(\pi(x)) \in \pi_{n+k}(S^{2n-1})$. Taking $k = 2s-1$, $n = 2s$ and $x = \partial\iota_{2s}$, then since $\pi(x) = 2\iota_{2s-1}$ we deduce $H(J(x)) = 2$, so the homomorphism $J : \pi_{2s-1}(SO_{2s}) \to \pi_{4s-1}(S^{2s})$ has rank 1.

(xviii) The image of the stable J homomorphism $J_k : \pi_k(SO) \to \pi_k^S$ was determined after heroic calculations by Adams [5]; a simpler proof was found in joint work with Atiyah [8].

(a) If $k \equiv 0$ or $k \equiv 1 \pmod 8$, the map J_k is a split monomorphism.

(b) If $k = 4m-1$ the image of J_k has order equal to $\mathrm{den}(B_m/4m)$, and is a direct summand of π_S^k.

(xix) It follows from (vi) that $\pi_r(SF_n)$ is finite for $r > 0$ except if n is even and $r = n-1$ when it is the direct sum of $\mathbb{Z}$ and a finite group.

In the exact sequence (B.2.1)

$$\ldots \to \pi_{r+n-1}(S^{n-1}) \to \pi_r(SG_n) \to \pi_r(S^{n-1}) \to \pi_{r+n-2}(S^{n-1}),$$

the final map is the Whitehead product with ι_{n-1}, so has infinite image if and only if n is odd and $r = n-1$. Thus $\pi_r(SG_n)$ is infinite if and only if either $r = n-1$ and n is even or $r = 2n-3$ and n is odd. The image of the map $\pi_r(SO_n) \to \pi_r(SG_n)$ has infinite order in each of these cases.

To summarise: the homotopy groups are finite except as follows:

Case	r	n	$rank(\pi_r(SO_n))$	$rank(\pi_r(SG_n))$
A	$4s+1$	$4s+2$	1	1
B	$4s-1$	$4s$	2	1
S	$4s-1$	$2s+1 < n \neq 4s$	1	0
C	$4s-1$	$2s+1$	1	1

(xx) If we take the exact sequence (B.2.1), increase n by 1, replace r by k, and compare with (B.3.1), we see that if $\pi_k(S^n)$ is stable, i.e. $2n \geq k+2$, we have an isomorphism $\pi_k(SG_{n+1}) \cong \pi_k(SF_{n+1})$.

The calculations in (xiv) can be compared with Haefliger's result [64] $\pi_r(F_n, G_n) \cong \pi_{r-n+1}(SO, SO_{n-1})$ for $r \leq 3n-6$, which he established by geometrical arguments.

B.4 Further techniques

We have defined CW complexes as built up from spheres by attaching cells. If these are attached in order of increasing dimension, a complex K has an n-skeleton $K^{(n)}$: the union of cells of dimension $\leq n$. The inclusion $i : K^{(n)} \to K$ has the HEP and is n-connected: the map $H_r(K^{(n)}) \to H_r(K)$ is an isomorphism for $r < n$ and an epimorphism for $r = n$; and the mapping cone $K \cup_i CK^{(n)}$ is n-connected.

There is also a dual approach. We may start with K, attach $(n+1)$-cells to K to kill $\pi_n(K)$; then $(n+2)$-cells to kill π_{n+1}, …, obtaining eventually an inclusion $j : K \to K_{(n)}$ with $\pi_r(j)$ an isomorphism for $r \leq n-1$ and $\pi_r(K_{(n)}) = 0$ for $r \geq n$. Denote the mapping fibre of j by $p^n : K^{\langle n \rangle} \to K$: then $K^{\langle n \rangle}$ is $(n-1)$-connected and $\pi_r(p^n)$ is an isomorphism for $r \geq n$. The pair $(K^{\langle n \rangle}, p^n)$ is called the $(n-1)$-connected cover of K, and is determined up to homotopy by these conditions.

It follows that, up to homotopy, there is for each k a fibration $K^{\langle k-1 \rangle} \to K^{\langle k \rangle} \to K(k, \pi_k(K))$. For any Y we have an induced map $[Y : K^{\langle k \rangle}] \to [Y : K]$; this is surjective if Y is k-connected, and bijective if Y is $(k+1)$-connected. The sequence of maps $\ldots \to K^{\langle 2 \rangle} \to K^{\langle 1 \rangle} \to K$ is called the Postnikov tower of K.

Given CW complexes K, L and a map $f : K^{(k-1)} \to L$ of the $(k-1)$-skeleton, the obstruction to extending f over a k-cell of K is an element of $\pi_{k-1}(L)$; collecting these over all k-cells gives a cochain on K, which is necessarily a cocycle. Its class in $H^k(K; \pi_{k-1}(L))$ is the obstruction to extending the restriction of f to $K^{(k-2)}$ over $K^{(k)}$.

If this obstruction vanishes, we can seek to extend over the $(k+1)$-skeleton, and so on. However, the later obstructions will in general depend on choices made at earlier stages. If k is the *least* integer such that $H^k(K; \pi_{k-1}(L))$ is non-zero, the obstruction in this group depends on no choices, and is called the *primary obstruction*.

If ξ is a vector bundle, and $\xi\langle k \rangle$ the associated bundle with fibre $V_{n,k}$, the primary obstruction to finding a section of $\xi\langle k \rangle$ is denoted $W_{n-k}(\xi)$; it lies in $H^{n-k}(B; \pi_{n-k}(V_{n,k}))$. The reduction modulo 2 of $W_{n-k}(\xi)$ is equal to the Stiefel–Whitney class $w_{n-k}(\xi)$.

Given $n \geq 1$ and a group π, abelian if $n \geq 2$, spaces $K(\pi, n)$ were constructed by Eilenberg and MacLane [49], with the property that $\pi_r(K)$ vanishes for $r \neq n$ and that $\pi_n(K) \cong \pi$: this determines $K(\pi, n)$ up to homotopy equivalence. For π abelian, there is a natural isomorphism $[X : K(\pi, n)] \cong H^n(X; \pi)$. It follows that a map $K(\pi, m) \to K(\rho, n)$ determines a natural transformation $H^m(X; \pi) \to H^n(X : \rho)$. Such a transformation is called a *cohomology operation*.

In particular, $[K(\mathbb{Z}_p, n) : K(\mathbb{Z}_p, n+k)] \cong H^{n+k}(K(\mathbb{Z}_p, n); \mathbb{Z}_p)$. Composing with an element of this group gives a natural transformation from $H^n(X; \mathbb{Z}_p)$ to $H^{n+k}(X : \mathbb{Z}_p)$. There are maps

$$H^{n+k}(K(\mathbb{Z}_p, n); \mathbb{Z}_p) \to H^{n+k+1}(K(\mathbb{Z}_p, n+1); \mathbb{Z}_p),$$

which are isomorphisms for $n > k$, so the groups with $n > k$ have a common value $H^k(K(\mathbb{Z}_p); \mathbb{Z}_p)$: elements of this give stable operations. Composition endows the set of these operations with a natural ring structure; this ring is known as the *Steenrod algebra* and denoted $\mathcal{S}_p$. Particular such operations are the Bockstein $\beta_p : H^n(X; \mathbb{Z}_p) \to H^{n+1}(X : \mathbb{Z}_p)$ and Steenrod's squares $Sq^i : H^n(X; \mathbb{Z}_2) \to H^{n+i}(X; \mathbb{Z}_2)$ and reduced pth powers $\mathcal{P}^r : H^n(X; \mathbb{Z}_p) \to H^{n+2r(p-1)}(X : \mathbb{Z}_p)$. These operations generate $\mathcal{S}_p$ and formulae for their composites (the Adem relations) are well known. There are rules (Cartan formulae) for evaluating these operations on the cup product of two classes. These define a diagonal map which furnishes $\mathcal{S}_p$ with the structure of a Hopf algebra. It thus has a canonical anti-automorphism, which is denoted χ.

It was shown by Milnor [93] that the dual algebra $\mathcal{S}_p^\vee$ is a polynomial algebra on a 1-dimensional generator b_p and generators c_r ($r \geq 1$) of degrees $2(p^r - 1)$. The quotient $\overline{\mathcal{S}}_p$ of $\mathcal{S}_p$ by the ideal generated by β_p has dual the polynomial algebra on the c_r. A careful and thorough account of this material is given in [145].

Steenrod squares are related to Stiefel–Whitney classes as follows. If ξ is a vector bundle, with projection $\pi : E \to B$ and Thom space $T(\xi)$, we have the Gysin isomorphism $\Phi : H^*(B : \mathbb{Z}_2) \to \tilde{H}^*(T(\xi); \mathbb{Z}_2)$, with $\Phi(1) = U$, say: then $Sq^i U = \Phi(w_i(\xi)) = w_i(\xi).U$. Classes $v_i \in H^i(B(O); \mathbb{Z}_2)$ are defined uniquely by the rule $w_i = v_i + \sum_{j=1}^{i-1} Sq^j v_{i-j}$, which may be written compactly as $w_* = Sq^* v_*$. In the special case of the tangent bundle of a manifold M^m, we have the formulae, known as Wu relations, $Sq^i x[M] = xv_i[M]$ for any $x \in H^{m-i}(M : \mathbb{Z}_2)$: these follow from the above and duality in M (see [103, IX, 5]).

As well as primary operations such as Steenrod squares there are secondary operations. The general idea is that if something vanishes for two independent reasons, this leads to a construction. Perhaps the simplest example: given maps $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} A_3$ such that $f_2 \circ f_1$ and $f_3 \circ f_2$ are nullhomotopic, choose homotopies $h_1 : A_0 \times I \to A_2$ and $h_2 : A_1 \times I \to A_3$: then both $h_1 \circ f_3$ and $h_2 \circ f_0$ are homotopies of $f_3 \circ f_2 \circ f_1 \circ f_0$ to a point. Glueing these together thus gives a map $SA_0 \to A_3$. This depends not only on the homotopy classes of the f_i but also on the choices of the homotopies, so (in the additive case) is unique up to adding elements of $f_3 \circ [SA_0 : A_2]$ and $[SA_1 : A_3] \circ Sf_1$.

For example, if $A_2 = K(G, m)$ and $A_3 = K(H, n)$, the map f_3 defines a cohomology operation $\phi : H^m(X; G) \to H^n(X; H)$. Thus if f_2 represents a class $\xi \in H^m(A_1; G)$ such that $f_1^* \xi = 0$ and $\phi(\xi) = 0$ we obtain an element of $H^n(SA_0; H) \cong H^{n-1}(A_0; H)$, which is denoted $\phi_{f_1}\xi$.

If p is a prime, we can localise a (finitely generated) abelian group A at p by forming the tensor product $A \otimes \mathbb{Z}_{(p)}$ with the group of integers localised at p (i.e. rational numbers with denominator prime to p). An Eilenberg–MacLane space $K(A, n)$ localises to $K(A \otimes \mathbb{Z}_{(p)}, n)$. Building up using fibrations, one can define the *localisation* $X_{(p)}$ at p of any simply-connected space X: it is unique up to homotopy, and $\pi_n(X_{(p)})$ is the localisation of $\pi_n(X)$ at p. See, for example, [23] for a textbook account. Similarly we can localise at any set S of primes. This permits calculations where we can ignore throughout the contribution of all primes not in S. This technique of 'mod $\mathcal{C}$' theory is due to Serre [136].

We define a *spectrum* $\mathbb{A}$ to be a sequence of (based) spaces A_n ($n \in \mathbb{Z}$) and maps $i_n : SA_n \to A_{n+1}$: equivalently, we may require maps $A_n \to \Omega A_{n+1}$. It is called an Ω-spectrum if the maps $A_n \to \Omega A_{n+1}$ are all homotopy equivalences.

The map i_n induces $\pi_{r+n}(A_n) \to \pi_{r+n+1}(SA_n) \to \pi_{r+n+1}(A_{n+1})$ and, for any $C, H_{r+n}(A_n; C) \to H_{r+n+1}(SA_n : C) \to H_{r+n+1}(A_{n+1}; C)$: the limits of these are defined to be $\pi_r(\mathbb{A})$ and $H_r(\mathbb{A}; C)$.

Proposition B.4.1 *Let $\mathbb{X}$ be a spectrum whose homology groups are finitely generated. Then the natural map $\pi_k^S(\mathbb{X}) \to H_k(\mathbb{X}; \mathbb{Z})$ has finite kernel and cokernel.*

This is proved using the methods of *mod $\mathcal{C}$* theory [136]. It is a very useful first step in calculation of bordism groups.

We give important examples of spectra. The *sphere spectrum* $\mathbb{S}$ is defined by the sequence S^n and $SS^n \simeq S^{n+1}$. The Eilenberg–MacLane spectrum $\mathbb{K}(A, k)$ is defined by the sequence $K(A, n+k)$ and the homotopy equivalences $K(A, n+k) \to \Omega K(A, n+k+1)$. The cohomology ring $H^*(\mathbb{K}(\mathbb{Z}_p, k); \mathbb{Z}_p)$ is free on one generator over $\mathcal{S}_p$; $H^*(\mathbb{K}(\mathbb{Z}, k); \mathbb{Z}_p)$ is free over $\overline{\mathcal{S}}_p$.

For J a stable group in the sense of §8.2, the sequence of maps $h_k : ST(J_k) \to T(J_{k+1})$ defines a spectrum, which we denote by $\mathbb{TJ}$.

A different example is obtained using the homotopy equivalence $\Omega U \to B(U)$ established by Bott: set $A_{2n} = B(U)$ and $A_{2n-1} = U$. This gives an Ω-spectrum $\mathbb{BU}$ with $\pi_{2k}(\mathbb{BU}) \cong \mathbb{Z}$ for *all* $n \in \mathbb{Z}$. Similarly using a homotopy equivalence $\Omega^8 O \to O$ we define a spectrum $\mathbb{BO}$. For any spectrum $\mathbb{A}$ we can define the $(k-1)$-connected cover $\mathbb{A}\langle k\rangle$: as for spaces, $\mathbb{A}^{\langle k\rangle}$ is $(k-1)$-connected and $\pi^k : \mathbb{A}^{\langle k\rangle} \to \mathbb{A}$ induces isomorphisms of the homotopy groups π_r for $r \geq k$. The spectrum $\mathbb{BO}\langle k\rangle$, which is a Ω-spectrum with 0-term $B(O)\langle k\rangle$, plays a role in Chapter 8.

A spectrum $\mathbb{A}$ is a ring spectrum if we are given a system of maps $A_m \wedge A_n \to A_{m+n}$ compatible with the i_n. There is a natural condition of associativity. For the above examples, $\mathbb{S}$ is a ring spectrum, a ring structure on A induces one on $\mathbb{K}(A, k)$, and $\mathbb{TJ}$ is a ring spectrum if (M) and (A) hold for $\mathbf{J}$.

Any spectrum $\mathbb{A} = \{A_n, i_n\}$ gives rise to a homology theory (satisfying the axioms discussed in §8.4) on defining

$$\begin{aligned} H_N(X; \mathbb{A}) &= \lim_{N\to\infty} \pi_{n+N}(A_n \wedge X^+) \\ H_N(X, Y; \mathbb{A}) &= \lim_{N\to\infty} \pi_{n+N}(A_n \wedge X^+, A_n \wedge Y^+) \\ &= \lim_{N\to\infty} \pi_{n+N}(A_n \wedge X, A_n \wedge Y). \end{aligned}$$

If $\mathbb{A}$ is a ring spectrum we obtain external products which are associative if the spectrum is.