3

Differentiable group actions

We begin by recalling the definitions of Lie groups, of group actions, and of smooth actions, and establish some elementary properties.

Although the centre of our interest is in actions of compact (including finite) groups, the geometrical properties extend to all proper group actions. A key step is the notion of slice. We establish the existence of slices for arbitrary proper actions. This leads at once to a local model for a proper smooth actions, which is the basis for all the subsequent results.

We show that the development of basic results in §1.1 can be parallelled in the group action situation: we have covers by coordinate neighbourhoods, partitions of unity, an approximation lemma, and invariant Riemannian metrics. There is also a theorem on the existence of an equivariant embedding in Euclidean space (with an orthogonal action), which applies when the group is compact.

We continue by defining orbit types, and the stratification of the manifold by orbit types. This stratification is locally finite and smoothly locally trivial. One consequence is that if the manifold is connected, one orbit type is dense and open: orbits of this type are called principal orbits. We give a model for a neighbourhood of a stratum, and proceed to an analysis of the case with two strata.

We conclude with examples.

3.1 Lie groups

We recall from §1.3 that a *Lie group* is a smooth manifold G, which is also a group, such that the group operations $g \mapsto g^{-1}$, $(g, h) \mapsto gh$ are smooth maps $G \to G$, $G \times G \to G$.

Important examples are the general linear groups $GL_m(\mathbb{R})$ and $GL_m(\mathbb{C})$ of nonsingular $m \times m$ real, respectively complex, matrices, which are open submanifolds of the vector space of all matrices. We also use the notation $GL(V)$ for the group of linear endomorphisms of the vector space V.

A *Lie subgroup* is a smooth submanifold which is also a subgroup. Any subgroup of a Lie group G which is a closed subset is a Lie subgroup. This result is not trivial: a proof is given, for example, in [146, Theorem 4.1] or in [148, §3.1].

Not every subgroup of a Lie group is a closed subset: a simple example is the additive subgroup $\mathbb{Q}$ of $\mathbb{R}$. However, the closure of any subgroup is also a subgroup, hence is a Lie subgroup. If H is a Lie subgroup of G, as H is locally closed, it is open in its closure and hence by homogeneity is equal to its closure.

Among the Lie subgroups of $GL_m(\mathbb{R})$ are the group $GL_m^+(\mathbb{R})$ of matrices with positive determinant, the group $SL_m(\mathbb{R})$ of matrices of determinant 1, the orthogonal group O_m (orthogonal matrices can be characterised by the equation $AA^t = I$), and $SO_m = SL_m(\mathbb{R}) \cap O_m$. Lie subgroups of $GL_m(\mathbb{C})$ include $SL_m(\mathbb{C})$, U_m (here we have $A\overline{A}^t = I$) and SU_m. Further important examples are the spinor groups $Spin_m$ (the double covering group of SO_m), and the symplectic group Sp_m, defined like U_m, but using the algebra $\mathbb{H}$ of quaternions. We identify SO_2 with the multiplicative group S^1 of complex numbers of modulus 1, and SU_2 (also $Spin_3$ and Sp_1) with the multiplicative group S^3 of unit quaternions.

There is a general classification of compact Lie groups, which has its origin in the work of Lie and Killing: a convenient recent account is given in [125] (see Theorem 10.7.2.4). Any connected compact Lie group G has a finite covering group which is a direct product of copies of groups of the type S^1, SU_m, $Spin_m$, Sp_m and five other groups denoted G_2, F_4, E_6, E_7, and E_8. We will not use this in this book, but it opens the way to enumerations of groups and group actions satisfying prescribed conditions.

For G a Lie group and $g \in G$, the map $\rho_g : G \to G$ defined by $\rho_g(x) = xg$ is a diffeomorphism, with inverse $\rho_{g^{-1}}$; it is called *right translation* by g. Left translation λ_g is defined similarly.

If G is a group and H a subgroup, we write G/H for the set of right cosets $\{gH \mid g \in G\}$ and $\pi : G \to G/H$ for the natural projection given by $\pi(g) = gH$. We also have left cosets $Hg := \{hg \mid h \in H\}$ and the coset space $H\backslash G := \{Hg \mid g \in G\}$.

If G is a Lie group and H a Lie subgroup, the coset space G/H (with the quotient topology) has a natural structure as a smooth manifold. For at any $g \in G$, choose a chart $\varphi : U \to \mathbb{R}^{p+q}$ such that the submanifold $U \cap gH$ corresponds to the subspace $\mathbb{R}^p$; then take the composite $\mathbb{R}^q \subset \mathbb{R}^{p+q} \to U \subset G \to G/H$ as a chart at $gH \in G/H$. It is easy to see that transformations between overlapping charts are smooth, using as guideline the fact that a function $f : G/H \to \mathbb{R}$ is smooth if and only if $f \circ \pi \in \mathcal{F}_G$ is smooth. A similar argument shows that the projection $G \to G/H$ is that of a smooth fibre bundle. More generally, if we have two Lie subgroups $H_1 \subset H_2 \subset G$, the projection $G/H_2 \to G/H_1$ is that of a fibre bundle, with fibre H_2/H_1.

If G is a Lie group, the tangent space $T_1(G)$ at the identity has the structure of a Lie algebra. We will not use this in this book.

If H is a Lie subgroup of G we may choose an additive complement Y to $T_1(H)$ in $T_1(G)$. Then the differential at $(0, 1)$ of the map $Y \times H \to G$ given by $(y, h) \mapsto exp(y)h$ is an isomorphism (here we may use any Riemannian metric on G to define the exponential map), so by Theorem 1.2.5 the map is a local diffeomorphism. We can thus choose open neighbourhoods U of 1 in $exp(Y)$ and V of 1 in H such that $U \times V \to G$ is an embedding. We will call U a *local section* of H in G.

Lemma 3.1.1 *There exist local sections U_1 such that the map $\mu : U_1 \times H \to G$ is an embedding.*

Proof The fact that the differential of μ at any $(u, h) \in U \times H$ is bijective follows since this holds at $(u, 1)$ by hypothesis, and (right) translation by h is a diffeomorphism. It follows (as in the proof of Theorem 2.3.2) from Corollary A.2.6 that there is a neighbourhood of $1 \times H$ such that the restriction of μ to it is an embedding and by Lemma A.2.4 that for ε small enough if U_1 is the ε-neighbourhood of 1 in U, the restriction of μ to $U_1 \times H$ is an embedding. □

It follows that the induced map $U_1 \to G/H$ is an embedding. We can also argue similarly for $H \times U \to G$ and $U \to H\backslash G$.

If G is a Lie group, the connected component of the identity is a subgroup G_0, as if $x(t)$ is a path from 1 to $g \in G$ and $y(t)$ a path from 1 to h, then $x(t)^{-1}$ gives a path from 1 to g^{-1} and $x(t)y(t)$ a path from 1 to gh. As G is a manifold, G_0 is an open subset. Any open subgroup G^* of G is closed, since its complement is a union of cosets of G^*, each open, hence is open. Now if $p : I \to G$ is any path, $p^{-1}(G^*)$ is open and closed in I, hence is either I or the empty set. Thus G^* contains all paths from $1 \in G$, hence contains G_0.

If N is a neighbourhood of 1 in G, then the subgroup G^* of G generated by N contains an open neighbourhood of 1, so by homogeneity is open, so $G_0 \subset G^*$. Thus if $N \subset G_0$ we also have $G^* \subset G_0$, so the two coincide.

The point G_0/G_0 of G/G_0 is open; since G acts by homeomorphisms, all points, hence all subsets, are open, so the coset space G/G_0 has the discrete topology. If G is compact, then G/G_0 also is compact, so is finite.

Proposition 3.1.2 *Let G be a compact Lie group and let $H \neq G$ be a Lie subgroup. Then either*
$\dim H < \dim G$ *or*
$\dim H = \dim G$*, and H has fewer components than G.*

Proof Since $H \subset G$ is a submanifold, we have $\dim H \leqq \dim G$.

Suppose $\dim H = \dim G$. Then H contains a neighbourhood of 1 in G, so H contains G_0. As H is a proper subgroup of G, H/G_0 is a proper subgroup of G/G_0, which is compact and discrete, hence finite. But the components of G are the cosets of G_0. □

If G is a compact topological group, there is an averaging operator on the space $C^0(G)$ of continuous functions on G: it is the unique linear map $\int_G : C^0(G) \to \mathbb{R}$ such that

(i) if $g.f : G \to \mathbb{R}$ is defined by $g.f(x) := f(gx)$, then $\int_G(g.f) = \int_G(f)$,
(ii) if f_c is given by $f_c(g) = c$ for all $g \in G$, then $\int_G(f_c) = c$,
(iii) if $f(g) \geq 0$ for all $g \in G$, then $\int_G(f) \geq 0$.

For the reader familiar with integration theory, we can give a quick account as follows. The bundle of differential n-forms on a smooth manifold M is defined to be the nth exterior power $\Lambda^n T_1^\vee M$. If M has dimension n, then for any section ω of this bundle with compact support we can integrate ω over M: the result is denoted $\int_M \omega$.

If G is a Lie group of dimension n, we choose a form ω_0 at the identity $1 \in G$ to be any element of the exterior power $\Lambda^n T_1^\vee G$. Now for any $g \in G$, left translation by g gives a diffeomorphism of G taking 1 to G and hence ω_0 to an n-form at $g \in G$; assembling these gives an n-form ω' on G invariant under left translations by elements $g \in G$. For any (smooth or even just continuous) function f of compact support on G we can now form the integral $\int_G f\omega'$.

In the case when G is compact we can now define $\int_G f := \int_G f\omega / \int_G \omega$; properties (i)–(iii) follow easily. We will not give the proof of uniqueness; however from uniqueness follows that if $f.g' : G \to \mathbb{R}$ is defined by $f.g'(x) := f(xg')$, then $\int_G(f.g') = \int_G(f)$. For since the averaging operator is unique, it suffices to show that $f \mapsto \int_G(f.g')$ satisfies (i)–(iii). But these follow from the same results for $\int_G$ by substitution. It follows similarly that if we define f^* by $f^*(g) = f(g^{-1})$, then $\int_G f^* = \int_G f$.

A proof of existence of an averaging operator for arbitrary compact groups (due to Haar) may be found in [68], also a theory of (left invariant) integration for any locally compact topological group.

3.2 Smooth actions

A (left) *action* of a group G on a set X is a map $\phi : G \times X \to X$ such that $\phi(1, x) = x$ for all $x \in X$ and $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $x \in X$ and $g, h \in G$. If the action is understood, it is frequently denoted by a dot: thus $\phi(g, x)$ becomes $g.x$; we will call X a G-space. If G is a Lie group, X a smooth manifold, and ϕ a smooth map, we speak of a *smooth action* and a smooth G-space.

If we have an action of a group G on a set X, and $x \in X$, the *isotropy group* of x is defined to be $G_x := \{g \in G \mid g.x = x\}$. It follows from the definition of group action that this is a subgroup of G; it is also sometimes called the stabiliser of x. The *orbit* of x is defined to be $\{g.x \mid g \in G\}$, and is denoted $G.x$. The action induces a bijection $G/G_x \to G.x$ since

$$g.x = h.x \Leftrightarrow h^{-1}g.x = x \Leftrightarrow h^{-1}g \in G_x \Leftrightarrow hG_x = gG_x.$$

Equivalently, the map $Op_x : G \to X$ defined by $Op_x(g) := g.x$ induces an injection of G/G_x into X.

The set of orbits of a left group action is denoted $G\backslash X$; in the case of continuous, in particular smooth actions, we give $G\backslash X$ the quotient topology and call it the *orbit space*. Even for a smooth action, this is only rarely a manifold.

For a smooth action, any isotropy group is a closed subgroup of G, hence is a Lie subgroup. A sufficient condition for the injection $G/G_x \to X$ to be a smooth embedding will be given in the next section.

A point $x \in X$ is *fixed* under G if $g.x = x$ for all $g \in G$, i.e. if $G_x = G$. The *fixed set* of the action is the set of all fixed points, and is denoted X^G. At the opposite extreme to the fixed set, an action is called *free* if $g.x = x$ implies $g = 1$: thus $\{1\}$ is the only isotropy group. The action is *semi-free* if the only isotropy groups are $\{1\}$ and G.

A subset $Y \subseteq X$ is *invariant* under G if $g.y \in Y$ for all $g \in G$ and $y \in Y$.

Given two actions $\phi : G \times X \to X$ and $\psi : G \times Y \to Y$, a map $f : X \to Y$ is *equivariant* (more precisely, G-equivariant) if, for all $g \in G$ and $x \in X$, we have $f(\phi(g, x)) = \psi(g, f(x))$.

Given a subgroup H of G and an action of H on X, we define $G \times_H X$ to be the quotient of $G \times X$ by the relation $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$ and $x \in X$: this is an equivalence relation since H is a subgroup. We denote the equivalence class containing (g, x) by $[g, x]$. Setting $g'.[g, x] := [g'g, x]$ defines an action of G on $G \times_H X$.

Lemma 3.2.1 *The isotropy group of* $[g, z] \in G \times_H X$ *is* gH_zg^{-1}.

For if $[g, z] \in G \times_H X$, we have $g'.[g, z] = [g'g, z]$, and this is equal to $[g, z]$ if and only if, for some $h \in H$, $g'g = gh$ and $h^{-1}.y = y$; so $h.y = y$ and $g' = ghg^{-1}$.

If these groups and actions are smooth, then by Lemma 3.1.1 we can pick a local section Y such that $Y \times H \to G$ is a diffeomorphism onto an open set. It follows that $Y \times X \to G \times_H X$ is a diffeomorphism onto an open set.

We will give many examples of smooth group actions at the end of this chapter, but offer two here.

If H is a subgroup of G, G acts on G/H by left translations: $g.g'H := gg'H$. If G is a Lie group and H a Lie subgroup, this action is smooth.

The group $GL(V)$ acts on the vector space V: for example, $GL_m(\mathbb{R})$ acts on the space $\mathbb{R}^m$ of column vectors by matrix multiplication. If G is any group and $f : G \to GL(V)$ a homomorphism, there is an induced action of G on V by linear maps; we refer to V as a linear G-space. The action is called a linear representation of G.

A classical theorem, known as the Peter–Weyl Theorem, states that for any compact group G there exist a (finite dimensional) real vector space V and an injective continuous homomorphism $G \to GL(V)$. Moreover, the function algebra $L^2(G)$ is a direct sum of finite dimensional invariant subspaces so, for example, any smooth function on G can be approximated by functions of the form $g \mapsto \ell(g.x)$, where $x \in V$ for some linear G-space V and $\ell : V \to \mathbb{R}$ is linear.

Lemma 3.2.2 *For any continuous linear action of a compact Lie group H on a vector space V, there is an inner product on V invariant under H.*

Proof Choose an inner product $V \times V \to \mathbb{R}$, and denote it $\langle x, y\rangle$. Define $\langle x, y\rangle_H := \int_H \langle g.x, g.y\rangle$. This is linear in each of x and y, and invariant in the sense that $\langle g.x, g.y\rangle_H = \langle x, y\rangle_H$ for all $g \in H$ and $x, y \in V$. Moreover we have $\langle x, x\rangle_H > 0$ if $x \neq 0$, so $\langle *, *\rangle_H$ is an inner product. □

The image of H in $GL(V)$ is a subgroup of the orthogonal group of V with respect to this product. Since any two inner products on V are equivalent under the general linear group $GL(V)$, it follows that any compact subgroup of $GL(V)$ is conjugate to a subgroup of $O(V)$. Extending Lemma 3.2.2, we have

Proposition 3.2.3 *For any smooth action of a compact Lie group H on a smooth manifold M, there is a Riemannian metric on M invariant under H.*

Proof Choose any Riemannian metric on M: we can regard it as a collection of inner products on all the tangent spaces T_xM. The action of $g \in H$ takes $g^{-1}.x$ to x and gives an isomorphism of $T_{g^{-1}.x}M$ on T_xM, so transporting the given inner product $\langle , \rangle$ on $T_{g^{-1}.x}M$ gives an inner product $\langle , \rangle_g$ on T_xM. Integrating over H as above gives a new family of scalar products giving a Riemannian metric invariant under H. □

Since the exponential map $Exp : \mathbb{T}(M) \to M$ was directly constructed from the metric, it follows that if we have a G-invariant metric on M, the corresponding exponential map is G-equivariant.

Corollary 3.2.4 *The fixed set M^H of a smooth action of a compact Lie group H on a smooth manifold M is a smooth submanifold of M.*

Proof By the Proposition, we can choose an H-invariant Riemannian metric on M. Let $x \in M^H$ be a fixed point, then the exponential map $T_xM \to M$ is a local diffeomorphism and is H-equivariant. Since H acts orthogonally on T_xM, the fixed set $(T_xM)^H$ is a linear subspace, and so a smooth submanifold. The result follows. □

3.3 Proper actions and slices

The main geometrical results about smooth group actions depend on compactness. The theory is usually written in terms of actions of a compact group G, but with a little effort, the results extend to arbitrary Lie groups, provided the action satisfies the following key condition.

An action $\phi : G \times X \to X$ is said to be *proper* if the map

$$(\phi, i) : G \times X \to X \times X$$

given by $(g, x) \mapsto (g.x, x)$ is a proper map.

Proposition 3.3.1 *Let $\phi : G \times X \to X$ be a proper group action and $x \in X$. Then*

(i) the isotropy group G_x is compact;
(ii) the map $Op_x : G \to X$ is proper;
(iii) the orbit $G.x$ is a closed subset of X;
(iv) the induced map $G/G_x \to G.x$ is a homeomorphism;
(v) for any compact subsets $K, L \subseteq X$, $\{g \in G \mid g.K \cap L \neq \emptyset\}$ is compact;
(vi) the orbit space $G\backslash X$ is Hausdorff.

A fuller discussion is given in §A.3. The above result is contained in Propositions A.3.1 and A.3.3.

By Lemma A.3.2(i), a smooth group action with G compact is always proper. So is the action on G by a Lie subgroup H by left translation. More generally, by (ii) of the Lemma, given two Lie subgroups $H,\ K$ of G with K compact, the natural action of H on the coset space G/K is proper.

To illustrate the importance of properness, we give examples where the condition fails, and the geometrical picture is very different from what we obtain below in the proper case.

First, we can consider $\mathbb{Q}$ as a discrete group and let it act additively on $\mathbb{R}$.

Second, take G as $\mathbb{R}$, $M := \mathbb{R}^2/\mathbb{Z}^2$ and let $\alpha \in \mathbb{R}$ be irrational: define an action by $\phi(t, [x, y]) := [x + t, y + \alpha t]$.

In these two cases, all isotropy groups are trivial but all orbits are dense in M. In general, a smooth action of $\mathbb{R}$ on M (also called a dynamical system) defines a vector field on M, and we saw in Theorem 1.4.2 that conversely any vector field defines a flow and subject to a completeness condition (see, for example, Proposition 1.4.4) gives a group action.

For a third example take $M = \mathbb{R}$ and $\frac{d\theta}{dt} = \sin\theta$ (which is certainly bounded). The fixed set of this action is the set of θ with $\sin\theta = 0$, so consists of integer multiples of π.

Theorem 3.3.2 (The Rank Theorem) *Let $f : \mathbb{R}^m \rightsquigarrow \mathbb{R}^n$ be a smooth map defined on a neighbourhood A of $a \in \mathbb{R}^m$ such that, for all $x \in A$, df_x has rank p, for some fixed $p > 0$. Then there exist open neighbourhoods $U \subset A$ of a, $V \supset f(U)$ of $f(a)$, and diffeomorphisms $u : U \to (\mathring{D}^1)^m$, $v : V \to (\mathring{D}^1)^n$ such that $f|U = v^{-1} \circ \pi \circ u$, where $\pi(x_1, \cdots, x_m) = (x_1, \cdots, x_p, 0, \cdots, 0)$.*

We regard this as an extension of Theorem 1.2.5 and, as for that result, proofs can be found in [40] and [52]. As for Theorem 1.2.5, the given statement refers only to a neighbourhood of a point in $\mathbb{R}^m$, but the result translates at once to one valid for any manifold.

Theorem 3.3.3 *For any smooth action of G on M and any $x \in M$, the induced map $j : G/G_x \to M$ is a smooth immersion with image $G.x$.*

If the action is proper, j is an embedding as a closed submanifold.

Proof We first apply Theorem 3.3.2 to the map $Op_x : G \to M$. We claim that it follows from the group action property that this map has the same rank at all points. For left translation ℓ_g by $g \in G$ is a diffeomorphism of G taking a neighbourhood of $1 \in G$ to a neighbourhood of g. The action of g is a diffeomorphism r_g of M taking x to $g.x$. The diagram

$$\begin{tikzcd} G \arrow[r, "Op_x"] \arrow[d, "\ell_g"'] & M \arrow[d, "r_g"] \\ G \arrow[r, "Op_x"] & M \end{tikzcd} \qquad \begin{tikzcd} h \arrow[r] \arrow[d] & h.x \arrow[d] \\ gh \arrow[r] & gh.x \end{tikzcd}$$

is commutative and the vertical maps are diffeomorphisms. Taking tangent spaces thus gives a commutative diagram with the vertical maps linear isomorphisms. Thus indeed dOp_x has the same rank at 1 and at g.

It follows from the rank theorem that the map Op_x is locally trivial. Hence the rank of dOp_x is equal to the dimension of the image, namely of the orbit $G.x$; and the rank of the kernel is equal to the dimension of the fibre, which is the isotropy group G_x. Thus the induced map $G/G_x \to G.x$ is an immersion.

By Proposition 3.3.1 (ii), if the action is proper, the map $j : G/G_x \to M$ is proper. It follows from Proposition 1.2.10 that j is an embedding as a closed submanifold. □

Although the basic idea of taking a slice is simple, the following definition is important; the existence of slices is key to the structure results that follow.

Given a smooth action of the G on M, and a closed subgroup H of G, a smooth H-*slice* to the action is a smoothly embedded submanifold V of M such that

(S1) For all $y \in V$, $T_yM = T_y(G.y) + T_xV$.

(S2) V is H-invariant.

(S3) If $s \in V$, $g \in G$ and $g.s \in V$, then $g \in H$.

The definition includes the case when V is a submanifold with boundary.

Theorem 3.3.4 *For any proper smooth action of G on M and any $x \in M$ there exists a smooth G_x-slice V to the action with $x \in V$.*

Proof Since the action is proper, the isotropy group G_x is compact; write $H := G_x$. By Proposition 3.2.3 we can choose a Riemannian metric of M invariant under H. Then $T_x(G.x)$ is a subspace of T_xM; write E for its orthogonal complement. Then E is also invariant under the induced action of H on T_xM.

Since the metric is H-invariant, the exponential map of M is H-equivariant. Denote by D_a, $\mathring{D}_a$ the closed and open discs of radius a in E, and write $V_a := \exp(D_a)$ and $\mathring{V}_a = \exp(\mathring{D}_a)$. As in the construction of tubular neighbourhoods, if a is small enough, the restriction of the exponential map to D_a is an embedding. We will show that for b small enough V_b, hence also $\mathring{V}_b$, is a smooth H-slice.

It follows from the construction that for any $b \leq a$, V_b is a smoothly embedded disc and is H-invariant, so satisfies (S2).

Now choose a local section U to H in G: then UH is an open neighbourhood of H in G, so its complement is closed. Since by Proposition 3.3.1(ii), Op_x is a closed map, $(G \setminus UH).x$ is a closed set. It does not contain x, so is at a positive distance 2ε from x.

We also have $T_x V_a = E$, so $T_x M = T_x(G.x) \oplus T_x V_a$. The action induces a smooth map $G \times V_a \to M$ whose differential is surjective at $(1, x)$. Hence it is also surjective on some neighbourhood of this point. If b is small enough, this neighbourhood contains $1 \times V_b$, thus V_b satisfies (S1).

Since $T_1 U \oplus T_1 H = T_1 G$, and it follows from the Rank Theorem that $T_1 G/T_1 H \cong T_x(G.x)$, the map $T_1 U \to T_x(G.x)$ is an isomorphism. Thus the map $U \times V_a \to M$ induces an isomorphism $T_1 U \oplus T_x V_a \to T_x M$ of tangent spaces, so induces a diffeomorphism of some neighbourhood; shrinking U if necessary, and taking b small enough, we may suppose this neighbourhood contains $U \times V_b$. Then $u \neq 1 \in U$ and $y \in V_b$ implies $u.y \notin V_b$.

Since the action is proper and V_a is compact, it follows from Proposition 3.3.1(v) that $K := \{g \in G : V_a g \cap V_a \neq \varnothing\}$ is compact; note that $H \subseteq K$. It follows from Lemma A.2.1 that for any ε we can find δ such that if $s \in V_a$, $g \in K$ and $\rho(s, x) < \delta$ we have $\rho(g.s, g.x) < \varepsilon$.

Now if $s \in V_b$ and $g.s \in V_b$, then

$$\rho(x, g.x) \leq \rho(x, g.s) + \rho(g.s, g.x) \leq b + \varepsilon < 2\varepsilon.$$

Hence $g \notin G \setminus UH$. i.e. $g \in UH$: say $g = uh$. Then $h.s \in V_b$ and $u.(h.s) \in V_b$. It now follows from the above that $u = 1$, so indeed $g = h \in H$. Thus V_b also satisfies (S3). □

We now derive a local model giving a description of the neighbourhood of an orbit in a proper group action.

Theorem 3.3.5 *Let V be an H-slice at x to a smooth proper action of G on M, with $H = G_x$. Then the action induces a smooth map $j : G \times_H V \to M$ giving an equivariant diffeomorphism onto a neighbourhood Y of $G.x$ in M.*

If V is a closed disc, this gives a tubular neighbourhood of $G.x$ in M.

Proof By (S2) V is H-invariant, so $G \times_H V$ is defined. The action ϕ now induces a smooth equivariant map j, and it follows from (S3) that j is injective and from (S1) that j is a submersion, hence a diffeomorphism.

We recall that a tubular neighbourhood of a (closed) submanifold F in M is defined to consist of a bundle B over F with fibre a disc and an embedding $\psi : B \to M$ (as submanifold with boundary) extending the map taking the centre of each disc to the corresponding point of V. Here we take $F = G.x$ and $B = G \times_H V$. A projection $G \times_H V \to G/H \cong G.x$ is given by $[g, s] \to gH \to g.x$:

we see at once that this is well defined, and its fibre is V. A local section U for H in G induces a local trivialisation. $\square$

For a smooth proper group action of G on M, by Theorem 3.3.5 there is a smooth map $j : G \times_H V \to M$ giving an equivariant diffeomorphism onto a G-invariant neighbourhood Y of $G.x$ in M. We constructed V as a metric disc in the orthogonal complement E of $T_x(G.x)$ in T_xM. Moreover, since we have a diffeomorphism of $\mathring{D}^m$ on $\mathbb{R}^m$ which is invariant under rotations, we may also replace V by E itself, and have an equivariant diffeomorphism of Y with $G \times_H E$. Here E is a real vector space on which H acts orthogonally. This choice gives a convenient local model, which we use for further analysis below.

3.4 Properties of proper actions

From now on, we suppose M a smooth proper G-space. By Proposition A.3.4, the quotient space $G\backslash M$ is Hausdorff, locally compact, and a countable union of compact sets. We can now parallel the development in §1.1.

First, we can apply Proposition 1.1.3 to express $G\backslash M$ as $\bigcup_n C_n$, where we have compact subsets C_n and open subsets $B_{n+\frac{1}{2}}$ such that for all $n \geq 1$, $C_n \subset B_{n+\frac{1}{2}} \subset C_{n+1}$.

The map $j : G \times_H V \to M$ of Theorem 3.3.5 induces a homeomorphism of $G\backslash(G \times_H V)$ onto a neighbourhood of the image $[x]$ of x in $G\backslash M$. We will regard such a map as a coordinate neighbourhood[1] for $G\backslash M$. Observe that $G\backslash(G \times_H V) \cong H\backslash V$, so this neighbourhood is a quotient of V. We will use the term 'nice neighbourhood' in the case when V is a disc D (we suppress the affix giving the dimension of the disc, which depends on the slice, and will be clear from the context). We think of this as a map $\overline{j} : H\backslash D \to G\backslash M$ coming from a map $j : G \times_H D \to M$.

Theorem 3.4.1 *We can find a set of nice coordinate neighbourhoods $\varphi_\alpha : \mathring{D}(3) \to G\backslash M$, with images denoted U_α, such that*

(i) The sets $\varphi_\alpha(\mathring{D})$ cover $G\backslash M$.

(ii) Each $P \in G\backslash M$ has a neighbourhood which meets only a finite number of sets U_α, i.e. the U_α are locally finite.

Moreover, the covering $\{U_\alpha\}$ may be chosen to refine any given covering of $G\backslash M$.

The proof of Theorem 1.1.4 goes through here with essentially no change.

[1] This differs from the notation of 1.1, where the map went from a neighbourhood in the manifold to one in Euclidean space.

It follows that the quotient space $G\backslash M$ is locally modelled by quotients $H\backslash D^k$ of discs D^k by compact subgroups H of O_k: although this is not smooth, it is a topological space with very good properties (triangulable, semi-algebraic, etc.).

We define a function $f : G\backslash M \to \mathbb{R}$ to be smooth: if the composite function $f \circ p : M \to \mathbb{R}$ is smooth.

Theorem 3.4.2 *For any covering $\mathcal{V}$ of M by G-invariant open sets, there is a smooth partition of unity by invariant functions strictly subordinate to it.*

Proof This is an analogue of Theorem 1.1.5, and the proof of the earlier result carries over with only minor change. The images of the elements of $\mathcal{V}$ define an open covering $\mathcal{U}$ of $G\backslash M$. By Theorem 3.4.1 there is a locally finite refinement of $\mathcal{U}$ by a set of coordinate neighbourhoods $\varphi_\alpha : \mathring{D}^k(3) \to G\backslash M$ such that the $\varphi_\alpha(\mathring{D}^k)$ cover $G\backslash M$. As in the earlier proof, we use these to construct smooth functions Ψ_α on $G\backslash M$, with Ψ_α supported on the image of φ_α, such that for each $P \in G\backslash M$, there is an α with $\Psi_\alpha(P) = 1$, and that each $P \in G\backslash M$ has a neighbourhood on which all but a finite number of functions Ψ_α vanish. Hence $\Sigma(P) := \sum_\alpha \Psi_\alpha(P)$ is defined, and is everywhere smooth. Thus the functions $\psi_\alpha(P) = \Psi_\alpha(P)/\Sigma(P)$ give a partition of unity; by construction it is strictly subordinate to $\mathcal{U}$. Now the functions $\psi_\alpha \circ p$ are smooth invariant functions on M giving the desired partition of unity. $\square$

Next we have an equivariant version of Proposition 1.1.7.

Proposition 3.4.3 *(i) Let f be a continuous positive invariant function on M. Then we can find a smooth invariant function g, with $0 < g(P) < f(P)$ for all $P \in M$.*

(ii) For any continuous invariant function f on M and any $\varepsilon > 0$ there exists a smooth invariant function h on M with $|h(x) - f(x)| < \varepsilon$ for every $x \in M$.

(iii) If $f : M \to \mathbb{R}$ is continuous and invariant, $\varepsilon > 0$, and F is a closed invariant subset of M such that f is smooth on some open invariant set $U \supset F$, we can find h such that also $h = f$ on an invariant neighbourhood of F.

We can carry over the whole proof of the earlier result: it suffices to work throughout in $G\backslash M$ rather than in M.

For actions of a compact group, it is shown in Proposition A.3.5 that any neighbourhood of an invariant set contains an invariant neighbourhood. This is *not* true in general for proper actions. For an example, consider the translation action of $\mathbb{R} \times \{0\}$ on $\mathbb{R}^2$. The subset $\mathbb{R} \times \{0\}$ is invariant, and the set $\{(x, y) \mid |xy| \le 1\}$ is a neighbourhood, but any invariant neighbourhood contains $\{(x, y) \mid |y| \le \varepsilon\}$ for some $\varepsilon > 0$.

We turn to the existence of an invariant metric. First we consider Riemannian metrics on G itself. A positive definite scalar product on the tangent space T_1G at the identity gives rise under left translation λ_g to a scalar product on T_gG; collecting these for all $g \in G$ gives a Riemannian structure on G, invariant under left translation by elements of G.

Inner automorphism $x \to g^{-1}xg$ by $g \in G$ is a diffeomorphism of G fixing the identity, so induces a linear automorphism of T_1G. Collecting these for all $g \in G$ gives a homomorphism $ad_G : G \to GL(T_1G)$. If H is a compact subgroup of G, we know by Lemma 3.2.2 that there is an inner product on T_1G invariant under H. If we begin with such an inner product, it follows that the Riemannian metric on G is also invariant under right translation by elements of H.

Theorem 3.4.4 *A smooth proper G-manifold M has a G-invariant Riemannian structure.*

Proof This is an analogue of Theorem 1.3.1, and the proof is again modelled on the previous one. As there, we begin with a cover by charts $\varphi_\alpha : \mathring{D}^k(3) \to G\backslash M$, associated to maps $j_\alpha : G \times_H \mathring{D}^k(3) \to M$, and a strictly subordinate partition ψ_α of unity.

We next construct a G-invariant metric on $Y := G \times_H E$. Since H is compact we can, as in Proposition 3.2.3, find an H-invariant Riemannian structure on the restriction of $\mathbb{T}(Y)$ to $E = H \times_H E$ (an explicit construction can be given using an H-invariant inner product on E, and a Riemannian metric on G). The action of G gives a unique G-invariant Riemannian metric on $\mathbb{T}(Y)$ extending this structure over E.

Pulling back this metric by j_α gives a metric m_α on $j_\alpha(G \times_H \mathring{D}^k(3))$. Then $\psi_\alpha m_\alpha$ extends to an invariant section over M of the Riemannian bundle which is supported in $j_\alpha(G \times_H D^k(2))$. Now consider $\sum_\alpha \psi_\alpha m_\alpha$. Since the U_α are locally finite, the sum is defined; since the partition was strictly subordinate to the cover, the sum is smooth. Since a linear combination of positive definite quadratic forms is again positive definite, the sum is everywhere positive definite. Thus it defines an invariant Riemannian structure on M^m. $\square$

I expect that the existence of a complete invariant metric can be established, but have not found a proof.

Under some restrictions, one can prove the existence of equivariant embeddings in Euclidean space. We first need a couple of results about linear actions.

Lemma 3.4.5 *Let G be a compact Lie group, H a Lie subgroup. Then*

(i) if V is a linear H-space, there exist a linear G-space W and an H-equivariant linear embedding $V \to W$;

(ii) there exist a linear G-space U and $u \in U$ with $G_u = H$.

We omit the proofs, which can both be deduced from the Peter–Weyl Theorem. For (i) we consider the vector bundle over G/H with fibre V. The space of L^2 sections is an infinite-dimensional linear G-space, and one needs to extract a finite dimensional subspace. For (ii) one similarly begins with the action of G on the space of functions on G/H (see [20, p. 105]).

Theorem 3.4.6 *For any smooth action of a compact group G on a compact manifold M, there exist a linear G-space E and a G-equivariant embedding $M \to E$.*

Proof By Theorem 3.4.1, we can cover $G\backslash M$ by a finite set of nice coordinate neighbourhoods $U_\alpha = j_\alpha(G \times_{H_\alpha} \mathring{D}_\alpha(3))$ coming from maps $\varphi_\alpha : \mathring{D}_\alpha(3) \to G\backslash M$, where $\mathring{D}_\alpha(3)$ is the disc of radius 3 in the H_α-space E_α. We define a smooth map $\Phi_\alpha : G\backslash M \to \mathbb{R}$ by $\Phi_\alpha(\varphi_\alpha(x)) = Bp(2 - \|x\|)$ for $x \in \mathring{D}_\alpha(3)$, $\Phi_\alpha(P) = 0$ otherwise.

By Lemma 3.4.5 (i) we can choose an H_α-linear embedding $f_\alpha : E_\alpha \to W_\alpha$ with W_α a linear G-space. By (ii) of the Lemma we can choose a linear G-space U_α and $u_\alpha \in U_\alpha$ with $G_{u_\alpha} = H_\alpha$. Now define $\phi_\alpha : G \times E_\alpha \to W_\alpha \oplus U_\alpha$ by $\phi_\alpha(g, s) = (g.f_\alpha(s), g.u_\alpha)$. Then for $h \in H_\alpha$, $\phi_\alpha(gh, s) = (gh.f_\alpha(s), gh.u_\alpha)$; since $G_{u_\alpha} = H_\alpha$, $h.u_\alpha = u_\alpha$ so $gh.u_\alpha = g.u_\alpha$; since f is H_α-equivariant, $gh.f_\alpha(s) = g.h.f_\alpha(s) = g.f_\alpha(h.s)$. Thus $\phi_\alpha(gh, s) = (g.f_\alpha(h.s), g.u_\alpha) = \phi_\alpha(g, h.s)$, so ϕ_α factors through $\psi_\alpha : G \times_{H_\alpha} E_\alpha \to W_\alpha \oplus U_\alpha$. By construction, ψ_α is a G-equivariant map.

The map ψ_α is injective since as $G_{u_\alpha} = H_\alpha$, $g.u_\alpha = g'.u_\alpha$ implies $g' = gh$ for some $h \in H_\alpha$; thus if $\phi_\alpha(g, s) = \phi_\alpha(g', s')$ then $\phi_\alpha(g, s) = \phi_\alpha(g, hs')$ so $g.f_\alpha(s) = g.f_\alpha(hs')$ and as f is injective, $s = hs'$. A corresponding argument on tangent spaces proves ψ_α a smooth embedding.

Now define $\Psi_\alpha : M \to W_\alpha \oplus U_\alpha \oplus \mathbb{R}$ by $\Psi_\alpha(Q) = (\Phi_\alpha(p(Q))\psi_\alpha([g, s]), \Phi_\alpha(p(Q)))$ if $Q = j_\alpha([g, s])$ with $s \in \mathring{D}_\alpha(3)$, and $\Psi_\alpha(Q) = 0$ otherwise. Since ψ_α is G-equivariant, so is this, where G acts trivially on $\mathbb{R}$. In view of the definition of Φ_α, Ψ_α is a smooth map. It now follows exactly as in the proof of Theorem 1.2.11 that the product map

$$\prod_\alpha \Psi_\alpha : M \to \bigoplus_\alpha (W_\alpha \oplus U_\alpha \oplus \mathbb{R})$$

is a smooth embedding. □

3.5 Orbit types

If we denote by ρ the (orthogonal) representation of H on E, then by Theorem 3.3.5 the structure of M in a neighbourhood of the orbit is determined by the pair $(H \subseteq G, \rho)$. In turn, this pair is determined by the action and the point $x \in M$. If we replace x by another point $g.x$ on the same orbit, $H = G_x$ is replaced by $G' = G_{g.x} = gHg^{-1}$ and ρ by an action ρ' of G' on E', where there is an isomorphism $\lambda : E \to E'$ with $\lambda(\rho(h).e) = \rho'(ghg^{-1}).\lambda(e)$ for all $h \in H,\ e \in E$. We will call two such pairs equivalent: then the equivalence class of the pair (H, ρ) depends only on the orbit $G.x$. We call it the *orbit type* of the orbit.

We may also define the *weak orbit type* of an orbit $G.x$ to be the conjugacy class of the isotropy group G_x of x. Since $G_{g.x} = g^{-1}G_x g$, this too is determined by the orbit. Two orbits have the same weak orbit type if and only if there is an equivariant bijection between them. Write $M^{\langle H \rangle} := \{x \in X \mid G_x = H\}$ for the set of points with isotropy group H. For M a proper smooth G-space, Theorem 3.3.5 describes the neighbourhood of an orbit as $Y = j(G \times_H E)$.

Lemma 3.5.1 *In the notation of Theorem 3.3.5,*

$$Y^H = Y^{\langle H \rangle} = j(N_G(H) \times_H E^H) = j((N_G(H)/H) \times E^H).$$

Thus $M^{\langle H \rangle}$ is an open submanifold of M^H.

Proof By Lemma 3.2.1, the isotropy group of $[g, z]$ is $gH_z g^{-1}$. For this to be conjugate to H, we need $H_z = H$, so $z \in E^H$; otherwise the isotropy group is strictly smaller (in the sense of Proposition 3.1.2). The calculation follows. □

The manifold $M^{\langle H \rangle}$ is not in general closed; nor need it be dense in M^H: if H does not itself occur as an isotropy group, the open subset $M^{\langle H \rangle}$ of M^H will be empty.

Different components of M^H, or of $M^{\langle H \rangle}$, may well have different dimensions. A simple example is given by the action of $\mathbb{Z}_2$ on the projective plane $P^2(\mathbb{R})$ defined by $T.(x_0 : x_1 : x_2) = (-x_0 : x_1 : x_2)$. The fixed point set consists of the point $(1 : 0 : 0)$ and the projective line $x_0 = 0$. Thus it is not convenient to partition M according to weak orbit type, and we focus on the study of orbit types.

Having the same orbit type is an equivalence relation on orbits, which we use to define partitions of $G\backslash M$ and of M. We will study these partitions, and begin with a key finiteness result.

Theorem 3.5.2 *Let φ be a proper smooth action of G on M. Then M has locally a finite number of orbit types.*

Proof We prove the result by induction on the dimension of M. If M is 0-dimensional, for each $x \in M$, the point $\{x\}$ is a neighbourhood of x and contains just one orbit type. So the assertion holds in this case. Now suppose M of dimension m and the result proved for manifolds of dimension $k < m$.

Let $x \in M$, set $H = G_x$ and let V be an H-slice at x. By Theorem 3.3.5, $G.x$ has an invariant neighbourhood diffeomorphic to $G \times_H V$. Every orbit in this neighbourhood meets V, so it is sufficient to show that there are only a finite number of orbit types in V.

We may also suppose $D^k \cong V \subset E$ a disc, and the action of H on E linear. All points on the same open radius have the same isotropy group, so the different orbit types occur at 0 and on the boundary of D^k, which is a sphere S^{k-1}, for some $k \leqq m$. By the inductive hypothesis, there are only a finite number of orbit types on S^{k-1}; and there is just one orbit type at 0. Thus $D^k \cong V$ has a finite number of orbit types. □

Let τ denote an orbit type, and write M^τ for the union of orbits of type τ.

Proposition 3.5.3 *M^τ is a smooth submanifold of M.*

Proof Let $x \in M^\tau$, and consider the neighbourhood $j(G \times_H V)$ of x constructed in Theorem 3.3.5. By Lemma 3.5.1 the points with the same weak orbit type as x in this neighbourhood form $j(G \times_H V^H)$, which is isomorphic to $(G/H) \times V^H$, and hence smooth. For $v \in V^H$, the translation in E by v is H-equivariant and takes a neighbourhood of O to one of v; thus we have the same orbit type. It follows that the set of points of orbit type τ in $j(G \times_H V)$ is also $j(G \times_H V^H)$. Thus M^τ is a smooth neighbourhood of each of its points. □

It follows from this proof that the orbit type is locally constant along $M^{\langle H \rangle}$, hence also along the space $G.M^{\langle H \rangle}$ of points of the same weak orbit type: thus is constant on each connected component of this set.

A *stratification* of a manifold is a locally finite partition into smooth submanifolds. The preceding two results show that given a smooth proper action of G on M, the partition by orbit types is a stratification. We next show that this partition has a local triviality property.

For a stratification to be used geometrically one usually imposes some condition on the way strata fit together; in particular on the behaviour of a bigger stratum near a smaller one. The strongest such condition is local triviality. We say a stratification $\mathcal{S} = \{S_\alpha\}$ of M is *locally trivial* if at each point $x \in M$ there is a neighbourhood W of x in M and a diffeomorphism $\phi : W \to A \times B$ with $A,\ B$ smooth manifolds such that if S_α is the stratum containing x, $\phi(S_\alpha \cap W) = A \times \{x_0\}$ for some $x_0 \in B$ and for any other stratum S_β, $\phi(S_\beta \cap W) = A \times B_\beta$ for some smooth submanifold B_β of B.

Theorem 3.5.4 *The stratification of M by orbit types is locally trivial.*

Proof Again we use the model given by Theorem 3.3.5, and work in a neighbourhood $Y = j(G \times_H E)$ of $G.x$ in M. The orbit type α of the point $[g, y]$ is determined by that of y under the action of H on V. Now split E as a direct sum $E^H \oplus E^\alpha$, where E^α is the orthogonal complement to E^H in E. Then the orbit type of y under H depends only on the component of y in E^α. The result follows, taking E^α as the B in the above definition. □

We defined above the stratification of M by orbit types: the strata M^α are smooth submanifolds of M, and are locally finite. Thus at least one must have the same dimension as M. More precisely,

Theorem 3.5.5 (Principal Orbit Theorem) *For any smooth proper group action on a connected manifold M, there is one orbit type stratum which is open and dense in M.*

Proof We use the model given by Theorem 3.3.5: an invariant neighbourhood of a point x with $H = G_x$ is equivariantly diffeomorphic to $G \times_H E$ for some H-vector space E. The set M^α of points with the same orbit type α as x locally form $G \times_H E^H$. Thus if $\dim E^\alpha \geq 2$, M^α has codimension at least 2 and does not separate M. Now consider the case $\dim E^\alpha = 1$. Since the action of H on E^α is orthogonal and non-trivial, there is a subgroup H^+ of index 2 which acts trivially, and H acts by reflection. In this case M^α does locally separate M, but points on opposite sides lie on the same orbit. So here also $G\backslash M^\alpha$ does not separate $G\backslash M$, so the complement of the union of the $G\backslash M^\alpha$ with $\dim E^\alpha > 0$ is connected, and so is a single orbit type stratum. □

There is a natural partial order on the set of orbit types which is defined as follows. An orbit type α determines (up to equivalence) a subgroup H_α of G and a linear H_α-space E_α. Then a neighbourhood of an orbit of this type is equivariantly diffeomorphic to $N_\alpha := G \times_{H_\alpha} E_\alpha$. If β is an orbit type occurring in N_α, we write $\beta \prec \alpha$.

Lemma 3.5.6 *The relation $\prec$ is a partial order. If $\beta \prec \alpha$ then $M^\alpha \subset \bar{M}^\beta$. For any α there are only finitely many types β with $\beta \prec \alpha$. For M connected, the principal orbit type of M is the least α with $M^\alpha \neq \emptyset$.*

Proof It follows from the definition that if $\beta \prec \alpha$ there is an equivariant embedding of N^β in N^α, so the relation is transitive. Moreover, by Proposition 3.1.2 H^β has either a lower dimension than H^α or the same dimension and fewer components, so the relation is antisymmetric.

The second clause also follows from the definition; the third from Theorem 3.5.2; and the fourth is the definition of 'principal'. □

There is scope for confusion here: if $\beta \prec \alpha$ then H^β is 'smaller' than H^α but the stratum M^β is 'larger' than M^α. If A is a set of orbit types we say that A is *closed* if $\alpha \in A$ and $\beta \prec \alpha$ imply $\beta \in A$: thus the set of orbit types with M^α non-empty is always closed. If A is a closed set of orbit types, then $\bigcup_{\alpha \in A} M^\alpha$ is an open subset of M. We observe that if α and β are distinct orbit types with the same class of isotropy groups $H^\alpha = H^\beta$, then neither precedes the other.

We return to the problem of equivariant embedding in a linear G-space L. We see from Theorem 3.5.2 that there are only finitely many orbit types for the action of G on L (they all appear in any neighbourhood of the origin), and it follows that there are only finitely many orbit types on any G-submanifold of L: thus the hypothesis in Theorem 3.4.6 that M be compact cannot simply be removed. However we do have

Theorem 3.5.7 *For any smooth action of a compact group G on M with only finitely many orbit types, there exist a linear G-space E and a G-equivariant embedding $M \to E$.*

Proof Write T for the set of orbit types α. As before, we cover $G\backslash M$ by a set of nice coordinate neighbourhoods $\{U_i \mid i \in I\}$, so there is a map $a : I \to T$ and for each $i \in I$, $U_i = j_i(G \times_{H_{a(i)}} D_{a(i)})$ coming from a map $\varphi_i : D_{a(i)} \to G\backslash M$, where $D_{a(i)}$ is a disc in the $H_{a(i)}$-space $E_{a(i)}$. Define $\Phi_i : G\backslash M \to \mathbb{R}$ by $\Phi_i(\varphi_i(P)) = Bp(2 - |P|)$ for $P \in \mathring{D}_\alpha(3)$, $\Phi_\alpha(Q) = 0$ otherwise.

In the former proof, for each α we chose an H_α-linear embedding $f_\alpha : E_\alpha \to W_\alpha$ with W_α a linear G-space, and a linear G-space U_α and $u_\alpha \in U_\alpha$ with $G_{u_\alpha} = H_\alpha$, and then formed the G-equivariant embedding $\psi_\alpha : G \times_{H_\alpha} E_\alpha \to W_\alpha \oplus U_\alpha$. Here we need to separate the different φ_i for the different i with the same $a(i) = \alpha$; the difficulty is that these neighbourhoods U_i overlap.

The images $\varphi_i(D_\alpha \cap E_\alpha^{H_\alpha})$ with $a(i) = \alpha$ give an open covering of B^α. Since B^α is finite dimensional, it follows from Proposition A.2.9 that this covering has a finite dimensional refinement. More precisely, there exist an open covering $\{S_j \mid j \in J\}$ of B^α, with each S_j contained in $\varphi_i(D_\alpha \cap E_\alpha^{H_\alpha})$ for some $i(j)$, and a map $d : J \to \{0, \ldots, N\}$ such that if $d(j) \neq d(j')$, then $\bar{S}_j \cap \bar{S}_{j'} = \emptyset$. Choose an open set C_j in D_α such that $C_j \cap E_\alpha^{H_\alpha} = \varphi_i^{-1}(S_j)$; then by shrinking the C_j if necessary, we may suppose that if $d(j) \neq d(j')$, then also $\overline{\varphi_{i(j)}(C_j)} \cap \overline{\varphi_{i(j')}(C_{j'})} = \emptyset$.

Now for each r with $0 \leq r \leq N$ we define a map $F_{\alpha,r} : E_\alpha \times d^{-1}(r) \to W_\alpha \times \mathbb{R}$ as follows. Choose an injective map $n : d^{-1}(r) \to \mathbb{Z}$ and set

$$F_{\alpha,r}(x, s) := (f_\alpha(x), \Phi_\alpha(x) + 3n(s)).$$

Since the f_α are injective and the values of Φ_α lie in $[0, 1]$, this is injective; since Φ_α is invariant, this is equivariant (where G acts trivially on $\mathbb{R}$). As above we can now form a G-equivariant embedding $\Psi_{\alpha,r} : G \times_{H_\alpha} (\bigcup_{d(j)=r}(C_j \times \{j\})) \to W_\alpha \oplus U_\alpha \oplus \mathbb{R}$.

Since we now have a finite set of embeddings, we can piece them together as before. □

There is also an equivariant embedding theorem when G is not compact. It is clear that some restriction on G is needed, as there exist Lie groups with no faithful finite dimensional linear G-space. A notorious example is the so-called Weil–Heisenberg group, which can be considered as the group of matrices $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ with $a,\ c \in \mathbb{R}$ and $b \in \mathbb{R}/\mathbb{Z}$.

Palais gives a result in [119] using only the hypothesis that there exists a faithful finite dimensional linear G-space.

If A^α is a stratum of the orbit type stratification of M, the quotient $B^\alpha := G\backslash A^\alpha$ is a smooth manifold. We next give a model for the action of G in a neighbourhood of A^α.

Theorem 3.5.8 *A neighbourhood of A^α in M is equivariantly diffeomorphic to a bundle over B^α with fibre $G \times_H E^\alpha$.*

Proof By Theorem 3.4.4 since the action is proper we can choose a G-invariant metric for M: this induces metrics on the submanifold A^α and, by Proposition A.3.6, on $G\backslash M$; it also induces a reduction of the structure group of the normal bundle N^α to the orthogonal group.

By Proposition 2.3.1, the exponential map e^α for the normal bundle $\mathbb{N}(M/A^\alpha)$ has non-zero Jacobian along the zero cross-section of N^α, so is a local diffeomorphism at A^α; since the metric is invariant, e^α is equivariant. We now follow the proof of Theorem 2.3.3: we know some neighbourhood of the zero cross-section A^α is embedded, but need an invariant one.

In the model given by Theorem 3.3.5, we can choose the slice at x as the image of the normal space by the exponential map; by equivariance, the same holds at each point on the orbit $G.x$. In the model $G \times_H (E^H \oplus E^\alpha)$, we can identify A^α with $G \times_H E^H$; by Lemma 3.5.1 the normal space at x to A^α can be identified with E^α, the normal bundle N^α is identified with the projection with kernel E^α; and e^α is represented by the identity map.

Now factor out G: $e^\alpha : N^\alpha \to M$ yields $G\backslash e^\alpha : G\backslash N^\alpha \to G\backslash M$. Near the zero cross-section $G\backslash A^\alpha = B^\alpha$ this too is represented by the identity map (of $E^H \times H\backslash E^\alpha$); thus it is a local homeomorphism. It follows from Corollary A.2.6 that there is a neighbourhood W of B^α on which $G\backslash e^\alpha$ is an embedding. Hence also the restriction of e^α to $Z := q^{-1}(W)$ is an embedding.

We have a function $\Phi : N^\alpha \to \mathbb{R}$ measuring the length of the normal vector. Define $f : A^\alpha \to \mathbb{R}$ by

$$f(x) = inf\{\rho(x, y) + \Phi(z) \mid z \in (N^\alpha \setminus Z), \pi(z) = y\}.$$

Since each $x \in A^\alpha$ has a neighbourhood disjoint from Z, we have $f(x) > 0$. It follows from the definition that $|f(x) - f(y)| \leq \rho(x, y)$, so f is continuous; and clearly f is invariant. By Proposition 3.4.3 there is a positive smooth invariant function F on A^α with $F(x) \leq f(x)$ for all x. The proof is completed, as for Theorem 2.3.3, by writing down a diffeomorphism of the bundle with fibre the unit disc to the submanifold $\Phi(z) \leq F(\pi(z))$. □

3.6 Actions with few orbit types

We can decompose a G-manifold M into orbit types and then build it up piece by piece. We begin with a stratum A^α of least dimension: this is a compact smooth manifold, and has a neighbourhood N^α given by a bundle over A^α with fibre E^α. The next piece A^β overlaps this bundle; the details are made precise by the local structure theorem. We now explore how M is built up in the case when there are at most two strata.

For the principal orbit type α we have $E^\alpha = 0$, A^α is open in M, and is equivariantly diffeomorphic to a bundle over B^α with group G and fibre G/H^α.

If there is only one stratum, it is necessarily principal: the orbit map $M \to G\backslash M$ is a fibration with fibre G/H. To regard this as a bundle, first consider the submanifold $M^{\langle H \rangle} = M^H$ of points with isotropy subgroup equal to H. This meets all orbits, and $g.M^H$ is equal to M^H if $g^{-1}Hg = H$, and is disjoint from M^H otherwise. The elements $g \in G$ satisfying $g^{-1}Hg = H$ form a subgroup of G, called the *normaliser* of H in G and denoted $N_G(H)$. The action of $N_G(H)$ on M^H factors through $N_G(H)/H$ (since H acts trivially here). We thus see that $N_G(H)/H$ acts freely on M^H and the quotient is just $G\backslash M$, so we have a principal bundle.

If in particular M is a sphere, we have a fibration of a sphere. The possibilities for fibrations of spheres are strictly limited: the standard examples are the Hopf fibrations $S^1 \to S^{2n-1} \to P^{n-1}(\mathbb{C})$, $S^3 \to S^{4n-1} \to P^{n-1}(\mathbb{H})$, and $S^7 \to S^{15} \to S^8$. It follows from a result of Browder [29] that for any non-trivial fibration of a sphere with connected fibres, the fibre is homotopy equivalent to S^1, S^3, or S^7. In the case of manifolds it follows from the generalised Poincaré conjecture (see §5.6 and discussion following) that the fibre is homeomorphic to a sphere and, except perhaps for S^7, diffeomorphic.

In the present situation we can be even more precise.

Theorem 3.6.1 *If H is a non-trivial compact connected Lie group, acting on S^n ($n \geq 2$) with just one weak orbit type, then either (a) the action is transitive or (b) H has rank 1 and the action is free.*

We refer to Borel [20, p 185] for the proof which, after several preliminaries, is homological in nature, so Borel's result is stated in more general terms.

It was shown by Poncet [122] that the only faithful transitive actions on spheres are the classical actions of SO_n and O_n on S^{n-1}, U_n and SU_n on S^{2n-1}, and Sp_n on S^{4n-1}; also three exceptional cases $S^6 = G_2/SU_3$, $S^7 = Spin_7/G_2$, and $S^{15} = Spin_9/Spin_7$.

Now consider groups H acting freely on spheres; first suppose H finite. Then (see [35, Chapter XII]) H has periodic cohomology, and hence all Sylow subgroups of G are cyclic or generalised quaternionic. The classification up to isomorphism of such groups is known: see [170], which also gives the latest known results about the classification of these actions.

In particular, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ cannot act freely on a sphere, hence neither can a torus $S^1 \times S^1$. Thus if H acts freely, it has rank at most 1. The only connected groups of rank 1 are S^1, S^3, and SO_3, and SO_3 has a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, so is excluded.

If $H \neq H_0 = S^1$ and $g \in H \setminus H_0$, conjugation of H_0 by g is an automorphism, hence is either the identity or the map $x \to x^{-1}$. If g centralises H_0, the subgroup $\langle H_0, g\rangle$ is isomorphic to a direct sum $S^1 \oplus \mathbb{Z}_k$ for some k, hence contains a subgroup $\mathbb{Z}_k \oplus \mathbb{Z}_k$; hence this case does not occur. Thus H/H_0 has order 2 and H is isomorphic to the subgroup $S^1 \cup jS^1$ of S^3. (This group can also be identified with the group Pin_2 of [15].)

If $H_0 = S^3$ and $g \in H \setminus H_0$, $g^{-1}S^1 g$ is a circle subgroup of S^3, hence conjugate in S^3 to S^1, so for some $h \in S^3$ gh normalises S^1. Arguing as above now yields a contradiction.

There are many free actions of S^1 on spheres; the classification is described, for example, in §14C of my surgery book [167]; a similar analysis holds for actions of S^3. The same methods could be applied to the $S^1 \cup jS^1$ case, but to the author's knowledge this has not been attempted.

We next consider the case of just two orbit types α (principal) and β. Choose $x \in M^\beta$ and set $H := G_x$ ($= H^\beta$). By Theorem 3.3.5, a neighbourhood of $G.x$ is equivariantly diffeomorphic to $G \times_H E$, where H acts orthogonally on E ($= E^\beta$) and the only fixed point is the origin. Thus there is only one orbit type for the action of H on the unit sphere S^{k-1} in E (where we choose an isomorphism of E with $\mathbb{R}^k$) and we can apply the classification just discussed; so by Theorem 3.6.1, either (a) H acts transitively on S^{k-1} or (b) H has rank at most 1.

In the present situation these H-spaces are the restrictions to S^{k-1} of linear H-spaces, so the list of cases is shorter. For (b) if H is finite a complete list of fixed point free representations (and of groups) was given by Wolf [182] (the list is repeated in a simpler notation in [170]). For $H = S^1$ any fixed point free representation is isomorphic (over $\mathbb{R}$) to the action on $\mathbb{C}^n$ for some n; and for S^3 and $S^1 \cup jS^1$ to $\mathbb{H}^n$.

By Theorem 3.5.8, a neighbourhood $N(M^\beta)$ of M^β in M is equivariantly diffeomorphic to a bundle over B^β with fibre $G \times_H E^\beta$: here M^β itself corresponds to choosing $0 \in E^\beta$. Choose $y \in M^\alpha$ to lie in the fibre over x corresponding to a point in $E^\beta \setminus \{0\}$, and set $K := G_y\,(= H^\alpha)$.

The isomorphism $E^\beta \to \mathbb{R}^k$ induces $E^\beta \setminus \{0\} \cong S^{k-1} \times]0, \infty[$. Thus we can identify $M^\alpha \cap N(M^\beta)$ with the bundle over B^β with fibre $(G \times_H S^{k-1}) \times]0, \infty[$. Factoring out G gives an identification of $B^\alpha \cap N(B^\beta)$ with the bundle over B^β with fibre $(H \backslash S^{k-1}) \times]0, \infty[$: note that this projection indeed has fibre G/K. Now B is the union of B^α and $N(B^\beta)$ modulo this identification on the intersection. Correspondingly, M is the union of M^α and $N(M^\beta)$ modulo an identification on the intersection of bundles with fibre G/K over the above. In principle, this reduces the classification problem to a problem about manifolds (with no group action) and bundles over them.

In case (a), H acts transitively on S^{k-1}: here $B = G\backslash M$ is a smooth manifold with boundary: B^β is the boundary and B^α its complement; the identification takes place over a collar neighbourhood of the boundary. This necessarily occurs if a principal orbit has codimension 2. Here some classifications have been effectively done. If also $M = \mathbb{R}^m$, it was shown by Borel (see [20, XIV]) that G has a fixed point P, so the whole action is modelled by the induced linear action on the tangent space at P.

Interesting examples were given by Bredon [25]. Begin with the linear action of SO_n on $\mathbb{R}^n \oplus \mathbb{R}^n$. Then (see example (vb) below) there are just two orbit types; the isotropy subgroups are SO_{n-1} and SO_{n-2}. Next restrict to $D^n \times S^{n-1}$. For $x \in S^{n-1}$ define $\theta_x \in O_n$ to be the reflection in the radius through x. Then the map of $S^{n-1} \times S^{n-1}$ given by $\psi_k(x, y) := ((\theta_x\theta_y)^k x, (\theta_x\theta_y)^k y)$ is a diffeomorphism equivariant for the action of SO_n; it acts on $H_{n-1}(S^{n-1} \times S^{n-1})$ by the identity (if n is odd) and by the matrix $\begin{pmatrix} 2k+1 & 2k \\ 2k & 1-2k \end{pmatrix}$ (if n is even). Now glue two copies of $D^n \times S^{n-1}$ together using the diffeomorphism ψ_k. We obtain a closed manifold M with an action of SO_n; it still has just the two orbit types. If n is odd, M has the homology of S^{2n-1}; if n is even, $H_{n-1}(M) \cong \mathbb{Z}_{2k+1}$. For $n = 3$ this coincides with the manifold denoted M_{2k+1} in §7.8.

In [26], Bredon goes on to give a classification of actions of compact Lie groups G on manifolds with the homology of S^m and just two orbit types, one with orbits of codimension 2, and one with orbits of lower dimension. He proves that $m = 2n - 1$ is odd and either $G = SO_n$ with one of the above actions (so if n is even we have $k = 0$); or we have the action restricted to the subgroup $Spin_7$ of SO_8 or the subgroup G_2 of SO_7. We do not give the proof: a large part of it is devoted to identifying the possibilities for the group G and the isotropy subgroups H^α and H^β.

3.7 Examples of smooth proper group actions

Most of the following examples are linear actions; each of these induces also an action on the unit sphere in the vector space, also one on the corresponding projective space. Write, for $n \in \mathbb{N}$, $\zeta_n := e^{2i\pi/n}$.

(ia) The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permutation of the coordinates. For each partition $\lambda : n = \lambda_1 + \lambda_2 + \ldots + \lambda_r$ (with $\lambda_1 \leq \lambda_2 \ldots$) there is an orbit type with isotropy subgroup $\prod_i \mathfrak{S}_{\lambda_i}$. The orbit type containing $(x_1, \ldots, x_n)$ is given by the partition defined by $i \sim j \Leftrightarrow x_i = x_j$. For a principal orbit, the x_i are distinct; each $\lambda_i = 1$; and the isotropy group is trivial.

The orbit space $\mathbb{R}^n/\mathfrak{S}_n$ can be identified with the subset $x_1 \leq x_2 \leq \ldots \leq x_n$.

(ib) If we replace $\mathbb{R}^n$ by $\mathbb{C}^n$ in example (i), the description of orbit types is the same, but now the orbit space $\mathbb{C}^n/\mathfrak{S}_n$ is isomorphic (using elementary symmetric functions) with $\mathbb{C}^n$.

(ic) The orthogonal group O_n acts on the space of symmetric $n \times n$ matrices by $P.A := PAP^t$ (where the affix t denotes transpose). Each orbit contains a diagonal matrix; to calculate the isotropy group we partition the eigenvalues (as above) into sets of equal ones: say this gives $n = \sum \lambda_i$. Then the isotropy group is (conjugate to) $\prod_i O_{\lambda_i}$. Principal orbits occur where all eigenvalues are distinct: here the isotropy group is $O_1^n = \{\pm 1\}^n$. The orbit space is as in (i) the simplicial cone $x_1 \leq x_2 \leq \ldots \leq x_n$. In this example, we can interpret the corresponding projective space as the space of (central) quadrics.

(id) A similar example is the action of the unitary group U_n on the set of self-adjoint $n \times n$ matrices over $\mathbb{C}$. Here the eigenvalues can be any non-zero complex numbers; the orbit space is all of $\mathbb{C}^n$.

(ie) The unitary group U_n acts on itself by conjugation: $x.y = xyx^{-1}$. As any unitary matrix is conjugate to a diagonal matrix, we again have a similar situation: here the eigenvalues satisfy $|\lambda| = 1$.

(iia) The circle group S^1 acts on the sphere S^2 by rotations, say $e^{i\theta}.(x, y, z) = (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, z)$. We have two fixed points at the poles $(0, 0, \pm 1)$, and the remaining orbits are principal, with trivial isotropy group. We can identify the orbit space with $[-1, 1]$ and $q : S^2 \to S^1\backslash S^2$ with $z : S^2 \to [-1, 1]$.

(iib) The group S^3 acts on itself by conjugation. The isotropy group of ± 1 is S^3; of other points in S^1 is S^1 and at other points is conjugate to S^1. The orbit space is $[-1, 1]$.

(iii) For any sequence $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ of integers, the circle group $S^1 := \{t \in \mathbb{C} \mid |t| = 1\}$ acts on $\mathbb{C}^{n+1}$ by $t.(z_0, z_1, \ldots, z_n) = (t^{a_0} z_0, t^{a_1} z_, \ldots, t^{a_n} z_n)$; the induced action on $P^n(\mathbb{C})$ is thus $t.(z_0 : z_1 : \ldots : z_n) = (t^{a_0} z_0 : t^{a_1} z_1 : \ldots : t^{a_n} z_n)$.

A point z is fixed under $t \in S^1$ if and only if, for all values of i with $z_i \neq 0$, the corresponding t^{a_i} are equal. Thus if t has multiplicative order r, we need the a_i for these i to be congruent mod r to each other; and, for the isotropy subgroup to have order r, no more. The isotropy action is then given by the $t^{a_j - a_i}$ for the j with $a_j \neq a_i$.

(iva) The quaternion group of order $4n$ has a presentation $\{t, u \mid t^{2n} = 1,\ u^2 = t^n,\ u^{-1} t u = t^{-1}\}$. There is a semi-free action on $\mathbb{C}^2$ with $t.(x_1, x_2) = (\zeta_{2n} x_1, \zeta_{2n}^{-1} x_2)$, $u.(x_1, x_2) = (x_2, -x_1)$. The ring of invariants is generated by $Y = x_1^{2n} + x_2^{2n}$, $Z = x_1^2 x_2^2$ and $W = x_1 x_2 (x_1^{2n} - x_2^{2n})$; these have the unique syzygy $Y^2 Z - W^2 = 4Z^{n+1}$.

(ivb) Let $G = \{u, v \mid u^7 = v^3 = 1, v^{-1} u v = u^2\}$. The subgroup $U = \langle u \rangle$ has a 1-dimensional representation $u \to \zeta_7$. The induced representation of G takes u to the diagonal matrix $(\zeta_7, \zeta_7^2, \zeta_7^4)$ and v to the matrix which cyclically permutes the coordinates. Thus v fixes the line $x_1 = x_2 = x_3$.

(ivc) Let $G = \{u, v \mid u^7 = v^9 = 1, v^{-1} u v = u^2\}$. The subgroup $U = \langle u, v^3 \rangle$ is cyclic and has a 1-dimensional representation $u \to \zeta_7,\ v^3 \to \zeta_3$. In this case, the induced representation of G on $\mathbb{C}^3$ is semi-free, and we have a free action of G on the unit sphere S^5.

(va) Consider the natural action of $SO_n \subset SO_{n+r}$ on $S^{n+r-1} \subset \mathbb{R}^n \times \mathbb{R}^r$. The isotropy subgroup of (x, y) is trivial, and the orbit an $(n-1)$-sphere unless $x = 0$, when we have a fixed point, so the action is semi-free. The orbit space is homeomorphic to D^r.

(vb) The diagonal subgroup $SO_n \subset SO_n \times SO_n$ acts on $S^{2n-1} \subset \mathbb{R}^n \times \mathbb{R}^n$. For $(x, y) \in S^{2n-1}$, if x and y are independent we have a principal orbit; the isotropy subgroup is (conjugate to) SO_{n-2} and the orbit a Stiefel manifold $V_{n,2}$. If x and y are linearly dependent, the isotropy subgroup is SO_{n-1} and the orbit S^{n-1}. The orbit space is homeomorphic to D^2.

(via) The group $SL_2(\mathbb{R})$ acts on the upper half-plane $\mathcal{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

This action is transitive, and the isotropy subgroup of i is the rotation group SO_2: thus we have a diffeomorphism of $SO_2 \backslash SL_2(\mathbb{R})$ on $\mathcal{H}^2$ and the action is proper. The action is not effective: $-I$ acts trivially, so the action factors through $PSL_2(\mathbb{R})$.

(vib) The restriction of the action in (via) to an action of $SL_2(\mathbb{Z})$ is thus also proper. There are only two non-principal orbits for this action: they are the orbits of i, with isotropy group of order 4, and of a cube root ζ_3 of 1, with isotropy group of order 6. The orbit space is usually identified with a sphere S^2 with one point deleted (puncture).

3.8 Notes on Chapter 3

§3.1 and §3.2 contain little more than basic definitions and terminology.

There are many introductory books on these subjects: for Lie groups: [37], for example, has an algebraic approach; and [6] gives an excellent account for topologists.

A good general reference for (compact) differentiable group actions is [27]. An early account is in [20], which is a good source for early references.

§3.3: Although slices in the sense of a submanifold transverse to an orbit had appeared long before, the use of 'slice' in the precise sense needed here perhaps appeared first in Montgomery and Yang [104], where existence is proved for actions of compact groups; for proper actions the result is due to Palais [119].

The concept of proper group action developed from special cases and seems to have been first formalised about 1960. It appears in the later revisions of Bourbaki (not yet in [24]): the first reference I have is [119]. (The volume [20] only considers actions of compact groups.)

We commented in §1.6 that (M4) was equivalent to various other conditions. A similar situation exists here. It is shown in Proposition A.3.1 that the action $\phi : G \times X \to X$ is proper if and only if

(i) the map $(\phi, \pi) : G \times X \to X \times X$ (where π denotes the projection) is a proper map;

(ii) (ϕ, π) is closed and all isotropy groups G_x are compact;

(iii) for any compact subsets $K, L \subseteq X$, $T_{K,L} := \{g \in G \mid g.K \cap L \neq \emptyset\}$ is compact;

further equivalent conditions are mentioned in Proposition 3.3.1:

(iv) for any compact subsets $K, L \subseteq X$, $\{g \in G \mid g.K \cap L \neq \emptyset\}$ is compact;

(v) the orbit space $G\backslash X$ is Hausdorff.

§3.4 Most of the results in this section are fairly easy for actions of compact groups; the extension to proper actions is again in [119], though his emphasis is on continuous actions on metric spaces.

§3.5 Several results on weak orbit type appear in [20]. The Principal Orbit Theorem is due to Montgomery and Yang [105]. However, orbit types in our sense are what is required in the study of cobordism of group actions. The Atiyah–Singer fixed point theorem gives formulae expressed in terms of sums where the character of the representation of H on E plays a role. The local finiteness theorem is due to Mostow [112]. The earlier literature does not explicitly mention the stratification.

There was an explosion of papers on group actions in the 1960s: see, for example, the conference proceedings [110].