Appendix A

Topology

A.1 Definitions

A *topology* on a set X is a collection $\mathcal{U}$ of subsets, called *open sets*, such that $X \in \mathcal{U}$, the union of any subfamily of $\mathcal{U}$ belongs to $\mathcal{U}$, and the intersection of two elements of $\mathcal{U}$ also belongs to $\mathcal{U}$. A topology can be defined by prescribing a set $\mathcal{V}$ of subsets of X to be a 'subbase' of open sets: then define $\mathcal{U}$ to consist of arbitrary unions of finite intersections of elements of $\mathcal{V}$. A set $\mathcal{W}$ is a *base* of open sets if every open set is a union of elements of $\mathcal{W}$.

A subset F of X is *closed* if its complement $X \setminus F$ is open. If A is any subset of X (in particular, if A is a point) a subset V of X is a *neighbourhood* of A if there is an open set U with $A \subseteq U \subseteq V$.

If $Y \subset X$ is a subset of a space X with a topology $\mathcal{U}$, the subspace topology on Y is given by taking as open sets the $U \cap Y$ with $U \in \mathcal{U}$.

A topology is said to be *Hausdorff* if for any $x_1 \neq x_2 \in X$ we can find $U_1, U_2 \in \mathcal{U}$ with $x_1 \in U_1$, $x_2 \in U_2$ and U_1, U_2 disjoint, i.e. $U_1 \cap U_2 = \emptyset$. This is a rather weak condition, and all spaces we will consider are Hausdorff. In a Hausdorff space, each point is a closed set. There are also stricter separation conditions (which hold for smooth manifolds): a topology is *completely regular* if any point x and closed set F not containing it are contained in disjoint open sets, and *normal* if disjoint closed sets F_1, F_2 are contained in disjoint open sets $U_1, U_2 \in \mathcal{U}$.

A mapping $f : X \to Y$ between two topological spaces is *continuous* if whenever V is open in Y, $f^{-1}(V)$ is open in X. It is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous. We call f an *embedding* if it is injective and gives a homeomorphism between X and $f(X)$ with the subspace topology.

An important condition on a topology is the existence of a countable base of open sets. This holds for $\mathbb{R}^n$ since we can take the balls with rational radii and centres having rational coordinates.

A set $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of subsets of X is a *covering* if $\bigcup_{\alpha \in A} U_\alpha = X$; it is an *open covering* if each U_α is open in X, and it is *locally finite* if each point of X has a neighbourhood intersecting only a finite number of the U_α. A covering $\mathcal{V} = \{V_\beta \mid \beta \in B\}$ of X refines $\mathcal{U}$ if for each β there is an α such that $V_\beta \subseteq U_\alpha$.

The space X is *compact* if for every open covering $\mathcal{U}$, a finite subset of $\mathcal{U}$ already covers X. It is *locally compact* if every neighbourhood of a point contains a compact neighbourhood. Since any point has a neighbourhood which is a disc, any manifold is locally compact. A space is *paracompact* if every open covering has a locally finite refinement by an open covering.

Any compact subset K of a Hausdorff space X is closed. For if $x \notin K$, then for each $k \in K$, x and k have disjoint neighbourhoods U_x, V_x. The $K \cap V_x$ form an open cover of K, so there is a finite subcover. The intersection of the corresponding U_x is an open neighbourhood of x disjoint from K.

If $\{U_a\}$ is a locally finite family of subsets of X and $K \subset X$ is compact, then K has a neighbourhood intersecting only finitely many of the U_a. For each point $k \in K$ has such an open neighbourhood N_k; we may choose a finite subset of the N_k which cover K, and their union is a neighbourhood of K with the desired property.

If $f : X \to Y$ is continuous and $K \subset X$ is compact, the image $f(K)$ is compact. For if $\{U_\alpha\}$ is an open cover of $f(K)$ we can write $U_\alpha = f(K) \cap V_\alpha$ with V_α open in Y. Since f is continuous, $f^{-1}(V_\alpha)$ is open in X, and these give an open covering of K. Taking a finite subcovering here gives a finite subcover of $\{U_\alpha\}$.

Thus if K is a compact space and $f : K \to Y$ is continuous, f takes closed sets to closed sets, so if f is bijective it is a homeomorphism; if f is injective, it is an embedding.

Lemma A.1.1 *If X is a locally compact space any neighbourhood of a compact set $K \subset X$ contains a compact neighbourhood of K.*

Proof Let U be the given neighbourhood of K: then U is a neighbourhood of each $x \in K$, so we can find neighbourhoods A_x, B_x, C_x of x in X with $C_x \subset B_x \subset A_x \subset U$ and A_x, C_x open and B_x compact. Since the open sets C_x cover the compact set K, there is a finite subcover $\{C_{x_n}\}$. The (finite) union of the B_{x_n} is compact and contains the open neighbourhood $\bigcup_n C_{x_n}$ of K. □

Taking K as a point $x \in X$, any open neighbourhood A_x of $x \in X$ contains a compact neighbourhood B_x, which contains an open neighbourhood C_x: and so on.

The product $\prod A_i$ of a family of spaces has a topology defined by the subbase consisting of products $\prod U_i$ with U_i open in A_i for each i and $U_i = A_i$ for all but finitely many. If each A_i is compact, so is $\prod_i A_i$.

The *inverse limit* $\varprojlim A_i$ of a sequence $A_{i+1} \xrightarrow{\alpha_i} A_i$ $(i \geq 1)$ is defined to be the subset of the product $\prod A_i$ with $\alpha_i(x_{i+1}) = x_i$ for each i. If the A_i are topological spaces, it inherits a topology as a subspace of the product.

A.2 Topology of metric spaces

A *metric* on a set X is a mapping $\rho : X \times X \to \mathbb{R}$ such that $\rho(x, y) \geq 0$ for all $x, y \in X$, $\rho(x, y) = 0$ if and only if $x = y$, and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all x, y and $z \in X$. This defines a topology with a base consisting of the sets $\{x \mid \rho(x, y) < d\}$ for all $y \in X$, $d > 0$. Equivalently, a subset $U \subseteq X$ is open if, for each $x \in U$, there exists $\varepsilon > 0$ such that $\rho(x, y) < \varepsilon$ implies $y \in U$.

We have seen in Theorem 2.1.1 that smooth manifolds are metric as topological spaces.

The prime example of a metric space is $\mathbb{R}^n$, with points $x = (x_1, \ldots, x_n)$ and distance function $\rho(x, y) = \|x - y\| = \sqrt{\sum_1^n (x_i - y_i)^2}$. The basic examples of topological spaces are subsets of $\mathbb{R}^n$ with the topology given by the induced metric. We are not concerned with arbitrary subsets: more typical are polyhedra, or subsets defined by vanishing of a certain number of polynomial functions. However, we will need the general terminology as we will also need to consider spaces of mappings.

In a metric space X, we define a sequence $\{x_n\}$ of points to converge to a limit x_∞ if $\rho(x_n, x_\infty) \to 0$ as $n \to \infty$. The limit, if it exists, is unique, since if y were another limit we would have $\rho(y, x_\infty) = 0$. We call a metric space X *complete* if it satisfies Cauchy's convergence condition, namely that for any sequence $x_n \in X$ such that $\rho(x_m, x_n) \to 0$ as $m, n \to \infty$ there exists a limit point $x_\infty \in X$ such that $\rho(x_n, x_\infty) \to 0$ as $n \to \infty$.

For metric spaces X, topological conditions can be expressed in terms of convergence of sequences; for example, $f : X \to Y$ is continuous iff for all $x_i \to x \in x$ we have $f(x_i) \to f(x)$.

If X is a metric space, $x \in X$, $F \subseteq X$ is closed, and $x \notin F$, then x has a neighbourhood disjoint from F, so there exists $\varepsilon > 0$ such that $y \in F$ implies $\rho(x, y) \geq \varepsilon$, so $\rho(x, F) := \inf\{\rho(x, y) \mid y \in F\}$ is strictly positive. For any $A \subset$

X, $\rho(x, A) = 0$ if and only if x is in the closure of A if and only if there is a sequence $a_i \in A$ with $a_i \to x$.

Clearly $|\rho(x, F) - \rho(y, F)| \leq \rho(x, y)$, so the map $x \mapsto \rho(x, F)$ is continuous. If F and F' are disjoint closed sets, there are disjoint open neighbourhoods $G := \{x \mid \rho(x, F) < \rho(x, F')\}$ of F and similarly for G'. Hence any metric space is normal. It may be that $\rho(F, F') = 0$: for example, consider $F = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ and $F' = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. However if K is compact and disjoint from F we have $\rho(F, K) > 0$, for the image of K by the continuous map $x \mapsto \rho(F, x)$ is a closed subset of $\mathbb{R}$ not containing $\{0\}$. Even if $\rho(F, F') = 0$, the formula $s(P) := \rho(P, F)/(\rho(P, F) + \rho(P, F'))$ defines a continuous map $s : X \to I$ with $s(F) = 0$ and $s(F') = 1$.

A metric space K is compact if and only if every sequence has a convergent subsequence. To see this, first observe that if $x_i \to y$, then the set whose elements are the x_i and y is compact, for given any open cover, one of the open sets of the cover contains y, hence all but finitely many of the x_i. Now if $\{x_i\}$ has no convergent subsequence, the set $\bigcup_i \{x_i\}$ is closed, its complement U is open, and $\{U \cup \{x_i\}\}$ is an open cover of K with no finite subcover. Conversely, if there is a cover with no finite subcover, there is a countable one $\{U_r\}$ and if we choose $x_n \notin \bigcup_{r \leq n} U_r$ if a subsequence converged to $y \in K$ we would have $y \in U_n$ for some n and then U_n would contain all but finitely many of the subsequence.

From this, or directly, it follows that the direct product of two, or indeed of any family of compact spaces is compact.

We will call a sequence $\{x_n\}$ with no convergent subsequence *discrete*. If $\{x_n\}$ is a discrete sequence, the set having these as elements is a closed set.

Lemma A.2.1 *Let $f : A \times B \to C$ be a continuous map of compact metric spaces. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(b, b') < \delta$ implies that $\rho(f(a, b), f(a, b')) < \varepsilon$ for all $a \in A$.*

Proof Suppose not. Then there exist $\varepsilon > 0$ and sequences b_n, $b'_n \in B$ with $\rho(b_n, b'_n) < \frac{1}{n}$ and $a_n \in A$ with $\rho(f(a_n, b_n), f(a_n, b'_n)) \geq \varepsilon$. In view of compactness, these all have convergent subsequences; passing to these, we may suppose $b_n \to b$, $b'_n \to b'$ and $a_n \to a$. It follows that $\rho(b, b') = 0$, so $b = b'$ and by continuity that $\rho(f(a, b), f(a, b')) \geq \varepsilon$, a contradiction. □

The notion of compactness for spaces is accompanied by the important notion of properness for maps.

Lemma A.2.2 *The following conditions on a map $f : X \to Y$ of metric spaces are equivalent:*

(i) f is closed and for each $y \in Y$, $f^{-1}(y)$ is compact;
(ii) every sequence $x_i \in X$ such that $f(x_i)$ converges has a convergent subsequence;
(iii) for each compact subset K of Y, $f^{-1}(K)$ is compact.

A map is said to be *proper* if it satisfies these conditions.

Proof $(i) \Rightarrow (ii)$ Suppose (i) holds, that $\{x_n\}$ is discrete, but that $f(x_n)$ converges to a limit y. Since $C = \{x_n \mid n \in \mathbb{N}\}$ is closed, so is $f(C)$, and since $f(x_n) \to y$, $y \in C$. The same argument shows that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ we have $y = f(x_{n_k})$ for some k. Thus $y = f(x_n)$ for all but finitely many n; hence $f^{-1}(y)$ contains a discrete sequence, contradicting its compactness.

$(ii) \Rightarrow (iii)$ Suppose (ii) holds, that $K \subset Y$ is compact, and that $f^{-1}(K)$ is not. Then $f^{-1}(K)$ contains a discrete sequence $\{x_n\}$. Since $\{f(x_n)\}$ lies in the compact set K, it has a convergent subsequence. It follows from (ii) that $\{x_n\}$ has a convergent subsequence, so is not discrete.

$(iii) \Rightarrow (i)$ It follows at once from (iii) that preimages of points are compact. Let C be closed in X and $f(x_n)$ be a sequence of points of $f(C)$ converging to a limit y. Then the set K consisting of y and the points $f(x_n)$ is compact, so by (iii) $f^{-1}(K)$ is compact. The sequence x_n of points in this compact set has a convergent subsequence x_{n_k} with limit x, say; as C is closed, $x \in C$. Thus $f(x_{n_k}) \to f(x)$; hence $y = f(x) \in f(C)$. □

It follows from the characterisation (iii) that the composite of two proper maps is proper. Also since the product of compact spaces is compact, for any X and compact K, the projection $K \times X \to X$ is proper. Since every closed subset of a compact space is compact, any continuous map $f : K \to Y$ with K compact is proper.

Lemma A.2.3 *A proper injective map $f : X \to Y$ of Hausdorff spaces is an embedding.*

Proof Replacing Y by $f(X)$, we may suppose f bijective. But now f takes closed sets to closed sets, hence also open sets to open sets, so is a homeomorphism. □

We now give some results for metric spaces which are useful for proving existence of embeddings when we weaken the requirement of compactness.

Lemma A.2.4 *(i) Let Y be a metric space, X a closed subset. For any open neighbourhood U of X in Y, there is a positive continuous function f on X such that if $x \in X$ and $\rho(x, y) < f(x)$, we have $y \in U$.*

(ii) If X is a compact subset of the metric space Y, any open neighbourhood U of X in Y contains an ε-neighbourhood for some $\varepsilon > 0$.

Proof (i) Define $f(x) = \rho(x, Y \setminus U)$: then $|f(x) - f(x')| \leq \rho(x, x')$, so f is continuous: it is non-zero and satisfies the condition.

(ii) Take $\varepsilon = \inf f$, where f is given by (i). □

We may apply this result in particular when $Y = X \times X$ with X embedded as the diagonal $\Delta(X)$. Thus if X is compact, there exists $\varepsilon > 0$ such that $\rho(x, y) < \varepsilon \Rightarrow (x, y) \in U$. Combining these ideas gives

Lemma A.2.5 *If X is a compact subset of the metric space Y, and U an open neighbourhood of $X \times X$ in $Y \times Y$, then for some $\varepsilon > 0$, if V is the ε-neighbourhood of X in Y, U contains $V \times V$.*

Proof Take $\varepsilon = \frac{1}{2}\rho(X \times X, (Y \times Y \setminus U))$. Then if $\rho(v_1, X) < \varepsilon$, $\rho(v_2, X) < \varepsilon$ we have $\rho((v_1, v_2), X \times X) < 2\varepsilon = \rho(X \times X, (Y \times Y \setminus U))$, so (v_1, v_2) does not lie in $Y \times Y \setminus U$. □

Corollary A.2.6 *Let Y be a metric space, $f : Y \to Z$ a map such that each $P \in Y$ has a neighbourhood U_P with $f|U_P$ an embedding, and $X \subset Y$ such that $f|X$ is injective. Then X has a neighbourhood V in Y such that $f|V$ is injective.*

If also each $f(U_P)$ is open, $f|V$ is an embedding.

Proof Let $D = \{(y_1, y_2) : y_1 \neq y_2, f(y_1) = f(y_2)\} \subset Y \times Y$. Since $f|X$ is injective, D is disjoint from $X \times X$. The closure $\bar{D}$ is contained in the closed subset defined by $f(y_1) = f(y_2)$, which is equal to $D \cup \Delta(Y)$. But by hypothesis, each point (P, P) has a neighbourhood $U_P \times U_P$ disjoint from D. Thus $\bar{D}$ is disjoint from $\Delta(Y)$, so D is closed. Now apply Lemma A.2.5, taking $U = Y \times Y \setminus D$: this gives a neighbourhood V of X such that $V \times V$ does not meet D, so $f|V$ is injective.

As each $f|U_P$ is an embedding, f induces a homeomorphism between U_P and $f(U_P)$ with the subspace topology. Thus the inverse map is continuous on $f(U_P)$, which is open in $f(V)$. Thus it is continuous at each point of $f(V)$. □

The following can be used to replace Theorem 1.1.4, which we proved for smooth manifolds.

Proposition A.2.7 *Suppose X locally compact and a countable union of compact subsets. Then there exist coverings by sets $F_a \subset G_a$ with each F_a compact, each G_a open, $\{G_a\}$ locally finite, and $\bigcup_a F_a = X$.*

Proof By Proposition 1.1.3, we can find compact subsets C_n and open subsets $B_{n+\frac{1}{2}}$ such that $X = \bigcup_n C_n$ and for all $n \geq 1$, $C_n \subset B_{n+\frac{1}{2}} \subset C_{n+1}$. It now suffices to set $F_n := C_{n+1} \setminus B_{n-\frac{1}{2}}$ and $G_n := B_{n+\frac{3}{2}} \setminus C_{n-1}$: these are locally finite since any $x \in X$ belongs to some $C_n \setminus C_{n-1}$, so the open set $B_{n+\frac{1}{2}} \setminus C_{n-1}$ is a neighbourhood of x, and meets G_N only if $n-2 \leq N \leq n+1$. □

The relation between paracompactness and countability is given by

Proposition A.2.8 *(i) If each component of X is open, X is paracompact if and only if each component is.*

(ii) A connected locally compact space X is paracompact if and only if it is a countable union of compact subsets.

Proof (i) is immediate since an open cover of X induces (and is induced by) open covers of each of its components.

(ii) If X is paracompact, the open covering by neighbourhoods of points with compact closures has a locally finite refinement. Since these sets have compact closures, each meets only finitely many others. Starting with one such set U_0, only finitely many others meet it; only finitely many meet one of the above, and so on. But since X is connected, each U_α is connected to U_0 by a finite chain. Thus there are only countably many U_α, and X is the union of their (compact) closures.

Conversely if $X = \bigcup_{n\geq 0} U_n$ is a countable union, setting $V_n := \bigcup_{0\leq i\leq n} U_i$, we may assume the sequence V_n increasing. Any compact subset is covered by the U_n, hence by a finite subset, hence is contained in some V_n. Each point of V_n has a compact neighbourhood; V_n is covered by these neighbourhoods, hence by finitely many. Their union is compact, so is contained in some V_m. Thus, passing to a subsequence, we may suppose that V_{n+1} contains an open neighbourhood of V_n. Now any open cover of X induces one of the compact set $V_{n+1} \setminus IntV_n$, which has a finite refinement. The union of all these refines the given cover, covers all of X, and is locally finite since any point is in some $V_{n+1} \setminus IntV_n$, so has a neighbourhood contained in V_{n+2} and disjoint from V_{n-1}. □

The following useful result has a different nature.

Proposition A.2.9 *If X is a finite dimensional metric space, any open covering $\{U_\alpha\}$ has a finite dimensional refinement. More precisely, there exist an open covering $\{S_j \mid j \in J\}$ of X, with each S_j contained in U_α for some α, and a map $d : J \to \{0, \ldots, N\}$ such that if $d(j) = d(j')$, $j \neq j'$ then $\bar{S}_j \cap \bar{S}_{j'} = \emptyset$.*

We omit the proof, which is given by Hurewicz and Wallman on [76, p. 54]. To understand the result, the reader should consider the picture of a simplicial complex K of dimension N: each simplex of dimension r admits coordinates $\{x_0, \ldots, x_r\}$ with $x_i \geq 0$, $\sum_i x_i = 1$, and $S_r(K)$ is a union of sets contained in the interior of each r-simplex. Replace this simplicial complex K by its barycentric subdivision K': each vertex V of this is labelled by the dimension $d(V)$ of the simplex of which V is the barycentre. Now map each point of K' to the nearest vertex: more precisely, define an open neighbourhood of the vertex V to be

$$N(V) := \{x \in K' \mid (\forall W \neq V)\rho(x, V) \geq \rho(x, W) - 2^{-N}\},$$

where W runs over the vertices of K'. Now set $S_r(K) := \bigcup\{N(V) \mid d(V) = r\}$: a disjoint union of the neighbourhoods $N(V)$ with $d(V) = r$. Now if $f : X \to K$, define $S_r(X) := f^{-1}(S_r(K))$ to obtain subsets with the desired properties.

Notes on this section. The results on compactness and proper maps can be extended to general (not metric) topological spaces (see [24, §12]).

A closer study of the notion of properness is also given in [47, §3.2], using the following concept. For any map $f : X \to Y$, define the *improper set* $Z(f)$ as the set of $y \in Y$ such that there is a discrete sequence $\{x_n \mid n \in \mathbb{N}\}$ on X with $f(x_n) \to y$. This is the smallest closed subset of Y such that the restriction of f to a map $X \setminus f^{-1}(Z) \to Y \setminus Z$ is proper: thus is empty if and only if f is proper.

A.3 Proper group actions

A (left) *action* of a group G on a set X is a map $\phi : G \times X \to X$ such that $\phi(1, x) = x$ for all $x \in X$ and $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $x \in X$ and $g, h \in G$. We usually denote $\phi(g, x)$ by $g.x$. We are really only interested in smooth group actions, so X will be a Hausdorff space throughout.

Given an action ϕ, the *isotropy group* of $x \in X$ is $G_x := \{g \in G \mid g.x = x\}$. The *orbit* of x is $G.x := \{g.x \mid g \in G\}$. The action induces a bijection $G/G_x \to G.x$ since

$$g.x = h.x \Leftrightarrow h^{-1}g.x = x \Leftrightarrow h^{-1}g \in G_x \Leftrightarrow hG_x = gG_x.$$

Equivalently, the map $\phi_x : G \to X$ defined by $\phi_x(g) := g.x$ induces an injection of G/G_x into X.

Given a left group action, we denote the set of orbits by $G\backslash X$ and the projection by $q : X \to G\backslash X$. We give $G\backslash X$ the quotient topology and call it the *orbit space*. The map q is open, for if U is open in X, $q^{-1}(q(U)) = \bigcup_{g \in G} g.U$, a union of open sets, hence open; by the definition of quotient topology, $q(U)$ is open.

Proposition A.3.1 *Let $\phi : G \times X \to X$ be a group action. Then the following are equivalent:*

(i) The map $(\phi, \pi) : G \times X \to X \times X$ *(where* π *denotes the projection) is a proper map;*

(ii) (ϕ, π) *is closed and all isotropy groups* G_x *are compact;*

(iii) for any compact subsets $K, L \subseteq X$, $T_{K,L} := \{g \in G \mid g.K \cap L \neq \emptyset\}$ *is compact.*

Proof $(i) \Rightarrow (ii)$ since $G_x \times \{x\}$ is the preimage of (x, x) under (ϕ, π).

$(ii) \Rightarrow (i)$ since the preimage of (y, x) is empty if $y \notin G.x$, and if $y = g.x$ is the coset gG_x, homeomorphic to G_x.

Now by Lemma A.2.2, (i) is equivalent to the condition that for any compact subset of $X \times X$, its preimage under (ϕ, π) is compact. It is sufficient to consider subsets of the form $L \times K$, where K and L are compact subsets of X. We have $(\phi, \pi)^{-1}(L \times K) = \{(g, x) \mid g.x \in L, x \in K\} \subseteq T_{K,L} \times K$. Thus if $T_{K,L}$ is compact, so is this (closed) subset of it; and if this set is compact, so is its projection on the first factor, which is $T_{K,L}$. □

A group action will be called *proper* if it satisfies the equivalent conditions of Proposition A.3.1. It is not true that for any proper group action ϕ itself is a closed map: consider, for example, $G = X = \mathbb{R}$ with action by translation.

Lemma A.3.2 *(i) A group action of a compact group is proper.*

(ii) Given two Lie subgroups H, K *of* G *with* K *compact, the natural action of* H *on the coset space* G/K *is proper.*

Proof (i) It will suffice to show that the preimage of a compact $C \subset X \times X$ is compact. The second projection C_2 of C is compact, and the preimage of C is a closed subset of the compact set $G \times C_2$.

(ii) It is enough to show that the action of G on G/K is proper. Any compact subset C of $G/K \times G/K$ is a subset of some $C_1 \times C_2$ with each C_i compact, and the preimage of C_i in G is a compact set B_i. The image of $B_1 \times B_2$ by the map $(x, y) \to xy^{-1}$ is a compact set B. Now the preimage of C in $G \times G/K$ is a closed subset of the compact set $B \times C_2$. □

Proposition A.3.3 *Let* $\phi : G \times X \to X$ *be a proper group action and* $x \in X$. *Then*

(i) the isotropy group G_x *is compact;*

(ii) the map $\phi_x : G \to X$ *given by* $\phi_x(g) = g.x$ *is proper;*

(iii) the orbit $G.x$ *is a closed subset of* X*;*

(iv) the induced map $G/G_x \to G.x$ *is a homeomorphism.*

Proof (i) $G_x \times \{x\}$ is the preimage of the point (x, x) (a compact set) under the proper map (ϕ, π).

(ii) This is a closed map as it is the restriction of (ϕ, π) to the closed subset $G \times \{x\}$. The preimage of a compact set K is the preimage of $K \times \{x\}$ under (ϕ, π), so is compact.

(iii) It is the image of G under the proper, hence closed map ϕ_x.

(iv) This map is bijective by construction, continuous since ϕ_x is, and by the definition of the quotient topology, and closed since ϕ_x is. $\square$

Proposition A.3.4 *Let $\phi : G \times X \to X$ be a smooth proper group action. Then the quotient space $G\backslash X$ is Hausdorff, locally compact, and paracompact.*

Proof Write Δ for the diagonal in $G\backslash X \times G\backslash X$. Since (ϕ, π) is closed, $C := \{(x, g.x) \,|\, x \in X\}$ is closed in $X \times X$. Now $C = (q, q)^{-1}(\Delta)$, and since $G\backslash X \times G\backslash X$ has the quotient topology, it follows that Δ is closed in $G\backslash X \times G\backslash X$. Thus $G\backslash X$ is Hausdorff.

By Theorem 3.3.5, any point of X has an invariant neighbourhood of the form $j(G \times_H V)$ with $H \subseteq G$ a compact subgroup and V a disc on which H acts orthogonally. Thus any point of $G\backslash X$ has a neighbourhood of the form $H\backslash V$, which is compact. So $G\backslash X$ is locally compact.

By Proposition A.2.8, paracompactness will follow provided $G\backslash X$ is a countable union of compact subsets. But this follows since X is such a union, and the image of a compact set is compact. $\square$

For the special case when G is compact, we have

Proposition A.3.5 *If $\phi : G \times X \to X$ is a group action with G compact, then*

(i) the map ϕ is a proper map;

(ii) the action is proper;

(iii) the map $q : X \to X/G$ is proper;

(iv) for any $Y \subset X$, any neighbourhood of Y contains a G-invariant neighbourhood.

Proof (i) Suppose F a closed subset of $G \times X$: we want to prove that any limit point x of $\phi(F)$ belongs to $\phi(F)$. Suppose $(g_i, x_i) \in F$ and $g_i.x_i \to x$. Since G is compact, $\{g_i\}$ has a convergent subsequence. Passing to this subsequence, we may write $g_i \to g$. Then $x_i = g_i^{-1}.(g_i.x_i) \to y := g^{-1}.x$. Thus $(g_i, x_i) \to (g, y)$, so $(g, y) \in F$ and $x = g.y \in \phi(F)$.

(ii) Similarly if $(g_i, x_i) \in F$ and $(g_i.x_i, x_i) \to (y, x)$ we have $x_i \to x$ and may suppose $g_i \to g$; thus $g.x = y$, $(g, x) \in F$ and $(y, x) = (\phi, \pi)(g, x)$.

(iii) The preimage under q of a point $q(x)$ is the orbit $G.x$, which is compact since G is. Now suppose F is closed in X: then $G \times F$ is closed in $G \times X$; since by (i) ϕ is proper, $G.F = \phi(G \times F)$ is closed in X. Now by the definition of quotient topology, $q(F)$ is closed in X/G since $q^{-1}(q(F)) = G.F$ is closed in X.

(iv) Let U be an open set containing Y. Then $W := X \setminus q^{-1}q(X \setminus U)$ is G-invariant and contained in U. Since q is proper $X \setminus U$ is closed, thus W is open. □

A similar argument shows that in general if $K \subset G$ is compact then the restriction of ϕ to $K \times X \to X$ is proper, and hence if $A \subseteq X$ is closed (compact) so is $K.A$.

Proposition A.3.6 *Let G act properly on M and ρ be a G-invariant metric on M. Define $\overline{\rho} : G\backslash M \times G\backslash M \to \mathbb{R}$ by $\overline{\rho}(G.x, G.y) := inf_{g\in G}\ \rho(x, g.y)$. Then $\overline{\rho}$ is a metric on $G\backslash M$.*

Proof Since the action is proper, the orbit $G.y$ is closed. Thus if $x \notin G.y$, $\rho(x, G.y) > 0$, i.e. $G.x \neq G.y$ implies $\overline{\rho}(G.x, G.y) \neq 0$.

For any $x, y, z \in M$ and any $\varepsilon > 0$ we can choose $g, g' \in G$ with $\rho(x, g.y) < \overline{\rho}(G.x, G.y) + \varepsilon$ and $\rho(y, g'.z) < \overline{\rho}(G.y, G.z) + \varepsilon$. Thus

$\overline{\rho}(G.x, G.z) \leq \rho(x, gg'.z) \leq \rho(x, g.y) + \rho(g.y, gg'.z)$,

and this is equal to

$\rho(x, g.y) + \rho(y, g'.z) < \overline{\rho}(G.x, G.y) + \overline{\rho}(G.y, G.z) + 2\varepsilon$.

Since this holds for any $\varepsilon > 0$, we have $\overline{\rho}(G.x, G.z) \leq \overline{\rho}(G.x, G.y) + \overline{\rho}(G.y, G.z)$, so the triangle inequality holds. □

Note As for the definition of proper maps, one can define and study a 'bad set'. If G is a locally compact group acting on a Hausdorff space X, then $x \in X$ is a *wandering point* if it has a neighbourhood V_x such that $\{g \in G \mid V_x.g \cap V_x \neq \varnothing\}$ has compact closure, or equivalently, if there exists a compact subset $K \subset G$ such that $g \notin K$ implies $V_x.g \cap V_x = \varnothing$. The set $\Omega(X)$ of all wandering points is open, and the action of G on $\Omega(X)$ is proper; the action on X is proper if and only if $\Omega(X) = X$. In the case when G is a discrete group, the term 'properly discontinuous' is often used instead of 'proper'.

A.4 Mapping spaces

We begin by discussing topologies on the set $C^0(X, Y)$ of continuous maps between two topological spaces X and Y. We are only interested here in the case when X and Y are manifolds, and hence metrisable.

Perhaps the most commonly used topology on function spaces is the so-called *compact-open topology*, which we call the C^0 topology. This is the topology on $C^0(X, Y)$ defined by taking the sets

$$A(K, U) := \{f \mid f(K) \subset U\} \quad \text{with } K \subset X \text{ compact, } U \subset Y \text{ open}$$

as a sub-base of open sets. It can be described as the topology of uniform convergence of f on compact sets.

There is also the *fine topology* (or fine C^0 topology), which we define by taking the

$$B(U) := \{f \mid (1 \times f)(X) \subset U\} \quad \text{with } U \text{ open in } X \times Y$$

as a base of open sets.

Lemma A.4.1 *(i) The sets $I(\{K_\alpha, U_\alpha\}) := \bigcap_\alpha A(K_\alpha, U_\alpha)$, with $K_\alpha \subset X$ compact, $U_\alpha \subset Y$ open, $\{K_\alpha\}$ locally finite, are a subbase for the fine topology.*

(ii) For $f \in C^0(X, Y)$ and ρ a metric on Y, the sets

$$J(f, k) := \{g \in C^0(X, Y) \mid (\forall x \in X)\, \rho(f(x), g(x)) < k(x)\},$$

with $k \in C^0(X, \mathbb{R}_{>0})$, are a base of neighbourhoods of f in the fine topology.

Proof We have $J(f, k) = B(U)$, where $U = \{(x, y) \in X \times Y \mid \rho(y, f(x)) < k(x)\}$, hence $J(f, k)$ is open. That these give a base of neighbourhoods of f follows by applying Lemma A.2.4 to neighbourhoods of the graph of f in $X \times Y$.

The set $A(K_\alpha, U_\alpha)$ is the preimage by $1 \times f$ of the open subset

$$((X \setminus K_\alpha) \times Y) \cup (X \times U_\alpha)$$

of $X \times Y$. Any finite intersection of these subsets is thus also open. But by hypothesis, any $x \in X$ has an open neighbourhood U_x intersecting K_α for only finitely many α. Thus the intersection of $U_x \times Y$ with $I(\{K_\alpha, U_\alpha\})$ is equal to its intersection with a finite number of the $A(K_\alpha, U_\alpha)$ and hence is open. It follows that $I(\{K_\alpha, U_\alpha\})$ is open.

For the converse, it will suffice to check that any neighbourhood $J(f, k)$ of f contains one of the form $I(\{K_\alpha, U_\alpha\})$. It will suffice if the K_α cover X and $z \in K_\alpha$, $y \in U_\alpha$ implies $\rho(f(z), y) < k(z)$. For each x, set $U_x := \{y \mid \rho(y, f(x)) < \frac{1}{3}k(x)\}$, choose a compact neighbourhood $K_x \subset f^{-1}(U_x) \cap \{z \mid k(z) > \frac{2}{3}k(x)\}$ now let $\{K_\alpha\}$ be a locally finite subcover of the sets K_x. □

If X is compact, the fine topology, the C^0 topology and the topology of uniform convergence are the same.

For when X is compact, the functions k in $J(f, k)$ have positive lower bounds, so the base of neighbourhoods $J(f, k)$ is equivalent to the base of neighbourhoods $J(f, c)$ (c constant), which defines the uniform topology.

Both topologies are Hausdorff; indeed completely regular.

We will shortly see that the C^0 topology is metrisable, hence Hausdorff and normal.

For the fine topology, any closed set C not containing f is disjoint from some $J(f, k)$, so $J(f, \frac{1}{2}k)$ is an open set containing f disjoint from the open set $\{g \in C^0(X, Y) \mid (\forall x \in X)\, \rho(f(x), g(x)) > \frac{1}{2}k(x)\}$ which contains C.

If X is not compact, the fine topology is very large, and the two topologies are distinct.

Proposition A.4.2 *(i) The space $C^0(X, Y)$ with the C^0 topology has a complete metric.*

(ii) A sequence of maps which converges in the fine topology is eventually constant outside a compact set.

(iii) If X is not compact, the fine topology on $C^0(X, Y)$ is not metrisable, and does not admit a countable base, even locally.

Proof (i) First suppose X compact, then choose a complete metric ρ on Y and take the uniform metric $\rho(f, g) = \sup_{x \in X} \rho(f(x), g(x))$. This is complete since if $\{f_n\}$ is a Cauchy sequence, so is each $\{f_n(x)\}$, which thus converges to a limit $f(x)$, and f is continuous as the uniform limit of $\{f_n\}$.

For X not compact, write $X = \bigcup_{i=1}^{\infty} X_i$ as a countable union of compact subsets. Then the topology for $C^0(X_i, Y)$ is defined by a complete metric ρ_i, hence also by the bounded metric $\rho'_i(f, g) := \min(\rho_i(f, g), 2^{-i})$. The metric $\rho := \sum_{i=1}^{\infty} \rho'_i$ defines the product topology on $\Pi_i C^0(X_i, Y)$, and hence the required topology on the subset $C^0(X, Y)$. Moreover, $C^0(X, Y)$ is a closed subset of the complete $\Pi_i C^0(X_i, Y)$ and is thus also complete.

(ii) Assume $f_n \to f$ and that for no compact $K \subset X$ is the sequence f_n eventually constant outside K. Choose an increasing sequence $\{K_n\}$ of compact subsets of X with union X. By hypothesis, there exist $x_n \in (X \setminus K_n)$ and $i_n > n$ with $f_{i_n}(x_n) \neq f(x_n)$. Set $\delta_n := \rho(f_{i_n}(x_n), f(x_n))$. Since the sequence x_n diverges, we can find a positive continuous function k on X such that $k(x_n) = \frac{1}{2}\delta_n$ for infinitely many n. For none of these n is $f_{i_n} \in J(f, k)$, contradicting the assumption that $f_n \to f$.

(iii) follows from (ii). □

Not only compactness of spaces, but properness of maps is important in discussing these topologies, and we have

Lemma A.4.3 *If Y is a locally compact, paracompact metric space, the set $C^0_{pr}(X, Y)$ of proper maps is open in $C^0(X, Y)$ in the fine topology.*

Proof By Proposition A.2.7, there exist coverings of Y by sets $F_a \subset G_a$ with each F_a compact, each G_a open, $\{G_a\}$ locally finite, and $\bigcup_a F_a = Y$.

For $f : X \to Y$ a proper map, the sets $K_a := f^{-1}(F_a)$ are compact. They are locally finite, since for any $x \in X$, $f(x)$ has a neighbourhood U_x meeting only finitely many of the G_a, so $f^{-1}(U_x)$ is a neighbourhood of x meeting only finitely many of the K_a. Hence by Lemma A.4.1, $I(\{K_a\}, \{G_a\})$ is a neighbourhood of f in the fine topology.

We claim that any $g \in I(\{K_a\}, \{G_a\})$ is proper. For any compact subset $L \subset Y$ meets only finitely many G_a, so $g^{-1}(L)$ is contained in the union of the corresponding K_a, so is compact. □

We turn to the question of continuity of the composition map.

Proposition A.4.4 *(i) The composition map $C^0(X, Y) \times C^0(Y, Z) \to C^0(X, Z)$ is continuous for the C^0 topologies.*

(ii) The map $C^0(Y, Z) \to C^0(X, Z)$ defined by composition with a continuous map f is continuous for the fine topologies if and only if f is proper.

(iii) The composition map $C^0_{pr}(X, Y) \times C^0(Y, Z) \to C^0(X, Z)$ is continuous for the fine C^0 topologies.

Proof (i) It will suffice to show that the preimage of a subbasic open set $A(K_X, U_Z)$ is open, and thus to show that if $g \circ f \in A(K_X, U_Z)$, it contains a neighbourhood of (f, g).

Since $g \circ f \in A(K_X, U_Z)$ and f is proper, $f(K_X)$ is a compact subset of Y and $g^{-1}(U_Z)$ is an open neighbourhood of it. By Lemma A.1.1, this contains a compact neighbourhood, so we can find a compact K_Y and an open U_Y with $f(K_X) \subset U_Y \subset K_Y \subset g^{-1}(U_Z)$.

It follows that the preimage of $A(K_X, U_Z)$ contains the open neighbourhood $A(K_X, U_Y) \times A(K_Y, U_Z)$ of (f, g).

(ii) If f is not proper, there is a discrete sequence $x_n \in X$ such that $f(x_n)$ converges to a limit $y_0 \in Y$. Let $g : Y \to Z$ be continuous, and consider a neighbourhood $J(g \circ f, k)$ of $g \circ f$. We want to show that for some k, $f^*J(g \circ f, k)$ is not open, in fact does not contain a neighbourhood $J(g, \ell)$ of g. For if it does, $\rho(g(y), h(y)) < \ell(y)$ for all y implies $\rho(g(f(x)), h(f(x))) < k(x)$ for all x.

Since x_n is discrete, we can choose k with $k(x_n) = n^{-1}$ for all n. Now as $f(x_n) \to y_0$, if $\rho(g(f(x_n)), h(f(x_n))) < k(x_n) = n^{-1}$ for all n, it follows that $\rho(g(y_0), h(y_0)) = 0$. Thus we do not have a neighbourhood of g.

(iii) We copy (i); so start with neighbourhood $I(\{K^X_\alpha, U^Z_\alpha\})$ of $g \circ f$: here as well as the U^Z_α being open, the K^X_α are locally finite. Any y has a compact neighbourhood B_y: then $f^{-1}(B_y)$ is compact (as f is proper), so meets only finitely many of the K^X_α. Hence the $f(K^X_\alpha)$ are locally finite.

We now have a locally finite family of compact sets $f(K_\alpha^X)$ with neighbourhoods $g^{-1}(U_\alpha^Z)$ and seek $f(K_\alpha^X) \subset U_\alpha^Y \subset K_\alpha^Y \subset g^{-1}(U_\alpha^Z)$ with the K's compact, the U's open and the K_α^Y locally finite. We restate the problem. First use countable compactness to say the set of α is countable. We have a locally finite family of compact sets A_n with open neighbourhoods D_n and seek $A_n \subset B_n \subset C_n \subset D_n$ with the C_n compact, the B_n open and the C_n locally finite.

Shrinking the D_n, we may suppose each meets only finitely many of the A_i. Now by Lemma A.1.1 we can find B_n and C_n as above, but have yet to make the C_n locally finite. Set

$$C'_n := C_n \setminus \bigcup \{D_r \mid r < n,\ D_r \cap A_n = \emptyset\}.$$

Since $A_n \subset C_n$ and we have only removed subsets disjoint from A_n, we have $A_n \subset C'_n$. If C'_r meets C'_n with $r < n$ then C'_n meets D_r so $D_r \cap A_n \neq \emptyset$. For each r this holds for finitely many n, and there are only finitely many $n < r$, so C'_r meets only finitely many C'_n. It remains only to take B'_n as a neighbourhood of A_n contained in C'_n.

In the original notation, it follows that the preimage of $I(\{K_\alpha^X, U_\alpha^Z\})$ contains $I(\{K_\alpha^X, U_\alpha^Y\}) \times I(\{K_\alpha^Y, U_\alpha^Z\})$, a product of neighbourhoods of f and g. □

We next discuss the Baire property, which is important for many of our applications.

Theorem A.4.5 (Baire's Theorem) *Let X be a complete metric space. The intersection of a countable family of dense open subsets of X is dense.*

Proof Let the given subsets be $\{U_i\}$, and let V be any non-empty open set. Then $V \cap U_1$ is non-empty and open, and so contains a metric neighbourhood $U(x_1, \varepsilon_1)$, say. Next, $U_2 \cap U(x_1, \varepsilon_{\frac{1}{2}})$ is non-empty and open, so contains some $U(x_2, \varepsilon_2)$. We can thus construct a decreasing sequence of neighbourhoods $U(x_i, \varepsilon_i)$ and have $\varepsilon_i \to 0$. Then $\{x_i\}$ is a Cauchy sequence, so has a limit point x, which lies in each $\bar{U}(x_i, \varepsilon_i)$ (since the later x_j do) and so in each U_i and in V. □

This result shows that any complete metric space has the Baire property. It follows from Proposition A.4.2 that $C^0(X, Y)$ with the C^0 topology has the Baire property. For the fine topology, we have to work harder.

Theorem A.4.6 *If X is paracompact and Y a complete metric space, then $C^0(X, Y)$ with the fine topology is a Baire space.*

Further, if $Q \subset C^0(X, Y)$ is closed in the C^0 topology, then Q with the fine topology is a Baire space.

Proof Let $\{U_i\}$ be a countable sequence of open dense sets and V a further open set. Choose $f_0 \in V$ and a neighbourhood $J(f_0, k_0)$ of f_0 with closure contained in V.

Now suppose inductively chosen functions $f_0, \ldots, f_r$ and neighbourhoods $J(f_i, k_i)$ $(0 \le i \le r)$ such that $f_i \in V$, $f_i \in J(f_j, k_j)$, $k_i < 2^{-i}$ and $\overline{J(f_i, k_i)} \subset U_i$ for $j \le i \le r$. Since U_{r+1} is dense, it meets the open set $\bigcap_{i=0}^r J(f_i, k_i)$: choose f_{r+1} in the intersection, and choose a neighbourhood $J(f_{r+1}, k_{r+1})$ with closure contained in it and with $k_{r+1} < 2^{-(r+1)}$.

Since ρ is a complete metric, and the sequence f_r converges uniformly, we can define f to be its limit. Since all f_i with $i > r$ belong to $J(f_r, k_r)$, f belongs to its closure, which is contained in U_r $(r > 0)$ or V $(r = 0)$. Thus $V \cap \bigcap_r U_r$ is non-empty, as required.

Given a countable sequence of open dense subsets W_i of Q, we can take $U_i := W_i \cup (C^0(X, Y) \setminus Q)$ and argue as above. We only need to note that since $Q \subset C^0(X, Y)$ is closed in the C^0 topology, the uniform limit f of the maps $f_i \in Q$ also belongs to Q. □

For smooth manifolds V^v and M^m, write $C^r(V, M)$ for the set of maps $V \to M$ whose restrictions in any local coordinates have continuous partial derivatives of all orders $\le r$; in particular, $C^\infty(V, M)$ is the set of smooth maps of V to M. Taking r-jets gives an injective map $j^r : C^r(V, M) \to C^0(V, J^r(V, M))$. The topology on $C^r(V, M)$ induced by regarding it as a subspace of $C^0(V, J^r(V, M))$ with the compact-open topology is called the C^r *topology*, and the topology induced from the fine topology is the *fine* C^r *topology*. The image of $j^r : C^r(M, N) \to C^0(M, J^r(M, n))$ is closed in the C^r topology.

The inclusion of $C^\infty(V, M)$ in $C^r(V, M)$ induces topologies on it, and we define the C^∞ *topology* to be the union of the C^r topologies, in the sense that a set is open if it is open in one of these topologies. Correspondingly, the fine C^∞ topology, which we christen the W^∞ *topology*, is the union of the fine C^r topologies.

The properties of these metrics are similar to those for the case $r = 0$, and the proofs run in parallel, though with complications of detail (the case $r = \infty$ requiring a little more effort), so we omit most of them. The discussion extends to manifolds with boundaries, corners, etc. The following statements hold for all $r \le \infty$: it is the case $r = \infty$ which is of prime interest to us.

We have equivalent characterisations of the fine C^r topology if the above conditions on the images of the maps are replaced by conditions on the r-jets. However if in the C^∞ version of $I(\{K_\alpha, U_\alpha\})$ we allow the U_α to be open in jet spaces $J^r(V, M)$ for varying values of r we obtain a new topology, the *very strong topology*, which we do not discuss further in this book.

Both topologies on $C^\infty(V, M)$ are completely regular. They agree if V is compact.

For the W^∞ topology, a convergent sequence of maps is eventually constant outside a compact set; hence the topology is neither metrisable nor even locally countable.

Theorem A.4.7 *With the C^∞ topology, $C^\infty(V, M)$ is a complete metric space.*

Proof First suppose V is compact. Each jet space $J^r(V, M)$ is a smooth manifold, and admits a complete Riemannian metric ρ^r, say. The distance function $\rho^r(f, g) := \sup_{P\in V} \rho^r(j^r f(P), j^r g(P))$ is well defined since V is compact, and defines the C^r topology on $C^\infty(V, M)$.

The same topology on $J^r(V, M)$ is given by the non-Riemannian metric $\rho'^r = \inf(\rho^r, 1)$, and the metric $\rho(f, g) = \sum_r 2^{-r} \rho'^r(f, g)$ defines the C^∞ topology on $C^\infty(V, M)$.

A Cauchy sequence $\{f_i\}$ in $C^\infty(V, M)$ must *à fortiori* be Cauchy with the metric ρ^r. Since $J^r(V, M)$ is complete, the maps $j^r f_i$ converge to a limit g^r, which is continuous, since the convergence was uniform.

The coordinates $u_{\omega,j}$ of $j^r f_i$ are the partial derivatives of the $u_{0,j}$. Let ω' be derived from ω by increasing ω_i by unity, and $|\omega'| \le r$: then $u_{\omega',j} = \partial u_{\omega,j}/\partial x_i$ and so $u_{\omega,j}$ is the indefinite integral with respect to x_i of $u_{\omega',j}$. Integration commutes with uniform limits, so the relation $u_{\omega',j} = \partial u_{\omega,j}/\partial x_i$ also holds for g^r. Thus the $u_{0,j} = y_j$ are r-times continuously differentiable, g^r is the r-jet of a C^r function g, independent of r, so g is smooth, and is the limit of the sequence.

For V not compact, write $V = \bigcup_{i=1}^\infty V_i$ as a countable union of compact submanifolds (with boundary). Then the topology for $C^\infty(V_i, M)$ is defined by a metric ρ_i, bounded by 1. Hence the metric $\rho = \sum_{i=1}^\infty 2^{-i}\rho_i$ defines the product topology on $\Pi_i C^\infty(V_i, M)$, and hence the required topology on the subset $C^\infty(V, M)$.

Now $C^\infty(V, M)$ is a closed subset of the complete $\prod_i C^\infty(V_i, M)$ which is thus also complete. □

Lemma A.4.8 *The set $C^\infty_{pr}(V, M)$ of proper smooth maps is open in $C^\infty(V, M)$ in the W^∞ topology.*

Proof Let $f : V \to M$ be a proper map, and $\{\varphi_\alpha : U_\alpha \to \mathring{D}^m(3)\}$ a locally finite open cover of M as in Theorem 1.1.4, so that M is covered by the compact sets $K_\alpha := \varphi_\alpha^{-1}(D^m(2))$. Since f is proper, $F_\alpha := f^{-1}(K_\alpha)$ is compact. Then $\mathcal{W} := \{g \mid \forall\alpha\ g(F_\alpha) \subset U_\alpha\}$ is an open neighbourhood of f. For any $g \in \mathcal{W}$ and any compact $L \subset M$, L meets only finitely many U_α, so $g^{-1}(L)$ is contained in the union of the corresponding F_α, so is compact. □

The composition map $C^\infty(V, M) \times C^\infty(M, N) \to C^\infty(V, N)$ is continuous for the C^∞ topologies; however for the W^∞ topologies this fails unless V is compact: more precisely, $C^\infty_{pr}(V, M) \times C^\infty(M, N) \to C^\infty(V, N)$ is continuous, and the map $C^\infty(M, N) \to C^\infty(V, N)$ defined by composition with $f : V \to M$ is continuous if and only if f is proper.

Theorem A.4.9 (see, for example, [73, 2.4.4], [57, 3.4]) *If F is any subspace of $C^\infty(V, M)$ which is closed in the C^∞ topology, then F (with either the C^∞ topology or the W^∞ topology) has the Baire property.*

For example, if $f \in C^\infty(V, M)$ and K is a closed subset of V, we can take $F = \{g \in C^\infty(V, M) \mid g|K = f|K\}$.

We also have

Theorem A.4.10 *If W is open in $C^\infty(V, M)$ with the C^∞ or W^∞ topology, then W has the Baire property.*

Proof Since $C^\infty(V, M)$ is completely regular, for any $f \in W$ we can choose a neighbourhood U of f whose closure $F \subset W$. If now the U_i are dense open subsets of W, the $U_i \cup (W \setminus F)$ are dense open subsets of X, hence their intersection $\bigcap U_i \cup (W \setminus F)$ is dense in X, and hence intersects U. Thus U meets $\bigcap U_i$, and since this holds for any neighbourhood of f contained in U, f is in the closure of $\bigcap U_i$. As this holds for all $f \in W$, $\bigcap U_i$ is dense in W. □