7

Surgery

In this chapter we discuss a method of constructing manifolds, or more precisely, of adapting a given manifold to satisfy certain conditions. This method is due to Milnor and Kervaire. In the paper [102] where they introduced the method, the objective was to simplify the homotopy type of the manifold, so the procedure was called 'killing homotopy groups'. However since the procedure can be seen as removing a piece of a manifold and replacing it by something else, it has come to be known as 'surgery'.

It was observed by Novikov that the method could be applied to the more general situation, given a manifold M and a map $f : M \to X$, to change both M and f to make f more like a homotopy equivalence, by killing the homotopy groups of f. The method was then codified and further extended by Browder and by the author.

In more detail, the manifold M will be changed by a cobordism. As we saw in §5.1, we may choose a handle decomposition of this cobordism, so the procedure is broken into a sequence of operations, each corresponding to a single handle. Although we may think of M as a closed manifold, the discussion will apply to any compact manifold M.

In the first section we analyse a single step in the procedure: both the conditions for performing the step and its effect. In §7.2, we show how to modify a map $f : M \to X$ to kill all homotopy groups of f in dimensions below the middle.

In view of duality, any change to the homology of M is reflected by a corresponding change in the dual dimension. We next discuss the algebraic results we need on bilinear and quadratic forms, then in §7.4 formulate duality in the setting of CW-complexes.

In order to perform surgery to make f a homotopy equivalence, we must also require X to satisfy duality and it is convenient to suppose f a 'normal map'. As in Chapter 5, we discuss in detail in this book only the case when X is simply-connected. We treat in turn the cases when the dimension of M is even (when there is an obstruction in $\mathbb{Z}$ or $\mathbb{Z}_2$ to performing surgery) or odd (when there is none).

The finer details of the results depend on deeper results in homotopy theory, which we give in §7.7. Here we show how Spivak's fibration theorem permits a reformulation of the classification of normal maps. We proceed to Brown's interpretation of the Kervaire invariant.

In §7.8 we apply the results to discuss the homotopy types of smooth manifolds: the aim (not quite fully accomplished) is to reduce the problem of classification of smooth manifolds entirely to homotopy theory.

The author has already written a monograph [167] on surgery, in which no restriction is placed on the fundamental group. The account here is intended to be introductory rather than complete but is, of course, informed by the same view of the topic.

7.1 The surgery procedure: a single surgery

Let M^m be a compact manifold M (perhaps with boundary), $\phi : S^{r-1} \times D^{m-r+1} \to M \setminus \partial M$ an embedding. The operation of removing the interior of the image of ϕ, and attaching $D^r \times S^{m-r}$ to the result by $\phi|(S^{r-1} \times S^{m-r})$ is called a simple surgery, or *spherical modification* of M, of type $(r, m-r+1)$. The aim of surgery is to perform a series of spherical modifications on M to simplify M in a way to be made explicit.

The effect of a spherical modification is determined by ϕ, and even by the diffeotopy class of ϕ (by Theorem 2.4.2). The modification gives a manifold M' with the same boundary as M: in particular, if M is closed so is M'.

The manifold $W = (M \times I) \cup_f h^r$ (with corner, if M has a boundary) thus has $\partial_- W = M$, $\partial_+ W = M'$: it is a cobordism between M and M', called the *supporting manifold* of the modification. Also, $\partial_c W = \partial M \times I$. If M' is obtained from M by a spherical modification of type $(r, m-r+1)$, we can obtain M from M' by one of type $(m-r+1, r)$. We have the same supporting manifold for both modifications. It follows from the existence (see §5.1) of handle decompositions that M and N are cobordant if and only if one may be obtained from the other by a series of spherical modifications.

The procedure begins with a manifold M and a continuous map $f : M \to X$. Let W be obtained by attaching an r-handle to $M \times I$, with attaching map $\phi : S^{r-1} \times D^{m-r+1} \to M \setminus \partial M$. If we can extend f to a map $F : W \to X$, we say that the manifold $M' := \partial_+ W$ and the map $f' := F|M'$ are obtained from (M, f) by surgery.

We write ϕ_0 for the restriction of ϕ to $S^{r-1} \times \{0\}$. Up to homotopy, W is obtained from M by attaching an r-cell, and extending f over the handle amounts to giving a map $g : D^r \to X$ whose restriction to the boundary is $f \circ \phi_0$. The pair (ϕ, f) defines an element of the relative homotopy group $\pi_r(f)$.

Conversely, suppose given an element $\xi \in \pi_r(f)$. For this to arise as above, the class $\partial\xi \in \pi_{r-1}(M)$ must be represented by an embedding of S^{r-1} in M. The existence of such embeddings is guaranteed by general position if $m > 2(r-1)$; otherwise more work is required. Moreover, we need to extend the embedding of S^{r-1} to an embedding of $S^{r-1} \times D^{m-r+1}$, so need the normal bundle of the embedded sphere to be trivial. Provided $m \geq 2r - 1$ this follows if the bundle is stably trivial, hence if the restriction to the sphere of the tangent bundle $\mathbb{T}(M)$ is trivial. Since the sphere is nullhomotopic in X, a neat way to ensure this is to require that $\mathbb{T}(M)$ is itself induced from a bundle over X. It is convenient to weaken this slightly, giving the following definition.

A *normal map* consists of a map $f : M \to X$, a vector bundle ν over X and a trivialisation T of the bundle $\mathbb{T}(M) \oplus f^*\nu$. A *normal cobordism* is a normal map $(g : W \to X, \nu, T)$ with the manifold W a cobordism. We can extend this definition in a natural way to the case of a manifold with boundary.

Theorem 7.1.1 *Let $(f : M \to X, \nu, T)$ be a normal map. Then any $\xi \in \pi_r(f)$ determines a regular homotopy class of immersions $\phi : S^{r-1} \times D^{m-r+1} \to M$, and given any embedding in this class we can do surgery to obtain another normal map.*

Proof Suppose ϕ an embedding whose restriction to $S^{r-1} \times \{0\}$ represents $\partial\xi$: then we can use ϕ to attach an r-handle to $M \times I$ and use ξ to extend f to a map $g : (M \times I) \cup h^r \to X$ (more precisely, we first use ξ to extend f to the union of $M \times I$ and the disc $D^r \times \{0\}$, and then use a retraction of $D^r \times D^{m-r+1}$ on $(S^{r-1} \times D^{m-r+1}) \cup (D^r \times \{0\})$ – see Figure 5.6 – to extend to the rest of the handle).

Since the handle $D^r \times D^{m-r+1}$ is contractible, the restriction to it of $g^*\nu$ is trivial, so extending $T \oplus 1$ to a trivialisation of $\mathbb{T}(W) \oplus g^*\nu$ is equivalent to trivialising the sum of a trivial bundle with the restriction to $S^{r-1} \times D^{m-r+1}$ of $\mathbb{T}(M)$. Using stability, such trivialisations correspond to those of this restriction, and hence to isomorphisms of it to $\mathbb{T}(S^{r-1} \times D^{m-r+1})$. But by Corollary 6.2.2, such isomorphisms correspond bijectively to regular homotopy classes of immersions $\phi : S^{r-1} \times D^{m-r+1} \to M$. □

There are two approaches to analysing the effect on homology of a spherical $(r, m-r+1)$-modification: we can use the supporting manifold $W = (M \times I) \cup_f h^r$ or the intersection $X = M \cap M'$ (obtained from M by removing the interior of the image of ϕ). Up to homotopy W is obtained from M by attaching an r-cell, and from M' by attaching an $(m-r+1)$-cell. On the other hand, M is obtained from X by attaching an $(m-r+1)$- and an m-cell, and M' from X by attaching an r- and an m-cell.

The inclusions $(M', X) \subset (W, M \times I) \supset (W, M)$ induce isomorphisms of relative homology groups in dimensions $\neq m$. For the inclusions

$$(D^r \times S^{m-r}, S^{r-1} \times S^{m-r}) \subset (M', X);$$

$$(D^r \times D^{m+1-r}, S^{r-1} \times D^{m+1-r}) \subset (W, M \times I)$$

induce homology isomorphisms by excision. Thus it suffices to consider

$$(D^r \times S^{m-r}, S^{r-1} \times S^{m-r}) \subset (D^r \times D^{m+1-r}, S^{r-1} \times D^{m+1-r}),$$

and here both relative groups vanish except in dimensions r, m; in dimension r we have an isomorphism. It follows that

Lemma 7.1.2 *Let $r \leq m - r$. Then M and M' have the same $(r-2)$-type (in particular, if $r \geq 3$, the same fundamental group). If $r < m - r$, and x is the homology class of the a-sphere $f(S^{r-1} \times 0)$ in M, then $H_{r-1}(M')$ is the quotient of $H_{r-1}(M)$ by the subgroup generated by x.*

We can now express the homology relations by a single diagram.

Proposition 7.1.3 *We have the following exact sequences for $i < m - 1$:*

$$\begin{tikzcd}
& H_{i+1}(M) \arrow[dr] \arrow[rr, bend left] & & H_{i+1}(W, M') \arrow[dr] \arrow[rr, bend left] & & H_i(M') \arrow[dr] \arrow[rr, bend left] & & H_i(W, M) \\
H_{i+1}(X) \arrow[ur] \arrow[dr] & & H_{i+1}(W) \arrow[ur] \arrow[dr] & & H_i(X) \arrow[ur] \arrow[dr] & & H_i(W) \arrow[ur] \arrow[dr] & \\
& H_{i+1}(M') \arrow[ur] \arrow[rr, bend right] & & H_{i+1}(W, M) \arrow[ur] \arrow[rr, bend right] & & H_i(M) \arrow[ur] \arrow[rr, bend right] & & H_i(W, M')
\end{tikzcd}$$

Proof Since we can identify $H_j(M', X) = H_j(W, M)$ and dually $H_j(M, X) = H_j(W, M')$ for $j \leq m - 1$, it suffices to write out the homology exact sequences of the four pairs (M, X), (M', X), (W, M), and (W, M'). □

7.2 Surgery below the middle dimension

We now show how we can perform surgery on a normal map $f : M \to X$ to make the homotopy type of M closer to that of X.

Theorem 7.2.1 *If X is a finite CW-complex and $m \geq 2k$, any normal map $(f : M \to X, \nu, T)$ is normally cobordant to a normal map $(f' : M' \to X, \nu, T')$ such that f' is k-connected.*

Proof Let X' be the mapping cylinder of f, obtained from the disjoint union $(M \times I) \cup X$ by identifying each point $(x, 1) \in M \times \{1\}$ with $f(x) \in X$. The inclusion $X \subset X'$ is a homotopy equivalence, so ν extends to a bundle ν' over X'. The inclusion of M as $M \times \{0\}$ is homotopic to f. Thus replacing X by X', ν by ν' and T by the induced trivialisation, we have made no essential change, but may now take f to be an inclusion.

Set $X_0 := M$, and let X_i be a sequence of subcomplexes of X formed by attaching one at a time to X_0 the cells of X of dimension $\leq k$ not already in X_0. As X is finite, this process terminates, in X_K, say.

We now show, by induction on i, that we can add to $M \times I$ a sequence of handles yielding manifolds N_i and extend the inclusion of M in X to homotopy equivalences $f_i : N_i \to X_i$ and normal cobordisms $(f_i' : N_i \to X, \nu, T_i)$, where f_i' is the composite of f_i and the inclusion. We have $\partial_- N_i = M$ and set $M_i := \partial_+ N_i$. We start the induction with $N_0 = M \times I$; f_0' is f composed with the projection; similarly for T_0.

Suppose inductively (N_i, f_i, T_i) already constructed; let X_{i+1} be obtained from X_i by attaching an r-cell. This cell defines an element of $\pi_r(X, X_i)$ hence, since f_i is a homotopy equivalence, of $\pi_r(f_i')$. Denote by $f_i'' : M_i \to X$ the restriction of f_i': we claim that the map $\pi_r(f_i'') \to \pi_r(f_i')$ is an isomorphism.

Since N_i is obtained from $M \times I$ by attaching handles of dimension $\leq k$, it is obtained from M_i by attaching handles of dimensions $\geq m + 1 - k$; hence (N_i, M_i) is $(m - k)$-connected and hence, since $r \leq k < m - k$, it is r-connected. The claim thus follows from the exact sequence

$$\pi_r(N_i, M_i) \to \pi_r(f_i'') \to \pi_r(f_i') \to \pi_{r-1}(N_i, M_i).$$

The r-cell thus defines an element of $\pi_r(f_i'')$ and hence, by Theorem 7.1.1, a regular homotopy class of immersions $S^{r-1} \times D^{m-r+1} \to M_i$. Since $m > 2(r - 1)$, it follows from Theorem 4.7.7 that this class contains embeddings. We may thus perform surgery to obtain a normal cobordism. Since the r-cell of the cobordism maps to the homotopy class of the cell in X_{i+1}, the homotopy equivalence $N_i \to X_i$ extends to a homotopy equivalence $N_{i+1} \to X_{i+1}$. The induction is complete.

At the final step, since X is obtained from X_K by attaching cells of dimensions $\geq k+1$, the map $X_K \to X$, hence also $f'_K : N_K \to X$, is k-connected. Since, as we have just seen, (N_K, M_K) is $(m-k)$-connected, the map f''_K is also k-connected. □

An important special case is when X is a point.

Corollary 7.2.2 *If $\mathbb{T}(M)$ is stably trivial, and $m \geq 2k$, we can perform surgery on M to make it $(k-1)$-connected.*

For we apply the theorem, taking X to be a point; then since $M' \to X$ is k-connected, M' is $(k-1)$-connected.

In general the tangent bundle of M is induced by a map $M \to B(O_m)$. Suppose given a bundle ν^s and a framing T of $\mathbb{T}(M) \oplus \nu$ (so ν is a normal bundle for M). Choose a classifying map $f : M \to B(O_s)$, so that if υ is the universal bundle over $B(O_s)$ we have $\nu \cong f^*\upsilon$. Then $(f : M \to B(O_s), \upsilon, T)$ is a normal map, so we can perform surgery on M to obtain a k-connected map $f' : M' \to B(O_s)$. The mod 2 Betti numbers of M' below the middle dimension thus coincide with those of $B(O_s)$, hence with those of $B(O)$; those above the middle are determined by duality. It follows, for example, that if $w \in H^j(B(O); \mathbb{Z}_2)$ (with $2j > m+1$) is such that for any $w' \in H^{m-j}(B(O); \mathbb{Z}_2)$ the Stiefel–Whitney number $f'^*(ww')[M']$ vanishes, then also $f'^*w = 0$. Corresponding remarks hold for oriented manifolds with $B(O)$ replaced by $B(SO)$ and the coefficient field $\mathbb{Z}_2$ by $\mathbb{Q}$.

A different type of application arises by fixing k. If our object is to make f' induce an isomorphism $\pi_1(M') \to \pi_1(X)$, it suffices to have f' 2-connected, and we can achieve this provided $m \geq 4$.

With a little more care, we can construct embeddings. The following result includes a characterisation of possible fundamental groups of complements in S^m of embedded copies of S^{m-2} (provided $m \geq 5$).

Theorem 7.2.3 *Let (K, L) be a CW pair of dimension $k \geq 3$ with K contractible and K obtained from L by adding a 2-cell. Then provided $m \geq 2k-1$ there exist a smooth embedding of S^{m-2} in S^m with complement C and an $(m-k)$-connected map $C \to L$.*

Proof Set $M := S^1 \times D^{m-1}$, $X = L$, and define $f : M \to X$ to be projection on S^1 composed with the attaching map of the 2-cell. We define a normal map by taking ν to be trivial and using a trivialisation of $\mathbb{T}(M)$.

Now apply the result proved inductively in Theorem 7.2.1. We obtain a manifold N formed by attaching handles of dimensions $\leq k$ to $M \times I$ and a normal cobordism $(g : N \to L, \nu, T)$ which is a homotopy equivalence. Set

$M' := \partial_+ N$. Since N is formed from $M' \times I$ by attaching handles of dimensions $\geq m+1-k \geq k \geq 3$, $\pi_1(M') \to \pi_1(N)$ is an isomorphism.

Define W by attaching $D^2 \times D^{m-1}$ along $M \times \{0\}$. Up to homotopy we have attached a 2-cell, so the homotopy equivalence $g : N \to L$ extends to a homotopy equivalence $W \to L \cup e^2 = K$; thus W is contractible. Now $\partial W = (D^2 \times S^{m-2}) \cup \partial_c N \cup M'$, so $\pi_1(\partial W) \cong \pi_1(M' \cup e^2) \cong \pi_1(N \cup e^2)$, so is trivial. Since $m \geq 5$, it follows from Corollary 5.6.3 that $W^{m+1} \cong D^{m+1}$.

We now have $S^{m-2} \times \{0\} \subset S^{m-2} \times D^2 \subset \partial W \cong S^m$, and its closed complement may be taken as $\partial_c N \cup M'$ or as M'; the inclusion $M' \subset N$ is $(m-k)$-connected and $g : N \to L$ is a homotopy equivalence. □

Corollary 7.2.4 *Given $m \geq 5$ and a group G, there exist a smooth embedding $f : S^{m-2} \to S^m$ and an isomorphism $G \to \pi_1(S^m \setminus f(S^{m-2}))$ if and only if G is finitely presented, $H_1(G) \cong \mathbb{Z}$, $H_2(G) = 0$, and there is an element $x \in G$ whose conjugates generate the whole group.*

Here, and in the proof, all homology has coefficient group $\mathbb{Z}$.

Proof If $f : S^{m-2} \to S^m$ is a smooth embedding and $C := S^m \setminus f(S^{m-2})$, then S^m is obtained from C by attaching a 2-cell and an m-cell. The 2-cell is attached by a map $S^1 \to C$ with homotopy class x, say: the fundamental group is changed by factoring out the normal closure of x (the m-cell has no effect), and becomes trivial. If $G := \pi_1(C)$, then $H_1(G) \cong H_1(C) \cong \mathbb{Z}$ and $H_2(G)$ is a quotient of $H_2(C)$, which is zero.

Conversely, given G and $x \in G$, choose a finite presentation of G and construct a CW-complex with L' with $\pi_1(L) \cong G$ by taking 1-cells given by generators and attaching 2-cells corresponding to relators. Adding a further 2-cell e^2 along x gives a simply-connected space K'; since this is 2-dimensional, it is homotopy equivalent to a bouquet of 2-spheres.

In the sequence $0 \to H_2(L') \to H_2(K') \to H_2(K', L') \to H_1(L')$ the group $H_2(K', L')$ is infinite cyclic, generated by the class of e, hence maps isomorphically to $H_1(L') = H_1(G)$. Hence $H_2(L') \cong H_2(K')$ is free abelian, and we can pick a free basis $\{y_i\}$. It now follows from the exact sequence $\pi_2(L') \to H_2(L') \to H_2(G) = 0$ that we can represent the y_i by maps $f_i : S^2 \to L'$. Define L by attaching 3-cells to L' by the f_i. Then $H_2(L)$, $H_3(L)$ and all higher homology groups vanish. The space $K = L \cup e^2$ is now simply-connected with vanishing homology, hence is contractible.

We can now apply Theorem 7.2.3, taking $k = 3$. This yields a smooth embedding of S^{m-2} in S^m with complement C and an $(m-k)$-connected map $C \to L$. Since $m \geq 2k-1$, $m-k \geq k-1 \geq 2$, so $\pi_1(C) \cong \pi_1(L) \cong G$ as required. □

7.3 Bilinear and quadratic forms

In order to proceed further with surgery, we need to take account of duality. In this section we will introduce the purely algebraic notions, and thus make a digression to discuss results about symmetric and skew-symmetric bilinear forms which will play a role below. I aim to give enough details for the discussion to make sense, but will not give full details of all proofs.

We consider abelian groups $G, G', \ldots$, a value group V, and bilinear maps $\lambda : G \times G' \to V$, which we sometimes call pairings. Denote by $G^\vee$ the dual group $\mathrm{Hom}(G, V)$, by $\lambda^t : G' \times G \to V$ the transpose, given by $\lambda^t(g', g) = \lambda(g, g')$ and by $A\lambda : G \to G'^\vee$ the associated homomorphism given by $A\lambda(g)(g') = \lambda(g, g')$. The map λ is called *nonsingular* if $A\lambda$ is an isomorphism.

If $G' = G$ and $\epsilon = \pm 1$, we call λ ϵ-symmetric if $\lambda^t = \epsilon\lambda$. From now on we consider only pairings which are either symmetric ($\epsilon = 1$) or skew-symmetric ($\epsilon = -1$).

We also suppose that the natural map $G \to (G^\vee)^\vee$ is an isomorphism: this holds, for example, in the following situations:

V a field, G a finite dimensional vector space,

$V = \mathbb{Z}$, G a finitely generated free abelian group,

$V = \mathbb{Q}/\mathbb{Z}$ or the circle group $\mathbb{R}/\mathbb{Z}$, G a finite abelian group.

We call $g, g' \in G$ *orthogonal* if $\lambda(g, g') = 0$; for any subgroup $H \subset G$, its *annihilator* is defined by $H^o := \{g \in G \mid \forall g' \in G, \lambda(g, g') = 0\}$. Thus $H \subseteq H^o$ if and only if $\lambda(H \times H) = 0$. If $H^o = H$, H is called *Lagrangian*. We have

Lemma 7.3.1 *If $\lambda : G \times G \to V$ is nonsingular and ϵ-symmetric, and $H \subset G$ such that $\lambda \mid H \times H$ is nonsingular, then G splits as $H \oplus H^o$.*

We say that the form λ is *even* if, for each $g \in G$, there exists $v \in V$ with $\lambda(g, g) = v + \epsilon v$.

Lemma 7.3.2 *If $\lambda : G \times G \to V$ is nonsingular, ϵ-symmetric and even, and $H \subset G$ is Lagrangian, then there is a Lagrangian subgroup H^* such that $G = H \oplus H^*$. We can identify H^* with $H^\vee$, so that λ is given by*

$$\lambda((g, h), (g', h')) = h(g) + \epsilon h'(g).$$

Thus in this case, the form is determined up to isomorphism by H.

For symmetric bilinear forms over $\mathbb{R}$, it is well known that one can choose a basis $\{e_i \mid 1 \le i \le r\}$ for G such that $\lambda(e_i, e_j) = 0$ for $i \neq j$; if the form is nonsingular then each $a_i = \lambda(e_i, e_i) \neq 0$, and the form is classified up to isomorphism by the signature, which is given by $\sigma(\lambda) = \#\{i \mid a_i > 0\} - \#\{i \mid a_i < 0\}$. There is a Lagrangian subspace if and only if $\sigma(\lambda) = 0$.

For symmetric bilinear forms over $\mathbb{Z}$, we can tensor with $\mathbb{R}$ to obtain a form over $\mathbb{R}$ and thus define a signature. The following are known.

Proposition 7.3.3 *A nonsingular symmetric bilinear form λ over $\mathbb{Z}$ has a Lagrangian subspace if and only if $\sigma(\lambda) = 0$.*

If λ is a nonsingular even symmetric bilinear form over $\mathbb{Z}$, then $\sigma(\lambda)$ is divisible by 8.

Necessity of the condition $\sigma(\lambda) = 0$ is trivial. An example of an even form with signature 8 is given by the form with the matrix (the 'E_8 matrix')

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \qquad (7.3.4)$$

The skew-symmetric case is easily handled.

Proposition 7.3.5 *Let λ be a skew-symmetric bilinear form over $\mathbb{Z}$. Then H has a basis $\{e_i, f_i \mid 1 \le i \le r\}$ $\{g_j\}$ such that $\lambda(x, y) = 0$ for all pairs of basis elements* except *that $\lambda(e_i, f_i) = a_i$ for each i.*

Proof Write $H^o := \{x \mid \forall y \in H, \lambda(x, y) = 0\}$ for the radical of λ. Since, for $0 \neq k \in \mathbb{Z}$, $nx \in H^o$ implies $x \in H^o$, H^o is a direct summand of H. We may thus take a basis $\{g_j\}$ of H^o and extend to a basis for H.

We have reduced to the case when H^o is trivial, so $A\lambda$ is injective, with finite cokernel. Choose $e_1 \in H$ with coset modulo $A\lambda(H)$ of maximal order a_1. Then a_1 is the highest common factor of the $\lambda(e_1, x)$ for $x \in H$. Choose $f_1 \in H$ with $\lambda(e_1, f_1) = a_1$. Now for any $x \in H$ we can write $\lambda(e_1, x) = pa_1$ and $\lambda(f_1, x) = qa_1$: then $x + qe_1 - pf_1$ is orthogonal to both e_1 and f_1. Thus H is the orthogonal direct sum of $\mathbb{Z}\langle e_1, f_1\rangle$ and its orthogonal complement and we can proceed by induction. □

In particular,

Lemma 7.3.6 *Any nonsingular skew-symmetric bilinear form over $\mathbb{R}$ or $\mathbb{Z}$ has a Lagrangian subspace.*

Thus we may take a basis $\{e_i, f_i \mid 1 \le i \le r\}$ of H such that $\lambda(e_i, e_j) = \lambda(e_i, f_j) = \lambda(f_i, f_j) = 0$ for all i, j *except* that $\lambda(e_i, f_i) = 1$ for each i: such a basis is called a *symplectic basis*.

Now suppose given a nonsingular skew-symmetric bilinear form λ on a free abelian group H together with a map $\mu : H \to \mathbb{Z}_2$ with $\mu(0) = 0$ and satisfying the identity

$$\mu(x + x') = \mu(x) + \mu(x') + \lambda(x, x') \quad (\lambda(x, x') \text{ taken mod } 2). \tag{7.3.7}$$

The classification is given by

Lemma 7.3.8 *Given (H, λ, μ) as above, choose a symplectic basis $\{e_i, f_i \mid 1 \le i \le r\}$ of (H, λ). Then the number $Arf(\mu) := \sum_i \mu(e_i)\mu(f_i) \in \mathbb{Z}_2$ is an invariant of (H, λ, μ), and two such triples (of the same rank) are isomorphic if and only if the invariants $Arf(\mu)$ agree.*

Moreover, $Arf(\mu) = 0$ if and only if H has a Lagrangian subgroup on which μ vanishes.

Proof If $\mu(e_i) = 1$ and $\mu(f_i) = 0$, replacing e_i by $e'_i := e_i + f_i$ changes $\mu(e_i)$ to 0 without affecting the other values; similarly with e_i and f_i interchanged; thus we may reduce to the case $\mu(e_i) = \mu(f_i)$ for each i.

If $\mu(e_1) = \mu(f_1) = \mu(e_2) = \mu(f_2) = 1$ we substitute $e'_1 := e_1 + e_2$, $f'_2 := -f_1 + f_2$, preserving λ, with $\mu(e'_1) = \mu(f'_2) = 0$, and then deal with $\mu(f_1)$ and $\mu(e_2)$ as above. We may thus reduce to a normal form where μ vanishes on all basis elements except perhaps e_1 and f_1.

To prove $\mathrm{Arf}(\mu)$ an invariant, we note that $\mu : H \to \mathbb{Z}_2$ factors through $H/2H = H \otimes \mathbb{Z}_2$, and can check using the normal form that the number of elements of $H/2H$ on which μ takes the value 1 is $2^{2r-1} + 2^{r-1}$ if $\mathrm{Arf}(\mu) = 1$ and $2^{2r-1} - 2^{r-1}$ if $\mathrm{Arf}(\mu) = 0$.

The final result follows by inspection. □

We now consider the case when G is a finite group and λ takes values in $\mathbb{Q}/\mathbb{Z}$. Here instead of working with a free basis, we write G as the direct sum of subgroups of prime power order; each of these is a direct sum of cyclic subgroups.

Proposition 7.3.9 *Given a nonsingular skew-symmetric form λ on a finite group G, we may express G as a direct sum of mutually orthogonal subgroups, each of which is either of order 2 with $\lambda(x, x) = \frac{1}{2}$ or is isomorphic to $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ (for some prime p and integer k), with generators x, x' satisfying $\lambda(x, x) = 0$ or $\frac{1}{2}$, $\lambda(x', x') = 0$, $\lambda(x, x') = p^{-k}$.*

Further, we may suppose there is at most one summand of order 2.

Proof For each p, choose $x \in G$ of order p^k with k maximal. Since the form is nonsingular, we can find $x' \in G$ with $\lambda(x, x') = p^{-k}$. If p is odd, since $\lambda(x, x) = \lambda(x', x') = 0$, the form λ is nonsingular on $\langle x, x' \rangle$, so by Lemma 7.3.1 (G, λ) is the orthogonal direct sum of this and another subgroup, so we can proceed by induction.

If $p = 2$ and $k > 1$, each of $\lambda(x, x)$ and $\lambda(x', x')$ may be either 0 or $\frac{1}{2}$, but the same argument applies; if each is $\frac{1}{2}$ we can substitute $x + x'$ for x' to reduce $\lambda(x', x')$ to 0.

If $p^k = 2$ and $\lambda(x, x) = 0$, we may proceed as above, but if $\lambda(x, x) = \frac{1}{2}$, λ is already nonsingular on $\langle x \rangle$, so we can split this off as an orthogonal direct summand.

Finally observe that if we have two such summands $\lambda(x, x) = \lambda(y, y) = \frac{1}{2}$ and $\lambda(x, y) = 0$, we can start with $z = x + y$ to reduce to the preceding case. □

Proposition 7.3.10 *Given a nonsingular symmetric form λ on a finite group G, we may express G as a direct sum of mutually orthogonal subgroups, each of which is either cyclic of prime power order or is isomorphic to $\mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$ (for some k), with generators x, x' satisfying $2^{k-1}\lambda(x, x) = 2^{k-1}\lambda(x', x') = 0$, $\lambda(x, x') = 2^{-k}$.*

Proof Again it suffices to consider the case when G is a p-group. Let k be the greatest integer such that G has an element of order p^k: choose such an element x. If $\lambda(x, x)$ has order p^k, the restriction of λ to the subgroup H generated by x is nonsingular, and we may apply Lemma 7.3.1. Otherwise, choose an element y with $\lambda(x, y) = p^{-k}$: then y has order p^k. If p is odd, either $z = y$ or $z = x + y$ is such that $\lambda(z, z)$ has order p^k and we may proceed as above.

If $p = 2$, it may be that $\lambda(y, y)$ and $\lambda(x + y, x + y)$ both have order $< 2^k$. In this case, the restriction of λ to the subgroup H generated by x and y is nonsingular, and we may again apply Lemma 7.3.1. □

One can now proceed to further analysis of each of these types of summand.

We define a nonsingular quadratic form on the finite group G to be a pair (λ, μ) with $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$ and $\mu : G \to \mathbb{Q}/2\mathbb{Z}$ satisfying

- λ is nonsingular symmetric bilinear,
- $\mu(0) = 0$, $\mu(-x) = \mu(x)$ and
- $2\lambda(x, y) = \mu(x + y) - \mu(x) - \mu(y)$.

It follows that $\mu(x) \equiv \lambda(x, x) \pmod 1$. Note that $\lambda(x, y) \in \mathbb{Q}/\mathbb{Z}$ determines $2\lambda(x, y)$ modulo 2. The classification of quadratic forms is close to that of symmetric bilinear forms (if $|G|$ is odd, there is no essential difference). In analogy with the above, call a subgroup $H \subset G$ Lagrangian if μ vanishes on H (and hence λ vanishes on $H \times H$) and $|G| = |H|^2$. As before, H then coincides with its annihilator under λ and $A\lambda$ induces an isomorphism of G/H on $H^\vee$.

Here there is a new feature. We define the Gauss sum $\mathfrak{G}(\mu) := \sum_{x \in G} e^{i\pi\mu(x)}$.

Theorem 7.3.11 *Suppose (λ, μ) a nonsingular quadratic form on the finite group G. Then $\mathfrak{G}(\mu)$ has the form $A(\mu)\sqrt{|G|}$, with $A(\mu)^8 = 1$. If there is a Lagrangian subgroup H, $\mathfrak{G}(\mu) = |H|$ and $A(\mu) = 1$.*

Proof We prove the second statement first. We split the sum over G into a sum over cosets of H. There are two cases. For the trivial coset, $\sum_{h \in H} e^{i\pi\mu(h)} = \sum_{h \in H} e^0 = |H|$. For any other coset, we have $\sum_{h \in H} e^{i\pi\mu(y+h)} = e^{i\pi\mu(y)} \sum_{h \in H} e^{i\pi\lambda(y,h)}$. Now if y has order k in G/H, as λ is nonsingular, there exists $z \in H$ with $\lambda(y, z) = \frac{1}{k}$, and as z varies, each value $\frac{j}{k}$ is taken $|H|/k$ times. But the sum $\sum_{j \bmod k} e^{2\pi i j/k}$ vanishes unless $k = 1$. Thus the sum over H vanishes. Summing over all cosets, we just have $|H|$.

Given two triples (G, λ, μ) and (G', λ', μ') we can form the direct sum $G'' := G \oplus G'$ and define $\lambda''((x, x'), (y, y')) := \lambda(x, y) + \lambda'(x', y')$ and $\mu''(x, x') := \mu(x) + \mu'(x')$. This has the above properties, and we see that $\mathfrak{G}(\mu'') = \mathfrak{G}(\mu)\mathfrak{G}(\mu')$.

Now take $G' = G$, $\lambda' = -\lambda$ and $\mu' = -\mu$: then $\mathfrak{G}(\mu') = \overline{\mathfrak{G}(\mu)}$. In the direct sum $G'' := G \oplus G'$, the diagonal is a Lagrangian subgroup. Hence $|\mathfrak{G}(\mu)|^2 = \mathfrak{G}(\mu)\overline{\mathfrak{G}(\mu)} = \mathfrak{G}(\mu'') = |G|$.

The calculation of the argument is more sophisticated. For p an odd prime, it was shown by Gauss that the sum $\sum_{j \bmod p} e^{2\pi i j^2/p}$ is equal to $i\sqrt{p}$: other cases when G is cyclic of order a power of p follow easily. The calculations for the case $p = 2$ can be done ad hoc: for example, if G has order 2, and $\lambda(x, x) = \frac{1}{2}$, $\mu(x) = \frac{1}{2}$, we have $\mathfrak{G}(\mu) = 1 + i = e^{i\pi/4}\sqrt{2}$. □

Lemma 7.3.12 *For (λ, μ) a nonsingular quadratic form on G, the function $\mu_z(x) := \mu(x) + 2\lambda(x, z)$ is a quadratic form if and only if $2z = 0$. We have $\mathfrak{G}(\mu_z) = e^{-i\pi\mu(z)}\mathfrak{G}(\mu)$.*

Proof For necessity note that $\mu_z(x) - \mu_z(-x) = \mu(x) - \mu(-x) + 2\lambda(x, 2z)$, and since λ is nonsingular, $2\lambda(x, 2z) \in 2\mathbb{Z}$ for all x if and only if $2z = 0$.

Now since $\mu_z(x) := \mu(x) + 2\lambda(x, z) = \mu(x + z) - \mu(z)$, we can write $\mathfrak{G}(\mu_z) = \sum_{x \in G} e^{i\pi\mu_z(x)} = \sum_{x \in G} e^{i\pi(\mu(x+z) - \mu(z))} = e^{-i\pi\mu(z)} \sum_{x \in G} e^{i\pi\mu(x+z)}$, which equals $e^{-i\pi\mu(z)}\mathfrak{G}(\mu)$. □

In Lemma 7.3.8 we considered pairs (λ, μ) with $\lambda : H \times H \to \mathbb{Z}$ a nonsingular skew-symmetric form and $\mu : H \to \mathbb{Z}_2$ satisfying (7.3.7). If we set

$G := H/2H$ and define $\lambda_S : G \times G \to \mathbb{Q}/\mathbb{Z}$ by $\lambda_S(g, g') = \frac{1}{2}\lambda(x, x')$ and $\mu_S : G \to \mathbb{Q}/2\mathbb{Z}$ by $\mu_S(g) = \mu(x)$ (where x, x' are lifts of g, g'), then (λ_S, μ_S) is a nonsingular quadratic form on the finite group G. Using a symplectic basis, we see at once that $A(\mu) = (-1)^{\mathrm{Arf}(\mu_S)}$.

Now consider an ϵ-symmetric pairing $\lambda : H \times H \to \mathbb{Z}$, but now suppose only that $A\lambda : H \to H^\vee$ is injective; denote by G its cokernel. We can tensor with $\mathbb{Q}$ to embed λ in a nonsingular pairing λ_Q on H_Q to $\mathbb{Q}$: denote by H' the subgroup of H_Q corresponding to $H^\vee$: thus G is identified with H'/H.

We can lift elements $g, g' \in G$ to elements $x, x' \in H'$ and form $\lambda_Q(x, x')$: denote its image in $\mathbb{Q}/\mathbb{Z}$ by $\overline{\lambda}(g, g')$.

Lemma 7.3.13 *The class of $\overline{\lambda}(g, g')$ in $\mathbb{Q}/\mathbb{Z}$ depends only on g, g'. This defines a nonsingular ϵ-symmetric pairing $\overline{\lambda} : G \times G \to \mathbb{Q}/\mathbb{Z}$.*

Proof It is immediate that $\overline{\lambda}$ is well defined, bilinear, and ϵ-symmetric. If $A\overline{\lambda}(g) = 0$, so for each $g' \in G$, $\overline{\lambda}(g, g') = 0$, then the lift x of g is such that for each $x' \in H'$ we have $\lambda(x, x') \in \mathbb{Z}$, so $A\lambda(x) \in H'^\vee$ so $x \in H$ and $g = 0$. Since $A\overline{\lambda} : G \to G^\vee$ is injective, and both these finite groups have the same order, it is an isomorphism. □

In the case $\epsilon = -1$, we note the additional property $\overline{\lambda}(g, g) = 0$ for all $g \in G$.

If the above form λ is even as well as symmetric, we can enhance $\overline{\lambda}$ by defining $\mu : G \to \mathbb{Q}/2\mathbb{Z}$ by $\mu(g) = \lambda(x, x) \pmod{2\mathbb{Z}}$, where x is a lift of g. It is immediate that μ is a quadratic form in the sense defined above. The following is a deeper result.

Theorem 7.3.14 *Let the even symmetric bilinear form λ on H induce the quadratic map $(\overline{\lambda}, \mu)$ on $G = H^\vee/H$ as above. Then $\mathfrak{G}(\mu) = e^{i\pi\sigma(\lambda)/4}\sqrt{|G|}$.*

This result is given by van der Blij [155], together with a short proof that depends on manipulation of divergent integrals. We observe that it ties in with the result in Proposition 7.3.3 that if λ is nonsingular (so G is trivial), then $\sigma(\lambda)$ is divisible by 8.

7.4 Poincaré complexes and pairs

As noted above, we cannot expect to improve the result of Theorem 7.2.1 without imposing some conditions on X. If we can construct a homotopy equivalence $M \to X$ with M a closed manifold, then X also must satisfy Poincaré duality as in Theorem 5.3.5. A corresponding conclusion applies for manifolds with boundary. We begin with a formal definition of the duality property, then explore some consequences in the form of pairings on its homology groups. We turn to the study of homological properties of maps of degree 1.

We make the following definitions. A *Poincaré complex* of formal dimension m consists of a finite CW-complex X and a homology class $[X] \in H_m(X; \mathbb{Z})$ (which we call the *fundamental class*) such that cap product with $[X]$ induces isomorphisms $H^r(X; \mathbb{Z}) \to H_{m-r}(X; \mathbb{Z})$ for all $r \in \mathbb{Z}$.

Thus a first necessary condition for a finite CW complex X to have the homotopy type of a closed manifold is that X be a Poincaré complex. We will see that, if X is simply-connected, there is a natural way to enhance this to obtain a sufficient condition.

We have defined Poincaré complexes in terms of homology. We will see in §7.8 how they can be defined from a homotopy theoretic viewpoint.

The above definition is not adequate if X is not simply-connected, and does not even include non-orientable closed manifolds. For the definition in the general case, see [164].

A *Poincaré pair* consists of a finite CW-pair (Y, X) and a homology class $[Y] \in H_{m+1}(Y, X)$ such that cap product with $[Y]$ induces isomorphisms

$$H^r(Y; \mathbb{Z}) \to H_{m+1-r}(Y, X; \mathbb{Z}), \quad H^r(Y, X; \mathbb{Z}) \to H_{m+1-r}(Y; \mathbb{Z})$$

for all r. It follows that if $[X] := \partial[Y] \in H_m(X; \mathbb{Z})$, then $(X, [X])$ is a Poincaré complex. Indeed, in view of the five lemma, the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
H^{r-1}(X) & \to & H^r(Y, X) & \to & H^r(Y) & \to & H^r(X) & \to & H^{r+1}(Y, X) \\
[X]\downarrow & & [Y]\downarrow & & [Y]\downarrow & & [X]\downarrow & & [Y]\downarrow \\
H_{m+1-r}(X) & \to & H_{m+1-r}(Y) & \to & H_{m+1-r}(Y, X) & \to & H_{m-r}(X) & \to & H_{m-r}(Y)
\end{array}$$

shows conversely that if we assume X a Poincaré complex, the two conditions defining Poincaré pairs are equivalent.

We regard Poincaré complexes and pairs as the homotopy-theoretic analogue to compact manifolds. Many theorems valid for manifolds have analogues in this context. Corresponding to the Disc Theorem 2.5.6, we have

Lemma 7.4.1 *Let Z be a Poincaré complex of formal dimension $n \geq 3$. Then there exist a Poincaré pair (Y, X) and a homotopy equivalence $f : S^{n-1} \to X$ such that the space $Y \cup_f e^n$ obtained by glueing (Y, X) to (D^n, S^{n-1}) is homotopy equivalent to Z.*

Proof ([164, Theorem 2.4]) We give details here only in the simply connected case. Then Z is homotopy equivalent to a finite CW-complex, and we may suppose that this has no cell of dimension greater than n, and only one n-cell. Now pick an embedding of D^n in the interior of this cell, and define

Y by deleting its interior. Thus inclusions induce isomorphisms $H_n(Z) \to H_n(Z, Y)$, $H_n(D^n, S^{n-1}) \to H_n(Z, Y)$ preserving the fundamental classes $[Z]$ and $[D^n, S^{n-1}]$. Cap products with the fundamental class give maps of the Mayer–Vietoris cohomology sequence of $(Z; Y, D^n; S^{n-1})$ to the homology sequence; these are isomorphisms for Z and for (D^n, S^{n-1}), hence also for (Y, S^{n-1}). $\square$

For any X, cup product gives a $(-1)^k$-symmetric bilinear pairing

$$H^k(X; \mathbb{Z}) \times H^k(X; \mathbb{Z}) \to H^{2k}(X, \mathbb{Z}).$$

If X is a Poincaré complex of formal dimension $2k$, we have $H^{2k}(X, \mathbb{Z}) \cong \mathbb{Z}$, and so a bilinear pairing of $H^k(X; \mathbb{Z})$, which is $(-1)^k$-symmetric. Since the map $[X]\cap : H^k(X; \mathbb{Z}) \to H_k(X; \mathbb{Z})$ is an isomorphism, we also obtain a pairing on $H_k(X; \mathbb{Z})$. The pairing is obtained by composing $[X]\cap$ with the natural map $H^k(X; \mathbb{Z}) \to \mathrm{Hom}(H_k(X; \mathbb{Z}), \mathbb{Z})$. If we extend coefficients from $\mathbb{Z}$ to $\mathbb{Q}$ this becomes an isomorphism, so the pairings become nonsingular. When X is a $2k$-manifold, the self-pairing of $H_k(X; \mathbb{Z})$ can be geometrically interpreted as intersection numbers.

When k is even, the question arises whether the form on $H^k(X; \mathbb{Z})$ is even, in the sense that for each $x \in H^k(X; \mathbb{Z})$, $x.x[X]$ is even. If we reduce mod 2, we obtain the cup product pairing on $H^k(X; \mathbb{Z}_2)$. We have the Wu relations (see §B.4) $x^2[X] = xv_k[X]$ for $x \in H^k(X; \mathbb{Z}_2)$. Thus the vanishing of the characteristic class v_k is necessary and sufficient for the form on $H^k(X; \mathbb{Z})$ to be even.

If (Y, X) is a Poincaré pair of formal dimension $2k + 1$, in the exact sequence

$$H^k(Y; \mathbb{Q}) \to H^k(X; \mathbb{Q}) \to H^{k+1}(Y, X; \mathbb{Q})$$

the two maps are dual to each other, so have the same rank, and the pairing vanishes on the image of $H^k(Y; \mathbb{Q})$ since this factors through the zero map $H^{2k}(Y; \mathbb{Q}) \to H^{2k}(X; \mathbb{Q})$: thus this image is a Lagrangian subspace. In the case when k is even, it follows from Lemma 7.3.2 that the pairings have signature $\sigma = 0$.

Now let X be a Poincaré complex of odd formal dimension $2k + 1$; first suppose $H_k(X; \mathbb{Q}) = 0$, so that $H_k(X; \mathbb{Z})$ is a finite group. Since by duality $H_{k+1}(X; \mathbb{Q}) = 0$, the map $H_{k+1}(X; \mathbb{Q}/\mathbb{Z}) \to H_k(X; \mathbb{Z})$ is an isomorphism, while by duality

$$H_{k+1}(X; \mathbb{Q}/\mathbb{Z}) \cong H^k(X; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_k(X; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).$$

Composing these maps gives a nonsingular pairing of $H_k(X; \mathbb{Z})$ with itself to $\mathbb{Q}/\mathbb{Z}$.

When X is a $(2k+1)$-manifold, this too can be interpreted geometrically. If $x \in H_k(X;\mathbb{Z})$ has order θ, we represent x by a k-cycle ξ; then $\theta\xi$ is a boundary, say $\theta\xi = \partial\zeta$. Given another class $y \in H_k(X;\mathbb{Z})$ represented by a cycle η disjoint from ξ, we may suppose η transverse to ζ and count the intersections. Then $\lambda(x, y) = \frac{1}{\theta}(\zeta.\eta) \pmod{\mathbb{Z}}$.

Either from the algebraic or geometric approach we can see that when the hypothesis $H_k(X;\mathbb{Q}) = 0$ is dropped we obtain a nonsingular pairing of the torsion subgroup Tors $H_k(X;\mathbb{Z})$ with itself to $\mathbb{Q}/\mathbb{Z}$. It follows from the symmetry property of cup products over $\mathbb{Q}$ that this form is $(-1)^{k+1}$-symmetric. Thus if k is even, $x \mapsto b(x, x)$ defines a homomorphism $c : \text{Tors}\, H_k(X;\mathbb{Z}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. It can be shown that we have $c(x) = \langle v_k, x\rangle$.

We saw in §7.1 that to facilitate surgery it is natural to consider normal maps $(f : M \to X, \nu, T)$. We now suppose X a Poincaré complex and impose the further condition $f_*[M] = [X]$ or, as we will say, that f has degree 1. Observe that if $(g : N \to X, \nu, U)$ is a normal cobordism of f to f' and f has degree 1, then so has f'. We thus study the homology of maps of degree 1.

Proposition 7.4.2 *Let $\phi : M \to X$ be a map of degree 1 of Poincaré complexes. Then the diagram*

$$\begin{tikzcd} H^r(M;\mathbb{Z}) \arrow[d, "{[M]\cap}"] & H^r(X;\mathbb{Z}) \arrow[l, "\phi^*"'] \arrow[d, "{[X]\cap}"] \\ H_{m-r}(M;\mathbb{Z}) \arrow[r, "\phi_*"] & H_{m-r}(X;\mathbb{Z}) \end{tikzcd}$$

is commutative, so $.[M]\cap$ induces an isomorphism of the cokernel $K^r(M;\mathbb{Z})$ of ϕ^* on the kernel $K_{m-r}(M;\mathbb{Z})$ of ϕ_*. In particular, if ϕ is k-connected, ϕ_* and ϕ^* are isomorphisms for $r < k$ and for $r > m-k$.

A map $\phi : (N, M) \to (Y, X)$ of degree 1 of Poincaré pairs induces split surjections of homology groups $M \to X$, $N \to Y$ and $(N, M) \to (Y, X)$ with kernels K_* and split injections of cohomology groups with cokernels K^*. The duality map $.[N]\cap$ induces isomorphisms

$$K^*(N) \to K_*(N, M), \qquad K^*(N, M) \to K_*(N).$$

The homology (cohomology) exact sequence of (N, M) is isomorphic to the direct sum of the sequence for (Y, X) and a sequence of groups $K_*(K^*)$.

Commutativity of the diagram follows from naturality of cap products. Since the vertical maps are isomorphisms, ϕ^* is a split injection and ϕ_* a split surjection. The other assertions are immediate consequences. The same holds if $\mathbb{Z}$ is replaced by any coefficient group.

If $\phi : (M, \partial M) \to (X, \partial X)$ is a map of degree 1 of Poincaré pairs inducing a homotopy equivalence $\partial M \to \partial X$, it follows that, as in the case of closed manifolds, $.[M] \cap$ induces an isomorphism of the cokernel $K^r(M; \mathbb{Z})$ of ϕ^* on the kernel $K_{m-r}(M; \mathbb{Z})$ of ϕ_*.

It follows that if M has dimension $2k$, we have a $(-1)^k$-symmetric bilinear form on $K_k(M; \mathbb{Z})$ to $\mathbb{Z}$. If moreover the map $f : M \to X$ is k-connected, so the groups K_* vanish in lower dimensions, this pairing is nonsingular. It follows from the commutative diagrams and the characteristic property of υ_k that $\upsilon_k(M) = \phi^* \upsilon_k(X)$. Hence if k is even, the self-pairing of $K_k(M; \mathbb{Z})$ is even.

If M has odd dimension $2k + 1$, we have a $(-1)^{k+1}$-symmetric bilinear form on Tors $K_k(M; \mathbb{Z})$ to $\mathbb{Q}/\mathbb{Z}$. Again, if the map $f : M \to X$ is k-connected, this pairing is nonsingular.

Now suppose given a Poincaré complex X of formal dimension m, and a normal map $(f : M \to X, \nu, T)$ of degree 1 (or more generally $(X, \partial X)$ a Poincaré pair and $f : (M, \partial M) \to (X, \partial X)$ inducing a homotopy equivalence $\partial M \to \partial X$). By Theorem 7.2.1, if $m \geq 2k$, we may perform surgery to make f k-connected. Then $K_r(M)$ vanishes for $r < k$. It now follows from Proposition 7.4.2, together with duality, that if $m = 2k$, $K_r(M)$ vanishes except if $r = k$, while if $m = 2k + 1$, the exceptions are $r = k,\ k + 1$.

Now let (Y, X) be a Poincaré pair of formal dimension n and $\phi : (N, M) \to (Y, X)$ a normal map of degree 1. We may first apply Theorem 7.2.1 to $\phi \,|\, M$, extend to a normal cobordism of ϕ, and then apply the Theorem to ϕ. This kills all the K groups except those in the sequence: if $n = 2k$,

$$0 \to K_k(M) \to K_k(N) \to K_k(N, M) \to K_{k-1}(M) \to 0 :$$

and if $n = 2k + 1$,

$$0 \to K_{k+1}(N) \to K_{k+1}(N, M) \to K_k(M) \to K_k(N) \to K_k(N, M) \to 0.$$

The following extension of Theorem 7.2.1 will be useful.

Proposition 7.4.3 *([167, p. 15]) Suppose $n = 2k + 1$, $k \geq 2$, and both X and Y are simply-connected; then ϕ is normally cobordant to a k-connected normal map such that $K_k(N, M) = 0$.*

Proof We may suppose ϕ k-connected. Since $k \geq 2$ and a 2-connected map induces an isomorphism of fundamental groups, both M and N are simply-connected. Thus $K_k(N, M) \cong \pi_{k+1}(\phi)$. Choose a finite set $\{e_i\}$ of generators. As in Theorem 7.1.1 we can represent each e_i by a framed immersion $f_i : (D^k, S^{k-1}) \to (N, M)$. By general position, we may suppose the f_i disjoint embeddings. We extend these to disjoint embeddings $F_i : (D^k, S^{k-1}) \times D^{k+1} \to (N, M)$.

Since these represent elements of $\pi_{k+1}(\phi)$, they are nullhomotopic in (Y, X), so ϕ is homotopic to a map taking the image of each F_i to a point. We obtain (N', M') from (N, M) by deleting the interiors of the images of the F_i and rounding the corners. We inherit a normal map $\phi' : (N', M') \to (Y, X)$, and a normal cobordism of ϕ' to ϕ is obtained from $\phi \times I$ by adjusting the corners as in Figure 8.1. Although this has not been described as a surgery, it can equivalently be obtained by first performing surgery on the boundary using the $F_i : S^{k-1} \times D^{k+1} \to M$ to add handles, then interior surgery on the k-spheres created by this.

Denote by A the free abelian group with basis $\{e_i\}$. We have an exact sequence $A \to K_k(N, M) \to K_k(N', M') \to 0$; since the first map is surjective, we have $K_k(N', M') = 0$. □

Lemma 7.4.4 *In the above situation, if $K_k(N, M) = 0$, $K_{k+1}(N, M)$ is a Lagrangian subspace of $K_k(M)$.*

Proof We again relativise the arguments of Lemma 7.1.1. We have an isomorphism $\pi_{k+2}(\phi) \to K_{k+1}(N, M)$, so each element α of $K_{k+1}(N, M)$ is represented by a map $g_\alpha : (D^{k+1}, S^k) \to (N, M)$ together with a nullhomotopy of the composed map to (Y, X). Now D^{k+1} is contractible, so has trivial tangent bundle, and the nullhomotopy shows that $g_\alpha^* \mathbb{T}(N)$ is trivial. We thus have a stable isomorphism of $\mathbb{T}(D^{k+1})$ with $g_\alpha^* \mathbb{T}(N)$, which restricts to a stable isomorphism of $\mathbb{T}(S^k)$ with $g_\alpha^* \mathbb{T}(M)$. By the remark following the proof of Theorem 6.2.1, such isomorphisms correspond bijectively to regular homotopy classes of framed immersions $i_\alpha : (D^{k+1}, S^k) \to (N, M)$.

We have now shown that α is represented by a framed immersion i_α. By Proposition 4.6.6, we may suppose this immersion self-transverse; the same goes for the immersion $i_\alpha \cup i_\beta$ of the union of two discs. The double point set is then a 1-manifold, so consists of a collection of embedded circles and arcs whose end points are the (self)-intersection points of the boundary spheres in M.

Now if $\alpha \neq \beta$, each intersection arc has two end points, which make contributions of opposite signs to the intersection number of the two spheres in M. It follows that $\partial\alpha.\partial\beta = 0$, so indeed the image of $K_{k+1}(N, M)$ in $K_k(M)$ is self-annihilating. For k even, this proves the result; if k is odd, $\mu(\alpha)$ is the number of self-intersection points of $i_\alpha(S^k)$, and this vanishes by the same argument. □

7.5 The even dimensional case

Suppose X a Poincaré complex of formal dimension $m = 2k$, and $(f : M \to X, \nu, T)$ a k-connected normal map, or more generally that $(X, \partial X)$ is a Poincaré pair and f a normal map inducing a homotopy equivalence $\partial M \to \partial X$. It follows from Proposition 7.4.2 that all $K_r(M)$ and $K^r(M)$ vanish except for $r = k$. Duality implies further that these groups are free abelian, and the isomorphism $[M]\cap$, together with the dual pairing, gives a nonsingular bilinear form

$$\lambda : K_k(M) \times K_k(M) \to \mathbb{Z},$$

which is symmetric if k is even and skew-symmetric if k is odd.

From now on we make the further assumption that X is simply-connected. Then the Hurewicz Theorem gives an isomorphism $h : \pi_{k+1}(f) \cong K_k(M; \mathbb{Z})$. By Theorem 7.1.1, any $\xi \in \pi_{k+1}(f)$ induces a regular homotopy class of immersions $S^k \times D^k \to M$, and given any embedding in this class we can perform surgery. Write $x := h(\xi)$. Then

Lemma 7.5.1 *In this situation, if* $k \geq 3$ *is even, surgery on* ξ *is possible if and only if* $\lambda(x, x) = 0$.

If k *is odd, there is an invariant* $\mu(x) \in \mathbb{Z}_2$, *and if* $k \geq 3$, *surgery on* ξ *is possible if and only if* $\mu(x) = 0$.

Proof We recall the discussion in §6.3.2 of immersions ϕ of S^k in $2k$-manifolds.

For k even, if $e(\phi)$ denotes the number given by the Euler class of the normal bundle and $I(\phi)$ the signed sum of the intersection numbers at points of self-intersection of $\phi(S^k)$ (which we may assume transverse), then by Lemma 6.3.5, we have $[\phi].[\phi] = e(\phi) + 2I(\phi)$. In the present situation, we have an immersion of $S^k \times D^k$, so $e(\phi) = 0$. Since $x := s(\xi)$ is the homology class $[\phi]$, we have $\lambda(x, x) = 2I(\phi)$. Finally by Theorem 6.3.2, provided $k \geq 3$, if $I(\phi) = 0$, ϕ is regularly homotopic to an embedding.

For k odd, there are two regular homotopy classes of immersions in each homotopy class of maps $S^k \to M^{2k}$, which are distinguished by the parity of the number $I(\phi)$ of self-intersection points of ϕ. Define $\mu(x)$ to be $I(\phi)$ mod 2, where ϕ is in the regular homotopy class determined by ξ. The conclusion again follows from the results in §6.3.2. □

We now calculate the effect of a surgery on homology. Let $(g : N \to X, \nu, T'')$ be a normal cobordism of $(f : M \to X, \nu, T)$ to $(f' : M \to X, \nu, T')$. Since we may regard g as a map $(N, M, M') \to (X \times I, X \times \{0\}, X \times \{1\})$, we may use the groups K_* defined above. Observe that $K_*(N, M) \cong H_*(N, M)$. Thus for a single surgery as above, the K_* exact sequence of (N, M) reduces to

$$0 \to K_{k+1}(N) \to K_{k+1}(N, M) \to K_k(M) \to K_k(N) \to 0,$$

in which $K_{k+1}(N, M) \cong \mathbb{Z}$, and a generator maps to $x \in K_k(M)$. Thus for $x \neq 0$, $K_k(N)$ is the quotient of $K_k(M)$ by the class of x.

We have an exact sequence

$$0 \to K_k(M') \to K_k(N) \to K_k(N, M') \to K_{k-1}(M') \to 0,$$

with $K_k(N, M') \cong \mathbb{Z}$. The map $K_k(N) \to K_k(N, M')$ can be identified with $K^{k+1}(N, \partial N) \to K^{k+1}(N, M)$, so is induced by intersection with the $(k+1)$-cell representing a generator of $K_{k+1}(N, M)$ or with x, its boundary. Thus provided for some $x' \in K_k(M)$ we have $\lambda(x, x') = 1$, this map is surjective, and hence $K_{k-1}(M') = 0$.

If k is even, the intersection pairing λ on $H_k(X; \mathbb{R})$ is symmetric, so determines a signature invariant $\sigma(X) \in \mathbb{Z}$. We saw above that if (Y, X) is a Poincaré pair, then $\sigma(X) = 0$. The main result here is

Theorem 7.5.2 *If $(f : M \to X, \nu, T)$ is a normal map of degree 1 with X a simply-connected Poincaré pair of formal dimension $2k$ with $k \geq 4$ even, and $\partial M \to \partial X$ a homotopy equivalence, then surgery to obtain a homotopy equivalence is possible if and only if $\sigma(M) = \sigma(X)$. Moreover, it then suffices to perform surgeries on spheres of dimension $\leq k$.*

Proof First suppose $(g : N \to X, \nu, T'')$ a normal cobordism of f to a homotopy equivalence $(f' : M' \to X, \nu, T')$. Since ∂N is the disjoint union of M' and M with orientation reversed, $0 = \sigma(\partial N) = \sigma(M') - \sigma(M) = \sigma(X) - \sigma(M)$.

Conversely, suppose $\sigma(M) = \sigma(X)$. We may suppose by Theorem 7.2.1 that f is k-connected. It follows from the proof of Theorem 7.4.2 that in the decomposition $H_k(M) \cong K_k(M) \oplus H_k(X)$ the two summands are mutually orthogonal for the intersection form. Hence the induced pairing λ on $K_k(M)$ has signature zero.

It follows from Proposition 7.3.3 that if λ is a nonsingular symmetric bilinear form on H over $\mathbb{Z}$, of signature zero, there exists a basis $\{e_i, f_i\}$ $(1 \leq i \leq r)$ of H such that $\lambda(e_i, e_j) = \lambda(e_i, f_j) = 0$ for all i, j *except* that $\lambda(e_i, f_i) = 1$ for each i.

Since $\lambda(e_1, e_1) = 0$, by Lemma 7.5.1, we can do surgery on e_1. It follows from the above calculations that for the resulting normal cobordism, $K_k(N)$ is the quotient of $K_k(M)$ by the class of e_1, and that $K_k(M')$ is the subgroup of $K_k(N)$ which is the quotient of the subgroup of $K_k(M)$ consisting of classes orthogonal to e_1: viz. with $\lambda(y, e_1) = 0$. Hence $K_k(M')$ looks like $K_k(M)$ but with base corresponding to $\{e_i, f_i\}$ $(2 \leq i \leq r)$. Thus at the end of r simple surgeries we have arrived at a situation with $K_k = 0$ and hence a homotopy equivalence. □

The details for k odd are somewhat subtler. We first need a closer study of μ, which we defined as a map $K_k(M) \to \mathbb{Z}_2$. First we have

Lemma 7.5.3 *For $x, x' \in K_k(M)$, we have $\mu(x + x') = \mu(x) + \mu(x') + \lambda(x, x')$, where $\lambda(x, x')$ has to be reduced modulo 2.*

Proof Set $x = h(\xi), x' = h(\xi')$: then ξ, ξ' determine immersions ϕ, $\phi' : S^k \times D^k \to M$ up to regular homotopy.

We may think of two disjoint $(k + 1)$-discs in $\mathbb{R}^{k+1}$ joined by a thickened arc: the boundary of the union is a model for the connected sum of two k-spheres. Now ξ, ξ' give maps of the spheres to M which extend to maps of the discs to X: joining along the arc gives maps representing $\xi + \xi'$. The ingredients (ν, T) of the normal maps also pass to the union. We conclude that an immersion ϕ'' representing $\xi + \xi'$ may be obtained from ϕ and ϕ' by joining the spheres along the neighbourhood of an arc (which may be taken disjoint from the spheres).

The self-intersections of ϕ'' thus consist of those of the two spheres together with their mutual intersections. The result follows by counting up. □

We thus have a nonsingular skew-symmetric bilinear form λ on a free abelian group $H := K_k(M)$ together with a map $\mu : H \to \mathbb{Z}_2$ satisfying the identity $\mu(x + x') = \mu(x) + \mu(x') + \lambda(x, x')$. According to Lemma 7.3.8, the classification of such triples (H, λ, μ) is given by the rank of H and the invariant $\mathrm{Arf}(\mu) \in \mathbb{Z}_2$.

We are now ready to give a first version of the main result for the case k odd.

Theorem 7.5.4 *If $(f : M \to X, \nu, T)$ is a normal map of degree 1 with X a simply-connected Poincaré pair of formal dimension $2k$ with $k \geq 3$ odd, and $\partial M \to \partial X$ a homotopy equivalence, then surgery to obtain a homotopy equivalence is possible provided that $\mathrm{Arf}(\mu) = 0$. If surgery is possible, it suffices to perform surgeries on spheres of dimension $\leq k$.*

Proof We may suppose after preliminary surgery that f is k-connected, so $K_k(M)$ is free abelian and supports a nonsingular skew-symmetric intersection form λ. Choose a symplectic basis $\{e_i, f_i \mid 1 \leq i \leq r\}$ of $(K_k(M), \lambda)$. Since $\mathrm{Arf}(\mu) = 0$, the proof of Lemma 7.3.8 shows that we may adjust this basis so that μ vanishes on each basis element.

Thus by Lemma 7.5.1, we may perform surgery on e_1. As in the proof of Theorem 7.5.2, $K_k(M')$ for the resulting manifold M' is the quotient by $\langle e_1 \rangle$ of the subgroup of $K_k(M)$ orthogonal to e_1, thus has basis $\{e_i, f_i \mid 2 \leq i \leq r\}$. The result now follows by induction on r. □

Theorem 7.5.4 is incomplete: we have neither given an à priori definition of $\mathrm{Arf}(\mu)$ nor proved necessity of its vanishing for completing surgery. We will offer an invariant form of Arf(μ) in §7.7 after introducing further concepts. We now take up the other point.

Consider a normal map $(f : M \to X, \nu, T)$ of degree 1 with X a simply-connected Poincaré pair of formal dimension $2k$ with k odd, and $\partial M \to \partial X$ a homotopy equivalence: we can perform surgery to replace f by a k-connected map $f' : M' \to X$, and then define λ and μ on $K_k(M')$ as above.

Proposition 7.5.5 *In the above situation, Arf(μ) is an invariant of the normal cobordism class of* (f, ν, T)*.*

Proof It suffices to consider two normally cobordant normal maps and show that the corresponding values of Arf are the same. Denote by $F : W \to X \times I$ the normal cobordism; we may suppose each of $\partial_- W \to X$ and $\partial_+ W \to X$ k-connected.

We next wish to use Proposition 7.4.3 to allow us to do surgery to kill $K_k(W, \partial W)$. There is a minor technical point: if X has no boundary, ∂W is disconnected. To deal with this, use Lemma 7.4.1 to write $X = X_0 \cup_g e^{2k}$, with (X_0, S^{2k-1}) a Poincaré pair and correspondingly delete the interior of an embedded disc from $\partial_\pm W$. This reduces to the case when $\partial X_0 = S^{2k-1}$, and now ∂W is connected.

The result of the surgery is a manifold W' with $K_k(\partial W')$ the orthogonal direct sum of $K_k(\partial_- W)$, $K_k(\partial_+ W)$ and a number of copies of $\mathbb{Z} \oplus \mathbb{Z}$, with one produced by each surgery on the boundary. Its Arf invariant is the sum of those of the summands, which are respectively equal to Arf$(\partial_- W)$, Arf$(\partial_+ W)$ and zero. But now $K_{k+1}(W', \partial W')$ provides a Lagrangian subspace, so the Arf invariant is zero. Hence indeed Arf$(\partial_- W) =$ Arf$(\partial_+ W)$ as required. □

The invariant Arf(μ) of a normal cobordism is known as the *Kervaire invariant*, and denoted Kerv(f, ν, T). By Theorem 7.5.4, if $k \geq 3$ it is the only obstruction to completing surgery.

7.6 The odd dimensional case

Let X be a simply-connected Poincaré complex of formal dimension $m = 2k + 1$, and suppose again that $(f : M \to X, \nu, T)$ is a k-connected normal map inducing a homotopy equivalence $\partial M \to \partial X$. Again the Hurewicz Theorem gives an isomorphism $h : \pi_{k+1}(f) \cong K_k(M; \mathbb{Z})$; by Theorem 7.1.1, any $\xi \in \pi_{k+1}(f)$ induces a regular homotopy class of immersions $S^k \times D^{k+1} \to M$, given any embedding in this class we can perform surgery, and we write

$x := h(\xi)$. In this case, for any $x \in K_k(M : \mathbb{Z})$ we can find such embeddings, but these are not unique up to diffeotopy. The object of this section is to prove:

Theorem 7.6.1 *Let $(f : M \to X, \nu, T)$ be a normal map of degree 1, inducing a homotopy equivalence $\partial M \to \partial X$, with X a simply-connected Poincaré complex of formal dimension $2k+1$, with $k \geq 2$. Then surgery to obtain a homotopy equivalence is possible. Moreover, it suffices to perform surgeries on spheres of dimension $\leq k$.*

We first calculate the effect of a single surgery on homology. If N is a normal cobordism of M to M' we have the diagram of Proposition 7.1.3, and by Proposition 7.4.2, we may simplify this by replacing terms H_* by K_*. The only ones which remain non-zero are those in the diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[dr] \arrow[rr, bend left] & & K_{k+1}(M) \arrow[dr] \arrow[rr, bend left] & & K_{k+1}(N,M') \arrow[dr] \arrow[rr, bend left] & & K_k(M') \arrow[dr] \arrow[rr, bend left] & & 0 \\
& K_{k+2}(N,\partial N) \arrow[ur] \arrow[dr] & & K_{k+1}(N) \arrow[ur] \arrow[dr] & & K_{k+1}(N,\partial N) \arrow[ur] \arrow[dr] & & K_k(N) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & K_{k+1}(M') \arrow[ur] \arrow[rr, bend right] & & K_{k+1}(N,M) \arrow[ur] \arrow[rr, bend right] & & K_k(M) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

Here the groups $K_{k+1}(N, M; \mathbb{Z})$ and $K_{k+1}(N, M'; \mathbb{Z})$ are isomorphic to $\mathbb{Z}$. If we take coefficients $\mathbb{Q}$, the isomorphism $[N, \partial N]\cap$, together with dual pairings, shows that groups symmetrically placed in the diagram have equal ranks. Writing r for the rank of $K_k(N; \mathbb{Q})$ we find just three possibilities for the ranks of all groups in the diagram: either

(i) $K_k(M)$, $K_k(M')$, $K_k(N)$ and their duals have rank r, $K_{k+1}(N, \partial N)$ and $K_{k+1}(N)$ have rank $r + 1$;

(ii) $K_k(M)$, $K_k(N)$ and their duals have rank r, $K_k(M')$, $K_{k+1}(N, \partial N)$ and their duals have rank $r + 1$; or

(ii') as (ii) but with M and M' interchanged.

Moreover the map $K_{k+1}(N) \to K_{k+1}(N, \partial N)$ induced by the intersection pairing has rank 1 in case (i) and 0 in cases (ii), (ii'). Since the intersection pairing is skew-symmetric if k is even, it follows that here case (i) cannot arise.

We now consider the torsion in $K_k(M)$ in more detail. In the following we use coefficients $\mathbb{Z}$ for homology throughout.

Lemma 7.6.2 *Let $x \in K_k(M)$ be indivisible, so that there is a homomorphism $\phi : K_k(M) \to \mathbb{Z}$ with $\phi(x) = 1$. Then if we perform surgery on x, the map*

$K_k(M') \to K_k(N)$ is an isomorphism, so $K_k(M')$ is the quotient of $K_k(M)$ by the class of x.

Proof We can identify ϕ with a class in $K^k(M)$, hence by duality with a class y in $K_{k+1}(M)$. We claim that the image of y in $K_{k+1}(N, M')$ is a generator, so that the map $K_{k+1}(N, M') \to K_{k+1}(N, \partial N)$ and hence also the map to $K_k(M')$ vanishes, which implies the result.

To calculate this image, we may consider the intersection of y with a generator z of $K_{k+1}(N, M)$. This equals the intersection in M of y with the boundary of z, namely x. But by hypothesis this is 1. □

We can now give the proof of Theorem 7.6.1 in the case when k is even.

Proof First perform preliminary surgeries to make f k-connected: as $k \geq 2$, all manifolds we encounter from now on are simply-connected. Next perform an induction on the rank r of $K_k(M)$. If $r > 0$, choose $x \in K_k(M)$ of infinite order and not divisible by an integer > 1, and perform surgery on x. By Lemma 7.6.2, $K_k(M')$ is isomorphic to the quotient of $K_k(M)$ by the class of x, so has lower rank. By induction, we may reduce the rank to 0.

We now have $K_k(M)$ finite, and conclude with a second induction on the order $|K_k(M)|$. Choose any non-zero $x \in K_k(M)$ and perform surgery. Here we have case (ii), so $K_k(N)$ is the quotient of $K_k(M)$ by the class of x, so has lower order than $K_k(M)$, and we have a short exact sequence

$$0 \to \mathbb{Z} \cong K_{k+1}(N, M') \to K_k(M') \to K_k(N) \to 0.$$

Thus $K_k(M')$ is isomorphic to the direct sum of $\mathbb{Z}$ and a finite group G, and as the map $G \to K_k(N)$ is injective, we have $|G| \leq |K_k(N)| < |K_k(M)|$.

Take a generator y of the summand $\mathbb{Z}$ of $K_k(M')$ and perform surgery on y. This yields a normal cobordism N', say, of M' to M''; $K_k(N')$ is the quotient of $K_k(M')$ by the class of y, so is isomorphic to G, and by Lemma 7.6.2, the map $K_k(M'') \to K_k(N')$ is an isomorphism. Since $K_k(M'')$ has lower order than $K_k(M)$, the desired result follows by induction. □

The case when k is odd requires further arguments. We consider the effect of a single surgery and recall the commutative diagram (7.6). The handle H is a smooth submanifold H of N with $H \cap M = \partial_- H \cong (S^k \times D^{k+1})$, $\partial_c H \cong (S^k \times S^k \times [-1, 1])$, and $H \cap M' = \partial_+ H \cong (D^{k+1} \times S^k)$ (compare Figure 5.5). Write V for the closure of $N \setminus H$; then $V \cong \partial_- V \times I$. We can extend the K_* notation to V, etc., by expressing X as the union of a complex X^* with a $2k$-cell attached, so that f maps V to X^* and H to the extra cell.

We have an exact sequence

$$K_{i+1}(N, V \cup M \cup M') \to K_i(V, \partial_- V \cup \partial_+ V) \to K_i(N, M \cup M') \\ \to K_i(N, V \cup M \cup M'),$$

and since by excision $H_i(N, V \cup M \cup M') \cong H_i(H, \partial H)$, so vanishes except for $i = 2k + 2$, we have an isomorphism $K_{k+1}(V, \partial_- V \cup \partial_+ V) \to K_{k+1}(N, M \cup M')$.

We may thus replace the term $K_{k+1}(N, \partial N)$ in (7.6) by $K_{k+1}(V, \partial_- V \cup \partial_+ V) \cong K_k(\partial_- V)$. We have an a-sphere $S^k \times \{0\} \subset S^k \times D^{k+1} \cong \partial_- H$ diffeotopic in H to $S^k \times * \times \{-1\} \subset \partial_c H$ which bounds a disc in M' and a b-sphere $\{0\} \times S^k \subset D^{k+1} \times S^k \cong \partial_+ H$ diffeotopic in H to $* \times S^k \times \{1\}$ which bounds a disc in M. Denote the corresponding classes in $K_{k+1}(N, \partial N)$ by x_a and x_b.

Further calculations depend on the self-pairing λ of $K_k(M; \mathbb{Z})$ to $\mathbb{Q}/\mathbb{Z}$.

Lemma 7.6.3 *Suppose we perform surgery on a sphere representing $x \in K_k(M; \mathbb{Z})$. Then in the exact sequence*

$$\mathbb{Z}\,(\cong K_{k+1}(N, M')) \to K_k(\partial_- V)\,(\cong K_{k+1}(N, \partial N)) \to K_k(M) \to 0,$$

the generator 1 of $\mathbb{Z}$ maps to x_b, and if $y \in K_k(M)$ has order q, $p/q \in \mathbb{Q}$ projects to $\lambda(x, y) \in \mathbb{Q}/\mathbb{Z}$, and $w \in K_k(\partial_- V)$ maps to y, we have $qw = p' x_b$ for some $p' \equiv p$ (mod q).

Proof Represent y by a k-cycle η and let $q\eta = \partial\phi$ for some $(k + 1)$-chain ϕ in M. The class x is represented by the a-sphere $S^k \times *$, and by definition of λ, this has p' intersections with ϕ for some $p' \equiv p\,(\mathrm{mod}\; q)$. We may suppose these transverse; then each one is the centre of a disc $* \times D^{k+1}$ (in H). Removing these discs gives a chain ϕ^* in $\partial_- V$ with boundary consisting of $q\eta$ and p' spheres $* \times S^k$ each parallel to the b-sphere, and with reversed orientation. Thus $qw - p' x_b$ vanishes in $H_k(\partial_- V)$. □

We now give the proof of Theorem 7.6.1 in the case when k is odd.

Proof As in the case when k is even, we may suppose preliminary surgeries performed so that f is k-connected and moreover that $K_k(M)$ is finite; again we proceed by induction on $|K_k(M)|$.

We will perform surgery on $x \in K_k(M)$. Since $r = 0$ we cannot have case (ii') above, so the first map in the exact sequence of 7.6.3 is injective. That Lemma calculates the extension, but does not determine the class x_a.

In fact, the class is not determined by the choice of x: the embedding $\phi : S^k \times D^{k+1} \to M$ is determined up to regular homotopy, but not up to diffeotopy. For any map $g : S^k \to SO_{k+1}$ we may form the twist ϕ_g by $\phi_g(x, y) := \phi(x, g(x)(y))$. If g is nullhomotopic in SO_{k+2}, ϕ and ϕ_g define homotopic embeddings of the tangent bundle of $S^k \times D^{k+1}$ in that of M, and hence, by Corollary 6.2.2, regularly homotopic immersions. Since $SO_{k+2}/SO_{k+1} = S^{k+1}$, we have an exact sequence

$$\pi_{k+1}(S^{k+1}) \to \pi_k(SO_{k+1}) \to \pi_k(SO_{k+2}),$$

so may twist by any element in the image of $\pi_{k+1}(S^{k+1}) \cong \mathbb{Z}$. Since k is odd, twisting by the image of $s \in \mathbb{Z}$ induces the self-map of $H_k(S^k \times S^k)$ with $x_a \mapsto x_a + 2sx_b,\ x_b \mapsto x_b$.

First suppose x of order q and that $\lambda(x, x) = p/q$ with p prime to q: then $K_k(M)$ is the direct sum of the group $\mathbb{Z}_q$ generated by x and its orthogonal complement, G, say. By Lemma 7.6.3, $K_k(\partial_- V)$ is the direct sum of an infinite cyclic group with generator z, say, where $qz = x_b$ and a group isomorphic to G. Write x_a in the form $mz + g$ with $m \in \mathbb{Z},\ g \in G$. Taking x_a for w and x for y in the lemma, we see $m \equiv p \pmod q$.

Twisting as above, we may suppose $|m| < q$. Now $K_k(M')$ is (isomorphic to) the quotient of $K_k(\partial_- V)$ by the class of x_a. It thus has order $m|G|$, so as $|m| < q$ we have reduced the order.

In view of Proposition 7.3.10, we see that it will now suffice to consider the case when $x,\ y \in K_k(M)$ have order 2^k, $\lambda(x, y) = \frac{1}{2^k}$, and $2^k\lambda(x, x) = 2^k\lambda(y, y) = 0$. Then $K_k(M)$ is the orthogonal direct sum of the group generated by x and y and a finite group G. Applying Lemma 7.6.3, we can write $K_k(\partial_- V)$ in the form $\mathbb{Z} \oplus \mathbb{Z}_{2^k} \oplus G$, where the first summand is generated by a class z which projects to y, the second by v say, we have $2^k z = x_b$, and $x_a = mz + v$ with m even. Twisting, we may suppose $|m| \leq 2^k$.

If $m = 0$, we obtain for $K_k(M')$ the direct sum of $\mathbb{Z}$ and G; by Lemma 7.6.2 a further surgery will kill the $\mathbb{Z}$, so we have reduced the order of K_k. If $m \neq 0$, $K_k(M')$ has order $2^k m|G|$, so if $|m| < 2^k$ we have again reduced the order. Finally if $m = 2^k$, although we have not reduced the order we have the direct sum of G and a cyclic group of order 2^{2k}, so we have reduced to the first case. □

7.7 Homotopy theory of Poincaré complexes

In this section we prepare for the reformulation in the next of the results of surgery in more general terms. We also complete our discussion of the Kervaire obstruction. The proofs of these results involve somewhat technical homotopy theory arguments. We will present an outline of the ideas, but give references for the detailed proofs.

First consider an orthogonal vector bundle ξ, with fibre dimension k and base B. Write A_ξ for the associated disc bundle: the set of vectors in the total space of ξ of length ≤ 1, and S_ξ for its boundary sphere bundle. The space obtained from A_ξ by identifying the subspace S_ξ to a point is called the *Thom space* of ξ and denoted $T(\xi)$. There is a natural isomorphism, called the *Gysin isomorphism*

$$H^r(X) \to H^{k+r}(A_\xi, S_\xi) \cong \tilde{H}^{k+r}(T(\xi)).$$

The class in $H^k(T(\xi); \mathbb{Z})$ corresponding to the unit in $H^0(X; \mathbb{Z})$ is called the *Thom class*, and traditionally denoted by U. Geometrically, if B is a CW complex, $T(\xi)$ has a natural decomposition with just one $(k + r)$-cell for each r-cell of B, as well as the base point.

If V is a submanifold of M, we can choose a tubular neighbourhood of V, which consists of an orthogonal vector bundle ξ over V together with an embedding h of A_ξ in M. Composing h^{-1} with the map $A_\xi \to T(\xi)$ gives a map of $h(A_\xi)$ which sends its boundary $h(S_\xi)$ to the base point, and hence extends to a map $M \to T(\xi)$ which sends the rest of M to this point. This is known as the Thom construction. As it is the foundation of the study of cobordism, we will treat it more fully in §8.1.

In particular, if V^v is a compact submanifold of Euclidean space $\mathbb{R}^{v+k}$ with normal bundle ν, since we can regard $\mathbb{R}^{v+k}$ as obtained from S^{v+k} by deleting a point, we obtain a map $F : S^{v+k} \to T(\nu)$; moreover, this map has degree 1 in the sense that it induces an isomorphism on the top non-vanishing homology group H_{v+k}.

If we start with a Poincaré complex, rather than a manifold, there are no immediately visible bundles. We generalise, replacing sphere bundles $S_\xi \to B$ by fibrations $\pi : X \to B$ with fibres homotopy equivalent to the sphere S^{k-1}. In general we use the term *spherical fibration* for a fibration with fibres homotopy equivalent to a sphere. The role of the disc bundle A_ξ is now played by the mapping cylinder of π, and we define $T(\pi)$ to be the mapping cone of π.

A decisive step is given by the following result of Spivak [142].

Theorem 7.7.1 *If X is a Poincaré complex of formal dimension m, and $k > m + 1$, there exist a fibration π^k over X with fibre homotopy equivalent to S^{k-1} and a map $F : S^{m+k} \to T(\pi)$ of degree 1. Moreover, the pair (π^k, F) is unique up to suspension and homotopy equivalence.*

We describe the construction in the simplest case when X is a finite simply connected CW complex. Choose an embedding $i : X \to \mathbb{R}^{m+k}$ for some k. Take a regular neighbourhood N of $i(X)$, and form the space EN of paths (continuous maps) $\alpha : I \to N$. As discussed in §B.1, the projection $p_0 : EN \to N$ given by $p_0(\alpha) = \alpha(0)$ is a homotopy equivalence, and its fibres are contractible. Thus if $P := p_0^{-1}(\partial N)$, p_0 gives a homotopy equivalence $P \to \partial N$.

Now define $p_1 : P \to N$ by $p_1(\alpha) = \alpha(1)$: this too is a fibration. The key lemma states that the fibres of p_1 are homotopy equivalent to S^{k-1} if and only if X satisfies Poincaré duality with formal dimension n. We omit the proof, which depends on an examination of the spectral sequence of the fibration. We can now either use the fact that N is homotopy equivalent to X or restrict to $p_1^{-1}(X)$ to obtain the desired fibration over X.

Now $T(\pi)$ is the mapping cone of π, hence is homotopy equivalent to the union of X, or equivalently, N, and the cone on P, which is homotopy equivalent to ∂N. Hence $T(\pi)$ is homotopy equivalent to the space formed from N by identifying ∂N to a point. But this is obtained from S^{m+k} by identifying ∂N and everything outside N to a point, so indeed we have a map $S^{m+k} \to T(\pi)$ of degree 1.

The existence proof in the general case depends on the same idea, but there are more details to check. The result also extends to Poincaré pairs. We will discuss the uniqueness shortly.

In general the Thom space of an external direct sum of two bundles is

$$\begin{aligned} T(\xi \oplus \eta) = \frac{A_{\xi\oplus\eta}}{S_{\xi\oplus\eta}} &= \frac{A_\xi \times A_\eta}{(S_\xi \times A_\eta) \cup (A_\xi \times S_\eta)} \\ &= \frac{T(\xi) \times T(\eta)}{(\{\infty\} \times T(\eta)) \cup (T(\xi) \times \{\infty\})}, \end{aligned}$$

and this is a space called the *smash product* of $T(\xi)$ and $T(\eta)$ and denoted $T(\xi) \wedge T(\eta)$. The same goes for spherical fibrations if we interpret $\oplus$ as the fibrewise join. Since the bundle ε^1 over a point has Thom space S^1, we have $T(\xi \oplus \varepsilon^1) = T(\xi) \wedge S^1$, which is the suspension of $T(\xi)$, which we denote $ST(\xi)$. In particular, a pair (π^k, F) as in Theorem 7.7.1 defines a suspended pair $(\pi^k \oplus \varepsilon^1, SF)$.

We introduce a related notation: for any space X, write X^+ for the disjoint union of X and a point $*$, which we take as base point. Then if Z has a base point ∞, we have

$$X^+ \wedge Z = \{(X \cup \{*\}) \times Z\}/\{(\{*\} \times Z) \cup (X \times \{\infty\})\} = (X \times Z)/(X \times \{\infty\});$$

in particular, $X^+ \wedge Y^+ = (X \times Y)^+$.

Now let X be a Poincaré complex of formal dimension m, ν a spherical fibration over X, and $F : S^{m+k} \to T(\nu)$ a map of degree 1. If $\nu = \alpha \oplus \beta$, we have a map $S^N \to T(\nu) = T(\alpha) \wedge T(\beta)$.

Theorem 7.7.2 *The Thom spaces $T(\alpha)$ and $T(\beta)$ are (S,N)-dual in the sense of Spanier and Whitehead [141].*

A proof appears in [14]. Taking slant product with the homology class of the image of the fundamental homology class $[S^N]$ induces a map $\tilde{H}^q(T(\alpha)) \to \tilde{H}_{n-q}(T(\beta))$. Somewhat as in the proof of Proposition 7.4.2, using the fact that we have maps in both directions, it can be deduced that these maps are isomorphisms. But this is the condition defining (S,N)-duality.

It is more usual to speak of Spanier–Whitehead duality. Here we adhere to the earlier terminology since it is useful to make the dualising dimension N explicit. A textbook account of this duality appears in Adams' book [7].

Essentially the same argument yields the relative case: if (Y, X) is a Poincaré pair, ν a bundle or spherical fibration over Y such that there is a map $S^N \to T(\nu)/T(\nu \,|\, X)$ of degree 1, and $\nu = \alpha \oplus \beta$, then $T(\alpha)$ is (S,N)-dual to $T(\beta)/T(\beta \,|\, X)$.

In the simply-connected case, the converse follows: if a map $S^N \to T(\nu)$ induces an S-duality, then X satisfies Poincaré duality.

S-duality is a duality in stable homotopy theory. If X and Y are spaces (finite CW complexes will suffice here) the set of (based) homotopy classes of maps $X \to Y$ is denoted $[X : Y]$. The set of morphisms in stable homotopy theory is the limit

$$\{X : Y\} := lim_n[S^nX : S^nY],$$

which is an abelian group. If X and X^*, and Y and Y^* are (S,N)-dual, there are isomorphisms

$\{X^* : Y^*\} \cong \{Y : X\}$;

$\{X : Y\} \cong \{S^N : X^* \wedge Y\}$.

If $\phi : (N, M) \to (Y, X)$ is a normal map of degree 1 of Poincaré pairs, the S-dual gives a map $\psi : T_{Y/X}(\nu) \to T_{N/M}(\nu)$. Composing with the Gysin isomorphism gives the map $\psi^* : H^*(N, M) \to H^*(Y, X)$ dual to the homology map ϕ_* which we used in Proposition 7.4.2.

We can now deal with the question of uniqueness in Theorem 7.7.1. Suppose we have spherical fibrations ν and ν' over X and maps $S^{m+k} \to T(\nu)$ and $S^{m+k'} \to T(\nu')$, both of degree 1. Since X is a finite CW-complex, if r is large enough, the suspension $\nu \oplus \varepsilon^r$ is fibre homotopy equivalent to $\alpha \oplus \beta$ with α fibre homotopy equivalent to ν'. Hence $T(\alpha)$ is (S,$(m + k + r)$)-dual to $T(\beta)$.

The map $S^{m+k'} \to T(\nu') \simeq T(\alpha)$ of degree 1 is (S,$(m + k + r)$)-dual to a map $T(\beta) \to S^{r+k-k'}$ inducing an isomorphism of the homology group $H^{r+k-k'}$. Now we can obtain $T(\beta)$ from the total space of the spherical fibration $\beta \oplus \varepsilon^1$ by identifying the cross-section in the summand ε^1 to a point. We thus have a map of this total space to a sphere which induces an isomorphism of $H^{r+k-k'}$, and hence a homotopy equivalence on each fibre. But this shows that $\beta \oplus \varepsilon^1$ is a fibre homotopically trivial bundle. As $\nu \oplus \varepsilon^{r+1}$ is fibre homotopy equivalent to $\nu' \oplus \beta \oplus \varepsilon^1$, the desired equivalence of ν and ν' (together with the maps of degree 1) is established.

We now turn to the Kervaire invariant. The following account is a simplified version of [34], with most proofs omitted. Define $W_k(n)$ to be the mapping fibre of the map $K(\mathbb{Z}_2, k) \to K(\mathbb{Z}_2, k+n+1)$ given by the cohomology operation $\chi(Sq^{n+1})$, and define a Wu orientation of a bundle ξ^n to be a map $T(\xi) \to W_k(n)$ such that the class in dimension k pulls back to the Thom class U.

If X^{2n} is a Poincaré complex, and ν its normal Spivak fibration, the Wu class $v_{n+1}(X) = 0$ since Sq^{n+1} vanishes on n-dimensional classes. It follows that $\chi(Sq^{n+1})U = 0$. Thus ν admits a Wu orientation.

It follows from the (S,$2n+k$)-duality between X^+ and $T(\nu)$ that there are bijections

$\{X^+ : S^n\} \to \{S^{2n+k} : S^n \wedge T(\nu)\}$ and

$\{X^+ : K(\mathbb{Z}_2, n)\} \to \{S^{2n+k} : T(\nu) \wedge K(\mathbb{Z}_2, n)\}$.

Homotopy calculations yield isomorphisms

$\{S^{2n} : K(\mathbb{Z}_2, n)\} \cong \mathbb{Z}_2$:

we denote by χ the image in $\{X^+ : K(\mathbb{Z}_2, n)\}$ of the non-zero element;

$z : \{S^{2n+k} : W_k(n) \wedge K(\mathbb{Z}_2, n)\} \cong \mathbb{Z}_4$.

Composing the map z with a Wu orientation α for ν gives a homomorphism $\{S^{2n+k} : T(\nu) \wedge K(\mathbb{Z}_2, n)\} \to \mathbb{Z}_4$ and hence, by (S,($2n+k$))-duality, a map $h : \{X^+ : K(\mathbb{Z}_2, n)\} \to \mathbb{Z}_4$. We now define $\phi : H^n(X) \to \mathbb{Z}_4$ by $\phi(u) := h(\{u\})$.

Further calculations show that, if $j : \mathbb{Z}_2 \subset \mathbb{Z}_4$ denotes the (unique) injective map, $h(\chi) = j(1) \neq 0$ (this key point depends on the Wu orientation); next that (cf. (7.3.7))

$$\phi(u+v) = \phi(u) + \phi(v) + j\{u.v[X]\}.$$

Thus ϕ is a quadratic form on $H^k(X; \mathbb{Z}_2)$ in the sense of Theorem 7.3.11. By that result, the Gauss sum $\mathfrak{G}(\phi) := \sum_{x \in H^k(X;\mathbb{Z}_2)} i^{\phi(x)}$ has the form $R(\phi)\eta_X$, where $R(\phi) = \sqrt{|H^k(X; \mathbb{Z}_2)|}$ and $\eta_X^8 = 1$. Writing η_X as $e^{2\pi i a(X)/8}$ defines an invariant $a(X) \in \mathbb{Z}_8$ of (X, α).

It is also shown that if (Y, X) is a Poincaré pair with a Wu orientation, ϕ vanishes on the image of $H^k(Y)$. Hence this image is a Lagrangian subgroup of $H^k(X)$. It follows from Theorem 7.3.11 that in this case $a(X) = 0$.

Now suppose given a normal map $(f : M \to X, \nu, T)$. A Wu orientation α for ν pulls back to one for M, and the above map h factors as $h : \{X^+ : K(\mathbb{Z}_2, n)\} \to \{M^+ : K(\mathbb{Z}_2, n)\} \to \mathbb{Z}_4$. Now $H^k(M; \mathbb{Z}_2) = K^k(M; \mathbb{Z}_2) \oplus H^k(X : \mathbb{Z}_2)$, and the restriction to $H^k(X:\mathbb{Z}_2)$ of the map ϕ_X coincides with the map ϕ_M defined in the same way for M. A further calculation shows that the restriction to $K^k(M;\mathbb{Z}_2)$ corresponds by duality to the map $\mu : K_k(M;\mathbb{Z}_2) \to \mathbb{Z}_2$ of Lemma 7.5.3. From this we deduce the equality

$$\eta_M = (-1)^{A(\mu)}\eta_X. \tag{7.7.3}$$

This gives a calculation of the Kervaire invariant which depends only on the Wu orientation of X.

It is also shown in [34] that choices of Wu orientation for X correspond bijectively to forms ϕ satisfying (7.3.7). Since the intersection form is nonsingular, it follows that a second such form can be written as $\phi_z(x) = \phi(x) + j(x.z)[X]$ for some z. It follows from Lemma 7.3.12 that $\mathfrak{G}(\phi_z) = e^{-i\pi\phi(z)}\mathfrak{G}(\phi)$. Now if M is given the induced Wu orientation, then changing ϕ to ϕ_z will multiply each of η_M and η_X by the same factor $e^{-i\pi\phi(z)}$. Thus the value of $A(\mu)$ is independent of this choice.

There exist Wu orientations for the universal $Spin_{4n+2}$ bundles, and (uniquely) for the universal SU_{4n+2} bundle. Choosing these give maps of the corresponding cobordism groups (studied in the next chapter) to $\mathbb{Z}_8$.

There is also a choice of a Wu orientation for the universal SO_{4n} bundle such that the corresponding invariant is just the signature mod 8.

An interesting special case is $n = 2$. In the case when ν has fibre dimension 1, there is a canonical Wu orientation. Geometrically, we have a framing of $\mathbb{T}(M) \oplus \nu^1$, hence an immersion $M^2 \to \mathbb{R}^3$. In this case, the map ϕ can be geometrically interpreted by representing $u \in H_1(M;\mathbb{Z}_2)$ by an immersion of S^1, and counting the number of half-twists of M in $\mathbb{R}^3$ as you go round the circle.

7.8 Homotopy types of smooth manifolds

A natural problem is to seek to characterise the set of homotopy types of closed smooth manifolds. The first necessary condition on X for being homotopy equivalent to a closed manifold is that it be a Poincaré complex. If X is itself a manifold, the identity map, together with the normal bundle ν of some embedding in Euclidean space and the induced trivialisation of $\mathbb{T}(X) \oplus \nu$ is a normal map of degree 1. Thus a second necessary condition is the existence of a normal map $(f : M \to X, \nu, T)$ of degree 1, with M a smooth manifold. We next reformulate this condition. Denote by $\Omega_m(X, \nu^k)$ the set of normal cobordism classes of normal maps $(f : M^m \to X, \nu^k, T)$. Surgery replaces a normal map by another normally cobordant to it, and thus in the same class in $\Omega_m(X, \nu^k)$.

Given a normal map $(f : M^m \to X, \nu^k, T)$, we can add a trivial bundle ϵ^1 to ν^k; then T together with the induced trivialisation of $f^*\epsilon^1$ induces a trivialisation T' of $\mathbb{T}(M) \oplus f^*(\nu^k \oplus \epsilon^1)$, defining a suspended normal map $(f : M^m \to X, \nu^k \oplus \epsilon^r, T')$. The same goes for normal cobordisms, so we have a suspension map $\Omega_m(X, \nu^k) \to \Omega_m(X, (\nu^k \oplus \epsilon^1))$. It is easily shown that this is an isomorphism for $k > m$.

Proposition 7.8.1 *For any finite CW-complex X and vector bundle ν^k over X, if $k > m + 1$ there is a natural bijection $\Phi_m(X, \nu^k)$ from $\Omega_m(X, \nu^k)$ to the homotopy group $\pi_{m+k}(T(\nu^k))$. If X is a Poincaré complex, $\Phi_m(X, \nu^k)$ preserves degree.*

Proof Given an element $\alpha \in \Omega_m(X, \nu^k)$, choose a representative $(f : M^m \to X, \nu^k, T)$, where T is a trivialisation of $\mathbb{T}(M) \oplus f^*\nu^k$. Let E^{m+k} be the total space of a disc bundle associated to $f^*\nu$: then T defines a trivialisation of $\mathbb{T}(E)$. By immersion theory (see Corollary 6.2.2), this corresponds to an immersion $E^{m+k} \to \mathbb{R}^{m+k} \subset S^{m+k}$. Since $k > m + 1$ we may suppose by general position (see, for example, Proposition 4.6.6) that this gives an embedding of M and hence of a neighbourhood of M in E, which we may choose to be given by a disc sub-bundle of $f^*\nu$. As above, identifying the boundary sphere bundle and all outside to a point gives a map $t : S^{m+k} \to T(f^*\nu^k)$. The map f induces a map $T(f) : T(f^*\nu^k) \to T(\nu^k)$; composing gives a map $T(f) \circ t : S^{m+k} \to T(\nu^k)$ defining $\beta \in \pi_{m+k}(T(\nu))$.

If we begin with a normal cobordism $(g : N^{m+1} \to X, \nu, T^*)$ between two normal maps, we can follow through the same construction leading to an immersion, then an embedding $F^{m+k+1} \to S^{m+k} \times I$ inducing the chosen embeddings in $S^{m+k} \times \{0\}$ and $S^{m+k} \times \{1\}$, and a map $S^{m+k} \times I \to T(g^*\nu^k) \to T(\nu^k)$, and conclude that the two maps $S^{m+k} \to T(\nu^k)$ are homotopic. We may thus define $\Phi_m(X, \nu^k)$ by setting $\Phi_m(X, \nu^k)(\alpha) = \beta$.

To prove $\Phi_m(X, \nu^k)$ surjective, we first choose a smooth manifold Y with a homotopy equivalence $h : Y \to X$: this is possible by Lemma 1.2.9. Under h a bundle ν over X induces a bundle ν' over Y.

A class $\beta \in \pi_{m+k}(T(\nu))$ is represented by a map $S^{m+k} \to T(\nu)$ and so by a map $\phi : S^{m+k} \to T(\nu')$. The space $T(\nu')$ is not a smooth manifold, but contains a point $*$ whose complement is an open disc bundle over Y and so is a smooth manifold. Using Proposition 2.3.4, we modify the map ϕ keeping it fixed on $\phi^{-1}(*)$ so as to make it smooth on the complement. Next by Proposition 4.5.10, modify ϕ further to make it transverse to Y, embedded as the zero cross-section.

Now set $M := \phi^{-1}(Y)$ and write $f = h \circ (\phi|M) : M \to X$. It follows from the basic property Lemma 4.5.1 of transversality that M is a smooth manifold of dimension m, and its normal bundle in S^{m+k} is the pullback of ν' and hence of ν: thus a framing of the tangent bundle of $\mathbb{R}^{m+k}$ induces one of $\mathbb{T}(M) \oplus f^*\nu$. We have thus constructed a normal map $(f : M \to X, \nu, T)$, defining $\alpha \in \Omega_m(X, \nu)$. Since we have effectively reversed the above construction, it follows that $\Phi_m(X, \nu^k)(\alpha) = \beta$.

We argue similarly to prove $\Phi_m(X, \nu^k)$ injective. Given two normal maps leading to homotopic maps $f^\nu \circ t$ and $g^\nu \circ t$, we take a homotopy $S^{m+k} \times I \to T(\nu')$ and, keeping it fixed at the ends, make it smooth away from the preimage of $*$ and then transverse to Y. The preimage of Y then gives a normal cobordism between the given normal maps.

The fact that corresponding maps have the same degree follows since $T(f) : T(f^*\nu^k) \to T(\nu^k)$ has the same degree as $f : M \to X$ and $t : S^{m+k} \to T(f^*\nu^k)$ has degree 1. $\square$

This result reduces us from the somewhat mysterious set of normal cobordism classes of normal maps to an explicit homotopy group. We now make a further reduction. There is a classifying space for vector bundles (see §B.2): isomorphism classes of vector bundles ν^k over X correspond to homotopy classes of maps $X \to B(O_k)$. There is a corresponding result for (fibre homotopy classes of) spherical fibrations, with a classifying space $B(G_k)$. By Spivak's Theorem 7.7.1, if X is a Poincaré complex, there is a well-defined spherical fibration π over X, which determines a map $\tau_X : X \to B(G_k)$ up to homotopy.

We write (following standard notation) G_k for the monoid of self-homotopy equivalences of S^{k-1}. For homotopy purposes, we can treat this as a topological group, and $B(G_k)$ as its classifying space.

A normal map $(f : M \to X, \nu^k, T)$ of degree 1 determines a class of maps $S^{m+k} \to T(\nu^k)$ of degree 1, and hence by the uniqueness in Theorem 7.7.1, a fibre homotopy equivalence $\nu \to \pi$. Thus the map $X \to B(O_k)$ classifying ν is a lift of the fixed map τ_X. More precisely, it follows that

Corollary 7.8.2 *There is a natural bijection between normal cobordism classes of normal maps $(f : M \to X, \nu^k, T)$ of degree 1 of smooth manifolds to X and homotopy classes of liftings of $\tau_X : X \to B(G_k)$ to $B(O_k)$.*

We write $\mathcal{T}(X, \nu)$ for the set of these homotopy classes of liftings, which can thus be identified with the subset of $\Omega_m(X, \nu)$ of classes of degree 1: we can regard these as tangential structures on X. Observe that a choice of lifting induces a bijection of the set $\mathcal{T}(X, \nu)$ to $[X : G_k/O_k]$.

We now restrict to the simply-connected case and also suppose X has formal dimension $m \geq 5$. We can summarise Theorems 7.5.2, 7.5.4, and 7.6.1 as stating that surgery on a normal map of degree 1 to obtain a homotopy equivalence to X^m is possible if and only if an obstruction belonging to L_m vanishes, where we define the surgery group L_m ad hoc by

$$L_{4k} = \mathbb{Z}, \quad L_{4k+1} = 0, \quad L_{4k+2} = \mathbb{Z}_2, \quad L_{4k+3} = 0.$$

The isomorphism of L_{4k} on $\mathbb{Z}$ is given by the signature divided by 8 (the 8 comes from Proposition 7.3.3) and of L_{4k+2} on $\mathbb{Z}_2$ by the Kervaire invariant.

More precisely, if m is odd, by Theorem 7.6.1 we can perform the desired surgery, so the above necessary condition is sufficient. If $m = 4p$ is divisible by 4, by Theorem 7.5.2, surgery to obtain a homotopy equivalence is possible if and only if $\sigma(M) = \sigma(X)$. Here $\sigma(X)$ is determined by the homology of X, but $\sigma(M)$ depends on the choice of lift. By Hirzebruch's signature theorem 8.6.7, there is a polynomial L_m in Pontrjagin classes such that if we take the classes of $\mathbb{T}(M)$, cap product with the fundamental class gives $\sigma(M) = \langle L_m(\mathbb{T}(M)), [M]\rangle$. We can choose a bundle τ over X such that $\nu \oplus \tau$ is trivial. Then $f^*\tau$ is stably equivalent to $\mathbb{T}(M)$, so $L_m(\mathbb{T}(M)) = f^*L_p(\tau)$. Since f has degree 1, $f_*[M] = [X]$, hence $\langle L_m(\mathbb{T}(M)), [M]\rangle = \langle L_m(\tau), [X]\rangle$. The desired equality of signatures thus holds if and only if $\sigma(X) = \langle L_m(\tau), [X]\rangle$.

If $m \equiv 2 \pmod 4$, by Theorem 7.5.4 surgery to obtain a homotopy equivalence is possible if and only if the Kervaire invariant $\kappa := \mathrm{Kerv}(\phi : M \to X, T) \in \mathbb{Z}_2$ vanishes. Here the choice of a Wu orientation of ν induces Wu orientations of X and M, so by (7.7.3) above we have invariants η_X and η_M with $\eta_M = (-1)^\kappa \eta_X$: thus surgery is possible if and only if $\eta_M = \eta_X$.

To show that all elements of the groups L_m effectively arise as obstructions, we need the plumbing construction, which is best seen in the simplest case, when X is the Poincaré pair (D^m, S^{m-1}) and we study normal maps $f : (M, \partial M) \to (D^m, S^{m-1})$ inducing a homotopy equivalence $\partial M \to S^{m-1}$. Here ν is necessarily a trivial bundle ϵ^r, so T is a framing of $\mathbb{T}(M) \oplus \epsilon^r$.

Proposition 7.8.3 *(i) There exists a framed manifold Z^{4k}, which is a handlebody obtained by adding eight $(2k)$-handles to D^{4k}, such that the intersection matrix on $H_{2k}(Z)$ is the E_8 matrix (7.3.4).*

(ii) There exists a framed manifold Z^{4k+2}, which is a handlebody obtained by adding two $(2k+1)$-handles to D^{4k+2}, such that the normal map to D^{4k+2} has Kervaire invariant 1.

Proof (i) Write A for the tangent D^{2k} bundle of S^{2k}: this has Euler class 2, so the self-intersection of the zero cross-section is 2. Since $\mathbb{T}(S^{2k}) \oplus \epsilon^1$ is trivial, there is a framing of $\mathbb{T}(A) \oplus \epsilon^2$.

The E_8 matrix P is a positive definite symmetric 8×8 matrix of determinant 1, with entries 2 on the diagonal and 0 or -1 elsewhere. Take 8 copies A_i of A indexed by the rows of P, and for each non-zero entry $p_{i,j}$ $(i < j)$ of P, choose

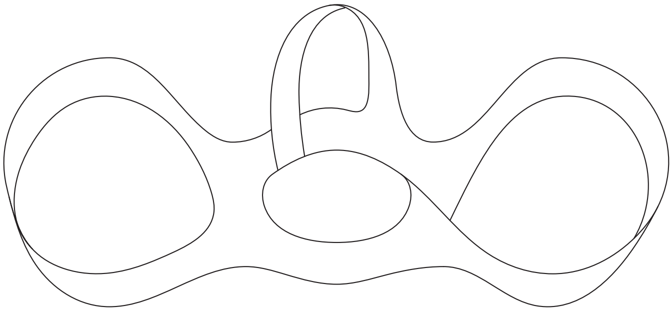

Figure 7.1 Plumbing

$(2k)$-discs $D_{i,j} \subset S_i^{2k}$, $D_{j,i} \subset S_j^{2k}$, ensuring that any two such discs in the same sphere S_i^{2k} are disjoint. The part of A_i lying over $D_{i,j}$ is a product bundle, so can be identified with a product $D_{i,j} \times D^{2k} \cong D_{i,j} \times D_{j,i}$. For each pair (i, j) we now identify $D_{i,j} \times D_{j,i}$ with $D_{j,i} \times D_{i,j}$ by the map interchanging the factors. This yields a manifold, containing copies of the A_i, but in which we have introduced intersections of the spheres so that the intersection matrix is 7.3.4 up to signs, but we can choose orientations of the basis elements to change all the signs to -1. The construction gives a manifold whose boundary has re-entrant corners, but these can be smoothed by the same techniques as in §2.6. The plumbing construction (but not this example) is illustrated in Figure 7.1.

(ii) Here the construction is simpler: we take two copies of the tangent disc bundle of S^{2k+1} and perform plumbing just once, so that the intersection matrix is just $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We need to choose a framing so that μ takes the value 1 on each basis element.

We recall the definition of μ for a $(2k+1)$-connected normal map $f : (M, \partial M) \to (D^{4k+2}, S^{4k+1})$. By Theorem 7.1.1, any $\xi \in \pi_{2k+2}(f) \cong K_{2k+1}$ determines a regular homotopy class of immersions $\phi : S^{2k+1} \times D^{2k+1} \to M$ with homology class x. By §6.3, there are two regular homotopy classes of immersions in each homotopy class of maps $S^{2k+1} \to M^{4k+2}$, which are distinguished by the parity of the number $I(\phi)$ of self-intersection points of ϕ. In Lemma 7.5.1, we defined $\mu(x)$ to be $I(\phi)$ mod 2.

Now in Proposition 6.3.3 we constructed an immersion $j : S^{2k+1} \to \mathbb{R}^{4k+2}$ with a single transverse self-intersection and with normal bundle $\mathbb{T}(S^{2k+1})$. Thus pulling back the standard framing of $\mathbb{R}^{4k+2}$ by j gives a framing of $\mathbb{T}(S^{2k+1})$ such that the preferred regular homotopy class of immersions indeed has $I(\phi) = 1$. $\square$

Since the matrices used have determinant ± 1, the manifolds constructed in Proposition 7.8.3 have boundaries with the homology of a sphere. It follows, except in low dimensions, that the boundary is simply-connected, hence is homotopy equivalent to a sphere. In fact it follows that the manifold Z^2 constructed above has boundary S^1; the boundary of Z^4 can be shown to be homeomorphic to Poincaré's dodecahedral space. In higher dimensions the boundary is a homotopy sphere, hence by Corollary 5.6.4 is homeomorphic to a sphere.

Write P_m for the set of normal cobordism classes of normal maps $(f : (M^m, \partial M) \to (D^m, S^{m-1}), T)$ with $f|_{\partial M} : \partial M \to S^{m-1}$ a homotopy equivalence. We observe that the structure of normal map amounts to giving a manifold M, a stable framing T of $\mathbb{T}(M)$, and the homotopy equivalence $\partial M \to S^{m-1}$.

Proposition 7.8.4 *If $m > 5$, the surgery obstruction gives a bijection $\beta : P_m \to L_m$.*

Proof Given a normal map, the surgery obstruction is well defined and belongs to L_m, so we have a map β.

Given two normal maps f_1, f_2 defining elements of P_m, we can form the boundary sum $M_1 + M_2$ and extend the map and framing. We claim that the surgery obstruction of the sum is the sum of the surgery obstructions. If m is odd, there is nothing to prove; if $m = 2k$, we first perform surgery below the middle dimension on each of f_1 and f_2: this induces surgeries on $f_1 + f_2$. We now have $K_k(M_1 + M_2) = K_k(M_1) \oplus K_k(M_2)$, and λ and μ split in a natural way. Since both σ and Arf are additive on direct sums, the claim follows. Similarly, we can define a normal map $\overline{f}$ by changing orientation, and $\beta(\overline{f}) = -\beta(f)$.

If $\beta(f_1) = \beta(f_2)$, we form $f_1 + \overline{f_2}$: by what we have just seen, $\beta(f_1 + \overline{f_2}) = 0$. We may thus perform surgery (keeping the boundary fixed) to construct a normal cobordism N^{m+1} of $M_1 + \overline{M_2}$ to a disc D^m.

The boundary sum may also be constructed as the union of $M_1, D^{m-1} \times I$ and M_2, since attaching $D^{m-1} \times [0, \frac{1}{2}]$ to M_1 by a collar on part of its boundary does not change the diffeomorphism class. Thus we write N as having $\partial_- N = M_1 \cup (D^{m-1} \times I) \cup M_2$ and $\partial_+ N = D^m$. Now adjusting corners we may rewrite N as N' with $\partial_- N' = M_2$, $\partial_+ N' = M_1$ and $\partial_c N' = D^m \cup \partial_c N \cup (D^{m-1} \times I)$. Since $\partial_c N \cong (S^{m-1} \times I)$, it follows that the same holds for $\partial_c N'$.

So we have a normal cobordism (keeping the boundary fixed) of M_1 to M_2. Thus β is injective.

It follows from Proposition 7.8.3 that the image of β contains a generator of L_m; now using sums and change of orientation as above it follows that β is surjective. □

Corollary 7.8.5 *Let Σ^{n-1} be a homotopy sphere which bounds a framed manifold N^n with $n \geq 6$. Then if n is odd, $\Sigma \cong S^{n-1}$; if $n \equiv 0$ (mod 4), Σ is determined up to diffeomorphism by $\sigma(N)$; if $n \equiv 2$ (mod 4), there are at most two diffeomorphism classes of such Σ.*

We return in Proposition 8.8.6 to the case $n \equiv 0$ (mod 4), and following that discuss the delicate question, the 'Kervaire invariant problem', of deciding for which $n \equiv 2$ (mod 4) Σ is unique (and so is diffeomorphic to S^{n-1}).

As well as seeking existence of a smooth manifold homotopy equivalent to X, we can investigate uniqueness by the same method. Consider pairs (M, f) with M a smooth manifold and $f : M \to X$ a homotopy equivalence, and let $(M, f) \sim (M', f')$ if there is a diffeomorphism $h : M \to M'$ with $f' \circ h \simeq f$. Write $\mathcal{S}(X)$ for the set of equivalence classes, which we may consider as smooth manifold structures on the homotopy type of X. There is a natural map $\mathcal{S}(X) \to \mathcal{T}(X)$.

Theorem 7.8.6 *For X simply-connected and $m \geq 5$ there is a sequence*

$$L_{m+1} \to \mathcal{S}(X) \to \mathcal{T}(X) \to L_m$$

which is 'exact': the image of $\mathcal{S}(X)$ in $\mathcal{T}(X)$ is the preimage of $0 \in L_m$, and the group L_{m+1} acts on $\mathcal{S}(X)$ and the orbits are the fibres of $\mathcal{S}(X) \to \mathcal{T}(X)$.

Proof The two latter maps, and exactness at $\mathcal{T}(X)$, are given by the above discussion.

Given an element $\alpha \in L_{m+1}$ and an element of $\mathcal{S}(X)$ represented by $f : M \to X$, by Proposition 7.8.4, α corresponds to an element of P_{m+1}, which we can represent by a normal map $g : (N, \partial N) \to (D^{m+1}, S^m)$ which defines a homotopy equivalence of the boundary. Choose embeddings of D^m in M and in ∂N (essentially unique by Theorem 2.5.6), and use them to glue N to $M \times I$ to give N', say. A retraction of N on D^m induces a map $G : N' \to M \times I \to X \times I$, and the restriction of G to $\partial_+ N'$ is a homotopy equivalence, so defines an element of $\mathcal{S}(X)$. Since N' inherits from N the structure of normal map, it gives a normal bordism between the two elements of $\mathcal{S}(X)$, so they map to the same in $\mathcal{T}(X)$. Any other choice of g representing α is normally cobordant to g; following this through gives an h-cobordism between the two choices for $\partial_+ N'$, so the element of $\mathcal{S}(X)$ is uniquely determined by α.

Conversely, given two elements of $\mathcal{S}(X)$ with the same image in $\mathcal{T}(X)$, there exists a normal bordism $G : N' \to X \times I$. If m is even, we can perform surgery (keeping the boundary fixed) to obtain a homotopy equivalence, and so have an h-cobordism: it follows that the two elements are equal. If $m = 2k - 1$, we can perform surgery to make the map G k-connected. It follows that N' can be obtained from $M := \partial_- N'$ by attaching k-handles. Moreover, the a-spheres of the handles are nullhomotopic in X and so in M. Since $k \geq 3$ we can perform a diffeotopy to take all of these attaching maps inside the disc D^m. But now N' is the boundary sum of $M \times I$ and a manifold defining an element of P_{m+1}. □

We have studied framed manifolds with homotopy sphere boundaries; if we weaken the 'homotopy sphere' hypothesis slightly, we still obtain strong results. Suppose M^{2n-1} an $(n-2)$-connected manifold which bounds a framed manifold N^{2n} with $n \geq 3$. Again we can do surgery to make N $(n-1)$-connected. It follows from the homology exact sequence that $H_r(N; \mathbb{Z}) \cong H^{2n-r}(N, M; \mathbb{Z})$ vanishes for $2n - r \leq k$, i.e. for $r \geq n+1$. The same holds for other coefficient groups; hence $H_n(N; \mathbb{Z})$ is free abelian. By Lemma 5.6.10, N is a handlebody, so by Theorem 5.6.12 is determined up to diffeomorphism by (H, λ, α) where $H := H_n(N; \mathbb{Z})$, λ is the $(-1)^n$-symmetric bilinear map $H \times H \to \mathbb{Z}$ given by intersection numbers, and α is a map $H \to \pi_{n-1}(SO_n)$, satisfying

(i) $\lambda(x, x) = \pi(\alpha(x))$ for $x \in H$, and

(ii) $\alpha(x + y) = \alpha(x) + \alpha(y) + \lambda(x, y)(\partial \iota_n)$ for $x, y \in H$.

Moreover since N is framed, $\alpha(x)$ maps to 0 in $\pi_{n-1}(SO)$. Thus if n is even, $\alpha(x)$ is determined by $\pi(\alpha(x))$, so the classification of N reduces to that of the symmetric bilinear form λ, which is even since $\pi(\alpha(x))$ is even.

In the case $n = 3$, as $\pi_2(SO_3)$ vanishes, N is determined up to diffeomorphism by the skew-symmetric bilinear form λ on the free abelian group $H_3(N; \mathbb{Z})$; hence, by Proposition 7.3.5, by the set of integers $\{a_i\}$. It follows from the construction that N is the boundary sum of terms of the form of handlebodies of two types:

(i) N_k say, having two handles with intersection number k, and

(ii) diffeomorphic to $S^3 \times D^3$, having just one handle.

This can be extended to a complete diffeomorphism classification of closed, simply connected 5-manifolds M. In general, the tangent group of M is stably trivial provided obstructions in $H^r(M; \pi_{r-1}(SO))$ vanish; in the present case all these groups vanish except $H^2(M; \mathbb{Z}_2)$, and the obstruction here is the Stiefel–Whitney class $w_2(M)$. If $w_2(M) = 0$, M is stably framed, so by Proposition 8.1.4, determines a class in π_5^S and bounds a framed manifold N if and only if this class vanishes, which it does since (see §B.3(x)), $\pi_5^S = 0$. (Alternatively we can argue that M has a spinor structure, and since the cobordism group $\Omega_5^{Spin} = 0$, M bounds a spinor manifold.) Now perform surgery on N; by Corollary 7.2.2, we may suppose N 2-connected. It follows that N is a handlebody. Hence M is a connected sum of manifolds M_k (the boundary of N_k) and $S^3 \times S^2$. This argument is due to Smale [140].

The case $w_2(M) \neq 0$ is more complicated. The invariants of M consist of a triple (H, b, w), where

$H = H_2(M; \mathbb{Z})$ is a finitely generated abelian group,

b is the linking form – a nonsingular skew-symmetric bilinear self-pairing of the torsion subgroup T of H to $\mathbb{Q}/\mathbb{Z}$ – and

$w : H \to \mathbb{Z}_2$ is the homomorphism given by cap product with $w_2(M)$; moreover for $x \in T$, $w(x) = b(x, x)$.

The invariants determine $w_2(M) \in H^2(M; \mathbb{Z}_2)$, hence also $w_3 = Sq^1 w_2$ and the Stiefel–Whitney number $w_2 w_3[M]$. The contributions to $w_3 \in H^3(M; \mathbb{Z}_2)$ come only from summands of H of order 2, and it follows that $w_2 w_3[M]$ is equal to the number (modulo 2) of such summands. The oriented cobordism group Ω_5^{SO} has order 2, and the class of M in it is determined by $w_2 w_3[M]$.

Theorem 7.8.7 *[16] Any system of invariants as above is the set of invariants of a simply-connected 5-manifold, and two such manifolds with isomorphic invariants are diffeomorphic.*

We will not give the full proof, but merely an outline of the argument. For existence, first recall that by Proposition 7.3.9, the triple (H, b, w) is a direct sum of triples with G either $\mathbb{Z}$, $\mathbb{Z}_k \oplus \mathbb{Z}_k$ or $\mathbb{Z}_2$; we can construct manifolds as connected sums correspondingly.

For $G = \mathbb{Z}$, we take M as an S^3 bundle over S^2: the product $S^3 \times S^2$ if $w = 0$, and the non-trivial bundle $S^3 \tilde{\times} S^2$ if not.

For $G = \mathbb{Z}_k \oplus \mathbb{Z}_k$, if $w = 0$, we have the manifold M_k constructed above. This is diffeomorphic to the manifold obtained from $S^2 \times S^3$ by surgery on a sphere representing kz, where the class of z generates $H_2(S^3 \times S^2)$; now replacing $S^3 \times S^2$ by $S^3 \tilde{\times} S^2$ gives a suitable manifold in the case $w \neq 0$.

The case $G = \mathbb{Z}_2$ is trickier, since here we need a manifold which is not a boundary. Begin with the Hopf bundle $S^1 \to S^3 \to S^2$: this bounds a bundle B_1 with fibre D^2, which is the tubular neighbourhood of the 2-sphere which is the zero cross-section, and hence diffeomorphic to the complement of a disc D^4 in $P^2(\mathbb{C})$. The associated bundle B_2 with fibre S^2 is thus split by the original copy of S^3 into two parts, each diffeomorphic to B_1, but with one diffeomorphism reversing the orientation; in turn, B_2 is the boundary of the associated bundle B_3 with fibre D^3. Now $P^2(\mathbb{C})$ admits a diffeomorphism φ_1 given by complex conjugation. This induces -1 on $H^2(P^2(\mathbb{C}); \mathbb{Z})$, but is orientation preserving, hence (by the disc theorem) isotopic to a diffeomorphism leaving a disc D^4 pointwise fixed, hence giving a diffeomorphism φ_1' of B_1. There is thus a diffeomorphism φ_2 of B_2 given by φ_1' on one copy of B_1 and by the identity on the other. Finally, define M_q by using the diffeomorphism φ_2 of $B_2 = \partial B_3$ to glue two copies of B_3 together. A short calculation shows that indeed $H_2(M_q) \cong \mathbb{Z}_2$.

To establish uniqueness, since we dealt above with the case $w_2 = 0$, one can suppose w_2 non-zero. Given manifolds M, M' and an isomorphism $\alpha : H_2(M) \to H_2(M')$ compatible with the pairings b and maps w, we know that there is an oriented cobordism W of M to M'. By Corollary 7.2.2, we can perform surgery on W to make the map $W \to B(SO)$ 2-connected, and so $H_2(W) \cong \mathbb{Z}_2$. The main part of Barden's argument now involves surgery on 3-spheres embedded in W to convert W to an h-cobordism. The result follows by Theorem 5.5.6.

7.9 Notes on Chapter 7

§7.1 The useful terminology of 'normal maps' is due to Browder [31]. In an early paper, Milnor thanks Thom for having described the technique of surgery to him.

§7.2 This account of surgery below the middle dimension follows that in my book [167]. As with handlebody theory, the idea is to copy for manifolds what happens for CW complexes.

The proof of Corollary 7.2.4 fails in lower dimensions. The problem of embeddings of S^2 in S^4 is much more delicate, and no simple result is known. For knots in S^3 it follows from Thurston's geometrisation principle that unless the knot is a torus knot or a companion knot, the fundamental group of the complement is isomorphic to a subgroup of $SL_2(\mathbb{C})$.

§7.3 The results for forms over $\mathbb{Z}$ are classical. A nice survey of nonsingular quadratic forms was given by Milnor [94].

A convenient reference for forms over finite groups is my paper [161], but there is a substantial literature; many of the results are older. The general concept of quadratic form is discussed in [165].

The Arf invariant was first introduced in [11]. Invariants of a quadratic form q on a vector space V over a field k can be extracted from its Clifford algebra $C(q)$. This admits a mod 2 grading, and if V has even dimension, the centre Z of the even Clifford algebra is a quadratic extension of k. Except in characteristic 2, we can write Z in the form $k[z]/\langle z^2 = a\rangle$, and the class of a in $k^\times/(k^\times)^2$ is the discriminant of q. In characteristic 2, define $\wp(x) := x^2 + x$: then $Z = k[z]/\langle \wp(z) = a\rangle$ (where, if the associated bilinear form to q has a symplectic basis $\{e_i, f_i\}$, we may take $z = \sum_i e_i f_i$), and the class of a in $k^+/\wp(k^+)$ is Arf's invariant in general.

§7.4 Poincaré complexes were first defined in [164]. The definition in the general case is a little more elaborate than for simply connected spaces. In this section we only need immediate consequences of duality and properties of maps of degree 1 following [167].

The self-pairing of the torsion subgroup is traditionally called the linking pairing, and was first introduced by Seifert [133].

In [167], the result corresponding to Proposition 7.4.3 was formulated in homotopy terms, and used to prove that surgery is always possible for Poincaré pairs (Y, X) such that the map $\pi_1(X) \to \pi_1(Y)$ is an isomorphism – the so-called '$\pi - \pi$ Theorem'.

§7.5, §7.6 Milnor's exciting paper [92] constructing differentiable manifolds homeomorphic but not diffeomorphic to S^7 aroused great interest in this area. His talk [102] at the 1958 International Congress exhibited interrelations of relevant homotopy groups. This was followed by a preprint of Milnor in 1959 introducing a programme: introduce the group Θ_n of homotopy spheres, then show any homotopy sphere is stably framed, then study the obstruction to bounding a framed manifold, then study the case when it does. There were preliminary publications [95] and [97]. These included the calculation of P_{4k}, and essentially that of P_{4k+2}. The final proof that $P_{2r+1} = 0$ was accomplished by myself [159] and in the full account by Milnor and Kervaire [79].

The idea that the method extended to arbitrary simply-connected manifolds was due to Novikov [114] and Browder in 1962. Fuller accounts appeared in [115], [163], and [30]; both Browder and I gave talks at the international congress in 1966, and wrote books [31] and [167].

§7.7 Spivak's 'homotopy normal bundle' brought clarity to several previous results of this nature.

S-duality was introduced and developed in [141].

After the introduction of the Kervaire invariant in 1960 in [78], progress was made successively in 1966 by Brown and Peterson, then in 1969 by Browder [30] (his book [31] appeared in 1972). Browder starts from a normal map and uses the Spanier–Whitehead dual map $\psi : T_{Y/X}(\nu) \to T_{N/M}(\nu)$. For each cohomology class $x \in H^k(N, M)$ with $\psi^*(x) = 0$, write h for the composite map $S^s x \circ \psi : T_{Y/X}(\nu) \to T_{N/M}(\nu) \to S^s K(\mathbb{Z}_2, k)$. Since $h^*\iota = Sq^{k+1}\iota = 0$, one can form the functional Steenrod square (see §B.4) $Sq_h^{k+1}(S^s\iota) \in H^{2k+s}(T_{Y/X}(\nu))$, and evaluate this on the fundamental class to obtain $\mu(x) \in \mathbb{Z}_2$. This gives a definition of the map μ and hence of its Arf invariant independent of any preliminary surgeries.

A comprehensive study was made in 1972 by Brown [34], which has the advantage of defining an invariant for Poincaré complexes (with a Wu orientation) rather than for normal maps. Our account is a simplified version of this.

§7.8 We have given the general form of the reduction of diffeomorphism classification of (simply-connected) smooth manifolds to homotopy problems: the account follows the one I gave in [167].

The plumbing construction seems to have been first introduced by Milnor [95].

For simply-connected 4-manifolds M, it was observed by Milnor [94] that the homotopy type is determined by the intersection form on $H_2(M)$. A further step was taken by the author [162] showing that if M and M' are homotopy equivalent, then they are h-cobordant, and deducing that they can be made diffeomorphic by taking connected sums with a number of copies of $S^2 \times S^2$. This is far from establishing diffeomorphism: at the time of writing, no criterion is known for proving two 4-manifolds diffeomorphic. Another tantalising problem is finding which quadratic forms appear. For spinor manifolds, these forms must be even, and it follows from the calculation of spinor cobordism that the signature is divisible by 16. This value is realised by so-called K3 surfaces (for example, nonsingular quartic surfaces in $P^3(\mathbb{C})$), but for such a surface M $H_2(M)$ has rank 22 and it is not known whether there exist surfaces with $\sigma = 16$ but lower rank. See §5.7 for a fuller discussion.

When we drop the hypothesis of simple connectivity, it is necessary, as in §5.7, to replace the coefficient group $\mathbb{Z}$ by $\mathbb{Z}[\pi]$. This leads to surgery obstruction groups $L_m(\pi)$, generalising the above group L_m which is $L_m(1)$. The exact sequence of Theorem 7.8.6 holds in general with L_m replaced by $L_m(\pi_1(X))$. There is, however, no direct analogue of Proposition 7.8.4.

The groups $L_m(\pi)$ can be defined in an abstract way. When $m = 2k$ is even, they can be interpreted by equivalence classes of $(-1)^k$-hermitian forms over $\mathbb{Z}[\pi]$, by a relatively minor modification of the geometry of the simply connected case, using the results of §6.3 on embeddings of m-spheres in $2m$-manifolds. The odd dimensional case requires a different approach, and the surgery groups can be interpreted as quotients of the stable unitary group of such forms. Some calculations of these groups can be made: if π is finite, by methods of algebraic number theory, and for some infinite groups π using geometrical arguments. A first version of all this was given in [167].

This theory was re-worked in a more satisfactory way by Ranicki in a series of papers from 1973 on. We refer to his book [128].