4

General position and transversality

We open our discussion of the deeper properties of smooth manifolds with Whitney's embedding theorem for two reasons. The first is historical: smooth manifolds were originally considered as submanifolds of Euclidean spaces, and this theorem reconciled this approach with the abstract form of definition which we prefer. Secondly, the proof is quite simple, and opens the way to our later discussion of the general transversality theorem.

In Chapter 5 we will give a method for describing compact manifolds up to diffeomorphism. The method consists in defining a smooth function $f : M^m \to \mathbb{R}$; and then we can regard M as 'filtered' by the subset $f^{-1}(-\infty, a]$ as a increases. In order to carry out this process in detail, it is necessary to suppose f non-degenerate. Thus we next give a direct proof of the existence of non-degenerate functions.

We proceed to techniques for moving a smooth map into 'general position'. The language of jet spaces, which is basic to the study of singularities of smooth maps, is introduced in §4.4. Jets are also used to define topologies on function space (we give some proofs of properties of these topologies in §A.4).

The fundamental technical general position result is the transversality theorem, which is stated and proved in §4.5, and extended in the following section to multitransversality, to deal with the interaction of two maps with a common target. The development of transversality as a tool is due to Thom [150]; the very flexible formulation of multitransversality is due to Mather [88].

The main theorems include 'general position' results which we will often use in later chapters. In particular, a map $f : V^v \to M^m$ may be supposed an embedding if $m > 2v$ (or an immersion if $m = 2v$); it may be deformed to avoid any subset of M of dimension $< (m - v)$, and to be transverse to any given submanifold of M.

However the results allow a much wider range of application: for example, dealing with transversality to submanifolds of jet space rather than just of M; and establishing that the set of smooth maps satisfying such conditions is open and dense in function space. We thus spend some time in §4.7 applying the main results to describe the singularities of a dense open set of maps when the target dimension is either small (≤ 2) or large ($\geq \frac{3}{2}m$). The main results also lead to local normal forms for smooth maps, and in §4.8 we obtain these in the same cases. The details here are somewhat technical, and the reader may prefer to pass over them and just read the statements of the theorems to get a feel for what can be proved.

4.1 Nul sets

We say that a subset A of $\mathbb{R}^n$ is *nul* if for each $\varepsilon > 0$, A can be enclosed in a countable union of discs of total volume (i.e. the sum of the volumes) $< \varepsilon$. The useful terminology 'nul' is now out of fashion; it is equivalent to saying that A has Lebesgue measure zero.

It is trivial that a countable union of nul sets is nul; also that a nul set has no interior: its complement is everywhere dense.

Lemma 4.1.1 *Suppose U open in $\mathbb{R}^n$, $f : U \to R^n$ smooth, and $A \subset U$ nul. Then $f(A)$ is nul.*

Proof Let K be a compact subset of U. Then in K the partial derivatives of f of first order are bounded, so infinitesimal lengths are multiplied by a bounded factor: let c be a bound. Then the image of a ball of radius r is contained in a ball of radius cr. If $A \subset K$ is nul, for any $\varepsilon > 0$ it is contained in a number of balls in K of total volume less than ε, so $f(A)$ is contained in a union of balls of total volume less than $c^n\varepsilon$, so is nul.

Now as in Theorem 1.1.4, we may find a countable set of discs $K_i = D^n_{x_i}(2\delta_i)$ contained in U, with the $\mathring{D}^n_{x_i}(\delta_i)$ covering U. As K_i is compact, and $A_i := A \cap K_i$ is nul, $f(A_i)$ is nul. Hence so is the countable union $f(A) = \bigcup_i f(A_i)$. □

We say that a subset A of a smooth manifold N is *nul* if, for each coordinate neighbourhood $\varphi : U \to \mathbb{R}^n$, $\varphi(U \cap A)$ is nul. Since by the lemma, nul sets are preserved by smooth maps, it is sufficient to verify the condition for a set $(U_\alpha, \varphi_\alpha)$ of coordinate neighbourhoods with the U_α covering N.

Corollary 4.1.2 *(i) If $A \subset N_1^n$ is nul, and $f : N_1^n \to N_2^n$ smooth, $f(A)$ is nul.*
(ii) Suppose U open in $\mathbb{R}^\upsilon$, $\upsilon < n$, $f : U \to \mathbb{R}^n$ smooth. Then $f(U)$ is nul.
(iii) If $\upsilon < n$ and $f : V^\upsilon \to N^n$ is smooth, $f(V)$ is nul.

Proof (i) follows at once from Lemma 4.1.1 and the definition. For (ii) define $F : U \times \mathbb{R}^{n-v} \to \mathbb{R}^n$ by $F(x, y) = f(x)$. Then $f(U) = F(U \times O)$, but $U \times O$ is nul in $\mathbb{R}^n$. Similarly for (iii). □

These give the basic properties of nul sets: we now go on to the deeper result which we will need. If $f : V^v \to M^m$ is a smooth map, a point $P \in V$ is a *regular point* of f if $df : T_{g(P)}V \to T_{f(P)}M$ has rank m. Otherwise P is a *critical point*, and $f(P)$ a *critical value* of f.

Theorem 4.1.3 (Sard's Theorem) *Let $f : V^v \to M^m$ be a smooth map. Then the set of critical values of f is nul.*

We give the proof here only for $v \leq m$. For $v > m$, we refer the reader to the original paper of Sard [132] or to Milnor's account [100].

Proof We observe that it is sufficient to consider values in a coordinate neighbourhood of M, and further that, since V is a countable union of coordinate neighbourhoods, we may also restrict attention to a coordinate neighbourhood of V. This reduces the proof to the case $M = \mathbb{R}^m$, V an open subset of $\mathbb{R}^v$. For $v < m$, the result follows by Corollary 4.1.2 (ii).

Now let $m = v$. If P is a critical point, the Jacobian determinant of f vanishes at P, so given δ, we can find a ball containing P with $|J(f)| < \delta$ in the ball. Hence the volume of the image is at most δ times the volume of the original ball, so it can be contained in balls of at most twice this total volume.

If K is a compact submanifold of $\mathbb{R}^v$, A the set of critical points in K, we enclose these in small balls of total volume less than $2\mu(K)$, say. Then $f(A)$ can be enclosed in balls of total volume less than $4\delta\mu(K)$. But δ is arbitrarily small, so $f(A)$ is nul. The set of critical values is a countable union of sets $f(A)$, hence also is nul. □

4.2 Whitney's embedding theorem

The proof of the embedding theorem 1.2.11 is very simple, but the result is rather weak. We shall now obtain a stronger version, with a bound on the dimension of the Euclidean space, and an approximation clause. It is possible by similar methods to give a proof for non-compact manifolds; we defer this extension till Corollary 4.7.8. First remark that the result extends to manifolds with boundary, as if M has boundary, form the double $D(M)$: then any embedding of $D(M)$ restricts to give an embedding of M.

Each non-zero vector in $\mathbb{R}^n$ determines the parallel unit vector from the origin, and hence its end-point, which lies on S^{n-1}. Define $\mathbf{u} : (\mathbb{R}^n \setminus \{0\}) \to S^{n-1}$ by $\mathbf{u}(x) := \frac{x}{\|x\|}$.

Lemma 4.2.1 *Let $f : M^m \to \mathbb{R}^n$ be an embedding. Then the set of points of S^{n-1} parallel to a tangent of M^m is nul if $n \geq 2m+1$, and the set of those parallel to a chord is nul if $n \geq 2m+2$.*

Proof Any tangent of M^m is parallel to a unit tangent. Let B be the sub-bundle of $\mathbb{T}(M)$ consisting of unit vectors. Then $df : \mathbb{T}(M) \to \mathbb{T}(\mathbb{R}^n)$ restricts to $df : B \to \mathbb{T}(\mathbb{R}^n)$, and the identification of tangent spaces to $\mathbb{R}^n$ with $\mathbb{R}^n$ defines a smooth map $\mathbb{T} : \mathbb{T}(\mathbb{R}^n) \to \mathbb{R}^n$. Moreover, since B consists of unit vectors, $\mathbb{T} \circ df$ maps B to S^{n-1}. Hence the set of points in S^{n-1} whose vectors are parallel to a tangent of M is the image of B under a smooth map. Since B has dimension $2m-1$, the first result follows from Corollary 4.1.2 (iii).

For chords we proceed similarly. Let $M \times M$ be the product manifold, $\Delta(M)$ the diagonal, and write $M^{(2)}$ for $M \times M \setminus \Delta(M)$: this is a smooth manifold. Since f is an embedding, any two distinct points have distinct images, so if we define $\Delta_f : M^{(2)} \to \mathbb{R}^n$ by $\Delta_f(P,Q) = f(P) - f(Q)$ (vector subtraction), the image does not contain O. Thus we can define $\delta_f := \mathbf{u} \circ \Delta_f : M^{(2)} \to S^{n-1}$. Again we see that the set of points of S^{n-1} whose vectors parallel to a chord of M is the image under a smooth map; this time of $M^{(2)}$. Since $M^{(2)}$ has dimension $2m$, the result follows as before. □

Theorem 4.2.2 (Whitney's Embedding Theorem) *Let M^m be a smooth compact manifold. Any map of M^m to $\mathbb{R}^{2m+1}$ may be approximated arbitrarily closely by an embedding.*

Since we have not yet discussed topologies for mapping spaces (see §4.4 below), approximation is here to be understood in the sense of pointwise convergence.

Proof Let $f_1 : M^m \to \mathbb{R}^{2m+1}$ be the given map; by Proposition 1.1.7 (applied to each component), we may suppose f_1 a smooth map. By Theorem 4.2.2, we can choose an embedding $f_2 : M^m \to \mathbb{R}^n$ for some n. The product map $f_3 : M^m \to \mathbb{R}^{2m+1+n}$ is an embedding, for since f_2 is an immersion and injective, so is f_3.

By Lemma 4.2.1, the set E of points of S^{2m+n} whose vector is parallel to a tangent or chord is nul, thus its complement is everywhere dense. Choose a point x, close to the unit point on the last axis, and not in E, and project $f_3(M)$ orthogonally in the direction x to $\mathbb{R}^{2m+n}$. The first $2m+1$ coordinates of the projected map f_4 differ from those of f_3, and hence of f_1, by an amount which can be made arbitrarily small by choice of x.

We claim that f_4 is an embedding. For since x is parallel to no chord of $f_3(M^m)$, no two distinct points of M have the same image under f_4; and since x is parallel to no tangent vector, there is no tangent vector which is mapped to zero by df_4. Thus f_4 is an immersion and injective, hence an embedding.

We may now repeat the projection process a further $(n-1)$ times, obtaining ultimately an embedding in $\mathbb{R}^{2m+1}$ with coordinates differing by arbitrarily little from those of f_1. $\square$

Theorem 4.2.3 *Any map of a compact smooth manifold M^m to $\mathbb{R}^{2m}$ may be approximated by an immersion.*

Proof As for Theorem 4.2.2, we obtain an embedding in $\mathbb{R}^{2m+1}$, and then choose $x \in S^{2m}$, arbitrarily close to the unit point on the last axis, and parallel to no tangent vector (which is possible, as before, using Lemma 4.2.1). Projecting parallel to x, we obtain the desired immersion. $\square$

4.3 Existence of non-degenerate functions

Let f be a smooth function on M, and P a critical point of f, so that $df(T_PM) = 0$. If we take local coordinates with P as origin, we have $f(O) = 0$ and $\partial f/\partial x_i$ vanishes at O for $1 \leq i \leq m$. It is now natural to consider the Hessian matrix $(\partial^2 f/\partial x_i \partial x_j)$ of second derivatives of f at O. We regard the Hessian as a symmetric bilinear form $H(f) : T_PM \times T_PM \to \mathbb{R}$, given in local coordinates by

$$H(f)\left(\sum a_i \frac{\partial}{\partial x_i}, \sum b_i \frac{\partial}{\partial x_i}\right) = \sum a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

We can also formulate an equivalent definition without referring to coordinates: given $u, v \in T_PM$, extend v to a local vector field $\mathbf{v}$ defined (at least) in a neighbourhood of P; then $H(f)(u, v) = u(\mathbf{v}(f))$ is independent of the extension $\mathbf{v}$ of v (since P is a critical point). (Recall here that a tangent vector is a mapping of functions on M to the reals, and a vector field maps functions to functions.)

We say that P is a *degenerate (resp. non-degenerate) critical point* of f if $H(f)$ is a singular (resp. nonsingular) bilinear form. Thus P is singular if and only if the matrix $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is; equivalently, if the rows are linearly dependent, i.e. if for some constants λ_i not all zero we have $\sum_i \lambda_i \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ for all j.

We call f *non-degenerate* if it has no degenerate critical point. Many authors call such functions 'Morse functions'.

For $i : M \to \mathbb{R}^n$ an embedding, since we identify $\mathbb{T}(\mathbb{R}^n)$ with $\mathbb{R}^n \times \mathbb{R}^n$, we may identify $\mathbb{N}(\mathbb{R}^n/M)$ with the submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$\mathbb{N}(\mathbb{R}^n/M) = \{(P, v) : P \in M, v \text{ orthogonal to } di(T_P M)\}.$$

Here the exponential map is given by $\exp(P, v) = P + v$ (vector addition).

In general, if M is a submanifold of the complete Riemannian manifold N, a critical value of $\exp : \mathbb{N}(N/M) \to N$ is called a *focus* of M; if the corresponding critical point is a vector at P, it is a focus of M at P. It follows from Sard's theorem 4.1.3 that the set of foci of M in N is nul.

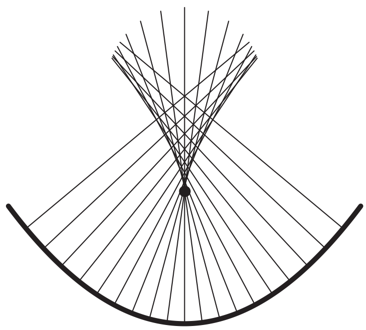

Figure 4.1 A focus

The existence of non-degenerate functions will now follow from the theorem below. Let M be a smooth submanifold of $\mathbb{R}^{m+n}$; for $P \in \mathbb{R}^{m+n}$, define $L_P : M \to \mathbb{R}$ by $L_P(Q) := \|P - Q\|^2$.

Theorem 4.3.1 *L_P has a critical point at $Q \in M$ if and only if the vector $Q - P$ is normal to M at Q. Q is a degenerate critical point if and only if P is a focus of M at Q.*

Proof The first statement is clear. For the second, first suppose M is a curve in $\mathbb{R}^2$. Then a focus must be a point of intersection of consecutive normals, i.e. a centre of curvature. But L_P has a degenerate critical point at Q if and only if $\|P - X\|^2$ is constant to the second order at $X = Q$, i.e. again if and only if P is the centre of curvature of M at Q. The notion of focus of a curve is illustrated in Figure 4.1.

The result holds in general for essentially the same reasons, but for clarity we calculate in convenient coordinates. We may suppose M given in the neighbourhood of Q as the graph B of a map $A : \mathbb{R}^m \to \mathbb{R}^n$ with $A(0) = 0$ and $d_0A = 0$: thus A has components a_r whose Taylor expansions at 0 begin $a_r = \sum_{i,j=1}^m p_r^{i,j} x_i x_j$, where $p_r^{i,j}$ is symmetric in i and j. Differentiating B with respect to x_i gives a vector α_i whose jth component is $\delta_{i,j}$ (i.e. 1 if $i = j$, 0 if not) and with rth component $\frac{\partial a_r}{\partial x_i}$. These span $T_x M$, hence a base for $N_x M$ is given by the vectors β_r with ith component $-\frac{\partial a_r}{\partial x_i}$ and sth component $\delta_{r,s}$.

We now have $\exp(x, \upsilon) = B(x) + \sum_r \upsilon_r \beta_r$. Its derivative with respect to υ_r is β_r; the derivative with respect to x_i has the last n coordinates zero. Thus at a singular point of exp there must be a linear relation of the form $\sum \lambda_i \frac{\partial}{\partial x_i} \exp(x, \upsilon) = 0$ with the λ_i not all zero. This reduces to $\sum_i \lambda_i \frac{\partial}{\partial x_i}(x_j - \sum_r \upsilon_r \frac{\partial a_r}{\partial x_j})$ for each j, so occurs at $x = 0$ if and only if $\lambda_j - 2\sum_{i,r} \lambda_i \upsilon_r p_r^{i,j} = 0$ for each j.

On the other hand, the square of the distance of $B(x)$ from a typical point on $N_Q M$, with coordinates $(0, \ldots, 0, c_1, \ldots, c_n)$ is $\sum_1^m x_i^2 + \sum_1^n (c_r - a_r(x))^2$, whose Taylor expansion at 0 is $\sum_1^n c_r^2 + \sum_1^m x_i^2 - 2\sum_{r,i,j} c_r p_r^{i,j} x_i x_j$.

The quadratic form $q(x) := \sum_1^m x_i^2 - 2\sum_{r,i,j} c_r p_r^{i,j} x_i x_j$ is degenerate if and only if, for some λ_i not all zero, the derivative $\sum_i \lambda_i \frac{\partial q}{\partial x_i}$ vanishes identically, i.e. $2\sum_i \lambda_i x_i - 4\sum_{r,i,j} \lambda_i c_r p_r^{i,j} x_j = 0$, i.e. $2\lambda_i - 4\sum_{r,j} \lambda_i c_r p_r^{i,j} = 0$ for each i. This coincides with the previous condition on setting $c_r = \upsilon_r$. The result follows. □

Corollary 4.3.2 *Any compact manifold M admits non-degenerate functions.*

Proof By Theorem 4.2.2, M can be imbedded in Euclidean space. By Sard's theorem, the set of foci, which are critical values of a smooth map, is nul. So we can choose $P \notin M$ not a focus, and then by the theorem L_P is a non-degenerate function. □

We remark that compactness is inessential, and also that using the approximation clause in Theorem 4.2.2, we could obtain one here.

If $P \notin M$, we can also replace $L_P = \|P - Q\|^2$ by the distance function $\|P - Q\|$.

4.4 Jet spaces and function spaces

We now introduce the methods for studying smooth mappings in general. We begin by introducing the language for describing a mapping locally, near a point.

Two functions f, g each defined on some neighbourhood of a point x of a topological space X have the same *germ* at x if there is a neighbourhood of x on which they take the same value. The definition applies whether the values are real numbers or lie in any space. We talk of germs, or map-germs at (X, x).

Lemma 4.4.1 *Let f, $g : \mathbb{R}^v \to \mathbb{R}^m$ be smooth map-germs at O such that the values of f and all its partial derivatives of orders $\leq r$ agree with those of g at O. Let φ, ψ be diffeomorphisms of $\mathbb{R}^v$, $\mathbb{R}^m$ keeping O fixed. Then the values of $\psi \circ f \circ \varphi$ and all its partial derivatives of orders $\leq r$ agree with those of $\psi \circ g \circ \varphi$ at O.*

Proof The result is an immediate consequence of the chain rules for differentiating a composite: 'a function of a function'. □

For $g, h : (V^v, P) \to M^m$ smooth map-germs, write $g \sim_r h$ at P if, with respect to some local coordinates at P and $g(P)$, we have $g(P) = h(P)$, and all partial derivatives of order $\leq r$ of g and h at P agree. By the lemma, this is independent of the chosen coordinate system. Clearly, $\sim_r$ is an equivalence relation for maps defined on a neighbourhood of P. An equivalence class is called an *r-jet* of maps from V to M at P. The set of all jets of maps of V to M is the *jet space* $J^r(V, M)$.

Each jet is a jet of a smooth map at some $P \in V$, so there is a natural projection $\pi_s : J^r(V, M) \to V$. Similarly (since $r \geq 0$), since two functions g, h with the same r-jet at P have $g(P) = h(P)$, there is another projection $\pi_t : J^r(V, M) \to M$. We call the point $\pi_s(j) \in V$ the *source* of the jet j and the point $\pi_t(j) \in M$ its *target*. The map (π_s, π_t) identifies $J^0(V, M)$ with the product $V \times M$. For any $k \geq r \geq 0$ there is a natural projection $\pi^k_r : J^k(V, M) \to J^r(V, M)$.

In terms of local coordinates $(x_1, \cdots, x_v)$ on V at P, $(y_1, \cdots, y_m)$ on M at Q, since two functions with the same partial derivatives define the same jet, we may take these partial derivatives as coordinates in $J^r(V, M)$. If $\omega = (\omega_1, \cdots, \omega_v)$ is a string of non-negative integers, write

$$\begin{aligned} x^\omega &= (x_1^{\omega_1} \cdots x_v^{\omega_v}), & \partial_\omega &= (\partial/\partial x_1)^{\omega_1} \cdots (\partial/\partial x_v)^{\omega_v}, \\ |\omega| &= \omega_1 + \cdots + \omega_v, & \omega! &= \omega_1! \cdots \omega_v!. \end{aligned}$$

Then if f is a smooth map-germ on (V, P) to M with target Q, its partial derivatives of order $\leq r$ are the numbers $u_j^\omega = \partial_\omega y_j$ $(0 \leq |\omega| \leq r, 1 \leq j \leq m)$, and these values determine the r-jet of f at P. We sometimes write y_j for the constant term u_j.

Conversely, given a set of numbers a_j^ω (where the point (a_j) must lie in the prescribed neighbourhood of Q), there exists a corresponding smooth map-germ: we may choose the polynomials $y_j = \sum_{0 \leq |\omega| \leq r} a_j^\omega x^\omega / \omega!$. Hence the set of r-jets j with source P and target Q is isomorphic to a Euclidean space. We can take (x_i, u_j^ω) as local coordinate system in $J^r(V, M)$, and coordinate changes are smooth (they exhibit, again, the chain rule for partial differentials: we shall spare the reader a detailed exhibition of them). We conclude that $J^r(V, M)$ is a smooth manifold, and the projections π_s and π_t are smooth maps.

The above polynomial is called the *polynomial representative* of the r-jet (in the given coordinates). It agrees with the sum of terms of degree $\leq r$ in the Taylor series expansion of f in the given coordinates. We are not concerned here with the question of convergence of this series.

For $f : V \to M$ a smooth map, the equivalence class of f at a point $P \in V$ is an r-jet at P, so f defines a cross-section $j^r f : V \to J^r(V, M)$ to π_s, which is smooth since f (and hence all its partial derivatives) is. Here the restriction to infinitely differentiable maps allows simpler statements: if g is a C^N map (with continuous partial derivatives of order $\leq N$), then $j^r g$ is C^{N-r} for $r \leq N$.

We can calculate the derivative of $j^r f$: the following result will be used explicitly below.

$$dj^1 f\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} + \sum_j u^i_j \frac{\partial}{\partial y_j} + \sum_{j,k} u^{ik}_j \frac{\partial}{\partial u^k_j}. \tag{4.4.2}$$

For $dj^2 f(\partial/\partial x_i)$ we add a further sum $\sum_{j,k,l} u^{ikl}_j \partial/\partial u^{kl}_j$, and so on.

Since $J^0(V, M) \cong V \times M$, $j^0 f$ is just the graph of f. A 1-jet with source P and target Q is determined by these points and a linear map $T_P V \to T_Q M$, and $j^1 f(P) = (P, f(P), df_P)$.

One can also consider jets at more than one point. We define ${}_rJ^k(V, M)$ to be the subset of the r-fold direct product $(J^k(V, M))^r$ consisting of r-tuples $(j_1, \ldots, j_r)$ such that the source points of the j_i are all distinct. We do not insist that the targets are distinct, and indeed we are largely interested in the case when they are not. Extending the notation $M^{(2)}$ of §4.2, write $V^{(r)}$ for the set (the configuration space) of ordered r-tuples of *distinct* points of V. Then the Cartesian power $(j^k f)^r : V^r \to (J^k(V, M))^r$ induces a map ${}_rj^k f : V^{(r)} \to {}_rJ^k(V, M)$. We call ${}_rJ^k(V, M)$ the *multijet space* and ${}_rj^k f$ the *multijet* of f.

We use jets to define topologies on spaces of smooth maps. One standard topology on function spaces is the so-called *compact-open topology*, which we call the C^0 topology. This is the topology on the space $C^0(X, Y)$ of continuous maps $X \to Y$ defined by taking the sets

$$A(K, U) := \{f \mid f(K) \subset U\} \quad \text{with } K \subset X \text{ compact, } U \subset Y \text{ open}$$

as a sub-base of open sets. It can be described as the topology of uniform convergence of f on compact sets.

There is also the *fine topology* (or fine C^0 topology), which we define by taking the

$$B(U) := \{f \mid (1 \times f)(X) \subset U\} \quad \text{with } U \text{ open in } X \times Y$$

as a base of open sets.

For smooth manifolds V^v and M^m, write $C^r(V, M)$ for the set of maps $V \to M$ whose restrictions in any local coordinates have continuous partial derivatives of all orders $\leq r$; in particular, $C^\infty(V, M)$ is the set of smooth maps of V to M. Taking r-jets gives an injective map $j^r : C^r(V, M) \to C^0(V, J^r(V, M))$. The topology on $C^r(V, M)$ induced by regarding it as a subspace of $C^0(V, J^r(V, M))$ with the compact-open topology is called the C^r *topology*, and the topology induced from the fine topology is the *fine* C^r *topology*.

The inclusion of $C^\infty(V, M)$ in $C^r(V, M)$ induces topologies on it, and we define the C^∞ *topology* to be the union of the C^r topologies, in the sense that a set is open if it is open in one of these topologies. Correspondingly, the fine C^∞ topology, which we christen the W^∞ *topology*, is the union of the fine C^r topologies.

Properties of these topologies are discussed in Appendix A.4. We summarise some key results:

Both topologies on $C^\infty(V, M)$ are completely regular. They agree if V is compact.

With the C^∞ topology, $C^\infty(V, M)$ is a complete metric space. However, a sequence of maps convergent for the W^∞ topology is eventually constant outside a compact set; hence this topology is neither metrisable nor even locally countable.

The space $C^\infty_{pr}(V, M)$ of proper C^∞ maps is open in $C^\infty(V, M)$ in the W^∞ topology.

The composition map $C^\infty(V, M) \times C^\infty(M, N) \to C^\infty(V, N)$ is continuous for the C^∞ topologies; however for the W^∞ topologies this fails unless V is compact: more precisely, for the W^∞ topologies, $C^\infty_{pr}(V, M) \times C^\infty(M, N) \to C^\infty(V, N)$ is continuous, and the map $C^\infty(M, N) \to C^\infty(V, N)$ defined by composition with $f : V \to M$ is continuous if and only if f is proper.

Lemma 4.4.3 *If U is open in $J^k(V, M)$, the set of $f : V \to M$ with $j^k f(V) \subset U$ is open in $C^\infty(V, M)$ in the W^∞ topology. If K is a compact subset of V, the set of $f : V \to M$ with $j^k f(K) \subset U$ is open in $C^\infty(V, M)$ in the C^∞ topology.*

This follows directly from the definitions of the topologies, and explains why we need the W^∞ topology. In particular, since immersions are just the maps whose 1-jet takes values in the open subset of $J^1(V, M)$ with df_P injective, it follows that the set $\mathrm{Imm}(V, M)$ of immersions is open in $C^\infty(V, M)$ in the W^∞ topology.

It can be shown (see, for example, [73, 2.1.4]) that the set $\mathrm{Emb}(V, M)$ of smooth embeddings is open in $C^\infty(V, M)$ in the W^∞ topology. We will see in Corollary 4.6.4 that the set of injective immersions is open, which will suffice for our purposes. It follows from this using the openness of $C^\infty_{pr}(V, M)$ that the set of closed embeddings is open, and hence taking $V = M$ that the set $\mathrm{Diff}(M)$ of diffeomorphisms of M is open.

The following result ties up the notion of approximation in function space with more geometrical notions of equivalence.

Proposition 4.4.4 *If V is a compact manifold and $f : V \to M$ an embedding, there is a neighbourhood $\mathcal{U}$ of f in $C^\infty(V, M)$ such that for any $g \in \mathcal{U}$, g is an embedding and f and g are ambiently diffeotopic.*

Proof Choose a neighbourhood W of $\Delta(M)$ in $M \times M$ and a map $H : W \times [0, 1] \to M$ as in Corollary 2.2.5. Now choose a neighbourhood $\mathcal{U}$ of f such that

(i) for all $g \in \mathcal{V}$ and all $P \in V$, $(f(P), g(P)) \in W$, so we can define a smooth map f_t by $f_t(P) = H(f(P), g(P), t)$,

(ii) with the same notation, for each $t \in [0, 1]$, f_t is a smooth embedding.

Then f_t is a diffeotopy of f to g, and by Theorem 2.4.2, this diffeotopy is ambient. □

A topological space W is said to have the *Baire property*, or to be a Baire space, if the intersection of any countable family of dense open subsets of W is dense. By Baire's Theorem A.4.5, any complete metric space has the Baire property.

In a Baire space, a countable intersection of dense open sets is called a *residual set*. It is not in general open: in examples, to prove openness, further work is required.

Since it has a complete metric, $C^\infty(V, M)$ with the C^∞ topology has the Baire property. The result for the fine C^∞ topology also holds: by Theorem A.4.9, if F is any subspace of $C^\infty(V, M)$ which is closed in the C^∞ topology then F, with either the C^∞ topology or the W^∞ topology, has the Baire property.

From now on, unless explicitly stated otherwise, we use the W^∞ topology on function spaces.

4.5 The transversality theorem

Let V^v, M^m be smooth manifolds, and let N^n be a submanifold of M^m. We say that a smooth map $f : V \to M$ is *transverse* to N if for every $P \in V$ such that $f(P) = Q \in N$, we have $df(T_P V) + T_Q N = T_Q M$. Equivalently, this states that df induces an epimorphism of $T_P V$ on $T_Q M/T_Q N$.

If $\dim V < \operatorname{codim} N$, the map df cannot be surjective: in that case transversality requires $f(V)$ to be disjoint from N.

The following result gives some indication of the geometrical meaning of the condition.

Lemma 4.5.1 *Let $f : V \to M$ be transverse to a submanifold N of M. Then $f^{-1}(N) = W$ is a submanifold of V, whose codimension equals that of N in M. Moreover, $df_P : T_P V \to T_{f(P)} M$ induces an isomorphism of the normal space $N_P(V/W)$ to W in V at P with the normal space $N_{f(P)}(M/N)$ of N in M at $f(P)$.*

Proof Let $P \in V$, $f(P) = Q \in N$, and let N be locally defined at Q by $x_1 = \cdots = x_c = 0$, where the x_i have linearly independent differentials at Q, and $c = \operatorname{codim} N$. Then, by transversality, the functions $x_1 \circ f, \cdots, x_c \circ f$ have linearly independent differentials at P, and their vanishing defines W near P. That W is a smooth submanifold follows using Corollary 1.2.6, as in the proof of Proposition 1.2.10. The same calculation gives the isomorphism of the normal spaces. □

We extend the concept as follows. Let N be a submanifold of $J^r(V, M)$. Then we say that f is transverse to N if $j^r f$ is so.

Lemma 4.5.2 *If K is a closed subset of V, and N a closed submanifold of $J^r(V, M)$, the set of maps which are transverse to N at all points of K is open in $C^\infty(V, M)$ in the W^∞ topology; if K is compact, it is also open in the C^∞ topology.*

Proof The differential of $j^r f$ is determined by the partial derivatives of f of order $\leq (r + 1)$, and hence by $j^{r+1} f$. Since the set of linear maps $\mathbb{R}^v \to \mathbb{R}^m$ which *fail* to be transverse to a given subspace of $\mathbb{R}^m$ is defined by the vanishing of some determinants, it is a closed subset. Thus the subset of $J^{r+1}(V, M)$ of jets of maps transverse to N is open. The conclusion now follows from Lemma 4.4.3. □

The transversality theorem states that the set of maps transverse to N is dense. The full proof is somewhat technical, but the following simple idea lies at its heart.

Lemma 4.5.3 *Let N be a submanifold of M, and let $F : V \times U \to M$ be transverse to N (for example, a submersion). Then for a dense set of $u \in U$ the map $f_u : V \to M$ given by $f_u(x) = F(u, x)$ is transverse to N.*

Proof Since F is transverse to N, by Lemma 4.5.1, $W := F^{-1}(N)$ is a submanifold of $V \times U$. Denote by φ the composite $W \subset V \times U \to U$. By Sard's Theorem 4.1.3, the set of critical values of φ is nul, so for a dense set of $u \in U$, u is a regular value of φ. We claim that for such u, f_u is transverse to N.

If u is a regular value of φ and $f_u(P) = Q$ lies in N, then $(P, u) \in W$, so $d\varphi(T_{(P,u)}W) = T_uU$. Thus W meets $V \times \{u\}$ transversely at (P, u). But this implies that f_u is transverse at P to N. □

This leads to a plan for proving the jet transversality result. First define the partial jet map $j_1^rF : V \times U \to J^r(V, M)$ of a family $F : V \times U \to M$ by $j_1^rF(v, u) := j^r f_u(v)$, where $f_u(v) := F(v, u)$. Then seek to embed f in a family $F : V \times U \to M$ such that the partial jet map j_1^rF is a submersion, and hence transverse to N. Then the set of u with f_u transverse to N is dense in U.

It is not so easy to construct such a family directly, but we can do it near a point, and will then be able to obtain the full result using the Baire property. We develop the local results in a lemma.

Lemma 4.5.4 *Let $f : V^v \to M^m$ be a smooth map, $j^r f(P) = Q$. Then we can find:*

a neighbourhood $\mathcal{W}$ of f in $C^\infty(V, M)$,
a coordinate neighbourhood (U_1, φ_1) of P in V,
and a coordinate neighbourhood (U_2, φ_2) of Q in $J^r(V, M)$,

such that for each $g \in \mathcal{W}$ there is a family $G : V \times Y \to M$ with $G_0 = g$, each $g_u \in \mathcal{W}$, and such that the restriction to $U_1 \times Y$ of the partial jet map j_1^rG takes values in U_2 and is a submersion.

Proof Choose coordinate neighbourhoods of P with $\bar{U}_1 \subset U_1'$ and a chart $\varphi_2 : U_2 \to \mathbb{R}^m$ of $f(P)$ in M. Let B be a C^∞ function on V to $[0, 1]$, vanishing outside U_1', and with $B(U_1) = 1$.

Let ε be such that $y \in \mathbb{R}^m$, $\|y\| < \varepsilon$ implies that y is in the image of φ_2. Let $\mathcal{W}_1$ be the set of $g \in C^\infty(V, M)$ such that for all $x \in U_1'$, $\|\varphi_2(f(x))\| < \varepsilon/3$.

Let Y be the set of polynomial maps $y : \mathbb{R}^v \to \mathbb{R}^m$ of degree $\leq r$, and let Y' be the subset such that for $x \in \varphi_1(U_1')$, we have $\|y(x)\| < \varepsilon/3$.

For $g \in \mathcal{W}$ define $G' : U_1' \times Y' \to \mathbb{R}^m$ by $G'(x, y) := g(x) + B(x)y(x)$. Since this takes values y with $\|y\| < \varepsilon$, it lifts under φ_2 to a map $G'' : U_1' \times Y' \to M$. Now define $G : V \times Y' \to M$ by $G(P, y) = G''(P, y)$ if $P \in U_1'$ and $G(P, y) = g(P)$ otherwise.

We claim that $j_1^r G$ restricts to a submersion of $U_1 \times Y'$ to $J^r(V, M)$. For on this subset, G is given in local coordinates by $G(x, y) := g(x) + y(x)$. At $x = 0$, this has Taylor series the sum of those of g and y. But by construction, the tangent space to Y' is Y, essentially the same as the fibre of $\pi_s : J^r(V, M) \to V$, so the derivatives with respect to the y-coordinates span the tangent space to the fibre. Since $j^r g$ is a section of π_s, the derivatives with respect to the x-coordinates span the tangent space to V. Thus the sum is indeed a submersion. The same result holds for points $x \neq 0$ since although the Taylor expansion at x is not the same as at 0, the space of all polynomials of degree $\leq r$ is the same. $\square$

Corollary 4.5.5 *Let $f : V^v \to M^m$ be a smooth map, and let N be a submanifold of $J^r(V, M)$ of codimension p. Let $j^r f(P) = Q \in N$. Then we can find a coordinate neighbourhood U_1 of P in V, a coordinate neighbourhood U_2 of Q in $J^r(V, M)$, and an open neighbourhood $\mathcal{W}$ of f in $C^\infty(V, M)$ such that*

(a) For $g \in \mathcal{W}$, $j^r g(\bar{U}_1) \subset U_2$.

(b) For every $g \in \mathcal{W}$, there are maps h arbitrarily close to g in $C^\infty(V, M)$ such that $j^r h|U_1$ is transverse to N.

Proof Define $\mathcal{W}$ and construct G as above. Since $j_1^r G$ gives a submersion of $U_1 \times Y'$ to $J^r(V, M)$, by Lemma 4.5.3, there exist $y \in Y$ arbitrarily close to 0 such that $j^r g_y \,|\, U_1$ is transverse to N. $\square$

We can now prove the general theorem.

Theorem 4.5.6 (Transversality Theorem) *Let N be a submanifold of $J^r(V, M)$. The set of maps $f : V \to M$ transverse to N is dense in $C^\infty(V, M)$; if N is closed, it is also open.*

Proof First let K be a compact subset of V. Then K can be covered by a finite number of the neighbourhoods U_1^α of the lemma. The intersection of the corresponding sets $\mathcal{W}_\alpha$ is an open neighbourhood $\mathcal{W}$ of f, and the subset of $\mathcal{W}$ of functions g with $g|U_1^\alpha$ transverse to N is dense in $\mathcal{W}$, by the lemma. Since by Theorem A.4.10 the open set $\mathcal{W}$ has the Baire property, the set of $g \in \mathcal{W}$ with $g|K$ transverse to N is also dense in $\mathcal{W}$. Since this holds for some neighbourhood $\mathcal{W}$ of any f, the set $\mathcal{T}$ of $g \in C^\infty(V, M)$ with $g|K$ transverse to N is dense in $C^\infty(V, M)$. Also, $\mathcal{T}$ is open by Lemma 4.5.2.

Since V may be covered by a countable family of compact sets K, the density result follows since $C^\infty(V, M)$ has the Baire property. Openness is given by Lemma 4.5.2. $\square$

The following addendum is often useful in applications, usually taking $X = \partial V$. For $f : V \to M$ and $X \subset V$ denote by $C^\infty(V, M; f, X)$ the set of $g \in C^\infty(V, M)$ with $g|X = f|X$.

Proposition 4.5.7 *Let N be a submanifold of $J^r(V, M)$, X a closed subset of V, $f : V \to M$ transverse to N along X. The set of maps $g \in C^\infty(V, M; f, X)$ transverse to N is dense in $C^\infty(V, M; f, X)$; if N is closed, it is also open.*

This follows from the same argument on making two changes. First, as well as the sets U_α above, we choose open sets U_β which cover X and are such that f is transverse to N along U_β: we then define $\mathcal{W}_\beta$ to be the open set of g transverse to N along U_β. Secondly, note that by Theorem A.4.9, $C^\infty(V, M; f, X)$ is a Baire space.

The Transversality Theorem is the general tool for proving 'general position' arguments in differential topology, and admits a wide variety of applications. We spend some time giving such examples, beginning with the simplest.

The following easy application seems worth formulating explicitly.

Corollary 4.5.8 *Given two embeddings $f : V \to M$ and $f' : V' \to M$, we can perturb f by an arbitrarily small diffeotopy to a map transverse to f'.*

In general, the set of f satisfying a transversality condition is residual; by further applications of Baire's theorem, we see that the set of f satisfying a finite, or even a countable, number of conditions of the above type is residual, hence dense. Thus given a countable family of submanifolds of various $J^k(V, M)$, the set of maps transverse to all of them is a residual set. Moreover, for those submanifolds of codimension $> v$, we know that transversality means that $j^k f$ avoids these submanifolds. In particular,

Lemma 4.5.9 *Given a finite or countable collection of submanifolds A_α of M, each of dimension $< (m - v)$, the set of maps $f : V^v \to M^m$ with $f(V) \cap \bigcup_\alpha A_\alpha = \emptyset$ is residual in $C^\infty(V, M)$.*

Any embedding $V \to M$ is diffeotopic to one avoiding all the A_α.

The first assertion is an immediate consequence of the theorem, since transversality to A_α implies that the two are disjoint. The second follows by Lemma 4.4.4.

The following was an early application of transversality.

Proposition 4.5.10 *Let $f : V \to M$ be a smooth map, N a submanifold of $J^k(V, M)$, and suppose F closed in V such that $f|F$ is transverse to N, then f can be approximated by g, transverse to N, and with $g|F = f|F$.*

Proof First, by Proposition 2.3.4 (i), we can approximate f by a smooth map g, and by (ii) of that result, we may suppose that g agrees with f on F.

The result now follows from Proposition 4.5.7. □

A case of particular importance is where V has boundary and we take $F = \partial V$. Even the case $k = 0$, where we seek transversality to submanifolds of M, is significant, and is useful for applications to cobordism theory.

In many cases, we can show that the intersection is not only dense, but open. Suppose we have a finite collection of submanifolds A_i of $J^k(V, M)$. To say that $j^k f$ is transverse to A_i can be regarded as having $j^{k+1} f$ avoid a certain subset, N_i, say, of $J^{k+1}(V, M)$. If the set $F := \bigcup_i N_i$ of non-transverse jets is closed, then by Lemma 4.5.2, the set of maps transverse to all the A_i is indeed open.

A collection of submanifolds A_i of a manifold B is said to be *A-regular* in the sense of Whitney if for each sequence $x_n \in A_i$ converging to a limit $y \in A_j$ and such that the tangent spaces $T_{x_n} A_i$ converge to a limit τ we have $T_y A_j \subset \tau$.

Lemma 4.5.11 *Suppose $\{A_i\}$ a finite A-regular collection of submanifolds of $J^k(V, M)$ with $\bigcup_i A_i$ closed. Then the set $F := \bigcup_i N_i$ of non-transverse jets is closed. Hence the set of maps in $C^\infty(V, M)$ transverse to all the A_i is open.*

Proof Suppose the condition is satisfied but that there is a sequence ξ_n of jets in F with limit $\eta \notin F$. Passing to a subsequence, we may suppose that all the $x_n = \pi_k^{k+1}(\xi_n)$ belong to the same submanifold A_i and that the sequence $T_{x_n} A_i$ of tangent spaces converges to a limit, τ say. Since $\bigcup_i A_i$ is closed, the limit $y = \pi_k^{k+1}(\eta)$ of the x_n belongs to A_j for some j. Since A-regularity holds, $T_y A_j \subset \tau$.

Now ξ_n induces a 1-jet of maps $V \to J^k(V, M)$ and hence a map $d\xi_n : T_{\pi_s(\xi_n)} V \to T_{x_n} J^k(V, M)$, and since $\xi_n \in N_i$, we have $d\xi_n(T_{\pi_s(\xi_n)} V) + T_{x_n} A_i \neq T_{x_n} J^k(V, M)$. Since the ξ_n converge to η, it follows that $d\eta(T_{\pi_s(\eta)} V) + \tau \neq T_y J^k(V, M)$. Hence a fortiori $d\eta(T_{\pi_s(\eta)} V) + T_y A_j \neq T_y J^k(V, M)$, thus $\eta \in N_j \subset F$, a contradiction. □

In the case $k = 0$, where we are given a collection of submanifolds of M, there is even a converse result. We do not give the statement; the crucial point is that any linear map $T_x V \to T_y M$ occurs as a 1-jet. It is however far from true that any linear map $T_x V \to T_y J^k(V, M)$ is induced by a $(k + 1)$-jet, on account of the symmetry of higher derivatives.

Stratifications give important examples of collections of submanifolds, and A-regularity is often defined in this context.

We now define some submanifolds of jet space: the most important are spaces of 1-jets. Recall that a 1-jet with source $P \in V^v$ and target $Q \in M^m$ is determined by the points P, Q and a linear map $g : T_P V \to T_Q M$. We partition these according to the rank of the linear map g: it is traditional to write $\Sigma^i(V, M)$ for the set of 1-jets (P, Q, g) such that the rank of g is $v - i$. Write also $\Sigma^i(f) := \{P \in V \mid j^1 f(P) \in \Sigma^i(V, M)\}$. Since the rank takes values from 0 to $\min(v, m)$, Σ^i is empty unless

if $v \geq m$, we have $v \geq i \geq v - m$;
if $v \leq m$, we have $v \geq i \geq 0$.

Lemma 4.5.12 *(i) The set of $(v \times m)$ matrices of rank $(v - i)$ is a smooth submanifold of codimension $i(m - v + i)$ in the space of matrices.*

(ii) $\Sigma^i(V, M)$ is a smooth submanifold of codimension $i(m - v + i)$ in $J^1(V, M)$.

Proof (i) In an open subset of the space of matrices, the first $v - i$ columns are linearly independent. The condition for rank $v - i$ is then that the remaining $m - v + i$ columns each lie in a subspace of $\mathbb{R}^v$ of codimension i. The same argument applies if we use a different set of columns.

(ii) Using local coordinates with $U_1 \subset V$ and $U_2 \subset M$, we see that the result holds in the preimage of any $U_1 \times U_2$. □

Thus the Σ^i form a stratification of matrix space, and the $\Sigma^i(V, M)$ a stratification of $J^1(V, M)$. We may think of the closure of Σ^i as a submanifold with singularities: it is the union of the Σ^j with $j \geq i$, and is a variety in the sense of algebraic geometry. A first step in putting a map f into general position is to make it transverse to the Σ^i. This is facilitated by

Lemma 4.5.13 *The stratification Σ^i is A-regular.*

Proof It suffices to consider the submanifolds of the space of matrices, since $J^1(V, M)$ is locally a product of V, M, and $\mathrm{Hom}(T_x V, T_y M)$.

We first show that the tangent space to Σ^i at a map $\phi \in \Sigma^i$ can be decomposed as a sum $S_1 + S_2$, where S_1 is the set of linear maps ψ with $\psi(\mathrm{Ker}\,\phi) = 0$ and S_2 the set of those with $\mathrm{Im}\,\psi \subset \mathrm{Im}\,\phi$. We can take coordinates such that the matrix of ϕ is in normal form. Then the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with A nonsingular and $r \times r$, has rank r if and only if $D = CA^{-1}B$. If we take $A - I$, B, C and D as infinitesimals, then to the first order this condition becomes $D = 0$. Thus we have the sum of the subspaces S_1 ($B = D = 0$) and S_2 ($C = D = 0$).

Consider a sequence $\psi_n \to \phi$ with all ψ_n of the same rank. We may suppose that both $\mathrm{Ker}\,\psi_n$ converges to a limit K and $\mathrm{Im}\,\psi_n$ converges to a limit L.

Then $K \subset \operatorname{Ker} \phi$ and $\operatorname{Im} \phi \subset L$. We need to show that the tangent space at ϕ is contained in the limit, which is the sum of the set of maps with kernel containing K and that with image contained in L. But this now follows. $\square$

Corollary 4.5.14 *The set of maps $f : V \to M$ with $j^1 f$ transverse to each Σ^i is open in $C^\infty(V, M)$.*

This follows from Lemmas 4.5.13 and 4.5.11.

4.6 Multitransversality

In general, applying the transversality theorem allows us to control the behaviour of a map $f : V \to M$ near a point of V. However, to describe the image of f we must contemplate pairs of points of V with a common image, and multitransversality is designed to enable us to do this.

An advantage of the above proof of the transversality theorem is that the version of Lemma 4.5.4 for multijets is an immediate consequence, so the same argument now leads to the multitransversality theorem.

Theorem 4.6.1 (Multitransversality Theorem) *Let N be a submanifold of ${}_rJ^k(V, M)$. The set of maps $f : V \to M$ such that ${}_rj^k f$ is transverse to N is residual in $C^\infty(V, M)$.*

Proof We follow the same plan as for Theorem 4.5.6.

Step 1: As for Lemma 4.5.4, given a smooth map $f : V^v \to M^m$ and points P_j $(1 \leq j \leq r)$ in V, write $j^k f(P_j) = Q_j$. By that lemma, we have neighbourhoods $\mathcal{W}_j$ of f in $C^\infty(V, M)$, coordinate neighbourhoods (U_{P_j}, φ_j) of P_j in V, and coordinate neighbourhoods (U_{Q_j}, ψ_j) of Q_j in $J^k(V, M)$ such that for each $g \in \mathcal{W}_j$ there is a family $G_j : V \times K_j \to M$ with $G_{j,0} = g$, each $G_{j,u} \in \mathcal{W}_j$, and such that the restriction to $U_{P_j} \times K_j$ of the partial jet map $j^k_1 G_j$ takes values in U_{Q_j} and is a submersion.

Since the P_j are distinct, we may suppose their neighbourhoods disjoint, and since G_j agrees with g outside a neighbourhood of P_j, for $g \in \mathcal{W} := \bigcap_j \mathcal{W}_j$ we may combine these deformations to $G : V \times K_0 \to M$ (with $K_0 := \prod_j K_j$), where the value near P_j is given by G_j. Then the restriction to $\prod_j U_{P_j} \times K_0$ of the partial jet map ${}_rj^k_1 G$ takes values in $\prod_j U_{Q_j}$ and is a submersion.

Step 2: follow Corollary 4.5.5. We are now given a submanifold N of ${}_rJ^k(V, M)$. Let ${}_rj^k f(P_1, \ldots, P_r) = (Q_1, \ldots, Q_r) \in N$. For $g \in \mathcal{W}$ we construct G as above. Now since ${}_rj^k_1 G$ gives a submersion to ${}_rJ^k(V, M)$, by Lemma 4.5.3 there exist $k \in K$ arbitrarily close to 0 such that $j^r g_k \mid \prod_j U_j$ is transverse to N.

Step 3: By Lemma 1.1.6(i) (adapted to r-tuples), a compact subset K of $V^{(r)}$ can be covered by a finite number of sets $\prod_j U_j^\alpha$ with the U_j^α compact and disjoint. The intersection of the corresponding sets $\mathcal{W}_\alpha$ is an open neighbourhood $\mathcal{W}$ of f, and the subset of $\mathcal{W}$ of functions g with $g|\prod_j U_j^\alpha$ transverse to N is dense and open in $\mathcal{W}$. It follows using the Baire property that the subset $\mathcal{T}$ of g with $g|K$ transverse to N is also dense in $\mathcal{W}$, and since this holds for some neighbourhood $\mathcal{W}$ of any f, is dense in $C^\infty(V, M)$; in fact, a residual set.

The result follows by another application of the Baire property. □

Unlike Theorem 4.5.6, the set given by an application of Theorem 4.6.1 is almost never open. In applications, we often want to prove we have an *open* subset of mapping space, not just a dense one. It is thus necessary in some way to 'fill in' the diagonal. This is usually accomplished by combining the multitransversality condition with a simple transversality condition.

Lemma 4.6.2 *Let A be a closed submanifold of ${}_2J^k(V, M)$ and U an open neighbourhood of $\Delta(V)$ in $V \times V$. Then the set of $f \in C^\infty(V, M)$ with ${}_2j^k f \mid (V^{(2)} \setminus U)$ transverse to A is open in $C^\infty(V, M)$.*

Proof By Lemma 1.1.6(ii), we can find a countable collection of pairs of disjoint compact sets (K_α, K'_α) in V such that $\{K_\alpha, K'_\alpha\}$ is locally finite in V, and such that the $\bigcup_\alpha (K_\alpha \times K'_\alpha) \supseteq V^{(2)} \setminus U$.

The condition that ${}_2j^k f$ is transverse to A at all points of the closed subset $(K_\alpha \times K'_\alpha) \setminus U$ defines an open set in $C^\infty(K_\alpha \times K'_\alpha, M)$ by Lemma 4.5.2, and hence in $C^\infty(V, M)$, since the restriction map $C^\infty(V, M) \to C^\infty(K_\alpha \times K'_\alpha, M)$ is continuous (for as $K_\alpha \times K'_\alpha$ is compact, its inclusion in V is proper).

Now we have a countable family of open conditions on the restrictions of f to members of a locally finite cover of V, so by the definition of the fine topology, the intersection again gives an open set. □

We now have

Proposition 4.6.3 *Suppose $\mathcal{W}$ an open subset of $C^\infty(V, M)$ and A a closed submanifold of ${}_2J^k(V, M)$; write $\mathcal{W}^*$ for the set of $f \in \mathcal{W}$ with ${}_2j^k f$ transverse to A.*

Suppose that, for each $f \in \mathcal{W}^*$, each $x \in V$ has a neighbourhood U_x such that $\{g \in \mathcal{W} \mid {}_2j^k g \mid U_x^{(2)}$ is transverse to $A\}$ *is a neighbourhood of f.*

Then $\mathcal{W}^*$ is open in $C^\infty(V, M)$.

Proof We first show that, for each $f \in \mathcal{W}^*$, there exist a neighbourhood U_f of $\Delta(V)$ in $(V \times V)$ and an open neighbourhood $\mathcal{W}_f$ of f in $\mathcal{W}$ such that, for all $g \in \mathcal{W}_f$, ${}_2j^k g(U_f \setminus \Delta(V)) \cap A = \emptyset$. By hypothesis we have a neighbourhood

U_x for each $x \in V$; we may suppose these open. Since they cover V, we can pick a locally finite refinement $\{U_\alpha\}$. We set $U_f := \bigcup_\alpha (U_\alpha \times U_\alpha)$. By hypothesis, the set of maps $g \in \mathcal{W}$ satisfying the condition on $U_x^{(2)}$ contains an open neighbourhood of f; the same follows for $U_\alpha^{(2)}$. But by the properties of the fine topology, the intersection $\mathcal{W}_f$ of a family of open sets defined by conditions on members of a locally finite family of subsets U_α of V is open in the W^∞ topology.

By Lemma 4.6.2, the set X_G of maps with ${}_2j^k f \mid (V \times V \setminus U_f)$ transverse to A is open in $C^\infty(V, M)$, so $\mathcal{W}_f \cap X_G$ is open. But this is a neighbourhood of f in $\mathcal{W}^*$. □

Corollary 4.6.4 *The set of injective immersions is open in $C^\infty(V, M)$.*

Proof We can take $\mathcal{W}$ as the set of immersions $V \to M$ and $\mathcal{W}^*$ as the set of injective immersions: then it suffices to show that, for each $f \in \mathcal{W}^*$, each $x \in V$ has a neighbourhood U_x such that $\{g \in \mathcal{W} \mid g \,|U_x$ is injective$\}$ is a neighbourhood of f.

But this is clear: we can take coordinates at x and $f(x)$ in which $f| \,U_x$ is the inclusion of the unit disc U in $\mathbb{R}^v$ into $\mathbb{R}^m$; then the maps whose restriction to a closed disc of smaller radius project immersively to $\mathbb{R}^v$ form an open set (we have a compact subset of V and an open subset of $J^1(V, M)$). □

Given two subspaces P_1, P_2 of a vector space Q, we say that they are transversal if $P_1 + P_2 = Q$: this condition is stable under perturbations. The corresponding condition for a set of several subspaces P_i of Q is less familiar. We require each P_i to be transverse to the intersection of the others. The neat formulation is that the set $\{P_i\}$ of linear subspaces of Q is *mutually transversal* if the diagonal map from Q to $\bigoplus_i (Q/P_i)$ is surjective; equivalently, if the map from $Q \bigoplus_i P_i$ to $\bigoplus_i Q$, where the first summand is mapped by the diagonal, is surjective.

All our explicit applications of the multitransversality Theorem 4.6.1 follow a common pattern. Suppose we have submanifolds A_i $(1 \le i \le r)$ of jet space $J^k(V, M)$: then define $(A_1, \ldots, A_r)_\Delta$ to be the submanifold of ${}_rJ^k(V, M)$ of multijets $(j_1, \ldots, j_r)$ with each $j_i \in A_i$, and all $\pi_t(j_i)$ equal. (For convenience, we take all submanifolds in the same jet space, but if $k < l$, the preimage of a submanifold $A \subset J^k(V, M)$ in $J^l(V, M)$ is a submanifold A^* of the same codimension, and f is transverse to A^* if and only if it is transverse to A.) Observe that

$$\operatorname{codim}(A_1, \ldots, A_r)_\Delta = \sum_i \operatorname{codim}(A_i) + (r-1)m.$$

For $f : V \to M$, write $A_i(f) := \{x \in V \mid j^k f(x) \in A_i\}$.

Lemma 4.6.5 *Suppose $P_i \in V$ with $j^k f(P_i) \in A_i$ and $j^k f$ transverse to A_i at P_i for each i, and each $f(P_i) = Q$. Then ${}_rj^k f$ is transverse at $(P_1, \ldots, P_r)$ to*

$(A_1, \dots, A_r)_\Delta$ *if and only if the subspaces* $df(T_{P_i}A_i f)$ *of* $T_Q M$ *are mutually transversal.*

Proof Write $j_i := j^k f(P_i)$. The tangent space at $(j_1, \dots, j_r)$ to $(A_1, \dots, A_r)_\Delta$ is the pullback of the diagonal under the projection $\bigoplus_i T_{j_i}A_i \to \bigoplus_i T_Q M$. Thus transversality holds, i.e. $T(A_1, \dots, A_r)_\Delta \bigoplus_i T_{P_i}V$ maps onto $\bigoplus_i (T_{j_i}J^k)$ if and only if the map $T_Q M \bigoplus T_{P_i}V \bigoplus T_{j_i}A_i \longrightarrow \bigoplus (T_Q M \oplus T_{j_i}J^k)$ is surjective.

Since transversality holds at each P_i, $T_{P_i}V \oplus T_{j_i}A_i$ surjects to $T_{j_i}J^k$, and we have $T_{P_i}(A_i(f)) = \mathrm{Ker}(T_{P_i}V \to T_{j_i}J^k / T_{j_i}A_i)$. Thus the condition holds if and only if $T_Q M \bigoplus T_{P_i}A_i(f)$ maps onto $\bigoplus_i T_Q M$, which is equivalent to the stated condition. □

Our first application is a simple general result.

Proposition 4.6.6 *The set of self-transverse immersions* $f : V \to M$ *is open and dense in* $\mathrm{Imm}(V, M)$.

Proof First consider the submanifold $(J^0, J^0)_\Delta$ of ${}_2J^0(V, M)$ consisting of pairs of 0-jets with a common target. By Theorem 4.6.1, the set of maps $f : V \to M$ with ${}_2 j^0 f$ transverse to $(J^0, J^0)_\Delta$ is dense in $C^\infty(V, M)$. By Lemma 4.6.5, ${}_2 j^0 f$ is transverse to $(J^0, J^0)_\Delta$ at a point (P_1, P_2) with $f(P_1) = f(P_2) = Q$ if and only if $df(T_{P_1}V) + df(T_{P_2}V) = T_Q M$, i.e. the branches of $f(V)$ at P_1 and P_2 meet transversely at Q.

Since $\mathrm{Imm}(V, M)$ is open in $C^\infty(V, M)$, it follows that the set of immersions f with this property is dense in $\mathrm{Imm}(V, M)$. Higher intersections are dealt with in the same way using $(J^0, \dots, J^0)_\Delta \subset_r J^0(V, M)$.

For openness we use Proposition 4.6.3. Again we give the details only for the case $r = 2$. The result will follow if for each self-transverse immersion f, each $x \in V$ has a neighbourhood U_x such that $\{g \in \mathcal{W} \mid {}_2 j^k g \mid U_x^{(2)}$ is transverse to $(J^0, J^0)_\Delta\}$ is a neighbourhood of f.

But since f is an immersion, each $x \in V$ has a neighbourhood U_x embedded by f. Since the set of embeddings is open, the set of maps of V restricting to an embedding of U_x is also open. □

4.7 Generic singularities of maps

In this section we apply the general theorems to reduce singularities of maps to general form. We first give applications of jet transversality, then deal with multijets. As well as showing that maps with a certain form are dense in the space of all maps, we also show they form an open set, so that the simplifications do not disappear under small perturbations. We first consider the case $m = 1$.

Theorem 4.7.1 *Non-degenerate functions are dense and open in $C^\infty(V, \mathbb{R})$.*

Proof As $m = 1$, Σ^i is empty unless $i = \upsilon$ or $i = \upsilon - 1$, and $\Sigma^{\upsilon-1}$ is smooth of codimension υ. By Theorem 4.5.6, the set of functions f which are transverse to $\Sigma^{\upsilon-1}(V, \mathbb{R})$ is dense and open.

Now $j^1 f(P) \in \Sigma^{\upsilon-1}$ if and only if $df_P = 0$: P is a critical point of f. We claim that $j^1 f$ is transverse to $\Sigma^{\upsilon-1}$ if and only if f is non-degenerate: this will imply the result.

Take local coordinates $\{x_i\}$ at P and y on $\mathbb{R}$, and write u_i for the coordinate on $J^1(V, \mathbb{R})$ corresponding to $\partial y/\partial x_i$. Now apply the calculation (4.4.2), which reduces here to $dj^1 f\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} + u^i \frac{\partial}{\partial y} + \sum_k u^{ik} \frac{\partial}{\partial u^k}$. Since $\Sigma^{\upsilon-1}$ is defined by the equations $u_i = 0$, its tangent space is spanned by $\partial/\partial y$ and the $\partial/\partial x_i$. These together with the $dj^1 f\left(\frac{\partial}{\partial x_i}\right)$ span $T_{j^1 f(P)} J^1(V, \mathbb{R})$ if and only if the matrix $u^{ik} = (\partial^2 f/\partial x_i \partial x_k)_P$ is nonsingular, i.e. P is a non-degenerate critical point of f. □

For the case $m = 2$, we have

Theorem 4.7.2 *Maps f with the following properties form a dense open subset of $C^\infty(V^\upsilon, M^2)$: $\Sigma^{\upsilon-2}(f)$ is empty, $\Sigma^{\upsilon-1}(f)$ is a smooth curve, and at each point of $\Sigma^{\upsilon-1}(f)$, there are local coordinates in which $j^2 f$ is given by either*

$(x_1, \sum_{i,j=2}^{\upsilon} b_{ij} x_i x_j)$ with $(b_{ij})_{i,j=2}^{\upsilon}$ nonsingular or

$(x_1, x_1 x_2 + \sum_{i,j=3}^{\upsilon} b_{ij} x_i x_j)$ with $(b_{ij})_{i,j=3}^{\upsilon}$ nonsingular;

in the latter case, the coefficient of x_2^3 in y_2 is non-zero.

Proof By Lemma 4.5.12, $\Sigma^{\upsilon-2}$ has codimension 2υ and $\Sigma^{\upsilon-1}$ has codimension $(\upsilon - 1)$. It thus follows from Theorem 4.5.6 that the set of maps $f : V^\upsilon \to M^2$ such that $\Sigma^{\upsilon-2}(f)$ is empty and f is transverse to $\Sigma^{\upsilon-1}$ is dense, and from Corollary 4.5.14 that this set is open.

Since f is transverse to $\Sigma^{\upsilon-1}$, $\Sigma^{\upsilon-1}(f)$ is a smooth curve in V. We now need to calculate. We choose local coordinates at a point of $\Sigma^{\upsilon-1}(f)$ such that the 1-jet of f is $(x_1, 0)$. The 2-jet is then of the form

$$\left(x_1 + \sum a_{ij} x_i x_j, \sum b_{ij} x_i x_j\right).$$

Essentially the same calculation as in the preceding proof using (4.4.2) shows that this 2-jet is transverse to $\Sigma^{\upsilon-1}(f)$ if and only if the vectors

$$\sum_{j=1}^{\upsilon} b_{ij} x_j (2 \le i \le \upsilon)$$

are independent.

There are now two cases. In general, the matrix $B := (b_{ij})_{i,j=2}^{\upsilon}$ is nonsingular. We may then make a linear substitution $x'_j = x_j + \lambda_j x_1$ to reduce the $b_{1,j}$ $(j > 1)$ to zero, and the further change of coordinates $x'_1 = x_1 + \sum a_{ij}x_ix_j$, $y'_2 = y_2 - b_{1,1}y_1^2$ reduces the 2-jet to $(x_1, \sum_{i,j=2}^{\upsilon} b_{ij}x_ix_j)$. We label this case $\Sigma^{\upsilon-1,0}$. For f in this form, the tangent space to $\Sigma^{\upsilon-1}(f)$ is the x_1-axis, and the restriction of f to $\Sigma^{\upsilon-1}(f)$ is an immersion.

Otherwise the matrix B has rank $\upsilon - 2$, so by a change of coordinates $x_2, \ldots, x_\upsilon$ we can reduce to the case when $b_{2,i} = 0$ for $2 \leq i \leq \upsilon$. The transversality condition now implies that $b_{1,2} \neq 0$. Coordinate changes as before allow us to reduce the 2-jet to the form $(x_1, x_1x_2 + \sum_{i,j=3}^{\upsilon} b_{ij}x_ix_j)$. We label this case $\Sigma^{\upsilon-1,1}$. For f in this form, the tangent space to $\Sigma^{\upsilon-1}(f)$ is the x_2-axis, and the restriction of f to $\Sigma^{\upsilon-1}(f)$ is not an immersion.

We have effectively defined $\Sigma^{\upsilon-1,1}$ as a subspace of $J^2(V, M)$: it has codimension 1 in the space of 2-jets defining maps transverse to $\Sigma^{\upsilon-1}$. A further application of the transversality theorem tells us that for a dense set of maps, $j^2 f$ is also transverse to this.

Since $\Sigma^{\upsilon-1,1}$ was defined as a subset of $\Sigma^{\upsilon-1}$ by the vanishing of $\det(B)$, f is transversal to it if and only if $dj^2 f(\partial/\partial x_1)$ maps onto the normal space to this. In the neighbourhood of a matrix B of rank $\upsilon - 2$ and with $b_{2,i} = 0$ for $2 \leq i \leq \upsilon$, the normal space is spanned by $b_{2,2}$. Since the tangent space to $\Sigma^{\upsilon-1}(f)$ is the x_2-axis, we need to evaluate $dj^2 f(\partial/\partial x_2)$. Again using (4.4.2), we see that the desired condition holds if and only if the coefficient of x_2^3 in y_2 is non-zero.

For openness it suffices by Lemma 4.5.11 to prove that the set of submanifolds of J^2 defined by $\Sigma^{\upsilon-2}$, $\Sigma^{\upsilon-1,0}$ and $\Sigma^{\upsilon-1,1}$ is A-regular; the only non-trivial case is a sequence in $\Sigma^{\upsilon-1,1}$ with limit in $\Sigma^{\upsilon-2}$. But as $\Sigma^{\upsilon-2}$ is the set of jets with zero 1-jet, the inclusion of tangent spaces follows. □

In the final case, the condition that the coefficient of x_2^3 in y_2 is non-zero implies that $f(\Sigma^{\upsilon-1}(f))$ has a simple cusp.

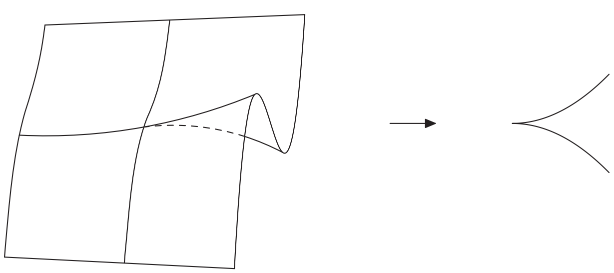

Figure 4.2 A cusp singularity

In Figure 4.2, we illustrate a cusp singularity of a map $M^2 \to \mathbb{R}^2$ as the projection of a surface M embedded in $\mathbb{R}^3$, together with the discriminant set $f(\Sigma^1(f)) \subset \mathbb{R}^2$.

Finally, we consider cases with m large compared to υ.

Theorem 4.7.3 *Maps f with the following properties form a dense open subset of $C^\infty(V^\upsilon, M^m)$:*

if $m \geq 2\upsilon$, f is an immersion,

if $2m \geq 3\upsilon - 3$, $\Sigma^2(f)$ is empty and f is transverse to Σ^1, so $\Sigma^1(f)$ is a smooth submanifold of V of dimension $2\upsilon - m - 1$,

if $2m \geq 3\upsilon - 1$, the 2-jet of f at any point of $\Sigma^1(f)$ can be reduced to the form

$$y_1 = \frac{1}{2}x_1^2,\ y_j = x_j\ (2 \leq j \leq \upsilon),\ y_{i+\upsilon-1} = x_1 x_i\ (2 \leq i \leq m - \upsilon + 1). \tag{4.7.4}$$

Proof By Theorem 4.5.6, the maps transverse to all the Σ^i form a dense set, and by Corollary 4.5.14 it is also open.

Since $m \geq \upsilon$, the codimension of Σ^1 is $m - \upsilon + 1$. Thus if $m \geq 2\upsilon$, the maps avoiding Σ^1, i.e. immersions, are dense and open in $C^\infty(M, V)$. This already sharpens Theorem 4.2.3.

Next, the codimension of Σ^2 is $2(m - \upsilon + 2)$, so provided this exceeds υ, i.e. $2m \geq 3\upsilon - 3$, for a dense open set of maps f, we have $\Sigma^2(f) = \emptyset$ and f is transverse to Σ^1. We choose local coordinates in which the 1-jet of f at P is given by $(0, x_2, \ldots, x_\upsilon, 0, \ldots, 0)$, thus $\partial/\partial x_1$ spans $\ker(df)$. Thus at $j^1 f(P)$, Σ^1 is locally the set of jets such that the first row of (u^i_j) is a linear combination of the rest, and the tangent space of Σ^1 is given by infinitesimal vanishing of u^1_1 and u^1_j for $\upsilon < j \leq m$.

From the calculation (4.4.2) we see that the coefficient of $\partial/\partial u^1_j$ in $dj^1 f(\partial/\partial x_i)$ is u^{1i}_j, i.e. $\partial^2 y_j/\partial x_1 \partial x_i$.

Now f is transverse to Σ^1 if and only if $dj^1 f(T_P V)$ spans the normal space to Σ^1, i.e. the matrix $(\partial^2 f_j/\partial x_1 \partial x_i)_{j=1, \upsilon<j\leq m, 1\leq i\leq \upsilon}$ has rank $m - \upsilon + 1$.

Consider the subvariety $\Sigma^{1,1}$ of $J^2(V, M)$ consisting of jets in Σ^1 at P such that $\ker(df)_P \subset T_P\Sigma^1(f)$. Since $\Sigma^1(f)$ has codimension $m - \upsilon + 1$, and we now impose $m - \upsilon + 1$ further conditions, $\Sigma^{1,1}$ has codimension $2(m - \upsilon + 1)$. By Theorem 4.5.6, provides this exceeds υ, i.e. $2m \geq 3\upsilon - 1$, the condition that $j^2 f(V)$ avoids $\Sigma^{1,1}$, i.e. at each point of $\Sigma^1(f)$, $dj^1 f(\ker df)$ is not tangent to Σ^1, holds for a dense set of maps f.

The condition for this tangency is that $dj^1 f(\partial/\partial x_1)$ lies in the subspace where the coefficients of $\partial/\partial u^1_1$ and $\partial/\partial u^1_j$ for $\upsilon < j \leq m$ vanish, i.e. that

$\partial^2 f_1/\partial x_1^2 = 0$ and $\partial^2 f_j/\partial x_1^2 = 0$ for $\upsilon < j \leq m$. If this condition does not hold, we may suppose, after replacing y_1 and the y_j for $\upsilon < j \leq m$ by linear combinations, that $\partial^2 f_1/\partial x_1^2 = 1$ and $\partial^2 f_j/\partial x_1^2 = 0$ for $\upsilon < j \leq m$.

It now follows that the matrix $(\partial^2 f_j/\partial x_1 \partial x_i)_{\upsilon < j \leq m, 2 \leq i \leq \upsilon}$ has rank $m - \upsilon$. We can thus make a linear transformation of the y_j with $\upsilon < j \leq m$ to arrange that $\partial^2 f_j/\partial x_1 \partial x_i = 1$ for $j = \upsilon - 1 + i$ and vanishes otherwise for $\upsilon < j \leq m$, $2 \leq i \leq \upsilon$. Thus the 2-jets take the form

$y_1 = \frac{1}{2}x_1^2 + Q_1(x_2, \cdots, x_\upsilon)$,
$y_j = x_j + Q_j(x_1, \cdots, x_\upsilon)$, for $2 \leq j \leq \upsilon$,
$y_{i+\upsilon-1} = x_1 x_i + Q_{i+\upsilon-1}(x_2, \cdots, x_\upsilon)$ for $2 \leq i \leq m - \upsilon + 1$,

where the Q_j are quadratic. Finally, if we make the coordinate changes

$x'_j = x_j + Q_j(x_1, \cdots, x_\upsilon)$,
$y'_1 = y_1 - Q_1(y_2, \cdots, y_\upsilon)$, and
$y'_{i+\upsilon-1} = y_{i+\upsilon-1} - Q_{i+\upsilon-1}(y_2, \cdots, y_\upsilon)$,

the quadratic terms drop out too.

For openness we could seek to show that A-regularity continues to hold when we throw in $\Sigma^{1,1}$. It is easier to apply the method of Proposition 4.6.3. Write $\mathcal{W}$ for the set of $f \in C^\infty(V, M)$ transverse to the Σ^i and $\mathcal{W}^*$ for the set of $f \in \mathcal{W}$ with $j^3 f$ transverse to $\Sigma^{1,1}$. Since for any $f \in \mathcal{W}$, $j^2 f(V)$ avoids Σ^2, it avoids a neighbourhood U_f of Σ^2 in $J^3(V, M)$, hence there is an open neighbourhood $\mathcal{W}_f$ of f in $\mathcal{W}$ such that, for all $g \in \mathcal{W}_f$, $j^2 g(V)$ avoids U.

In the complement of U we only need to consider $\Sigma^{2,0}$ and $\Sigma^{2,1}$, and here the A-regularity condition trivially holds (the latter is a smooth submanifold of codimension 1 in Σ^2 and the former is its complement). By Lemma 4.5.11, transversality defines an open subset of U_f. It follows that $\mathcal{W}^* \cap U_f$ is open in U_f, so $\mathcal{W}^*$ contains an open neighbourhood of f. □

We now give applications of multitransversality: we treat the cases in the same order, so begin with functions $f \in C^\infty(V)$. Recall that the critical values of f are the $f(P)$ with P a critical point of f.

Proposition 4.7.5 *Non-degenerate functions with all critical values distinct form a dense open set in $C^\infty(V)$.*

Proof By Theorem 4.7.1, functions with only non-degenerate critical points are dense. As the submanifold $(\Sigma^{\upsilon-1}, \Sigma^{\upsilon-1})_\Delta$ of ${}_2J^1(V, \mathbb{R})$ of pairs of singular jets with the same image has codimension $2\upsilon + 1$, it follows from the Multitransversality Theorem 4.6.1, that functions avoiding it are also dense. As these are both residual sets, so is their intersection.

For openness it suffices, by Proposition 4.6.3, to show that for each non-degenerate function f with distinct critical values, each $x \in V$ has a neighbourhood U_x such that the set of non-degenerate functions g whose restriction to U_x has distinct critical values is a neighbourhood of f. Choose a coordinate neighbourhood U'_x so that f has at most one critical point on U'_x, and let U_x be the neighbourhood defined by a disc of half the radius. Then the set of non-degenerate functions on V with at most one critical point in U_x is open. □

We come to target dimension 2.

Theorem 4.7.6 *For any V^v, M^2, maps with the following properties form a dense and open subset of $C^\infty(V, M)$:*

the singular set of f is a smooth curve $\Sigma(f)$ embedded in V,

$f \mid \Sigma(f)$ is a smooth embedding except that

(a) for a discrete set of points $P \in \Sigma(f)$, the curve $f(\Sigma(f))$ has a cusp at $f(P)$,

(b) for a discrete set of pairs (P, Q) of points in $\Sigma(f)$ (all distinct from the cusps), $f(\Sigma(f))$ has a transverse self-intersection at $f(P) = f(Q)$.

Proof We give the proof of density: openness is more technical and is best established using methods described in the Notes §4.9.

Most of the conclusions were obtained in Theorem 4.7.2, but we have yet to consider double points of $f(\Sigma(f))$.

First apply the multitransversality theorem to the submanifold $(\Sigma^{v-1,1}, \Sigma^{v-1})_\Delta$ of ${}_2J^2(V, M)$. This has codimension $v + (v-1) + 2$, so is avoided by a dense set of maps; thus cusps will not be double points.

Now apply the theorem to $(\Sigma^{v-1}, \Sigma^{v-1})_\Delta$. This has codimension $(v-1) + (v-1) + 2$, so occurs at isolated points. It follows by Lemma 4.6.5 that the self-intersection of $f(\Sigma(f))$ is transverse at such points. □

For the cases of large target dimension, we have

Theorem 4.7.7 *Maps f with the following properties form a dense subset of $C^\infty(V^v, M^m)$ if V is compact, or of $C^\infty_{pr}(V, M)$ in general:*

(i) If $m \geq 2v + 1$, f is an embedding.

(ii) If $m = 2v$, f is an immersion with isolated points of transverse self-intersection.

(iii) If $2m \geq 3v - 3$, $\Sigma^2(f)$ is empty and f is transverse to Σ^1, so $\Sigma^1(f)$ is a smooth submanifold of V of dimension $2v - m - 1$.

(iv) If $2m > 3v$, f is an embedding except as follows. There are double points, forming a submanifold $D(f)$ of dimension $2v - m$, and singular points, forming a submanifold $\Sigma^1(f)$ of dimension $2v - m - 1$. Near $\Sigma^1(f)$, f is given locally by (4.7.4). Hence the closure $\bar{D}(f)$ of $D(f)$ is $D(f) \cup \Sigma^1(f)$ and is smooth, and $f(\bar{D}(f))$ is a submanifold of M with boundary $f(\Sigma^1(f))$.

(v) If $2m = 3v$ the same holds, except that now $D(f)$ is immersed with transverse self-intersection, and $f(D(f))$ can have triple points with transverse self-intersection.

Proof We extend the results of Theorem 4.7.3. For (i), we may suppose f an immersion, and apply multijet transversality to $(J^0, J^0)_\Delta$. Since this has codimension $2v$, it is avoided by a dense set of maps. Thus injective immersions are dense in $C^\infty(V, M)$; now any proper injective immersion is an embedding by Proposition 1.2.10.

Now (ii) follows using Proposition 4.6.6.

We make three further applications of the multitransversality Theorem 4.6.1. First consider the subvariety $(\Sigma^1, J^1)_\Delta$ of ${}_2J^1(V, M)$ consisting of pairs of jets with the same image, one of which (say the first) is singular. As this has codimension $m + (m - v + 1)$, if $2m \geq 3v$, the set of maps avoiding it is dense.

Next consider $(J^0, J^0)_\Delta$: by Lemma 4.6.5, ${}_2j^0 f$ is transverse to this at (P_1, P_2) if and only if $df(V_{P_1}) + df(V_{P_2}) = M_Q$. By the previous paragraph, neither P_1 nor P_2 is a singular point, so we have a transverse intersection of smooth pieces of the image, giving the set $D(f)$ of double points of f.

Finally consider the subvariety $(J^0, J^0, J^0)_\Delta$ of ${}_3J^0$ of triples of jets with the same image. Since this has codimension $2m$, if $2m > 3v$ it follows by multitransversality that the set of maps avoiding it is dense. If $2m = 3v$, this will appear at isolated points, and by Lemma 4.6.5, the three branches at such points are mutually transverse.

We have seen that $D(f)$ is an immersed submanifold; when there are no triple points it is imbedded. That $D(f)$ remains a manifold near $\Sigma^1(f)$, with $\Sigma^1(f)$ as its frontier, follows from the equations (4.7.4). Now $D(f)$ is simply given by $x_i = 0$ $(2 \leq i \leq m - v + 1)$ (modulo higher terms). Moreover $f(D(f))$ is also a submanifold, except perhaps near $f(\Sigma^1(f))$; but there it is locally given by $y_1 \geq 0$, $y_i = 0$ $(2 \leq j \leq m - v + 1)$ and $(v + 1 \leq j \leq m)$.

To prove openness it again suffices by Proposition 4.6.3 to show that, for each f satisfying the conditions, each $x \in V$ has a neighbourhood U_x such that the set of maps g whose restriction to $\Sigma^1(g) \cap U_x$ is injective is a neighbourhood of f.

By Theorem 4.7.3, we may suppose that at the point x, either f is an immersion (in which case the immersions give a neighbourhood of the desired type), or the 2-jet of f has the form (4.7.4): $y_1 = \frac{1}{2}x_1^2$, $y_j = x_j$ for $2 \leq j \leq v$, and

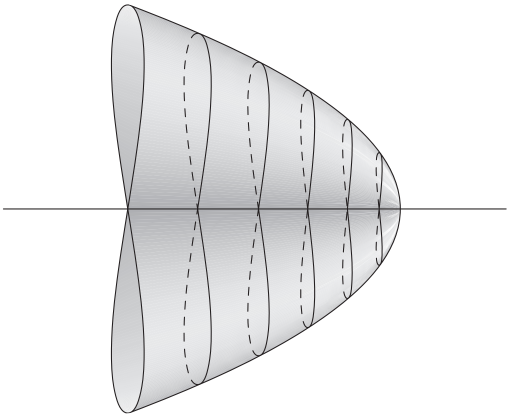

Figure 4.3 A Whitney umbrella

$y_{i+v-1} = x_1 x_i$ for $2 \leq i \leq m - v + 1$; so $\Sigma^1(f)$ is given (to the first order) by $x_i = 0$ for $1 \leq i \leq m - v + 1$. Restricting to a small neighbourhood U we see that for any nearby g, the coordinates x_i for $i > m - v + 1$ are independent on $\Sigma^1(g)$ and define an injective map of it. $\square$

We can now give a fuller statement of Whitney's Embedding Theorem.

Corollary 4.7.8 *For any smooth manifold V^v there exist proper smooth embeddings $V^v \to \mathbb{R}^m$ whenever $m > 2v$. The image of such an embedding is a closed submanifold of $\mathbb{R}^m$.*

The existence of proper maps $V \to \mathbb{R}^m$ is given by Corollary 2.2.10 and of proper smooth maps follows from Proposition 1.1.7; it follows by the theorem that there exist proper smooth embeddings. The final statement follows from Proposition 1.2.10.

It also follows that for a dense open set of maps of a compact smooth surface to 3-dimensional space, the possible types of singularity of the image are the curves D of (transverse) self-intersections of the surface, triple points where three sheets meet transversely, and a set S of isolated singular points, where the map is locally of the form (modulo higher terms, but we will see in Theorem 4.8.5 that these are unnecessary)

$$f(x_1, x_2) = (x_1^2, x_2, x_1 x_2),$$

so here the image is defined by $y_3^2 = y_1 y_2^2$ and D is the curve $y_2 = y_3 = 0,\ y_1 > 0$. Points of this type are known as *Whitney umbrella* points: an example is pictured in Figure 4.3.

Although the results using multitransversality always give a partial description of the picture of the map in the target manifold M, this should be treated with caution unless we restrict to the space $C^\infty_{pr}(V, M)$ of proper maps. We already saw this in §1.2 when discussing the notion of submanifold. Moreover, though we have proved that the set of such 'excellent' maps is open, we have used the W^∞ topology, which is somewhat counterintuitive. For example, it is possible to construct a non-degenerate function with distinct critical values which are dense in $\mathbb{R}$: maps nearby in the C^∞ topology need no longer have distinct critical values.

4.8 Normal forms

We show in this section that in each of the cases studied in the preceding section, we can choose local coordinates to reduce the map f to a precise normal form.

We begin by showing that a mutually transverse set of submanifolds has as local normal form a set of linear subspaces of a vector space.

Lemma 4.8.1 *Suppose the submanifolds V_i of M each contain a point P, and suppose that the subspaces T_PV_i of T_PM are mutually transverse. Then there exists a chart $\varphi : (U, P) \to (\mathbb{R}^m, 0)$, with U a neighbourhood of P in M, such that each $\varphi(V_i \cap U)$ is an open subset of a coordinate subspace of $\mathbb{R}^m$.*

Proof For each i, if V_i has codimension r_i, we know that there is a set of r_i smooth functions on M, each vanishing on V_i, whose differentials at P are linearly independent.

It follows from the definition of mutual transversality that the differentials of all these functions at P are linearly independent, so we can extend them to a basis of $T_P^\vee M$ by adjoining the differentials of a further $m - \sum_i r_i$ smooth functions. It follows from the Inverse Function Theorem that the set of all these functions defines a chart at P, and by construction, this has the desired property on some neighbourhood of P. □

A normal form theorem for non-degenerate functions is proved as follows. First take local coordinates with source $O \in \mathbb{R}^m$ and target $0 \in \mathbb{R}$; then by linear algebra reduce the 2-jet of f to the form $\sum_1^m \varepsilon_i x_i^2$, with each $\varepsilon_i = \pm 1$.

Proposition 4.8.2 (Morse Lemma) *Let f be a smooth function on a neighbourhood of 0 in $\mathbb{R}^n$ with 2-jet $\sum_1^n \varepsilon_i x_i^2$, where each $\varepsilon_i = \pm 1$. Then there is a smooth coordinate change $y = y(x)$ such that $y(0) = 0$, $\left.\frac{\partial y}{\partial x}\right|_0 = I_n$, and near 0, $f(x) = \sum_1^n \varepsilon_i x_i^2$.*

Proof We have $f(0) = 0$, so by Lemma 1.2.3 there exist near 0 smooth functions f_i with $f(x) = \sum x_i f_i(x)$. Also, $f_i(0) = \frac{\partial f}{\partial x_i}\Big|_0 = 0$, so we can apply the result again to obtain h_{ij} with $f_i(x) = \sum x_j h_{ij}(x)$. Write $g_{i,j}(x) = \frac{1}{2}(h_{ij}(x) + h_{ji}(x))$. We think of $f(x) = \sum_{ij} g_{i,j}(x)x_i x_j$ as a quadratic form, and diagonalise. Note that

$$g_{i,j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_0 = \begin{cases} 0 & i \neq j \\ \varepsilon_i & i = j. \end{cases}$$

Set $y_1 = (\varepsilon_1 g_{11}(x))^{-1/2}(\sum_{j=1}^n g_{1j}x_j)$. Then

$$\frac{\partial y_1}{\partial x_1} = \pm 1, \frac{\partial y_1}{\partial x_i} = 0 \quad \text{if } i > 1, \quad \text{and} \quad f(x) = \pm y_1^2 + \sum_{i,j=2}^n g'_{i,j}(x)x_i x_j.$$

We now repeat the reduction, observing only that although $g'_{i,j}(x)$ depends on x_1 we can express x_1 by y_1, and the dependence is smooth. Eventually we obtain the required result. □

For the remaining cases we require further machinery, which is provided by the Malgrange Preparation Theorem. To formulate this, we need some notation. Denote by $\mathcal{E}_n$ the ring of germs at 0 of smooth functions on $\mathbb{R}^n$ under pointwise addition and multiplication. This is a local ring with maximal ideal $\mathfrak{m}_n$ consisting of germs of functions vanishing at 0. This is closely related to our introduction of jets: it follows from Lemma 1.2.3 by a simple induction that a function-germ f on $(\mathbb{R}^n, 0)$ has zero r-jet: $f \sim_r 0$: if and only if $f \in \mathfrak{m}_n^{r+1}$.

Theorem 4.8.3 (Malgrange Preparation Theorem) *For $u : \mathbb{R}^m \to \mathbb{R}^n$ a map-germ and $f_1, \dots f_p \in \mathcal{E}_m$, the following are equivalent:*
the f_i generate $\mathcal{E}_m$ as module over $\mathcal{E}_n$,
the images of the f_i generate $\mathcal{E}_m / u^*\mathfrak{m}_y.\mathcal{E}_m$ as real vector space.

We omit the proof: see Notes §4.9 for references.

By Theorem 4.7.2, for a dense open set of maps $f : V^v \to M^2$, local coordinates can be taken at any point $P \in V$ such that we have either a submersion, a map with 2-jet $(x_1, \sum_{i,j=2}^v b_{ij}x_i x_j)$ with (b_{ij}) nonsingular, or a map with 2-jet $(x_1, x_1 x_2 + \sum_{i,j=3}^v b_{ij}x_i x_j)$ with (b_{ij}) nonsingular, and a non-zero coefficient of x_1^3 in y_2.

Theorem 4.8.4 *For a dense open set of maps $f : V^v \to M^2$, local coordinates can be taken at any point $P \in V$ such that f takes one of the forms*

(x_1, x_2),
$(x_1, \sum_{i=2}^{\upsilon} \varepsilon_i x_i^2)$, *or*
$(x_1, x_1x_2 + x_2^3 + \sum_{i=3}^{\upsilon} \varepsilon_i x_i^2)$.

We give the proof only for $\upsilon = 2$.

Proof In each case, y_1 has 1-jet x_1. First simplify by taking $x_1' = y_1(x_1, x_2)$, $x_2' = x_2$. By the Inverse Function Theorem 1.2.5, this is an allowed coordinate change, and it reduces us to the case $y_1 = x_1$.

We recall that by Lemma 1.2.3, if g is a smooth function and $g(0) = 0$, there exist near 0 smooth functions g_i with $g(x) = \sum x_i g_i(x)$. We can thus write $y_2 = x_1A_1 + x_2A_2$. As y_2 has 2-jet x_2^2, each of A_1 and A_2 vanishes at 0, so applying the lemma again gives $y_2 = x_1^2A_{11} + x_1x_2A_{12} + x_2^2A_{22}$.

Thus the ideal $f^*\mathfrak{m}_2.\mathcal{E}_\upsilon = \langle y_1, y_2\rangle = \langle x_1, x_1^2A_{11} + x_1x_2A_{12} + x_2^2A_{22}\rangle = \langle x_1, x_2^2A_{22}\rangle$. But $A_{22}(0) \neq 0$, so A_{22} is invertible, hence the ideal coincides with $\langle x_1, x_2^2\rangle$, and the quotient $\mathcal{E}_\upsilon/u^*\mathfrak{m}_2.\mathcal{E}_\upsilon$ is generated by $\{1, x_2\}$. In case (iii) a similar argument shows that the ideal is equal to $\langle x_1, x_2^3\rangle$, and the quotient is generated by $\{1, x_2, x_2^2\}$.

In case (ii), it follows by Theorem 4.8.3 that $\mathcal{E}_\upsilon$ is generated over $\mathcal{E}_2$ by $\{1, x_2\}$. Thus we can write $y_2(x_1, x_2) - \varepsilon x_2^2 = A(y_1 \circ u, y_2 \circ u) + x_2B(y_1 \circ u, y_2 \circ u)$ for some C^∞ functions A, B. Now change coordinates first by $x_2' = x_2 + \frac{\varepsilon_2}{2}B(y_1 \circ u, y_2 \circ u)$ to eliminate B; then by $y_2' = y_2 - A(y_1, y_2)$ to achieve the desired normal form.

In case (iii), $\mathcal{E}_\upsilon$ is generated over $\mathcal{E}_2$ by $\{1, x_2, x_2^2\}$. We can thus write

$$x_2^3 = (A \circ f) + x_2(B \circ f) + 3x_2^2(C \circ f),$$

where we omit the explicit dependence of A, B and C on y_1 and y_2. So

$$(x_2 - C)^3 = (A + BC + 2C^3) + (x_2 - C)(B + 3C^2).$$

If we can substitute

$$x_1' = (B + 3C^2) \circ f,\ x_2' = x_2 - C \circ f,\ y_1' = B + 3C^2,\ y_2' = A + BC + 2C^3,$$

we indeed obtain

$$y_1' = x_1, \quad y_2' = x_2^3 - x_1x_2.$$

Equating successively coefficients of x_1, x_1x_2 and x_2^3 shows that the 1-jet of A is y_2 and the 1-jet of B has the form $-y_1 + \alpha y_2$. Hence by the Inverse Function Theorem, the change of y coordinates is legitimate. Now the 1-jet of $B \circ f$ is $-x_1$ and the 1-jet of $C \circ f$ has the form βx_1, so also the change of x coordinates is legitimate. □

In Theorem 4.7.3 we saw that if $2m \geq 3v - 1$, maps f with the following properties form a dense open subset of $C^\infty(V^v, M^2)$: $\Sigma^2(f)$ is empty, f is transverse to Σ^1, and the 2-jet of f at any point of $\Sigma^1(f)$ can be reduced to the form

$$y_1 = \frac{1}{2}x_1^2, \quad y_j = x_j \text{ for } 2 \leq j \leq v, \quad y_{i+v-1} = x_1 x_i \text{ for } 2 \leq i \leq m - v + 1.$$

Theorem 4.8.5 *There exist local coordinates in which f takes precisely this form.*

Proof Here Theorem 4.8.3 gives generators $\{1, x_1\}$. So we can write

$y_1 = \frac{1}{2}x_1^2 + x_1 A_1(y) + B_1(y)$,
$y_j = x_j + x_1 A_j(y) + B_j(y)$ for $2 \leq j \leq v$, and
$y_{i+v-1} = x_1 x_i + x_1 A_{i+v-1}(y) + B_{i+v-1}(y)$ for $2 \leq i \leq m - v + 1$;

moreover equating terms of order 2 shows that the B_* have zero 1-jet, and the 1-jet of each A_i is a linear combination of the y_i with $i = 1$ or $i > v$.

First substitute $x_1' = x_1 + (A_1 \circ f)$; this reduces the map to a map of the same form, but with A_1 absent. We continue to write A_j, B_j, etc. for the new terms.

Next substitute $x_j' = x_j + x_1(A_j \circ f) + (B_j \circ f)$ for $2 \leq j \leq v$; this eliminates A_j and B_j but gives $y_{i+v-1} = x_1(x_i' - x_1 A_i(y) - B_i(y)) + x_1 A_{i+v-1}(y) + B_{i+v-1}(y)$ for $2 \leq i \leq m - v + 1$. Now set $y_{i+v-1}' = y_{i+v-1} + 2y_1 A_i(y)$ to eliminate the term in x_1^2, and renotate as before.

Thirdly write $x_i' = x_i + (A_{i+v-1} \circ f)$ for $2 \leq i \leq m - v + 1$. We now have

$y_1 = \frac{1}{2}x_1^2 + B_1(y)$,
$y_j = x_j - A_{i+v-1}(y)$ for $2 \leq j \leq v$, and
$y_{i+v-1} = x_1 x_i + B_{i+v-1}(y)$ for $2 \leq i \leq m - v + 1$.

By the Inverse Function Theorem 1.2.5, the equations $y_1' = y_1 - B_1(y)$, $y_j' = y_j + A_{i+v-1}(y)$, $y_{i+v-1}' = y_{i+v-1} - B_{i+v-1}(y)$ can now be solved to give a coordinate transformation; making these substitutions reduces f to the stated form. □

4.9 Notes on Chapter 4

§4.1 Sard's work followed that of Brown [32] which obtained a weaker result sufficient for most applications. There is a neat account of the proof in Milnor's little book [100].

§4.2 Whitney's great paper [175], although written in terms of explicit inequalities, effectively also introduced the W^∞ topology on the space of maps.

§4.3 The elementary argument here essentially goes back to Monge. A similar account is given by Milnor [98, I]. The importance of non-degenerate functions will appear in §5.1.

§4.4 Jets were first introduced by Ehresmann [48]. Their application to singularities was pioneered by Whitney, and systematically promoted by Thom [153].

A general discussion of these function space topologies, with references for proofs, is given, for example, in [121, §3.4]. A number of proofs are given in [73, §2.1] (but with a number of errors); another useful reference is [57]. The account in §A.4 includes proofs for the C^0 cases, which can be adapted to the C^∞ case.

There is no general agreement on terminology for these topologies. Some authors refer to the Thom topology for C^∞ and to the Whitney topology for W^∞, though neither of these authors formally introduced these topologies. Indeed, the first formal use of W^∞ seems to be in [88]. A discussion of their origins is given on [47, p. 59].

§4.5 The idea and use of transversality was introduced by Thom in [150]. The original proof was somewhat clumsy, but soon evolved to essentially the one presented here. An abstract form of the argument was given by Abraham [1].

Direct construction of families allowing use of Lemma 4.5.3 was given in many cases in [168].

The submanifolds Σ^i were first introduced by Thom, as were extensions to higher orders. A precise account, with the notations $\Sigma^{i,j}$ etc., was given by Boardman [19]. Whitney's regularity conditions were first formulated in [180].

The transversality Theorem 4.5.6 can be adapted to obtain results about 1-parameter families of mappings. We consider such a family as a map $F : V \times \mathbb{R} \to M \times \mathbb{R}$ of the form $F(x, t) = (f(x, t), t)$: F is compatible with projection on $\mathbb{R}$; we say that it is level-preserving. If N is a submanifold of $J^r(V \times \mathbb{R}, M \times \mathbb{R})$, we wish to make $j^r F$ transverse to N allowing only perturbation of F through level-preserving maps.

The idea of the proof of Theorem 4.5.6 is to embed g in a family $G : V \times U \to M$ such that the partial jet map $j_1^r G : V \times U \to J^r(V, M)$ is a submersion, and then apply Lemma 4.5.3; moreover we constructed G by piecing together maps locally constructed as $G' : X \times Y \to \mathbb{R}^m$ defined by $G'(x, y) := g(x) + B(x)y(x)$, where X is a coordinate chart for V and Y is the set of polynomial maps $y : \mathbb{R}^v \to \mathbb{R}^m$ of degree $\leq r$.

To adapt this to 1-parameter families, we must replace Y by the set Y^{lp} of level-preserving polynomial maps $\mathbb{R}^{v+1} \to \mathbb{R}^{m+1}$ of degree $\leq r$. We then need to require that N is a submanifold of $J^r(V \times \mathbb{R}, M \times \mathbb{R})$ transverse to the set of level-preserving jets. In practice, it is more efficient to use the methods of Mather mentioned below.

Consider in particular $M = \mathbb{R}$ and $N = \Sigma^{v-1}$. A generic map f meets this only at its (isolated) critical points, all non-degenerate. It can be shown that a generic homotopy F can be locally put in one of the forms of Theorem 4.7.2: (x_1, x_2), $(x_1, \sum_{i=2}^{v} \pm x_i^2)$ or $(x_1, x_1x_2 + x_2^3 + \sum_{i=3}^{v} \pm x_i^2)$, with $x_1 = t$ the parameter in $\mathbb{R}$. In the first case, f_t has no critical point; in the second there is a critical point at the origin; in the third, there are no critical points if $t > 0$, but if $t = -3u^2$ there are two critical points, at $(\pm u, 0, \ldots, 0)$. This gives the model for the deformation of a function corresponding to the handle cancellations considered in §5.4.

In Lemma 4.5.13 I offered a direct proof of A-regularity: however, it follows from the fact that the strata are the orbits of the natural action of $GL(V) \times GL(M)$ that the stratification is locally trivial, which is stronger than A-regularity.

§4.6 Versions of transversality involving several source points were current in the early 1960s (and indeed examples were given in the original version of these notes) but the formulation in terms of multitransversality is due to Mather [88] III in 1969. Some of the openness lemmas are new.

§4.7 We have just focussed on the examples needed later. Proving openness, as well as density, is harder than is often given credit for. A useful general criterion was given by Looijenga (see [56, p. 146], [47, Theorem 3.4.11]).

We have presented the results in three stages, following the natural progression. Thom used the term ‘source genericity’ for the results obtained from transversality (for example, Theorem 4.7.3) and ‘target genericity’ for those using multitransversality (for example, Theorem 4.7.7); we go on to normal forms (for example, Theorem 4.8.5). These cases ($2m > 3v$) are due to Haefliger, who used them in his original proof [60] of Theorem 6.4.11.

§4.8 In general, given v and m, we can think of a generic map of V^v to M^m as one which satisfies all the transversality conditions which can be stated in terms of v, m alone (using no special facts about V, M).

This vague idea is made precise in §4.7 and §4.8 above in the cases $m = 1$, $m = 2$, and $2m \geq 3v$. In each of these cases we have a class of maps with the four properties of characterisation, local normal form, density, and stability. It can be shown [88] that (at least if V is compact) maps with these properties are C^∞ stable in the sense that nearby maps are equivalent up to diffeomorphisms of the source and target, and that in these dimensions, C^∞ stable maps are dense in $C^\infty(V, M)$. The case $v = m = 2$, motivating the search for results in higher dimensions was obtained by Whitney [179], and the case $m = 2v - 1$ by Whitney [176].

A general survey and discussion was given by Thom in [153]. However, he found that to describe general map-germs $\mathbb{R}^{16} \to \mathbb{R}^{16}$ a finite list of normal forms does not suffice: one needs to allow a parameter. The simplest example is for $\mathbb{R}^8 \to \mathbb{R}^6$: for a general map in these dimensions, $\Sigma^4 f$ consists of isolated points, and to describe the 2-jet of f at such a point involves a homogeneous quadratic map $\mathbb{R}^4 \to \mathbb{R}^2$; and the classification of such maps involves a parameter.

The above method of direct reduction to normal form is somewhat clumsy. A more general approach was introduced by Mather [88] III. Here one uses the Malgrange preparation theorem to construct vector fields, and then integrates these to find changes of coordinates. I have written an expository account of this approach in [169]. Malgrange gave a proof of his preparation theorem in Cartan seminars in 1962–63; full details appear in his book [86]. There have been many further proofs: four appear in the volume [166].

Mather's work created a full theory of C^∞ stability: see [88], also [121]. The final conclusion is that stable maps are dense in $C^\infty(V^v, M^m)$ (V compact), and a finite explicit list of normal forms analogous to the above can be given, if and only if the pair (v, m) belongs to the so-called nice dimensions, which are given if $m - v \geq 4$ by $7v < 6m + 8$; otherwise by

$$\begin{array}{c|ccccccc} m-v & 3 & 2 & 1 & 0 & -1 & -2 & \leq -3 \\ m & <30 & <23 & <16 & <9 & <8 & <6 & <7. \end{array}$$