8

Cobordism

We have already defined the word ‘cobordism’ in §5.1: recall that if W is a manifold, and $\partial_- W$ and $\partial_+ W$ are disjoint manifolds with union ∂W, we call the pair $(W, \partial_- W)$ a cobordism and the pair $(W, \partial_+ W)$ the dual cobordism; and also call W a cobordism of $\partial_- W$ to $\partial_+ W$ and say that $\partial_- W$, $\partial_+ W$ are cobordant.

In the earlier chapter, we were concerned with the geometry of a particular cobordism. We now observe that being cobordant is an equivalence relation amongst diffeomorphism classes of manifolds. For $M \times I$ is a cobordism of M to itself; if W is a cobordism from M_0 to M_1 then the same manifold, but with $\partial_\pm W$ interchanged, is a cobordism from M_1 to M_0; and if W_0 is a cobordism from M_0 to M_1 and W_1 is a cobordism from M_1 to M_2, then glueing W_0 to W_1 along M_1 gives a cobordism from M_0 to M_2. For this relation not to be vacuous, we insist throughout that the manifolds W in question be compact: otherwise the product $M \times [0, 1)$ would give a cobordism of any manifold M to the empty set.

The simple definition just given already leads to interesting results, but the concept of cobordism lends itself to a wide variety of possible generalisations and restrictions, and these lead to a flexible tool in the study of manifolds.

For example, we may choose to restrict the manifolds (*and* cobordisms) to be oriented, weakly complex, or k-connected (for a fixed k); we may add the structure of a map to a fixed space X; if X is a manifold, we may further require this map to be an embedding, or an immersion. We may consider pairs (M, V) with V a submanifold of M and then cobordisms (N, W) with W a submanifold of N (and $\partial_- W = V$, $\partial_- N = M$), where we may also fix the group of the normal bundle.

Next we consider pairs (M, φ), where M is a manifold and $\varphi : M \times G \to M$ defines a smooth action of the compact Lie group G on M. We may also restrict the orbit types of the action to lie in an assigned closed set of orbit types - an extreme example is the class of fixed-point-free actions.

We may also allow M to be a manifold with boundary – and then a cobordism is a manifold W with $\partial W = \partial_- W \cup \partial_c W \cup \partial_+ W$ – and impose one restriction on M and W and another on ∂M and $\partial_c W$. These variants of the definition may now be combined ad lib.

Lemma 8.0.1 *Disjoint union defines an addition which turns the set of cobordism classes (of a given dimension) into an abelian group.*

Proof The other kinds of structure pass at once to the disjoint union. Union is compatible with cobordism: if V, W are cobordisms of $\partial_- V$ to $\partial_+ V$, $\partial_- W$ to $\partial_+ W$, then the disjoint union $V \cup W$ is a cobordism of $\partial_- V \cup \partial_- W$ to $\partial_+ V \cup \partial_+ W$. Thus we have a binary operation on the set of cobordism classes, which is commutative and associative since disjoint unions are. The empty manifold acts as zero.

We obtain an inverse to W whenever $M \times I$ may be regarded as a cobordism of the disjoint union $(M \times 0) \cup (M \times 1)$ to the empty set (the induced structure on $M \times 0$ must coincide with that on M: on $M \times 1$ it can be different). $\square$

For k-connected cobordism, we show in Lemma 8.8.1 that disjoint union can be replaced by connected sum.

In this chapter, vector bundles will be denoted by lower case Greek letters, so we write τ_M for the tangent bundle of M in place of $\mathbb{T}(M)$; normal bundles will usually be denoted by ν; and the trivial bundle of fibre dimension r by ε^r.

In the first section, we describe the basic Thom construction, leading to a bijection between certain sets of homotopy classes and certain bordism sets, and give an application to the problem of realising homology classes by submanifolds. Then we focus on the structure group on the normal bundle, and stabilisation, and define cobordism groups and rings.

The framework of cobordism lends itself to the construction of exact sequences, and we next describe this technique, which we will use many times. Then we treat cobordism of pairs; this leads to an interpretation of some relative groups.

The next section treats bordism as a homology theory, checks the axioms, introduces spectra, and dual notions of bordism and cobordism.

We then discuss equivariant cobordism, and show how the techniques of the preceding sections yield methods of calculation of the equivariant cobordism groups.

After a brief review of homology of classifying spaces, we describe the calculations of the unoriented bordism ring, and the unitary bordism ring. We hope to provide enough detail for the reader to follow the ideas, but refer to the original papers for details of calculations. We then attempt the same for oriented bordism and SU-bordism, with a detour to obtain the Hirzebruch signature theorem. We discuss k-connected cobordism, and then pull many results together in final calculations of groups of homotopy spheres and of knots.

Part of the use of cobordism theory is to make calculations, and in this chapter we will assume significantly more knowledge of homotopy theory than in earlier chapters. We attempt to provide enough background definitions and results for this discussion in Appendix B.

8.1 The Thom construction

We introduce the main tool in cobordism theory by considering the example which occurs in the earliest work on the subject: the study of submanifolds M^m of a fixed ambient manifold E^{m+k}.

Let ξ be an orthogonal vector bundle. As in §7.7, write V_ξ for the total space and B_ξ for the base; A_ξ for the subspace of V_ξ of all vectors of length $\leqq 1$ and S_ξ for its boundary, consisting of vectors of length 1. The Thom space $T(\xi)$ of ξ is obtained from A_ξ by identifying S_ξ to a point (denoted ∞): thus $T(\xi) = A_\xi / S_\xi$. We may identify B_ξ with the zero cross-section of the bundle, and hence with a subspace of $T(\xi)$. In the same section we met a special case of the Thom construction. Also Proposition 7.8.1 gave a preview of the next result.

If B_ξ is a smooth manifold, we can give ξ the structure of smooth vector bundle, and V_ξ and $T(\xi) \setminus \{\infty\}$ then also acquire the structure of smooth manifolds. Note that if B_ξ is a finite CW complex, so is $T(\xi)$; more precisely, if ξ has fibre dimension k, over each r-cell e^r of B_ξ we have a $(k+r)$-cell in A_ξ part of whose boundary lies over ∂e^r and part in S_ξ, so this gives a $(k+r)$-cell of $T(\xi)$, and all cells outside ∞ arise in this way.

Now let M^m be a submanifold of the compact manifold E^{m+k}, ν be its normal bundle. By Theorem 2.3.8 we can find an imbedding $h : A_\nu \to E$ defining a tubular neighbourhood of M in V.

The collapsing map $A_\nu \to T(\nu)$ defines a map $h(A_\nu) \to T(\nu)$ which extends to a continuous map $c_M : E \to T(\nu)$ which takes everything outside the tubular neighbourhood to ∞. This idea is due to Thom [150], and is called the *Thom construction*. Observe that if B_ν is identified with the zero cross-section of ν, we have $M = c_M^{-1}(B_\nu)$.

We introduce one more ingredient. Let $M^m \subset E^{n+k}$ have normal bundle ν, let ξ be a bundle whose base space B_ξ is a smooth manifold, and let $\phi : \nu \to \xi$ be a map of (orthogonal) vector bundles, hence inducing maps $B_\phi : B_\nu \to B_\xi$ and similarly for A, S and T. As above, identify A_ν with a tubular neighbourhood of M; write $c_M : E \to T(\nu)$ for the collapsing map, and form the composite $F_M := T_\phi \circ c_M : E \to T(\xi)$.

The first significant result in cobordism theory is that the Thom construction can, in a sense, be reversed. Define a cobordism of submanifolds of E to be a smooth compact submanifold W of $E \times I$, with $M_0 = \partial_- W = W \cap (E \times \{0\})$ and $M_1 = \partial_+ W = W \cap (E \times \{1\})$.

Proposition 8.1.1 *The Thom construction induces a bijection τ from the set of cobordism classes of submanifolds $M \subset E$ with normal bundle induced from ξ to the set of homotopy classes of maps $E \to T(\xi)$.*

Proof The construction takes a submanifold M with a bundle map $\phi_M : \nu_M \to \xi$ and gives a map $F_M := T_{\phi_M} \circ c_M : E \to T(\xi)$. To show we have a well-defined map τ we must show that cobordant submanifolds give rise to homotopic maps. Let $W^{m+1} \subset (E \times I)$ be a cobordism, with normal bundle ν_W induced via $\phi_W : \nu_W \to \xi$, and suppose the construction already performed for M_0 and M_1. It follows from the tubular neighbourhood theorem 2.5.5 that the chosen tubular neighbourhoods of M_0 and M_1 can be extended to a tubular neighbourhood of W in $E \times I$. Thus the collapsing map c_W for this neighbourhood extends those on the boundary. Hence $T_{\phi_W} \circ c_W$ is a homotopy between the maps obtained from M_0 and M_1. We thus have a well-defined map τ from cobordism classes to homotopy classes.

To show τ is surjective, suppose given a map $F : E \to T(\xi)$. Since $T(\xi) \setminus \{\infty\}$ is a smooth manifold, it follows from Proposition 2.3.4 that we can approximate F by a map F' agreeing with F on $F^{-1}(\infty)$ and which is smooth on a neighbourhood of $F^{-1}(B_\xi)$. If the approximation is close enough, $F' \simeq F$. Next by Theorem 4.5.6, we can further approximate F' by a map F'' transverse to B_ξ, and also suppose $F'' \simeq F'$. Now set $M := F'^{-1}(B_\xi)$. By Lemma 4.5.1, the normal bundle of M is induced from ξ by a map $\phi_M : \nu_M \to \xi$. If we now perform the Thom construction on M , the resulting $h : E^{m+k} \to T(\xi)$ agrees with F'', together with its first derivatives, on M^m. After a small homotopy, then, we can suppose $F'' = h$ on a neighbourhood of M. But the complement of such a neighbourhood is mapped, both by F'' and by h, to $T(\xi) \setminus B_\xi$, which is contractible. It follows that $h \simeq F'' \simeq F$, as desired.

That τ is injective follows by relativising the same arguments. Suppose given $M_0 \subset E \times 0, M_1 \subset E \times 1$ giving rise by the Thom construction to maps $f_0, f_1 : E \to T(\xi)$, and a homotopy $F : E \times I \to T(\xi)$ between f_0 and f_1. As above, we can replace F (keeping it fixed on $E \times \partial I$) by a homotopy F' of f_0 to f_1, which is smooth and transverse to B_ξ. Then $W := F'^{-1}(B_\xi)$ is a submanifold of $E \times I$, and provides a cobordism of M_0 to M_1; moreover the normal bundle of N is induced from ξ. □

A key point in the above arguments is where we first approximate the map F by a smooth map, and then make it transverse to a smooth submanifold of the target. Since we will use this idea several times below, in future we omit the references to Proposition 2.3.4 and Theorem 4.5.6.

We have described the Thom construction directly in a geometric context. We next relax the condition that B_ξ be a smooth manifold. There is a space $B(O_k)$ and a vector bundle γ_k over $B(O_k)$ such that, for any k-vector bundle ξ over a space X, there is a map $p : X \to B(O_k)$ such that ξ is equivalent to $p^*\gamma_k$, and this induces a bijection between isomorphism classes of vector bundles ξ and homotopy classes of maps p. (See §8.6 for more about classifying spaces). Moreover, we may construct $B(O_k)$ as the union of Grassmann manifolds $Gr_{m,k}$, and the map $Gr_{m,k} \to B(O_k)$ is m-connected. The bundle γ_k has associated disc bundle AO_k, say, and Thom space $T(O_k)$.

Lemma 8.1.2 *The Thom construction gives a bijection between cobordism classes of submanifolds $M^m \subset E^{m+k}$ and homotopy classes of maps $E \to T(O_k)$.*

Proof We apply Proposition 8.1.1 taking $Gr_{m,k}$ in place of B_ξ. For any submanifold M^m of E^{m+k}, the normal bundle is induced by a map to the classifying space $B(O_k)$, but we may replace these by maps to $G_{m,k}$. Since the map $G_{m,k} \to B(O_k)$ is m-connected, up to homotopy we can obtain $B(O_k)$ from $Gr_{m,k}$ by attaching cells of dimension $> m$. It follows that up to homotopy we can obtain $T(O_k)$ from the Thom space of $Gr_{m,k}$ by attaching cells of dimension $> m + k$. The result follows. □

We next replace O_k by an arbitrary structure group J (for example, J could be a Lie group), furnished with a homomorphism $J \to O_k$. There is (again see §8.6) a classifying space $B(J)$, and isomorphism classes of (vector) bundles over a space X with structure group J correspond bijectively to homotopy classes of maps $X \to B(J)$. There is an induced map $B(J) \to B(O_k)$ of classifying spaces. There is a universal bundle ξ_J over $B(J)$ and a map $f : X \to B(J)$ corresponds to the bundle $f^*\xi_J$. We write $A(J)$ for the disc bundle, $S(J)$ for its boundary sphere bundle and $T(J)$ for the Thom space $A(J)/S(J)$.

In fact we do not need J at all: only a space X playing the role of $B(J)$, and a map $X \to B(O_k)$ (here we can interpret the loop space $\Omega(X)$ as playing the role of J). However we adhere to the notation with J.

As in the case $J = O_k$, although $B(J)$ is rarely itself a smooth manifold, we can find a sequence of smooth manifolds $B(J^{(r)})$ and r-connected maps

$B(J^{(r)}) \to B(J^{(r+1)}) \to B(J)$. The same argument as for Lemma 8.1.2 now yields

Theorem 8.1.3 *The Thom construction induces a bijective map of the set of cobordism classes of pairs (E^{m+k}, M^m), with E fixed and J as structure group of the normal bundle, onto the set of homotopy classes $[E : T(J)]$.*

We can state this as a slogan: the extra structure defined on E by a submanifold whose normal bundle has group J is equivalent to the extra structure consisting of a map to $T(J)$.

Taking $J = SO_k$ in particular yields a natural bijection between cobordism classes of submanifolds $M^m \subset E^{m+k}$ with oriented normal bundle and homotopy classes of maps $E \to T(SO_k)$. Even more simply, taking J to be the trivial group gives

Proposition 8.1.4 *There is a natural bijection between cobordism classes of submanifolds $M^m \subset E^{m+k}$ with framed normal bundle and homotopy classes of maps $E \to S^k$.*

One application of Theorem 8.1.3 is to the problem of representing homology classes by manifolds. It seems that this problem, raised by Steenrod, was part of what led Thom to introduce the notion of cobordism. A first formulation is: let X be a space and $x \in H_n(X; \mathbb{Z})$: do there exist a closed oriented manifold M^n and a map $f : M \to X$ such that $f_*[M] = x$? We can vary this by using $\mathbb{Z}_2$ as coefficient group and not having an orientation. We can also take X as a manifold and require f to be an embedding: by general position results, this makes no difference if dim $M > 2n$. An affirmative result for manifolds implies one for spaces, since we can replace X by a manifold E homotopy equivalent to it; since we can apply such a result to the double $D(E)$ of E, it will suffice to consider the case of closed manifolds.

Given an oriented orthogonal vector bundle ξ over E with fibre dimension k, we have the Thom class $U \in H^k(T(\xi); \mathbb{Z})$.

Proposition 8.1.5 *Suppose E^{n+k} a closed oriented manifold. Then given a class $x \in H_k(E; \mathbb{Z})$, there is an oriented submanifold M^k of E whose fundamental homology class maps to x if and only if there is a map $F : E \to T(SO_n)$ with F^*U the Poincaré dual of x.*

Proof Since E is oriented, orientations of a submanifold M correspond to orientations of its normal bundle. We already know the correspondence between submanifolds of E and maps $F : E \to B(SO_k)$. It will thus suffice to show that $F^*U = [E] \cap x$.

In the situation of the Thom construction we have a commutative diagram

$$\begin{tikzcd} H^k(T,\partial T) \arrow[d] & H^k(TSO_k) \arrow[l] \arrow[d] \\ H^0(M) & H^0(BSO_k) \arrow[l] \end{tikzcd},$$

so F^*U is the image in $H^k(E)$ of the class in $H^k(T, \partial T)$ corresponding to $1 \in H^0(M)$. Since the Thom class in $H^k(T, \partial T)$ is dual to the image of $[M]$ in $H_m(T)$, F^*U is indeed the cohomology class dual to the image of $[M]$ in $H_m(E)$. □

In the cases $k = 1, 2$ the condition is automatically satisfied: SO_1 is trivial and $T(SO_1)$ is D^1 with the boundary collapsed to a point, so can be identified with S^1, of type $K(\mathbb{Z}, 1)$; similarly $T(SO_2)$ and $K(\mathbb{Z}, 2)$ can both be identified with infinite complex projective space $P^\infty(\mathbb{C})$.

In his paper [149] Thom used his results on cobordism to prove that for any homology class $x \in H_n(X; \mathbb{Z}_2)$, there exist a closed manifold M^n and a map $f : M \to X$ such that $f_*[M] = x$. However for integer coefficients, while any $x \in H_n(X : \mathbb{Z})$ with $n \leq 6$ is the image of the fundamental class of a closed orientable manifold, this fails for any $n \geq 7$: there is an obstruction, obtained using the Steenrod reduced cube $\mathcal{P}^1$ (see §B.4).

8.2 Cobordism groups and rings

If we are interested in the manifold M^m but not the embedding in an ambient manifold E, it is natural to take E to be Euclidean space of large dimension $m + k$: by Whitney's embedding Theorem 4.2.2 we know that for $k > m + 1$ such embeddings exist and are unique up to diffeotopy. To apply the preceding section, we need E to be compact. Since embeddings in S^{m+k} yield ones in $\mathbb{R}^{m+k}$ by deforming M away from the point at infinity, we can take E as S^{m+k}.

Identifying $\mathbb{R}^{m+k}$ as a hyperplane in $\mathbb{R}^{m+k+1}$ leads us to identify S^{m+k} with a great sphere in S^{m+k+1}, and use the composite embedding $M^m \to S^{m+k} \to S^{m+k+1}$ to obtain independence of k. We may thus calculate the set Ω_m^O of cobordism classes of closed m dimensional manifolds by applying the theory of the preceding section to manifolds contained in spheres of large enough dimension.

We must also discuss the normal bundles. If ν^k is the normal bundle of M^m in S^{m+k}, the normal bundle in S^{m+k+1} is $\nu^k \oplus \varepsilon^1$. Before developing the theory more fully we present axioms for the ‘stable groups’ which will play the role of structure groups of the normal bundles.

A *stable group* **J** is given by a commutative diagram of groups and homomorphisms

$$\begin{tikzcd} \dots \arrow[r] & J_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\varphi_{n-1}"] & J_n \arrow[r, "i_n"] \arrow[d, "\varphi_n"] & J_{n+1} \arrow[r] \arrow[d, "\varphi_{n+1}"] & \dots \\ \dots \arrow[r] & O_{n-1} \arrow[r] & O_n \arrow[r] & O_{n+1} \arrow[r] & \dots \end{tikzcd} \tag{8.2.1}$$

where the inclusions in the lower row are natural. We impose the stability condition

(S): There is a function q_n of n, increasing (in the weak sense) and tending to infinity, such that i_n is q_n-connected.

We also need products and impose the following further conditions.

(M): We have a family of maps $\psi_{m,n} : J_m \times J_n \to J_{m+n}$ such that the following diagrams commute up to conjugating by an element in the component of the identity:

$$\begin{tikzcd} J_m \times J_n \arrow[r, "\psi_{m,n}"] \arrow[d, "i_m \times 1"] & J_{m+n} \arrow[d, "i_{m+n}"] & J_m \times J_n \arrow[l, "\psi_{m,n}"'] \arrow[d, "1 \times i_n"] \\ J_{m+1} \times J_n \arrow[r, "\psi_{m+1,n}"] & J_{m+n+1} & J_m \times J_{n+1} \arrow[l, "\psi_{m,n+1}"'] \end{tikzcd} \tag{8.2.2}$$

$$\begin{tikzcd} J_m \times J_n \arrow[r, "\psi_{m,n}"] \arrow[d, "\varphi_m \times \varphi_n"] & J_{m+n} \arrow[d, "\varphi_{m+n}"] \\ O_m \times O_n \arrow[r] & O_{m+n} \end{tikzcd} \tag{8.2.3}$$

(A): The following diagram also commutes (in the same sense)

$$\begin{tikzcd} J_l \times J_m \times J_n \arrow[r, "\psi_{l,m} \times 1"] \arrow[d, "1 \times \psi_{m,n}"] & J_{l+m} \times J_n \arrow[d, "\psi_{l+m,n}"] \\ J_l \times J_{m+n} \arrow[r, "\psi_{l,m+n}"] & J_{l+m+n} \end{tikzcd} \tag{8.2.4}$$

(C): We have commutativity in the diagram

$$\begin{tikzcd} J_m \times J_n \arrow[r, "T"] \arrow[d, "\varphi_{m+n}\psi_{m,n}"] & J_n \times J_m \arrow[d, "\varphi_{m+n}\psi_{n,m}"] \\ O_{m+n} \arrow[r, "T'"] & O_{m+n} \end{tikzcd} \tag{8.2.5}$$

where T is the natural interchange of factors, and T' means conjugation by an element whose determinant has sign $(-1)^{mn}$.

The important examples of stable groups **J** are the classical groups **O**, **SO**, **Spin**, **U**, **SU** and **Sp**, and the trivial group $\{1\}$. The above properties are immediate in these cases. Of interest also are the groups **Spin**^{c}, **Pin**, and **Pin**^{c} of [15, pp. 7–10]; however **Pin** fails to satisfy (M). Further examples can easily be constructed: for example, products of the above with each other or with any group of linear operators on a finite dimensional vector space.

We have presented the axioms in a geometrical setting, but note here that it would in fact suffice to have maps of classifying spaces $B(J_k)$ throughout; the map $B(J_k) \to B(O_k)$ induces an orthogonal vector bundle γ^k over $B(J_k)$ which is all we will need for our constructions.

An embedding $M^m \to S^{m+k}$ with J_k as structure group of the normal bundle now gives an embedding $M^m \to S^{m+k+1}$ with normal bundle with group J_{k+1}. It is however more natural to consider the tangent bundle. A *weak* **J**-*structure* on M^m is prescribed by choosing an integer r and reduction (e, f) of the group of $\tau_M \oplus \varepsilon^r$ to J_{m+r}; (r, e, f) and (r', e', f') are *equivalent* if the reductions (e, f) and (e', f') of $\tau_M \oplus \varepsilon^s$ are so for some $s \geqq r, r'$. When $J = U$ we call this a weakly complex structure.

We now show that if (S) holds, we can pass between the structure group on the stable tangent bundle and the structure group on a normal bundle. This fails for Pin: if M has a Pin normal bundle, the tangent bundle is not necessarily Pin: we have $\overline{w_2} = 0$ but $w_2 = w_1^2$.

Lemma 8.2.6 *Suppose in the diagram (8.2.3) that the map $\psi_o : J_r \to J_{r+s}$ induced by $\psi_{r,s}$ is c-connected. Let K be a CW complex of dimension $d \leqq$* $\min(c, r-2)$*, and ξ^r, η^s vector bundles over K, with a J_s-structure on η^s. Then the function f induce by ψ from J_r-structures on ξ^r to J_{r+s}-structures on $\xi^r \oplus \eta^s$ is bijective.*

Proof Let X_i be the classifying space for $J_i (i = r, s$ or $r + s)$; E_i the total space of the principal bundle with fibre O_i induced over X_i by φ_i. Write E_ξ, E_η, $E_{\xi\oplus\eta}$ for the spaces of the corresponding principal bundles over K. Then J_r-structures of ξ correspond to sections of the bundle over K with total space $E_\xi \times_{O_r} E_r$; similarly for $\xi \oplus \eta$. But the J_s-structure of η induces a fibrewise map

$$E_\xi \times_{O_r} E_r \to E_{\xi\oplus\eta} \times_{O_{r+s}} E_{r+s} \tag{8.2.7}$$

and the induced map $E_r \to E_{r+s}$ of fibres is at least $\min(c+1, r-1)$-connected since $X_r \to X_{r+s}$ is $(c+1)$-connected and $O_r \to O_{r+s}$ is $(r-1)$-connected. Thus (8.2.7) is at least $(d+1)$-connected, so any map of K to the second term can be factorised (up to homotopy) through the first, and f is surjective; moreover, the result is unique up to homotopy, so f is bijective. (It follows from the CHP that sections of a bundle are homotopic only if they are homotopic through sections.) □

Corollary 8.2.8 *Let $M^m \subset \mathbb{R}^{m+N}$ have a weak $\mathbf{J}$-structure, where the stable group $\mathbf{J}$ satisfies (S), and $q_N \geqq m$. Then the normal bundle has a J_N-structure; conversely, this implies a weak $\mathbf{J}$-structure on M.*

Proof In this case, $\xi \oplus \eta$ has a standard framing, and hence J-structure. We use (A) only to identify the ψ_0 of the lemma with a composite of maps i_n. □

The definition of a cobordism W of manifolds with weak $\mathbf{J}$-structures demands a reduction of the structure group of τ_W. Now $\tau_W|\partial W \cong \tau_{\partial W} \oplus \varepsilon^1$, so the induced structure of the boundary is a reduction of the group of $\tau_{\partial W} \oplus \varepsilon^1$ rather than of $\tau_{\partial W}$ itself. Here we make the convention (necessary to obtain an equivalence relation) that the positive vector ε^1 is to be identified with the inward normal to $\partial_- W$ in W, but with the outward normal on $\partial_+ W$. Now a weak $\mathbf{J}$-structure on a cobordism W induces weak $\mathbf{J}$-structures on $\partial_- W$, $\partial_+ W$: we call it a cobordism between these manifolds with the induced structures.

We denote by Ω_m^J the set of cobordism classes of m-manifolds with a weak $\mathbf{J}$-structure.

Lemma 8.2.9 *If $\mathbf{J}$ satisfies (S), and $N \geq m+2$, $q_N \geqq m+1$, there is a natural bijection of Ω_m^J to the set of cobordism classes of manifolds $M^m \subset S^{m+N}$ with J_N as group of the normal bundle, and hence to $\pi_{m+N}(T(J_N))$.*

Proof The first statement follows from Corollary 8.2.8, and the second from the results in the preceding section. □

Now let $\mathbf{J}$ be a stable group, with γ^k the universal bundle over $B(J_k)$. The inclusion $i_k : J_k \to J_{k+1}$ induces a bundle map $\phi_k : \gamma^k \oplus \varepsilon^1 \to \gamma^{k+1}$ over $Bi_k : B(J_k) \to B(J_{k+1})$. Write $B(\mathbf{J})$ for the limit of this sequence (we can regard the Bi_k as inclusions and form the union). In view of the identification

$T(\xi \oplus \varepsilon^1) = T(\xi) \wedge S^1$, the bundle map ϕ_k lifts to a map $h_k : ST(J_k) \to T(J_{k+1})$. The sequence of maps $h_k : ST(J_k) \to T(J_{k+1})$ defines a spectrum, which we denote by $\mathbb{TJ}$.

Theorem 8.2.10 *For* **J** *a stable group we have a bijection*

$$\Omega_m^J \cong \lim_{k\to\infty} \pi_{m+k}(T(J_k)) = \pi_m^S(\mathbb{TJ}).$$

Proof An embedding $i : M^m \to \mathbb{R}^{m+k} \subset S^{m+k}$ can be regarded as lying in a hyperplane (or great sphere) giving an embedding $i_1 : M^k \to \mathbb{R}^{m+k+1} \subset S^{m+k+1}$. Applying the Thom construction to the first gives a map $F : S^{m+k} \to T(J_k)$, and to the second gives its suspension $SF : S^{m+k+1} \to T(J_{k+1})$.

By definition, possession of a **J**-structure is equivalent to having a normal J_k-structure in S^{m+k} for some k. If we fix k, then by Lemma 8.2.9 we obtain the group $\pi_{m+k}(T(J_k))$. The desired group is the direct limit of these under the natural injection maps. □

If **J** satisfies (S), the suspension map $\pi_{m+k}(T(J_k)) \to \pi_{m+k+1}(T(J_{k+1}))$ is an isomorphism for $k > m + q_m$, so no limiting process is necessary.

The cobordism set Ω_m^J has a natural group structure: the sum of the classes of disjoint manifolds M, M' is defined to be the class of $M \cup M'$. Any M' is diffeomorphic (hence cobordant) to a manifold disjoint from M. The sum is well defined since the disjoint union of cobordisms of M with N and of M' with N' is a cobordism of $M \cup M'$ to $N \cup N'$. Commutativity and associativity are immediate. Since $\partial(M \times I) = (M \times \{0\}) \cup (M \times \{1\})$, we have an inverse (note that the normal bundle is different in the two cases).

The bijections of Lemma 8.2.9 and Theorem 8.2.10 are group isomorphisms since both are induced by the Thom construction. We can take manifolds M and M' to lie in distinct discs in S^{m+k}. The map given by the Thom construction takes the boundaries of these discs to ∞. If we then remove discs, and glue the two spheres together, we obtain the usual sum of homotopy classes.

Products are compatible with cobordism: if W is a cobordism from $\partial_- W$ to $\partial_+ W$, then $W \times M$ is a cobordism from $\partial_- W \times M$ to $\partial_+ W \times M$. Also, products are associative, and distributive over disjoint union, and there is a natural diffeomorphism of $M' \times M$ on $M \times M'$, which gives rise to a form of commutativity of multiplication.

If G, H are groups of orthogonal operators on $\mathbb{R}^q$, $\mathbb{R}^r$, then $B(G) \times B(H)$ is a classifying space for $G \times H$, and $\xi_G \times \xi_H$ is a universal bundle. As observed above, $T(G \times H) = T(G) \wedge T(H)$.

If $\mathbf{J}$ satisfies (M), the products $\psi_{m,n} : J_m \times J_n \to J_{m+n}$ induce maps $\psi'_{m,n} : T(J_m) \wedge T(J_n) \to T(J_{m+n})$, and if (A) holds, these associate up to homotopy. This provides $\mathbb{TJ}$ with the structure of a ring spectrum.

Theorem 8.2.11 *If $\mathbf{J}$ is a stable group satisfying (M), we have a bilinear product $\Omega_m^J \times \Omega_n^J \to \Omega_{m+n}^J$ which corresponds to the pairing in homotopy groups induced by the maps $T(J_m) \wedge T(J_n) \to T(J_{m+n})$. The product is associative if $\mathbf{J}$ satisfies (A), and defines a commutative graded ring if $\mathbf{J}$ satisfies (C).*

Proof The product of submanifolds $M \subset V$ and $N \subset W$ gives a submanifold $M \times N \subset V \times W$. Using the Thom construction as in Theorem 8.1.3 these determine elements of $[V : T(J_m)]$ and $[W : T(J_n)]$ and the product is given by

$$[V : T(J_m)] \times [W : T(J_n)] \to [V \times W : T(J_m \times J_n)].$$

The conclusion follows by taking V and W to be Euclidean spaces (or rather spheres) and stabilising. □

A case of particular simplicity is $\mathbf{J} = \{1\}$: each J_k consists only of the unit element, so we can take $B(J_k)$ to be a point; then $T(J_k) = S^k$. For each bundle occurring we have a specified isomorphism with a trivial bundle, i.e. a framing, and for clarity write Ω^{fr} for the cobordism group.

Corollary 8.2.12 *Framed cobordism groups are isomorphic to stable homotopy groups of spheres: $\Omega_n^{fr} \cong \lim_{k\to\infty} \pi_{n+k}(S^k)$.*

This, due to Pontrjagin [123], was the first theorem in the subject.

8.3 Techniques of bordism theory

In this section we introduce a couple of techniques, variants of which will often be used below. The first is a general method of constructing exact sequences. Recall from §5.1 that a cobordism of the bounded manifolds M_0 and M_1 is a manifold W with corner $\angle W$ which divides ∂W into three parts, with disjoint interiors: $M_0 = \partial_- W$, $\partial_c W$ and $M_1 = \partial_+ W$, with M_0 and M_1 disjoint. Thus $\partial_c W$ is a cobordism of ∂M_0 to ∂M_1.

By itself, this definition gives nothing: any manifold M with boundary is cobordant to the empty set by the manifold W obtained from $M \times I$ by rounding corners at $M \times \{1\}$. The interesting cases are those in which an extra condition is imposed on the cobordism $\partial_c W$.

Suppose two kinds of structure specified, which we call an α-structure and a β-structure, with the latter stronger than the former. For example, we may consider structure groups G_1 and $G_2 \subset G_1$, or maps to spaces X_1 and $X_2 \subset X_1$, or actions of groups H_1 and $H_2 \supset H_1$, or k_1-connectivity and $k_2 (> k_1)$-connectivity.

By Ω_n^α and Ω_n^β we denote the cobordism groups of manifolds with α- (resp. β-) structure; and by $\Omega_n^{\alpha,\beta}$ the cobordism group of bounded manifolds with α-structure, whose boundaries have a β-structure including the given α-structure. We suppose there are natural group structures, though all we need is the zero element provided by the empty manifold.

Lemma 8.3.1 *There is an exact sequence* $\ldots \to \Omega_n^\beta \xrightarrow{i_n} \Omega_n^\alpha \xrightarrow{j_n} \Omega_n^{\alpha,\beta} \xrightarrow{\partial_n} \Omega_{n-1}^\beta \xrightarrow{i_{n-1}} \Omega_{n-1}^\alpha \to \ldots$

Proof The maps i_n and j_n are the natural ones; ∂_n is induced by taking the boundary.

Exactness at Ω_{n-1}^β follows since a β-manifold M^{n-1} represents an element z of Ker i_{n-1} if and only if, as α-manifold, it bounds some N^n. But then N represents an element $y \in \Omega_n^{\alpha,\beta}$ with $\partial_n y = z$.

We have $\partial_n \circ j_n = 0$ since an element of Ω_n^α is represented by a manifold with empty boundary. Now suppose N^n represents $y \in \Omega_n^{\alpha,\beta}$ with $\partial_n y = 0$. Then ∂N bounds a β-manifold N'. We form a closed manifold N'' by glueing N to N' along their common boundary. The α-structures on N and N' induce a α-structure on the union N''. We now define a cobordism W by taking $N'' \times I$ and introducing a corner along $\partial N \times \{0\}$, so that $\partial_- W = N \times \{0\}$, $\partial_c W = N' \times \{0\}$ and $\partial_+ W = N'' \times \{1\}$. Here $\partial_c W$ has a β-structure, so $N'' = \partial_+ W$ also represents y and is in the image of j_n.

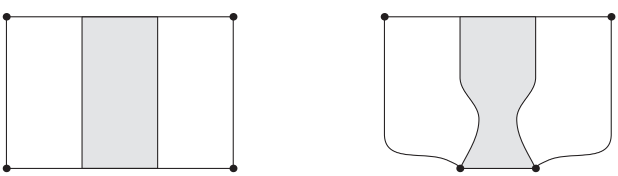

Figure 8.1 A new cobordism obtained by changing the corner

We have $j_n \circ i_n = 0$ since for any β-manifold M, we can interpret $N = M \times I$ as an (α, β) cobordism to the empty set by setting $\partial_-(M \times I) = M \times \{0\}$ and $\partial_c(M \times I) = M \times \{1\}$. Finally, if the closed α-manifold M has class x with $j_n x = 0$, there is a (α, β)-cobordism W of M to the empty set. Thus $\partial_- W = M$, $\partial_+ W = \varnothing$, and $N := \partial_c W$ is a closed β-manifold. Now letting V be the cobordism, diffeomorphic to W but with $\partial_- V = M$ and $\partial_+ V = N$, we see N is α-cobordant to M, so if N has class z we have $i_n z = x$. $\square$

This procedure of changing the corner is itself a useful technique: we met it in §7.4 and will encounter it again.

The following addendum is easily proved by the same method.

Lemma 8.3.2 *Suppose given three kinds of structure: an α-structure, a β-structure, and a γ-structure, with γ stronger than β which in turn is stronger than α. Then there is a commutative diagram including the exact sequences corresponding to the three inclusions and one with the relative terms.*

$$\begin{tikzcd}
\Omega_n^{\gamma} \arrow[dr] \arrow[rr, bend left] & & \Omega_n^{\alpha} \arrow[dr] \arrow[rr, bend left] & & \Omega_n^{\alpha,\beta} \arrow[dr] & \\
& \Omega_n^{\beta} \arrow[ur] \arrow[dr] & & \Omega_n^{\alpha,\gamma} \arrow[ur] \arrow[dr] & & \Omega_{n-1}^{\beta} \\
\Omega_{n+1}^{\alpha,\beta} \arrow[ur] \arrow[rr, bend right] & & \Omega_n^{\beta,\gamma} \arrow[ur] \arrow[rr, bend right] & & \Omega_{n-1}^{\gamma} \arrow[ur] &
\end{tikzcd}$$

Lemma 8.3.1 is often applied together with a method of calculating $\Omega_n^{\alpha,\beta}$. To illustrate this, suppose any manifold W^n with α-structure has an induced β-structure except on a closed submanifold M^m, and define γ to be the type of structure induced on M by an α structure on a tubular neighbourhood V of M in W. This is imprecise; the details need to be clarified in each case where this is applied.

Lemma 8.3.3 *Inclusion induces an isomorphism $\Omega_m^{\gamma} \to \Omega_n^{\alpha,\beta}$.*

Proof The map is defined by taking the class of M in Ω_m^{γ} to that of $(V, \partial V)$ where V is the disc bundle over M which is part of the γ structure: by the definition of γ, this pair has an (α, β)-structure.

To prove the map surjective, take any $(W, \partial W)$ with (α, β)-structure, and construct M, V as above. Then $(V, \partial V)$ has a (α, β)-structure since ∂V is disjoint from M so we have an induced β structure on it. An (α, β)-cobordism X from $(V, \partial V)$ to $(W, \partial W)$ is obtained from $W \times I$ by rounding the corner at $\partial W \times \{0\}$ (using Proposition 2.6.2), and introducing a corner at $\partial V \times \{0\}$ (using Lemma 2.6.3) as in Figure 8.1: thus $\partial_c X = ((W \setminus \mathring{V} \times \{0\}) \cup (\partial W \times I)$.

Now suppose M such that $(V, \partial V)$ is (α, β)-cobordant to the empty set: write X for a cobordism. The α-structure on X gives a β-structure except on a compact submanifold L with boundary M. Then L has an induced γ-structure, so the class of M in Ω_m^γ is zero. □

Let V be a submanifold of M; then we call (M, V) a pair. If (N, W) is a pair of manifolds with boundary, and N is a cobordism of $\partial_- N$ to $\partial_+ N$, we set $\partial_- W = W \cap \partial_- N$, $\partial_+ W = W \cap \partial_+ N$. Our definition of submanifold then implies that W is a cobordism of $\partial_- W$ to $\partial_+ W$, and we shall call the pair (N, W) a cobordism of the pair $(\partial_- N, \partial_- N)$ to the pair $(\partial_+ N, \partial_+ W)$. Rather than restrict the structure groups of the stable tangent bundles of M and V independently; we usually restrict the structure group of the normal bundle of V in M: here there is no need to speak of weak structures.

We study cobordism of pairs by establishing a principle of 'extension of cobordism' (analogous to homotopy extension). This is illustrated in the next lemma.

Consider pairs $(M^{\upsilon+q}, V^\upsilon)$, where M has a weak $\mathbf{J}$-structure and the normal bundle an H_q-structure; more generally, consider $V^\upsilon \subset M^{\upsilon+q} \subset S^{\upsilon+q+r}$, where the structure groups of the normal bundles are H_q and J_r. Then the normal bundle $V^\upsilon \subset S^{\upsilon+q+r}$ has an $H_q \times J_r$-structure. Here we only consider the stable case $r > \upsilon + q + 1$ where the imbedding of M in S is irrelevant, so may replace J_r by $\mathbf{J}$.

Let $\mathbf{J}$ be a stable group, and H_q a group mapping to O_q. Then setting $(\mathbf{J} \times H_q)_n = J_{n-q} \times H_q$ defines a stable group $\mathbf{J} \times H_q$, which satisfies (**S**) if J does.

Lemma 8.3.4 *The pair $(M^{\upsilon+q}, V^\upsilon)$ is $(\mathbf{J}, H_q)$-cobordant to the empty pair if and only if $M^{\upsilon+q}$ is $\mathbf{J}$-cobordant to zero and V^υ is $\mathbf{J} \times H_q$-cobordant to zero.*

Proof The necessity of the condition is evident. To prove sufficiency we give a construction to extend a $\mathbf{J} \times H_q$-cobordism of V^υ to the empty set to a $\mathbf{J} \times H_q$-cobordism of (M, V) to a pair (M', ψ). Since cobordism is an equivalence relation, it follows that M' is $\mathbf{J}$-cobordant to φ, say by N'; then (N', φ) is the required $(\mathbf{J}, H_q)$-cobordism of (M', ψ) to (ψ, φ).

Let $W^{\upsilon+1}$ be the given $\mathbf{J} \times H_q$-cobordism of V to φ: then there is an induced bundle over W with fibre D^q, whose total space we denote by $L^{\upsilon+q+1}$. Note that the restriction to V of this bundle is the normal bundle of V in M; hence we can identify a tubular neighbourhood of V in M with part of the boundary of L. We form $M \times I$, and attach L to $M \times 1$ by this identification, giving N. Since L and $M \times I$ have $\mathbf{J}$-structures, which agree (by hypothesis, W is a cobordism of V with the $\mathbf{J} \times H_q$-structure induced from M) on the pair identified, $N^{\upsilon+q+1}$

has a weak **J**-structure. Also, $V \times I \cup W = W'$ is a submanifold whose normal bundle has group H_q.

Set $M \times 0 = \partial_- N$. Then (N, W') is a $\mathbf{J} \times H_q$-cobordism, and $W' \cap \partial_+ N = \varnothing$. This completes the proof of the lemma. $\square$

Lemma 8.3.5 *The cobordism group of pairs* (M^{v+q}, V^v)*, where* M *has a weak* **J***-structure and the normal bundle an* H_q*-structure is isomorphic to* $\Omega^J_{v+q} \oplus \Omega^{J\times H_q}_v$.

Proof In Lemma 8.3.4 we defined a map to the direct sum, and proved it a monomorphism; it clearly respects additive structure. The map to $\Omega^{J\times H_q}_v$ is onto, for given a $(J \times H_q)$-manifold V^v, we construct as above a bundle over V with fibre D^q, and can take M as the double of this manifold. Finally, the image contains $\Omega^J_{v+q} \oplus 0$: we need only consider pairs with V empty. $\square$

8.4 Bordism as a homology theory

For **J** a stable group, and X any space, we denote by $\Omega^J_m(X)$ the cobordism group of (closed) manifolds M^m together with a weak **J**-structure and a map to X. The arguments of the preceding section generalise easily to this situation.

In fact we go further: given a pair of spaces $Y \subseteq X$ we define $\Omega^J_m(X, Y)$ to be the set of cobordism classes of (compact) manifolds M^m with a weak **J**-structure and a map $f : M \to X$ with $f(\partial M) \subset Y$. The definition of the cobordism relation is implicit in the above: a cobordism is a compact manifold W with corner, with a weak **J**-structure inducing the given weak **J**-structures on $\partial_\pm W$ (with the above convention), together with a map $g : (W, \partial_c W) \to (X, Y)$. Generalising Theorem 8.1.3, we have

Theorem 8.4.1 *If* **J** *is a stable group, the Thom construction induces isomorphisms*

$$\Omega^J_m(X, Y) \cong \lim_{k\to\infty} \pi_{m+k}((X^+ \wedge T(J_k), Y^+ \wedge T(J_k))).$$

Proof Given M^m, it follows from Whitney's embedding theorem and the refinements of Theorem 4.7.3 that for $k > m$ we can find embeddings of $(M, \partial M)$ as a submanifold of $(D^{m+k}, \partial D^{m+k})$, and that for $k > m + 1$ any two such embeddings are diffeotopic. It follows from Lemma 8.2.6 that for k large enough the weak **J**-structure on M induces a J_k-structure on the normal bundle ν of the embedding. Write ∂A_ν for the part of the disc bundle A_ν lying over ∂M. Then we have a map $A_\nu \to A(J_k) \to T(J_k)$ and also maps $(A_\nu, \partial A_\nu) \to$

$(M, \partial M) \to (X, Y)$. Taking the product, we have a map $(A_\nu, \partial A_\nu) \to (X \times T(J_k), Y \times T(J_k))$, which takes $(S_\nu, \partial S_\nu)$ to $(X \times \{\infty\}, Y \times \{\infty\})$.

Now collapse everything in D^{m+k} outside A_ν to a point, giving a map of $(D^{m+k}, \partial D^{m+k})$ to $(T(\nu), \partial T(\nu))$, and hence to

$$(X \times T(J_k)/X \times \{\infty\}, Y \times T(J_k)/Y \times \{\infty\}).$$

Recalling that $X \times T(J_k)/X \times \{\infty\} = X^+ \wedge T(J_k)$, we see as in Theorem 8.1.3, that this construction define a map

$$\Omega^J_m(X, Y) \to \lim_{k\to\infty} \pi_{m+k}((X^+ \wedge T(J_k), Y^+ \wedge T(J_k))).$$

The proof of the result now also closely follows that of Theorem 8.1.3. To establish surjectivity, we start with a map f and let K be the inverse image of ∞. Then f defines a map of $D^{m+k} \setminus K$ to $A(J_k) \times X$. We alter the first component by a small homotopy, to make it smooth and transverse to $B(J_k)$. This defines also a homotopy of f, say to f'. Now set $M^m = f'^{-1}(B(J_k) \times X)$; then f' induces a map $(M^m, \partial M^m) \to (X, Y)$, and the normal bundle of M has group reduced to J_k. It follows that the bordism class defined by M maps to the homotopy class of f.

Again, injectivity follows by a similar but simpler argument, and the proof that the bijection preserves group structures and the passage to the limit work as before. □

In particular we have an isomorphism $\Omega^J_m(X) \cong \lim_{k\to\infty} \pi_{m+k}(X^+ \wedge T(J_k))$. It follows as in Corollary 8.2.12 that

Corollary 8.4.2 *Under the isomorphism of Theorem 8.4.1, the external products $\Omega^J_m(X) \times \Omega^J_n(Y) \to \Omega^J_{m+n}(X \times Y)$ correspond to the homotopy pairings induced by $(X^+ \wedge T(J_k)) \wedge (Y^+ \wedge T(J_l)) \to (X^+ \wedge Y^+) \wedge T(J_{k+l})$.*

The maps $T(J_k) \wedge T(J_l) \to T(J_{k+l})$ give the limit $\mathbb{TJ}$ the structure of a ring spectrum (see §B.4). We can immediately extend Theorem 8.2.11 to

Theorem 8.4.3 *The Thom construction induces a natural equivalence between the functor Ω^J_* and homology theory with coefficients in the spectrum $\mathbb{TJ}$; this respects products in the multiplicative case.*

We have shown that Ω^J_* defines a homology theory. We prefer to present also a direct proof of this fact.

Theorem 8.4.4 *The groups $\Omega^J_*(X)$, $\Omega^J_*(X, Y)$ satisfy the Eilenberg–Steenrod axioms [50] for a homology theory.*

We first recall these axioms:

I, II: Ω^J_* is a functor from the category of pairs of spaces (X, Y) and continuous maps to the category of graded abelian groups. We denote $\Omega_*(X, \varnothing)$ by $\Omega_*(X)$.

III: For any pair (X, Y) there is a map $\partial : \Omega_m(X, Y) \to \Omega_{m-1}(Y)$ which is natural for maps of pairs.

IV: For any pair (X, Y), if $i : Y \to X$ and $j : (X, \varnothing) \to (X, Y)$ denote the inclusions, we have an exact sequence

$$\cdots \to \Omega^J_m(Y) \xrightarrow{i_*} \Omega^J_m(X) \xrightarrow{j_*} \Omega^J_m(X, Y) \xrightarrow{\partial_m} \Omega^J_{m-1}(Y) \to \cdots$$

V: Homotopic maps φ_0 and $\varphi_1 : (X_1, Y_1) \to (X_2, Y_2)$ induce the same map in bordism: $\varphi_{0,*} = \varphi_{1,*} : \Omega^J_*(X_1, Y_1) \to \Omega^J_*(X_2, Y_2)$.

VI: If $U \subset X$ has its closure in the interior of Y, then inclusion induces an isomorphism $\Omega^J_*(X \setminus U, Y \setminus U) \cong \Omega^J_*(X, Y)$.

Proof I, II: If $\varphi : (X_1, Y_1) \to (X_2, Y_2)$ is a map, M has a weak **J**-structure, and $f : (M, \partial M) \to (X_1, Y_1)$ represents a class $z \in \Omega^J_m(X_1, Y_1)$, then $\varphi \circ f$ represents $\varphi_*(z)$. This is well defined since if F defines a cobordism of f then $\varphi \circ F$ defines a cobordism of F. It is clear that the construction respects unions, so the map is additive.

III. If $f : (M, \partial M) \to (X, Y)$ gives a bordism class of (X, Y), then $f|\partial M$ gives a bordism class of Y. If $F : (W, \partial_c W) \to (X, Y)$ is a cobordism, then $F|\partial_c W$ is a cobordism between the boundary maps of $F|\partial_- W$ and $F|\partial_+ W$: thus restriction induces a map $\partial_m : \Omega^J_m(X, Y) \to \Omega^J_{m-1}(Y)$ which is compatible with disjoint union and hence a homomorphism. It is immediate that the construction is natural for maps of pairs.

IV. This is our first illustration of Lemma 8.3.1: here all manifolds have weak **J**-structures, and an α-structure consists of a map to X and a β-structure of a map to $Y \subset X$. Observe that in this case, if we form N'' by glueing manifolds N, N' with α-structure along their common boundary, both the weak **J**-structures and the maps to X fit to define a α-structure on N''.

V: If $\Phi : \varphi_0 \simeq \varphi_1$, then for any $f : (M, \partial M) \to (X, Y)$, we can regard $\Phi \circ f$ as defining a cobordism between $\varphi_0 \circ f$ and $\varphi_1 \circ f$.

VI: To prove surjectivity, we let $f : (M, \partial M) \to (X, Y)$ represent an element of $\Omega^J_m(X, Y)$. It is convenient first to alter f (if necessary) by a homotopy on a collar neighbourhood of ∂M so that some smaller neighbourhood is mapped into Y. Then $A = f^{-1}(X \setminus Y)$ and $B = \partial M \cup f^{-1}(U)$ have disjoint closures, so (see §A.2) we can find a continuous map $s : M \to I$ with $s(A) = 0$ and $s(B) = 1$.

We now approximate s by a smooth map, and make it transverse to $\frac{1}{2}$. Then $N := s^{-1}[-\frac{1}{2}, \frac{1}{2}]$ is a smooth submanifold of M, and $f|N$ determines an element of $\Omega^J_m(X \setminus U, Y \setminus U)$. But N and M determine the same class in $\Omega^J_m(X, Y)$. For a cobordism W, we use $f \times 1_I : M \times I \to X$ with a corner introduced at $\partial N \times 0$ and the corner at $\partial M \times 0$ rounded (as in the proof of Lemma 8.3.3). Since $(M \setminus N) \subset s^{-1}[1/2, 1]$, it is disjoint from A, and $f(M \setminus N) \subset Y$, so we can safely adjoin $(M \setminus N) \times 0$ to $\partial_c W$.

The proof of injectivity is similar. If $f : (W, \partial_c W) \to (X, Y)$ is a cobordism of $f|\partial_- W : (\partial_- W, \angle_- W) \to (X \setminus U, Y \setminus U)$ to $\partial_+ W = \emptyset$, we first adjust f so that $A = f^{-1}(X \setminus Y)$ and $B = \partial_c W \cup f^{-1}(U)$ have disjoint closures. Next choose a smooth $s : (W, A, B) \to (I, 0, 1)$, transverse to $\frac{1}{2}$, and set $V = s^{-1}[0, \frac{1}{2}]$. Then V is a cobordism of $\partial_- V$ to zero in $\Omega^J_m(X \setminus U, Y \setminus U)$: a cobordism of $\partial_- V$ to $\partial_- W$ is obtained exactly as above. This completes the proof of the theorem. □

Various standard properties of homology now follow.

Proposition 8.4.5 *(i) For any non-empty X, the maps $\{*\} \to X \to \{*\}$ induce a direct sum split $\Omega^J_*(X) \cong \Omega^J_* \oplus \tilde{\Omega}^J_*(X)$, where $\tilde{\Omega}^J_m(X) \cong \lim_{k\to\infty} \pi_{m+k}(X \wedge T(J_k))$. If CY is the cone on Y, $\tilde{\Omega}^J_*(CY) = 0$, and $\partial : \Omega^J_m(CY, Y) \cong \tilde{\Omega}^J_{m-1}(Y)$.*

(ii) If (X, Y) is a CW pair, or more generally if it has the homotopy extension property (HEP), $\Omega^J_*(X, Y) \cong \Omega^J_*(X/Y, pt) \cong \tilde{\Omega}^J_*(X/Y)$.

(iii) If $X \supset Y \supset Z$ is a triple, we have an exact sequence

$$\cdots \to \Omega^J_m(Y, Z) \to \Omega^J_m(X, Z) \to \Omega^J_m(X, Y) \to \Omega^J_{m-1}(Y, Z) \to \cdots$$

(iv) $\tilde{\Omega}^J_m(S^p) \cong \Omega^J_{m-p}$.

(v) Let $X = Y_1 \cup Y_2$, $Z = Y_1 \cap Y_2$, and suppose inclusion induces isomorphisms $\Omega^J_*(Y_i, Z) \cong \Omega^J_*(X, Y_{1-i})$ (by (i), this holds if the pairs (Y_i, Z) have the HEP). Then we have the exact sequences

$$\cdots \to \Omega^J_m(Z) \to \Omega^J_m(Y_1) \oplus \Omega^J_m(Y_2) \to \Omega^J_m(X) \to \Omega^J_{m-1}(Z) \to \cdots$$
$$\cdots \to \Omega^J_m(Z) \to \Omega^J_m(X) \to \Omega^J_m(X, Y_1) \oplus \Omega^J_m(X, Y_2) \to \Omega^J_{m-1}(Z) \to \cdots$$

(vi) $\Omega^J_*(X \cup Y) \cong \Omega^J_*(X) \oplus \Omega^J_*(Y)$ for disjoint union; $\tilde{\Omega}^J_*(X \vee Y) \cong \tilde{\Omega}^J_*(X) \oplus \tilde{\Omega}^J_*(Y)$.

(vii) If (X, Y) is a CW pair, $\Omega^J_m(X^p \cup Y, X^{p-1} \cup Y) \cong C_p(X, Y; \Omega^J_{m-p})$.

Proof (i) The splitting follows as we have an additive functor; the isomorphism follows as we have the same split on both sides of the equation. The next assertion follows from the homotopy axiom, the final one from the exact sequence.

(ii) Under the hypothesis, X/Y has the homotopy type of X with a cone on Y attached; by excision, this modulo the cone has the same groups as X modulo Y.

(iii) This is a standard exercise in diagram chasing.

(iv) Follows by induction from (ii) and (iii).

(v) These follow by another standard argument (the same for both).

(vi) Here $X \vee Y$ denotes the union of spaces X and Y with a single common point. Since $Z = \varnothing$, we can apply (v).

(vii) By (i), $\Omega^J_m(X^p \cup Y, X^{p-1} \cup Y) \cong \Omega^J_m(X^p/(X^{p-1} \cup (X^p \cap Y)))$. But $X^p/(X^{p-1} \cup (X^p \cap Y))$ is a wedge of p-spheres. Now apply (iv) and (vi). □

These results all illustrate how we can begin to calculate the groups $\Omega^J_m(X, Y)$ in terms of the Ω^J_m. We can formalise this process as a spectral sequence.

Theorem 8.4.6 *Let* (X, Y) *be a CW pair. Then there is a first quadrant* Ω^J_**-module spectral sequence, converging strongly to* $\Omega^J_*(X, Y)$*, which starts with* $E^2_{pq} = H_p(X, Y; \Omega^J_q)$.

Proof By Proposition 8.4.5 (iii), the triple $(X^p \cup Y, X^q \cup Y, X^r \cup Y)$ $(r < q < p)$ has an exact bordism sequence. All the maps are induced by inclusions and boundary homomorphisms, so all expected diagrams commute. Such a collection of exact sequences defines a spectral sequence. We write $X^\infty = X$, $X^{-\infty} = \varnothing$: then the limit term is $\Omega^J_*(X, Y)$. The module structure is induced by natural products $\Omega^J_m \times \Omega^J_n(X^p \cup Y, X^q \cup Y) \to \Omega^J_{m+n}(X^p \cup Y, X^q \cup Y)$: if M^m is a closed manifold, and $f : (N, \partial N) \to (X^p \cup Y, X^q \cup Y)$, then we use the manifold $M \times N$ (with induced J-structure) and the map induced by first projecting on N.

By Proposition 8.4.5 (vii), the E^1 term is

$$E^1_{pq} = \Omega^J_{p+q}(X^p \cup Y, X^{p-1} \cup Y) \cong C_p(X, Y; \Omega^J_q).$$

The boundary d^1 is induced by taking the boundary of a manifold: it is easy to verify that this coincides with the usual boundary in the chain complex of (X, Y). It follows that $E^2_{pq} = H_p(X, Y; \Omega^J_q)$ and hence (since $\Omega^J_q = 0$ for $q < 0$) we have a first quadrant spectral sequence.

As to convergence, we note that

$$\Omega^J_n(X^{-\infty} \cup Y) = \Omega^J_n(X^p \cup Y) \quad \text{for all } p < 0$$
$$\Omega^J_n(X^p \cup Y) = \Omega^J_n(X^\infty \cup Y) \quad \text{for all } p > n,$$

the first since $X^{-1} = \varnothing = X^{-\infty}$ and the second since (by the cellular approximation theorem) any map of an n-manifold into X is homotopic to a map into X^n. These two isomorphisms imply strong convergence of the sequence. □

We have now discussed the homology theory associated with the spectrum $\mathbb{TJ}$ and so with the stable group $\mathbf{J}$. There is also an associated cohomology theory, defined by

$$\Omega^n_J(X) := H^n(X; \mathbb{TJ}) := \lim_{N\to\infty}[S^N X : T(J_{N+n})].$$

The geometric content of this definition arises again by Theorem 8.4.1. If M is a closed smooth manifold, the suspension $S^N M$ is obtained from $\mathbb{R}^N \times M$ by adding a single point ∞. It follows from the Theorem that $[S^N M : T(J_{N+n})]$ corresponds bijectively to cobordism classes of submanifolds of $\mathbb{R}^N \times M$ whose normal bundles have group reduced to J_{N+n}.

Theorem 8.4.7 *Let* $\mathbf{J}$ *satisfy (S). Let* M^m *have a weak* $\mathbf{J}$*-structure. Then* $\Omega^n_J(M) \cong \Omega^J_{m-n}(M, \partial M)$.

Proof In this case, $\mathbb{R}^N \times M^m$ also has a weak $\mathbf{J}$-structure. By Corollary 8.2.8, a J_{N+n}-structure on the normal bundle of V^{m-n} in $\mathbb{R}^N \times M^m$ induces a weak $\mathbf{J}$-structure on the tangent bundle of V, and conversely if N is large enough. We thus have a bijective correspondence between $\Omega^n_J(M)$ and cobordism classes of manifolds V^{m-n} with weak $\mathbf{J}$-structure and an imbedding in $\mathbb{R}^N \times M^m$, for large enough N. But if N is large, any map to $\mathbb{R}^N \times M^m$ is homotopic to an embedding, homotopic embeddings are diffeotopic, and a diffeotopy gives a cobordism. Hence specifying an imbedding in $\mathbb{R}^N \times M^m$ up to diffeotopy is equivalent to specifying a map to $\mathbb{R}^N \times M^m$, or indeed to M^m, up to homotopy. It remains only to note that if M has boundary, ∂V is imbedded in $\mathbb{R}^N \times \partial M$, so we must insist that it be mapped to ∂M. □

This result shows that a manifold with weak $\mathbf{J}$-structure is orientable for the homology theory Ω^J_*. We also have a form of the Gysin isomorphism theorem.

Theorem 8.4.8 *Let* $\mathbf{J}$ *be a stable group satisfying (S),* (**M**)*;* X *a topological space,* ξ *a* J_k*-bundle over* X*. Then* $\Omega^J_n(X) \cong \tilde{\Omega}^J_{n+k}(T(\xi))$.

Proof Let $f : X \to B(J_k)$ classify ξ, and f_N denote the composite

$$B(J_N) \times X \xrightarrow{1\times f} B(J_N) \times B(J_k) \xrightarrow{B\psi_{N,k}} B(J_{N+k}).$$

Write F_N for the map $B(J_N) \times X \to B(J_{N+k}) \times X$ whose components are f_N and projection on the second factor. F_N is covered by a bundle map of $\gamma^N \oplus \xi$ to γ^{N+k}. Also, $B(J_N)$ is mapped by the natural injection i to $B(J_{N+k})$, and we have a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & \pi_r(B(J_N)) & \to & \pi_r(B(J_N)\times X) & \to & \pi_r(X) & \to & 0 \\ & & \downarrow i_* & & \downarrow F_{N*} & & \| & & \\ 0 & \to & \pi_r(B(J_{N+k})) & \to & \pi_r(B(J_{N+k})\times X) & \to & \pi_r(X) & \to & 0. \end{array}$$

Thus F_{N*} is an isomorphism in the limit as $N \to \infty$. We have an induced map of Thom spaces

$$T(J_N) \wedge T(\xi) \to T(J_{N+k}) \wedge X^+,$$

which then also in the limit gives homotopy isomorphisms. Thus

$$\begin{aligned} \Omega_n^J(X) &\cong \lim_{N\to\infty} \pi_{N+n+k}(T(J_{N+k}) \wedge X^+) \\ &\cong \lim_{N\to\infty} \pi_{N+n+k}(T(J_N) \wedge T(\xi)) \\ &\cong \tilde{\Omega}_{n+k}^J(T(\xi)). \end{aligned}$$ □

The calculation in Lemma 8.3.5 of cobordism of pairs involves the groups $\Omega_*^{J\times H_q}$, which admit a natural Ω_*^J-module structure. We can now 'compute' them directly using bordism groups.

Lemma 8.4.9 *We have* $\Omega_n^{J\times H_q} \cong \Omega_{n+q}^J(T(H_q))$*, and more generally*

$$\Omega_n^{J\times H_q}(X) \cong \Omega_{n+q}^J(T(H_q) \wedge X^+).$$

Proof By Theorem 8.4.1, we have

$$\begin{aligned} \Omega_n^{J\times H_q}(X) &= \lim_{N\to\infty} \pi_{n+N}(T(J\times H_q)_N \wedge X^+) \\ &= \lim_{N\to\infty} \pi_{n+N}(T(J_{N-q}\times H_q) \wedge X^+) \\ &= \lim_{N\to\infty} \pi_{n+N}(T(J_{N-q}) \wedge T(H_q) \wedge X^+) \\ &= \Omega_{n+q}^J(T(H_q) \wedge X^+). \end{aligned}$$ □

As with any homology theory, we can define bordism theory with coefficients. If $n > 1$ and $r > 1$ are natural numbers, write e_n^r for a space obtained from S^n by attaching an $(n+1)$-cell by a map $S^n \to S^n$ of degree r; thus $\tilde{H}_N(e_n^r;\mathbb{Z})$ is isomorphic to $\mathbb{Z}_r$ if $N = n$ and is zero otherwise.

We can now define $\Omega_N^J(X;\mathbb{Z}_r) := \tilde{\Omega}_{N+n}^J(X \wedge e_n^r)$. Elementary properties of this definition are easily deduced from homotopy properties of the spaces e_n^r. We will not go into further details.

The following construction is also sometimes useful. Let $\mathbf{J}$ be a stable group and H be any topological group. Then we can define a stable group $\mathbf{J} \rhd H$ by setting $(\mathbf{J} \rhd H)_n := J_n \times H$, operating on $\mathbb{R}^n$ via its projection on J_n. (This is *not* the same as the $\mathbf{J} \times H$ defined above.)

We have $B(J_n \rhd H) = B(J_n) \times B(H)$ and $T(J_n \rhd H) = T(J_n) \wedge B(H)$. In particular, if X is any CW complex, the loop space ΩX is equivalent to a topological group, and we have

$$\begin{aligned}\Omega_n^{J \rhd \Omega X} &= \lim_{N\to\infty} \pi_{n+N}(T(J_{n+N} \times \Omega X)) \\ &= \lim_{N\to\infty} \pi_{n+N}(T(J_{n+N} \wedge X)) = \tilde{\Omega}_n^J(X).\end{aligned}$$

Thus the groups $\tilde{\Omega}_*^J(X)$ may be considered as the coefficient groups of the homology theory $\Omega_*^{J \rhd \Omega X}$.

8.5 Equivariant cobordism

The object of this section is to give a method for reducing the calculation of equivariant cobordism groups to that of the bordism groups of certain classifying spaces.

We begin by formulating the definitions of equivariant cobordism groups. First define $I_m^O(G)$ to be the cobordism group of manifolds with a smooth action of the compact Lie group G. Next, let A be a closed collection of orbit types (in the sense of §3.5), and write $I_m^O(G; A)$ for the cobordism group of those actions such that all orbit types belong to A. Here we identify a type defined by a pair (H, E) with the type defined by $(H, E \oplus \mathbb{R})$, where H acts trivially on $\mathbb{R}$, to be able to use the same list of types for manifolds and for cobordisms.

We also wish to incorporate a structure group. Let J be a stable group satisfying (**M**), (**A**) and (**S**), and M have a J-structure (on its stable tangent bundle). We say that a smooth action of G on M *respects the J-structure* if the following condition is satisfied. For some n, we are given an action of G on a principal J_n-bundle P which defines the J-structure, lifting the given action of G on M. This defines actions of G on the associated bundles; in particular, on the principal J_{n+1}-bundle, so the condition is independent of n. To avoid technicalities we restrict to three cases: J may be O, SO, or U: in the second case all the bundles are orientable; in the third all are unitary, in particular E is acted on by a unitary group which we denote $U(E)$.

Write $I_m^J(G; A)$ for the group of cobordism classes of manifolds M^m with J-structure and a smooth G-action which respects it, and such that each orbit type belongs to A.

First consider the case of free actions: we have the single orbit type when H is trivial; denote it by 'free'. In this case, the projection $M \to G\backslash M$ is a fibre bundle with group G. Such bundles over X are classified by (homotopy classes of) maps $X \to B(G)$, where $B(G)$ is the classifying space of G.

Lemma 8.5.1 *The cobordism group $I^J_m(G; free)$ is isomorphic to the bordism group $\Omega^J_{m-g}(B(G))$, where $g = \dim G$.*

Proof We have just observed that a free action on M leads to a map $G\backslash M \to B(G)$; the converse is also immediate. The same remarks apply to manifolds with boundary, so the correspondence goes over to cobordism. □

Now let A be a closed set of orbit types, $\alpha \in A$ a maximal element, and write $A' := A \setminus \{\alpha\}$: since α is maximal, this too is closed. There is a natural map $I^J_m(G; A') \to I^J_m(G; A)$.

Lemma 8.5.2 *There is a natural exact sequence*

$$\ldots I^J_m(G; A') \to I^J_m(G; A) \to I^J_m(G; (A, A')) \to I^J_{m-1}(G; A') \to I^J_{m-1}(G; A) \ldots$$

Proof This is a direct application of the general principle of Lemma 8.3.1. Here the third term is defined as the group of cobordism classes of cobordisms W^m with J-structure and a smooth G-action which respects it, and such that each orbit type belongs to A and those on ∂W to A'. □

This formal result is only of value once we have a way to compute the third term.

We recall that an orbit type α is associated to a subgroup H^α of G and a representation of H^α on a Euclidean space E^α (both defined up to conjugacy). Since α is maximal in A, W^α is a closed submanifold of W, and by hypothesis is disjoint from ∂W. By Theorem 3.5.8, a neighbourhood of W^α in M is equivariantly diffeomorphic to a bundle over $X^\alpha = W^\alpha/G$ with fibre $G \times_{H^\alpha} E^\alpha$. Note that $\dim X^\alpha = \dim W^\alpha - \dim G + \dim H^\alpha$, and that $\dim M - \dim W^\alpha = \dim E^\alpha$.

According to Lemma 8.3.3, the third term $I^J_m(G; (A, A'))$ is isomorphic to the bordism group of G-manifolds W^α together with a G-bundle $\pi : N^\alpha \to W^\alpha$ with fibre E^α on which H^α acts as indicated. To proceed, we let P be the principal $J(E^\alpha)$-bundle associated to π: P is the set of isometries of E^α on fibres of π. On P we have the natural (right) action of $J(E^\alpha)$, also an induced (left) action of G which commutes with it, hence an action of $G \times J(E^\alpha)$: this action has only a single orbit type. The isotropy group is the set of elements

$$H^* = \{(h^{-1}, \rho(h)) \mid h \in H^\alpha\} \subseteq G \times J(E^\alpha).$$

Recalling the discussion in §3.5 of the structure of a G-manifold with just one orbit type, we now consider the submanifold P^{H^*} and the induced action on it of

$$N(H^*) = \{(g, r) \in G \times J(E^\alpha) : \rho(g^{-1}hg) = r^{-1}\rho(h)r \text{ for all } h \in H^\alpha\}.$$

Denote by L^α the quotient group $N(H^*)/H^*$. Then P^{H^*} is a principal L^α-bundle over X. The mechanism of classifying spaces tells us that such bundles correspond to homotopy classes of maps $X \to BL^\alpha$.

Theorem 8.5.3 *We have*

$$I^J_m(G; (A, A') \cong \Omega^J_{m-c}(BL^\alpha),$$

where $c = \dim G - \dim H^\alpha + \dim E^\alpha$.

Proof As in the proof of Lemma 8.5.1, the homotopy class of $X \to BL^\alpha$ determines the isomorphism class of X and hence W with all its structure. Since the same applies to bounded manifolds, we can pass to cobordism classes. □

This argument extends trivially to the case when A has two maximal elements α, β such that neither $\alpha \prec \beta$ nor $\beta \prec \alpha$ (for example, the subgroups H^α and H^β are conjugate), then if $A'' := A \setminus \{\alpha, \beta\}$ then

$$I^J_m(G; (A, A'')) \cong I^J_m(G; (A'' \cup \{\alpha\}, A'')) \oplus I^J_m(G; (A'' \cup \{\beta\}, A'')),$$

for the orbit types M^α and M^β have disjoint closures. Similarly we can deal with further such summands.

The simplest example is the group $G = \mathbb{Z}_2$ with $J = O$. Any action is semi-free; the possible non-trivial orbit types have $H = \mathbb{Z}_2$ and $E = \mathbb{R}^k$ for some k, with the antipodal action. In this case, H^* has order 2 and is central in $G \times O_k$, so its normaliser is the whole group and $L = N(H^*)/H^*$ is isomorphic to O_k. Denote by A the set of all orbit types. It follows from Theorem 8.5.3, together with the remark following it, that we have

$$I^O_m(\mathbb{Z}_2; (A, free) \cong \bigoplus_k \Omega^O_{m-k}(B(O_k)).$$

Theorem 8.5.4 *There is a split short exact sequence*

$$0 \to I^O_m(\mathbb{Z}_2; A) \to \bigoplus\nolimits_k \Omega^O_{m-k}(B(O_k)) \to \Omega_{m-1}(B\mathbb{Z}_2) \to 0.$$

Proof By Lemma 8.5.2 we have an exact sequence

$$I^O_m(\mathbb{Z}_2; free) \to I^O_m(\mathbb{Z}_2; A) \to I^O_m(\mathbb{Z}_2; (A, free)) \to I^O_{m-1}(\mathbb{Z}_2; free) \to I^O_{m-1}(\mathbb{Z}_2; A).$$

We claim that the map $I^O_m(\mathbb{Z}_2; free) \to I^O_m(\mathbb{Z}_2; A)$ is trivial. Indeed, given a free action of $\mathbb{Z}_2$ on M, the mapping cylinder of the projection $M \to \mathbb{Z}_2 \backslash M$ can be identified with a bundle over $\mathbb{Z}_2 \backslash M$ with fibre the interval $[-1, 1]$. This is a smooth manifold, with a $\mathbb{Z}_2$-action given by -1 in each fibre, and has boundary M.

The long sequence thus breaks into short exact sequences, and we can substitute $I^O_{m-1}(BZ_2; free) \cong \Omega_{m-1}(B\mathbb{Z}_2)$ by Lemma 8.5.1 and the value of

$I_m^O(\mathbb{Z}_2; (A, free))$ from the calculation preceding the theorem. The fact that the sequence splits follows since the middle group has exponent 2. □

The same approach may be applied to the case $G = \mathbb{Z}_p$, with p an odd prime, but here the details do not simplify. Here it is more natural to take the structure group J as SO or U. It is still true that any action is semifree, and we have a calculation of the bordism group of free actions; although the calculation of $\Omega^{SO}(B\mathbb{Z}_p)$ and $\Omega^U(B\mathbb{Z}_p)$ is less easy than for $\Omega^O(B\mathbb{Z}_2)$, this may be explicitly done, using the results quoted in the following section. A non-free orbit type has $H = \mathbb{Z}_p$, but to describe the isotropy action we must specify for each $r = 1, 2, \ldots, p-1$ a multiplicity $a_r \geq 0$ and then have an action on $\mathbb{C}^{n_\alpha}$ with $n_\alpha = \sum_r a_r$ where the generator t of G is represented by the diagonal action with the eigenvalue ζ_p^r repeated a_r times. In each case, H^* is isomorphic to $\mathbb{Z}_p$, but the calculation of its normaliser $N(H^*)$ depends very much on α; only in the case when all a_r except one vanish is the corresponding group H^* central. Nor is there any reason for the map $I_m^{SO}(\mathbb{Z}_p; free) \to I_m^{SO}(\mathbb{Z}_p; A)$ to be trivial.

8.6 Classifying spaces, Ω_*^O, Ω_*^U

We first describe the cohomology of the classifying spaces, and begin with the unitary group, where the structure is simplest.

The group U_n has a subgroup consisting of diagonal matrices; this is a torus T^n, a product of n copies of the circle group $U_1 = S^1$. The classifying space $B(S^1)$ can be taken to be infinite complex projective space $P^\infty(\mathbb{C})$ and $H^*(B(S^1); \mathbb{Z})$ is the polynomial ring $\mathbb{Z}[t]$ on a single generator $t \in H^2(B(S^1); \mathbb{Z})$. Thus $H^*(B(T^n); \mathbb{Z})$ is the polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$.

The inclusion induces maps $B(T^n) \to B(U_n)$ and $H^*(B(U_n); \mathbb{Z}) \to H^*(B(T^n); \mathbb{Z})$. It is well known that the map $H^*(B(U_n); \mathbb{Z}) \to H^*(B(T^n); \mathbb{Z})$ is injective, and that its image is the subring of polynomials invariant under the action of the Weyl group W. The Weyl group W of U_n is the symmetric group, and acts by permutations: the invariants form the ring of symmetric functions in the t_i. We can identify this with the polynomial ring generated by the elementary symmetric functions c_i $(1 \leq i \leq n)$, which can be defined by the formal identity $\prod_1^n (x - t_i) = x^n + \sum_1^n (-1)^r x^{n-r} c_r$. The class c_i is known as the *Chern class*. An additive basis of $H^*(B(U_n); \mathbb{Z})$ is given by the elements $s_{i_1, i_2, \ldots}$ $(i_1 \geq i_2 \ldots)$, defined as the sum of all the *distinct* monomials formed from $t_1^{i_1} t_2^{i_2} \cdots$ by permuting the variables. To distinguish these from similar calculations below, we sometimes write $s_I(c)$ for emphasis.

Taking the limit as $n \to \infty$ gives $H^*(B(U); \mathbb{Z})$ as the polynomial ring in an infinite sequence of variables $c_1, c_2, \ldots$. There is an additive isomorphism of $H^*(B(U); \mathbb{Z})$ to the cohomology of the Thom spectrum $\mathbb{TU}$.

The multiplicative structure for $\mathbb{TU}$ appears in homology, and is induced by direct sum, which gives maps $B(U_m) \times B(U_n) \to B(U_{m+n})$ and $TU_m \wedge TU_n \to TU_{m+n}$. To evaluate this in cohomology, observe that it comes from the identity $T^m \times T^n = T^{m+n}$ and hence $H^*(T^m; \mathbb{Z}) \otimes H^*(T^n; \mathbb{Z}) \cong H^*(T^{m+n}; \mathbb{Z})$; we can identify these as polynomial rings with generators, say, $t_1, \ldots, t_m, u_1, \ldots, u_n$. The induced map $\nabla : H^*(B(U_{m+n}); \mathbb{Z}) \to H^*(B(U_m); \mathbb{Z}) \otimes H^*(B(U_n); \mathbb{Z})$ is given by $\nabla(s_I) = \sum s_{I_1} \otimes s_{I_2}$, where the sum is extended over all partitions of the set $I = \{i_1, i_2, \ldots\}$ as a *disjoint* union $I = I_1 \cup I_2$. This is compatible with the inclusion maps which increase m and n, so we can pass to the limit, giving a diagonal map $\nabla : H^*(B(U); \mathbb{Z}) \to H^*(B(U); \mathbb{Z}) \otimes H^*(B(U); \mathbb{Z})$.

Dualising gives an algebra structure on $H_*(B(U); \mathbb{Z})$. If we define $\{\tau_I\}$ to be the dual basis to $\{s_I\}$ it follows that if I_1 and I_2 are disjoint we have $\tau_{I_1}\tau_{I_2} = \tau_{I_1, I_2}$ and hence $\tau_I = \tau_1^{i_1}\tau_2^{i_2}\cdots$. Thus $H_*(B(U); \mathbb{Z})$ is a polynomial ring with the τ_r as generators.

It follows from the Gysin isomorphism theorem that we have an isomorphism $H_*(B(U_m); \mathbb{Z}) \to \tilde{H}_*(TU_m : \mathbb{Z})$ of degree m. Since the diagram

$$\begin{array}{ccccc} H_*(B(U_m); \mathbb{Z}) \otimes H_*(B(U_n); \mathbb{Z}) & \cong & H_*(B(U_m) \times B(U_n); \mathbb{Z}) & \to & H_*(B(U_{m+n}); \mathbb{Z}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \tilde{H}_*(TU_m; \mathbb{Z}) \otimes \tilde{H}_*(TU_n; \mathbb{Z}) & \cong & \tilde{H}_*(TU_m \wedge TU_n; \mathbb{Z}) & \to & \tilde{H}_*(TU_{m+n}; \mathbb{Z}) \end{array}$$

is commutative, we have an induced isomorphism of $H_*(B(U); \mathbb{Z})$ on the stable homology ring $\tilde{H}_*(TU; \mathbb{Z})$.

The structure for the orthogonal group O_n is very similar. The subgroup X_n of diagonal matrices is a product of n copies of $O_1 \cong S^0$: it is a maximal elementary 2-subgroup. The classifying space $B(O_1)$ can be taken to be infinite real projective space $P^\infty(\mathbb{R})$ and $H^*(B(O_1); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[t]$ on a single generator $t \in H^1(B(O_1); \mathbb{Z}_2)$. Thus $H^*(B(X_n); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[t_1, \ldots, t_n]$. The inclusion induces maps $B(X_n) \to B(O_n)$ and $H^*(B(O_n); \mathbb{Z}_2) \to H^*(B(X_n); \mathbb{Z}_2)$; the image is the subring of polynomials invariant under the action of the group of permutations of the t_i, so is the ring of symmetric functions, and hence the polynomial ring generated by the elementary symmetric functions. In this case, the class defined by the ith elementary symmetric function is known as the *Stiefel–Whitney class*, and is denoted w_i, and we write $s_I(w)$ for the symmetric functions defined as above. We refer to [103] for a good general introduction to Stiefel–Whitney and other characteristic classes.

Thus $H^*(B(O); \mathbb{Z}_2)$ is the polynomial algebra in classes w_i. It is additively isomorphic to $\tilde{H}^*(\mathbb{TO}; \mathbb{Z}_2)$. Direct sum of vector bundles induces a diagonal map for these, hence a multiplication on $\tilde{H}_*(\mathbb{TO}; \mathbb{Z}_2)$, given by essentially the same formulae as in the unitary case.

Returning to the unitary case and applying Proposition B.4.1, we obtain our first calculation.

Proposition 8.6.1 *The ring $\Omega_*^U \otimes \mathbb{Q}$ is a polynomial ring with one generator in each even dimension, and each group Ω_n^U is finitely generated.*

Since we had dual bases above, s_n is orthogonal to all the τ_I such that I has more than one part, and hence to all decomposable classes in $H_*(B(U); \mathbb{Z})$ (i.e. classes which can be expressed as sums of products of classes of lower degree). Thus $z \in H_n(B(U); \mathbb{Z})$ is decomposable if and only if $\langle z, s_n(c)\rangle = 0$.

Since $H_*(B(U); \mathbb{Z})$ and $\tilde{H}_*(TU; \mathbb{Z})$ are polynomial rings, any ring homomorphism $\tilde{H}_*(TU; \mathbb{Z}) \to \mathbb{Z}$ is determined by its values on the generators τ_r, and these values may be chosen arbitrarily. A corresponding statement holds with coefficients $\mathbb{Q}$ in place of $\mathbb{Z}$. We seek a formula to express this.

Any additive homomorphism $\phi : H_*(TU; \mathbb{Q}) \to \mathbb{Q}$ is given by taking inner product with an element Φ of the direct product $H^{**}(TU; \mathbb{Q}) \cong H^{**}(B(U); \mathbb{Q})$ of the groups $H^n(B(U); \mathbb{Q})$. Dualising, it follows that ϕ is a ring homomorphism if and only if $\nabla(\Phi) = \Phi \otimes \Phi$. As above, it is convenient to consider Φ as a symmetric element of the power series ring in infinitely many variables t_i. Since $\nabla(t_i^r) = t_i^r \otimes 1 + 1 \otimes t_i^r$, we see that for any coefficients a_r, the infinite product

$$\Phi := \prod_i \left(1 + \sum_{r=1}^{\infty} a_r t_i^r\right)$$

has the desired property. Since this formula allows one arbitrary coefficient at each stage, it allows independent choices for the $\phi(\tau_r)$, so Φ is necessarily of this form.

We have seen that if M^{2n} has a weak **U**-structure given by a lift $f : M \to B(U)$ of the map inducing τ_M, the class of M in $\Omega_*^U \otimes \mathbb{Q}$ is decomposable if and only if $s_n(M) := \langle M, f^* s_n(c)\rangle = 0$. It will be useful to have some calculations of these numbers. Denote by $Y_{m,n}^{\mathbb{C}}$ a nonsingular hypersurface of degree $(1, 1)$ in $P^m(\mathbb{C}) \times P^n(\mathbb{C})$, for example, that given by $\sum_{i=0}^{min(m,n)} x_i y_i = 0$. These examples were introduced by Milnor [96]. Write also $Y_{m,n}^{\mathbb{R}}$ for a nonsingular hypersurface of degree $(1, 1)$ in $P^m(\mathbb{R}) \times P^n(\mathbb{R})$.

Proposition 8.6.2 *We have $s_n(P^n(\mathbb{C})) = n + 1$ and $s_{m+n-1}(Y_{m,n}^{\mathbb{C}}) = -\binom{m+n}{m}$.*

Proof Write λ for the canonical line bundle over $P^n(\mathbb{C})$. Then $\tau_{P^n(\mathbb{C})} \oplus \varepsilon \equiv (n+1)\lambda$, so the characteristic classes of $\tau_{P^n(\mathbb{C})}$ agree with those of $(n+1)\lambda$, so are induced by the map $P^n(\mathbb{C}) \to B(T^{n+1}) \to B(U_{n+1})$.

Since $c_1(\lambda)$ is the generator x of $H^2(P^n(\mathbb{C}); \mathbb{Z})$, the generators t_i of the cohomology groups $H^2(;\mathbb{Z})$ of the factors BT of BT^{n+1} all map to x. Thus s_n, which is the image of $\sum t_i^n$, maps to $(n+1)x^n$, and evaluating this on $[P^n(\mathbb{C})]$ gives $(n+1)$.

Set $M := P^m(\mathbb{C}) \times P^n(\mathbb{C})$ and write $\lambda_1,\ \lambda_2$ for the line bundles induced from the two factors. We have $H^2(M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$: write $x_1,\ x_2$ for the generators coming from the two factors. Thus $\tau_M \oplus 2\varepsilon \equiv (m+1)\lambda_1 \oplus (n+1)\lambda_2$ and in calculating characteristic classes we may take $(m+1)$ of the t_i equal to x_1 and $(n+1)$ equal to x_2. These all pull back by the inclusion $i : Y^{\mathbb{C}}_{m,n} \subset M$, and the normal bundle ν_Y of Y in M is the pullback of $\lambda_1 \otimes \lambda_2$, with first Chern class $i^*(x_1 + x_2)$.

Since s_n is defined as a sum of contributions coming from summands, we have in general $s_n(\xi \oplus \eta) = s_n(\xi) + s_n(\eta)$. Now as

$$\tau_Y \oplus \nu_Y \oplus 2\varepsilon \equiv (m+1)i^*\lambda_1 \oplus (n+1)i^*\lambda_2,$$

and as $i^*x_1^{m+n-1}[Y] = 0$, it follows that

$$s_n(Y) = -s_n(\nu_Y)[Y] = -i^*(x_1+x_2)^{m+n-1}[Y] = -(x_1+x_2)^{m+n-1}[i_*Y]$$
$$= -(x_1+x_2)^{m+n}[M] = -\binom{m+n}{m}. \qquad \square$$

The same calculations yield

Corollary 8.6.3 *We have* $s_n(w)(P^n(\mathbb{R})) \equiv n+1$ *(mod 2) and* $s_{m+n-1}(w)(Y^{\mathbb{R}}_{m,n}) \equiv \binom{m+n}{m}$ *(mod 2).*

The calculations for (special) orthogonal and symplectic groups are similar to the unitary case, provided for the orthogonal group we localise away from the prime 2. The groups SO_{2n+1} and Sp_{2n} each contain a maximal torus T^n, but in these cases the action of the Weyl group includes, as well as permutations, the inversions in each factor. (For SO_{2n} we only allow an even number of inversions.) The ring of invariants thus consists of the symmetric functions in the variables t_i^2 (also for SO_{2n} the product $\prod t_i$). The class defined by the ith elementary symmetric function is known as the *Pontrjagin class*, and is denoted p_{4i}. The same arguments apply here to calculate the dual. It now follows as before from Proposition B.4.1 that

Proposition 8.6.4 *The rings* $\Omega_*^{SO} \otimes \mathbb{Q}$ *and* $\Omega_*^{Sp} \otimes \mathbb{Q}$ *are polynomial rings with one generator in each dimension divisible by 4.*

The unitary structure on $P^n(\mathbb{C})$ induces an SO-structure, and the dummy variables t_i play the same role as before. As s_n is given by $\sum t_i^n$, we see that if n is even, the formula for s_n in terms of Chern classes is the same as the formula in terms of Pontrjagin classes, thus it follows from Proposition 8.6.2 that

Lemma 8.6.5 *We have* $s_{2n}(p)(P^{2n}(\mathbb{C})) = 2n + 1$.

Thus we can take the manifolds $P^{2k}(\mathbb{C})$ as generators of $\Omega_*^{SO} \otimes \mathbb{Q}$.

It follows from Proposition 8.6.4 that any ring homomorphism $\Omega_*^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$ is determined by its values on a list of generators. The most celebrated example of this is the signature. We defined the signature $\sigma(M)$ of an oriented manifold M of dimension $4k$ in §7.5 as the signature of the quadratic form given by intersection numbers on $H_{2k}(M; \mathbb{R})$. We saw in that section that $\sigma(M)$ vanishes if M is an oriented boundary, and σ is clearly additive on disjoint unions, hence defines an additive homomorphism $\sigma : \Omega_*^{SO} \to \mathbb{Z}$.

Lemma 8.6.6 *The signature* σ *is multiplicative for products, hence defines a ring homomorphism* $\sigma : \Omega_*^{SO} \to \mathbb{Z}$.

Proof Consider the product $M^m \times N^{4k-m}$ of two oriented manifolds. We have $H^{2k}(M \times N) = \oplus_i H^i(M) \otimes H^{2k-i}(N)$. Under cup product the term $H^i(M) \otimes H^{2k-i}(N)$ is dually paired with $H^{m-i}(M) \otimes H^{2k-m+i}(N)$, so only the term $m = 2i$ can contribute to the signature. The self-pairing of $H^i(M) \otimes H^{2k-i}(N)$ with itself to $H^m(M) \otimes H^{4k-m}(N) \cong \mathbb{R}$ is the tensor product of the self-pairings of $H^i(M)$ and $H^{2k-i}(N)$ to $\mathbb{R}$. If i is odd, there is a Lagrangian subspace K of $H^i(M)$, so $K \otimes H^{2k-i}(N)$ is a Lagrangian subspace of $H^i(M) \otimes H^{2k-i}(N)$ and this has signature zero. If i is even, we can diagonalise the quadratic forms on $H^i(M)$ and $H^{2k-i}(N)$, and the calculation is trivial. □

The value is given by Hirzebruch's signature theorem [74]. First recall the expansion

$$\frac{t}{tanh(t)} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k t^{2k},$$

which we may use to define the Bernoulli numbers B_k

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66},$$
$$B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \ldots$$

let us write this as $t/tanh(t) = 1 + \sum_{k=1}^{\infty} \beta_k t^{2k}$. Now define the class $L_* \in H^{**}(B(O); \mathbb{Q})$ by the formula $L_* := \prod_i (1 + \sum_{k=1}^{\infty} \beta_k t_i^{2k})$, where the t_i are the auxiliary variables introduced above.

Theorem 8.6.7 *For* M *oriented, the signature is given by* $\sigma(M) = f^* L_m[M]$, *where* $f : M \to B(SO)$ *induces* τ_M.

Proof We have seen that we can take the manifolds $P^{2k}(\mathbb{C})$ as generators of $\Omega_*^{SO} \otimes \mathbb{Q}$. Thus two ring homomorphisms agreeing on these coincide, so it suffices to verify the formula for $M = P^{2k}(\mathbb{C})$. We see at once that each manifold $P^{2k}(\mathbb{C})$ has signature 1.

Since $p(\tau_M) = (1+\alpha^2)^{2k+1}$, we have

$$\begin{aligned} L_*[P^{2k}(\mathbb{C})] &= \left(\frac{\alpha}{tanh\,\alpha}\right)^{2k+1} [P^{2k}(\mathbb{C})] \\ &= \textit{coefficient of } \alpha^{2k} \textit{ in } \left(\frac{\alpha}{tanh\,\alpha}\right)^{2k+1} \\ &= \frac{1}{2\pi i}\oint \frac{dz}{(tanh\,z)^{2k+1}} \\ &= \frac{1}{2\pi i}\oint \frac{du}{u^{2k+1}(1-u^2)} \quad (\textit{substituting } u = tanh(z)) \\ &= Res_0(1+u^2+u^4+\ldots)/u^{2k+1} \\ &= 1. \end{aligned}$$ □

Explicit formulae for the L classes may be calculated: for example,

$$L_1 = \frac{1}{3}p_1, \ \ L_2 = \frac{7p_2 - p_1^2}{45}, \ \ L_3 = \frac{62p_3 - 13p_1p_2 + 2p_1^3}{945}.$$

We will use the below formula for the leading coefficient (see, for example, [103]).

Lemma 8.6.8 *The coefficient of p_k in L_k is $2^{2k}(2^{2k-1}-1)B_k/(2k)!$.*

For A_* a graded vector space over a field F, we count the dimensions by the *Poincaré series*

$$P(A_*; F)(t) := \sum_0^\infty \dim_F(A_n)t^n.$$

Thus if A is a polynomial algebra with the degrees of generators in a set S, we have $P(A)(t) = \prod_{i\in S}(1-t^i)^{-1}$. In particular, by Proposition 8.6.1,

$$P(\Omega_*^U \otimes \mathbb{Q})(t) = P(H_*(U;\mathbb{Q})(t) = \prod_{i=1}^\infty (1-t^2)^{-1},$$

and by Proposition 8.6.4,

$$P(\Omega_*^{SO} \otimes \mathbb{Q})(t) = P(H_*(SO;\mathbb{Q})(t) = \prod_{i=1}^\infty (1-t^4)^{-1}.$$

Thom's great achievement [150] was the calculation

Theorem 8.6.9 *The ring Ω_*^O is a polynomial ring over $\mathbb{Z}_2$ with one generator in each dimension* not *of the form $2^k - 1$. The bordism class of a manifold M^m*

is determined by the Stiefel–Whitney numbers of M. *Moreover* M *qualifies as a generator if and only if* $s_m(w)[M] \neq 0$.

We outline the steps in the proof. Since $M \times I$ can be regarded as a cobordism of the union of two copies of M to the empty set, any element of Ω_*^O has order 2. It thus suffices to perform calculations in mod 2 cohomology.

The next step is to calculate the action of the Steenrod algebra $\mathcal{S}_2$ on $H^*(B(O);\mathbb{Z}_2)$ and hence that on $H^*(\mathbb{TO};\mathbb{Z}_2)$. This shows that the latter is a free $\mathcal{S}_2$-module, and hence that there is a map $\psi : \mathbb{TO} \to \prod \mathbb{K}(\mathbb{Z}_2, n)$ to a product of Eilenberg–MacLane spectra which induces a cohomology isomorphism, hence is a (stable) homotopy equivalence, so induces an isomorphism of Ω_*^O to a sum of copies of $\mathbb{Z}_2$. This can be formulated as follows. For any M^m and cohomology class $k \in H^m(B(O);\mathbb{Z}_2)$, the classifying map $\phi_M : M \to B(O)$ of τ_M induces $\phi_M^* k \in H^m(M;\mathbb{Z}_2)$ and hence a number $\phi_M^* k[M] \in \mathbb{Z}_2$, called a *Stiefel–Whitney number* of M. The result implies that these numbers determine the class of M in Ω_m^O.

More generally, ψ induces, for any X, an isomorphism $\Omega_*^O(X) \to \prod H_n(X;\mathbb{Z}_2)$: given a map $f : M \to X$ and a cohomology class $k \in H^n(B(O);\mathbb{Z}_2)$, the map ϕ_M induces $\phi_M^* k \in H^n(M;\mathbb{Z}_2)$, hence a dual homology class $[M] \cap \phi_M^* k \in H_{m-n}(M;\mathbb{Z}_2)$ and a class $f_*([M] \cap \phi_M^* k) \in H_{m-n}(X;\mathbb{Z}_2)$. Now the composed map $\Omega_m^O(X) \to \oplus_n H_{m-n}(X;\mathbb{Z}_2)$ is a natural isomorphism.

Further, $H_*(\mathbb{TO};\mathbb{Z}_2)$ is a free comodule over the dual $\mathcal{S}^2$ of $\mathcal{S}_2$, and this is a polynomial ring with one generator in each dimension of the form $2^k - 1$. Thus

$$P(\mathcal{S}_2)(t) = \prod_k (1 - t^{2^k - 1})^{-1}.$$

It follows that

$$P(\Omega_*^O;\mathbb{Z}_2)(t) = \prod_{i \text{ not of form } 2^n - 1} (1 - t^i)^{-1}.$$

For the multiplicative structure we can argue abstractly using the fact that $H_*(\mathbb{TO};\mathbb{Z}_2)$ is a polynomial ring, or we can argue as follows.

If M is such that $s_m(w)[M^m] \neq 0$, the class of M in $H_m(\mathbb{TO};\mathbb{Z}_2)$ is indecomposable, hence so is the class of M in Ω_m^O. If m is even, we can take M as $P^m(\mathbb{R})$; otherwise if $m+1$ is not a power of 2, write $m + 1 = 2^{r-1}(2s+1)$ with $s > 0$, then by Corollary 8.6.3 we can take $M = Y^{\mathbb{R}}_{2^{r-1}, 2^r s}$.

Since we have exhibited manifolds M^m with $s_m(w)[M] \neq 0$ for each m not of the form $2^k - 1$, these indecomposables generate a polynomial ring, and the above counting argument shows that this is the whole of Ω_*^O.

Alternative choices are as follows. Write $P(m, n)$ for the bundle over $P^m(\mathbb{R})$ with fibre $P^n(\mathbb{C})$ where the structure group $\mathbb{Z}_2$ acts by complex conjugation.

Lemma 8.6.10 *If $N = 2^{r-1}(2s+1)$ with $s > 0$, set $V^{2N-1} := P(2^r - 1, 2^r s)$. Then $s_{2N-1}(w)[V^{2N-1}] = 1$.*

We omit the proof (an elementary calculation) which, like the construction of the V^{2N-1}, is due to Dold [41].

A similar, but more elaborate argument gives the result in the unitary case, which is due to Milnor [96] and Novikov [113]. Introduce the notation r_n by

$$\begin{aligned} r_n :&= p \text{ if } n \text{ is a power of the prime } p, \\ &= 1 \text{ if } n \text{ is not a prime power.} \end{aligned}$$

Theorem 8.6.11 *The ring Ω_*^U is a polynomial ring with one generator in each even dimension. The bordism class of a manifold M^{2m} with weak* **U***-structure is determined by the Chern numbers of M. Moreover M qualifies as a generator if and only if $s_m(c)[M] = \pm r_{m+1}$.*

The argument includes the same steps, but encounters additional technical difficulties. One must analyse the Steenrod algebras $\mathcal{S}_p$ for each prime p and the corresponding actions on $H^*(B(U); \mathbb{Z}_p)$ and hence on $H^*(\mathbb{TU}; \mathbb{Z}_p)$. This time the modules are not free, since the Bockstein β_p acts trivially, but are free over the quotient $\overline{\mathcal{S}}_p$ of S_p by the ideal generated by β_p. It follows that, for each p, there is a map of $\mathbb{TU}$ to a product of Eilenberg–MacLane spectra $\mathbb{K}(\mathbb{Z}, n)$ which induces an isomorphism of (mod p) cohomology.

For the multiplicative structure we find that $H_*(\mathbb{TU}; \mathbb{Z}_p)$ is a polynomial ring, it is a free comodule over the dual $\overline{\mathcal{S}}^p$ of $\overline{\mathcal{S}}_p$, and that this is a polynomial ring with one generator in each dimension of the form $2(p^k - 1)$.

Additional calculations are needed first, to ensure that we can fit these together for all primes p to obtain a map which is a stable homotopy equivalence, and then to make an analysis of the multiplicative structure.

Again some of this can be bypassed using explicit constructions of manifolds. By Proposition 8.6.2 we have $s_{m+n-1}(Y_{m,n}^{\mathbb{C}}) = -\binom{m+n}{m}$. Thus for manifolds of dimension N we have values of $s_N[M]$ taking all values $-\binom{N+1}{m}$ with $1 \leq m \leq N$. The highest common factor of these is just r_{N+1}.

8.7 Calculation of Ω_*^{SO} and Ω_*^{SU}

We consider two cases:

$\mathbf{J} = \mathbf{O}$, $\mathbf{SJ} = \mathbf{SO}$, $\mathbf{J}/\mathbf{SJ} = \{\pm 1\}$, $d = 1$, $\mathbb{K} = \mathbb{R}$,

$\mathbf{J} = \mathbf{U}$, $\mathbf{SJ} = \mathbf{SU}$, $\mathbf{J}/\mathbf{SJ} = S^1$, $d = 2$, $\mathbb{K} = \mathbb{C}$;

we will present the two theories in parallel as far as possible. We will omit many details (the account of these results occupies the whole of the memoir

[39] and 140 pp. of Stong's book [147]) but aim to describe all the geometrical ideas involved.

We will focus on geometrical arguments, and begin with certain exact sequences. Some of the arguments will apply to other cases satisfying similar conditions: for example, taking (**J**, **SJ**) to be (**Pin**, **Spin**) or ($\mathbf{Spin^c}$, **Spin**) or ($\mathbf{U} \times H$, $\mathbf{SU} \times H$) with H compact.

In each case, we have $J_n/SJ_n \cong \mathbf{J}/\mathbf{SJ} \cong S^{d-1}$. We will write P for $P^\infty(\mathbb{K})$ and P^k for $P^k(\mathbb{K})$. We write γ^n for the standard vector bundle over $B(J_n)$ or $B(SJ_n)$ and η for the standard line bundle over P^k. We regard P as the classifying space for the group S^{d-1}, so the map $\mathbf{J} \to \mathbf{J}/\mathbf{SJ} \cong S^{d-1}$ induces $\pi : BJ \to P$.

Lemma 8.3.1 gives us an exact sequence in which the third term is the cobordism group $\Omega_m^{J,SJ}$ of bounded J-manifolds with a weak SJ-structure on the boundary. We first interpret this relative term using Lemma 8.3.3.

Theorem 8.7.1 *We have a natural isomorphism* $\Omega_m^{J,SJ} \cong \Omega_{m-d}^{SJ}(P)$.

Proof This is an instance of the general method of Lemma 8.3.3, but there are many details to clarify.

We will specify the J-structure of a manifold M by the classifying map of its stable normal bundle, $\nu_M : M \to BJ$. We have a fibration $B(SJ) \to B(J) \xrightarrow{\pi} P$, and an SJ-structure of M is determined by a nullhomotopy of $\pi \circ \nu_M$ which is thus covered by a homotopy of ν_M to a map into $B(SJ)$.

The standard line bundle over P has a J-structure, classified by $P \xrightarrow{\eta} BJ_d \xrightarrow{\iota} BJ$; we may assume that $\pi \circ \iota \circ \eta$ is the identity map 1_P of P. The section $\iota \circ \eta$, together with the group action, shows that the fibration $B(SJ) \to B(J) \to P$ is trivial.

Write $(-1)_P : P \to P$ for the negative of the identity: this is given in the real case by the identity, and in the complex case by complex conjugation. Moreover P is an H-space, and the diagram

$$\begin{array}{ccc} BJ \times BJ & \xrightarrow{B\psi} & BJ \\ \downarrow \pi \times \pi & & \downarrow \pi \\ P \times P & \to & P \end{array}$$

is homotopy commutative; we may choose our model of BJ to make it commutative.

Now suppose M^m a J-manifold such that ∂M is an SJ-manifold. Consider the map $\pi_M := \pi \circ \nu_M : M \to P$; up to homotopy, we may suppose that this maps M to a finite dimensional projective subspace P^k. We can make this map smooth and transverse to the submanifold P^{k-1}, whose preimage will then be a smooth submanifold V^{m-d} of M^m, with normal bundle induced from η. As ∂M

has an SJ-structure, $\pi \circ \nu_M$ is trivial on ∂M (which has trivial normal bundle in M), so may be assumed to avoid P^{k-1}. Thus V lies in the interior $\mathring{M}$ of M, and is closed.

The stable normal bundle ν_V of V is the sum of the bundles induced from ν_M and from η. We give the second summand *minus* the obvious structure. So the normal bundle ν_V is induced by

$$V \xrightarrow{\nu_M|V} BJ \xrightarrow{1\times\pi} BJ \times P \xrightarrow{1\times -1} BJ \times P \xrightarrow{1\times\eta} BJ \times BJ \xrightarrow{B\psi} BJ.$$

The composite $\pi \circ \nu_V$ is thus induced by

$$V \xrightarrow{\nu_M|V} BJ \xrightarrow{\pi} P \xrightarrow{(1,-1)} P \times P \to P.$$

Thus a null-homotopy of the composite map $P \to P$ defines one for $\pi \circ \nu_V$, and hence an SJ-structure for V.

Now choose a tubular neighbourhood W of V in M: this is a bundle over V, with fibre D^d, associated to $(\pi \circ \nu|V)^*\eta$. It follows as in Lemma 8.3.3 that $(M, \partial M)$ is (J, SJ)-cobordant to $(W, \partial W)$. We need to verify that the SJ-structure on ∂M extends to $(M \setminus \mathring{W})$: this follows since $\pi \circ \nu_M$ takes $(M \setminus \mathring{W})$ to the contractible set $(P^k \setminus P^{k-1})$.

Thus the (J, SJ)-cobordism class of $(M, \partial M)$ agrees with that of $(W, \partial W)$, hence is determined by the class of $(V, \pi \circ \nu|V)$ in $\Omega_{m-d}^{SJ}(P)$. The formula which determines it is as follows. Let η' be the bundle induced from η. Then $\nu_V = \nu_M + \bar{\eta}'$, where the bar recalls the sign change above. Thus $\nu_V + \eta' = \nu_M + \bar{\eta}' + \eta' = \nu_M + \varepsilon^2$.

Conversely, given any element of $\Omega_{m-d}^{SJ}(P)$, represented say by (V, f), we can take the bundle E with fibre D^d associated to $f^*\eta$ and give it a J-structure. The stable normal bundle $\nu_{\partial E}$ of the boundary ∂E is the restriction of ν_E. But $\pi \circ \nu_E$ is essentially f, by definition, and is covered by a bundle map over V of E to the disc bundle $D(\eta)$ associated to η, and hence of ∂E to the corresponding sphere bundle $S(\eta)$. But $S(\eta)$ is contractible, so we have a null-homotopy of $\partial E \to S(\eta) \to P$, and so an SJ-structure on ∂E. Since all our constructions carry over to cobordisms, we have indeed an isomorphism $\Omega_m^{J,SJ} \cong \Omega_{m-d}^{SJ}(P)$. □

We remarked above that for cobordism theory, the extra structure provided by a submanifold is equivalent to the extra structure provided by a map to its Thom space. Moreover P is homeomorphic to the Thom space of η. This leads to

Theorem 8.7.2 *We have a natural isomorphism* $\Omega_{m-d}^{SJ}(P) \cong \Omega_{m-d}^{SJ} \oplus \Omega_{m-2d}^{J}$.

Proof Given an SJ-manifold V^{m-d} and a map $\chi_V : V \to P$, we may suppose χ_V maps V into P^{k-1}. We make this transverse to P^{k-2}, and write $B = \chi_V^{-1}(P^{k-2})$.

Then $\nu_B = \nu_V|B + (\chi_V|B)^*\eta$, and we use this formula to give B a J-structure. Since the cobordism class of (V, f) determines the cobordism classes of V and B, we have a homomorphism $\Omega^{SJ}_{m-d}(P) \to \Omega^{SJ}_{m-d} \oplus \Omega^{J}_{m-2d}$.

Conversely, the class of (V, f) is determined by those of V, B, and the map $B \to P$ inducing the normal bundle of B in V. By Lemma 8.3.5 we can separate the contributions of B and V, provided the stable normal bundle of B is induced by $B \to B(SJ) \times P$. But since the fibration $B(SJ) \to B(J) \to P$ is trivial, $B(SJ) \times P$ is homotopy equivalent to $B(J)$. □

By Lemma 8.3.1 we have an exact sequence

$$\dots \Omega^{SJ}_m \to \Omega^{J}_m \to \Omega^{J,SJ}_m \to \Omega^{SJ}_{m-1} \cdots$$

and combining Theorem 8.7.1 and Theorem 8.7.2 gives a natural isomorphism $\Omega^{J,SJ}_m \cong \Omega^{SJ}_{m-d} \oplus \Omega^{J}_{m-2d}$. We now study the maps in the sequence obtained by making this substitution.

Theorem 8.7.3 *There is an exact sequence*

$$\Omega^{SJ}_n \xrightarrow{i_S} \Omega^{J}_n \xrightarrow{(d_1,d_2)} \Omega^{SJ}_{n-d} \oplus \Omega^{J}_{n-2d} \xrightarrow{\binom{\times\alpha}{0}} \Omega^{SJ}_{n-1} \to \cdots$$

where α is the class of S^{d-1} with a twisted framing. Also, there exists $s_2 : \Omega^{J}_{n-2d} \to \Omega^{J}_n$ with $(d_1, d_2) \circ s_2 = (0, 1)$.

Proof Write (d_1, d_2) for the components of the map $\Omega^{J}_m \to \Omega^{SJ}_{m-d} \oplus \Omega^{J}_{m-2d}$, so that the image of the class of M by d_1 (resp. d_2) is determined by V (resp. B) in the notation above; also write (q_1, q_2) for the components of the map $\Omega^{SJ}_{m-d} \oplus \Omega^{J}_{m-2d} \to \Omega^{SJ}_{m-1}$.

As to q_1, we can suppose B empty and χ_V trivial. Then the disc bundle defining the class in $\Omega^{J,SJ}_m$ is trivial, and has boundary $V \times S^{d-1}$. Since we have a product bundle, we obtain multiplication by the class, α say, of S^{d-1} with appropriate SJ-structure. To determine this, we can take V to be a point and M a disc D^d. Recall that V was constructed from M by making π_M transverse to P^{k-1}. Now π_M maps $\partial M = S^{d-1}$ to a point, so induces a map of $S^d = M/\partial M$ which meets P^{k-1} transversely in just one point. This coincides (up to homotopy) with the inclusion of a projective line P^1. So α is the class of S^{d-1}, with SJ-structure defined by a framing of the normal bundle, twisted in this way.

We now construct a map $s_2 : \Omega^{J}_{m-2d} \to \Omega^{J}_m$ and show that $d_1 \circ s_2 = 0$ and $d_2 \circ s_2 = id$. From this, and the exactness of the sequence it follows that $q_2 = 0$.

To define s_2, suppose that B^{m-2d} is a J-manifold, and form $(\pi \circ \nu_B)$, which we may suppose a map $B \to P^k$ for some k. Write Q_{k+2} for the projective bundle over P^k associated to $\eta \oplus \varepsilon^2$. Let M^m be the induced bundle over B, V^{m-d} the sub-bundle corresponding to $\eta \oplus \varepsilon^1$, and identify B itself with the sub-bundle of V corresponding to η.

Write f for the map $M \to B$ and g for the composite map $M \to B \to P^k$. Then there is a natural splitting $\tau_M \oplus g^*\eta \cong f^*\tau_B \oplus g^*(\eta \oplus \varepsilon^1) \oplus (-1 \circ g)^*\varepsilon^1$. We give all these bundles the induced J-structures. Since the construction passes to cobordisms, we have a well-defined map of cobordism classes with $s_2(B) := (M)$.

Consider the decomposition $\mathbb{K}^{k+3} = \mathbb{K}^{k+1} \oplus \mathbb{K}^2$. Then P^k is the projective space of $\mathbb{K}^{k+1}$, so a point $x \in P^k$ corresponds to a line $\ell_x \subset \mathbb{K}^{k+1}$ which we can identify with the fibre of η over x. We now identify the fibre of $\eta \oplus \varepsilon^2$ over x with $\ell_x \oplus \mathbb{K}^2$, and so Q_{k+2} with the subspace of $P^k \times P^{k+2}$ of pairs (x, y) with $\ell_y \subset \ell_x \oplus \mathbb{K}^2$.

Now define $\zeta : Q_{k+2} \to P^{k+2}$ to be the map induced by projection on P^{k+2}. Then $\zeta^{-1}(P^{k+1})$ is the set of pairs (x, y) with $\ell_y \subset \ell_x \oplus \mathbb{K} \oplus 0$ and $\zeta^{-1}(P^k)$ is the set of pairs (x, x) with $x \in P^k$. Thus $\zeta : Q_{k+2} \to P^{k+2}$ is transverse to P^{k+1} and P^k, and these have preimages the sub-bundles associated to $\eta \oplus \varepsilon^1$ and η.

We claim that $\zeta \circ \beta \simeq \pi \circ \nu_M$. Since the target of these maps is the Eilenberg–MacLane space P, this only needs checking on the level of the cohomology class. It follows that this map is transverse to P^{k+1} and P^k, and these have preimages V and B. Hence we have $d_2 \circ s_2 = id$.

To see $d_1 \circ s_2 = 0$, we must find an SJ-manifold with boundary V. But V is a $P^1(= S^d)$-bundle over B, with structure group $Z(= S^{d-1})$, so bounds the associated disc bundle, which is topologically the product by I of the mapping cylinder of the principal bundle. Since the principal bundle was obtained from $\pi \circ \nu_B$, this has an SJ-structure. □

When $d = 1$ we have $S^{d-1} = S^0$, but each point has the positive orientation: this twists the standard framing of ∂D^1 by changing a sign. In this case $J = O$ and the map $\Omega_{m-1}^{SO} \to \Omega_{m-1}^{SO}$ is multiplication by 2. If $d = 2$, we have $S^{d-1} = S^1$, and the twisted framing differs from the standard one. Here (see §B.3(x)) homotopy theory tells us that $\alpha \in \pi_1^S$ is the non-zero element η_2, and $2\eta_2 = 0$.

We now define **RJ** as the stable group given by the pullback diagram

$$\begin{array}{ccccc} B(SJ) & \to & B(RJ) & \to & B(J) \\ \downarrow & & \downarrow & & \downarrow \\ P^0 & \to & P^1 & \to & P \end{array} .$$

Proposition 8.7.4 *(i) There is a split short exact sequence*

$$0 \to \Omega_n^{RJ} \xrightarrow{i_R} \Omega_n^J \xrightarrow{d_2} \Omega_{n-2d}^J \to 0 \tag{8.7.5}$$

split by a map $s_0 : \Omega_n^J \to \Omega_n^{RJ}$ *with* $i_R \circ s_0 = 1$.

(ii) The following sequence also is exact:

$$\Omega_n^{SJ} \xrightarrow{i_{SR}} \Omega_n^{RJ} \xrightarrow{d_1} \Omega_{n-d}^{SJ} \xrightarrow{\times\alpha} \Omega_{n-1}^{SJ} \to \cdots \tag{8.7.6}$$

Here i_{SR}, i_R and $i_S = i_R \circ i_{SR}$ are the maps induced by the natural inclusions $SJ \subset RJ \subset J$.

Proof (i) It follows from the definition of RJ that $d_2 \circ i_R = 0$, and we have already proved d_2 surjective. For exactness at Ω_J^n, suppose M defines an element of Kerd_2. Thus we may take $\pi \circ \nu_M$ as a map to P^k, make it transverse to P^{k-2}, and write B for the preimage: then by hypothesis B is cobordant to the empty set. As in Lemma 8.3.5 we may extend this cobordism to one of $(M, \pi \circ \nu_M)$, and thus suppose $\pi \circ \nu_M$ a map to $P^k \setminus P^{k-2}$. But this is homotopic to a map into P^1. Thus M defines a class in Ω_n^{RJ}.

It remains to define s_0 and prove $i_R \circ s_0$ the identity. We begin as usual with $M \xrightarrow{\nu_M} BJ \xrightarrow{\pi} P$; again as usual we may replace the target by P^k. We thus have a map $M \times P^1 \to P^k \times P^1 \to P^{2k+1}$, where the final map is the Veronese embedding

$$((x_0, \dots, x_k), (y_0, y_1)) \to (x_0y_0, \dots, x_ky_0, x_0y_1, \dots, x_ky_1).$$

The composite is transverse to a generic linear subspace L, say given by $x_0y_1 = x_1y_0$, of P^{2k} (at a point where transversality failed we would have $y_0 = y_1 = 0$), and we define $s_0[M]$ to be the class of the preimage M' of L.

Adapting the above proof that $\pi \circ \nu_V$ is nullhomotopic shows in this case (where an extra factor P^1 appears) that $\pi \circ \nu_{M'}$ is homotopic to a map to P^1, so M' defines a class in Ω_n^{RJ}. Finally, if M itself defines such a class, we may take $k = 1$ above, so that the projection of M' on M is a diffeomorphism. □

Since $d_2 \circ s_2 = i$, $1 - s_2d_2$ retracts Ω_n^J on the kernel of d_2, which we can now identify with Ω_n^{RJ}. We denote this map by $\rho : \Omega_n^J \to \Omega_n^{RJ}$.

There are alternative presentations of the above material. One can *define* Ω_n^{RJ} as the kernel of d_2. One can also show (cf. Theorem 8.7.1) that there is a natural isomorphism $\Omega_m^{J,RJ} \cong \Omega_{m-2d}^J$. It can also be shown that $\Omega_n^{RJ} \cong \bar{\Omega}_{n+d}^{SJ}(P^2)$.

We observe that $\Omega_*^J(P)$ is a free Ω_*^J-module with base the classes x_j defined by the inclusions of P^j in P.

We define a module endomorphism Δ of $\Omega_*^J(P)$ as follows. Given a class represented by $f : M \to P^k \subset P$, we make f transverse to P^{k-1}, set $L := f^{-1}(P^{k-1})$ and define $\Delta(M, f) := (L, f|L)$. It follows that $\Delta(x_j) = x_{j-1}$.

Write $\varepsilon : \Omega_*^J(P) \to \Omega_*^J$ for the augmentation and $\mu : \Omega_*^J \to \Omega_*^J(P)$ for the map sending $[M]$ to the class of $(M, \pi \circ \nu_M)$. The map $P \times P \to P$ which classifies the tensor product of line bundles induces a multiplication in $\Omega_*^J(P)$ with respect to which μ is a ring homomorphism.

We observe that $\varepsilon \circ \Delta \circ \mu \circ i_R = i_S \circ d_1$ and that $\varepsilon \circ \Delta^2 \circ \mu = d_2$. The retraction s_0 is given in this notation by $s_0(z) = \varepsilon\Delta(\mu(z).x_1)$ for any $z \in \Omega_*^J$.

We find that Ω_*^{RJ} is a subring of Ω_*^J if $J = O$ but not if $J = U$, so we define a multiplication on Ω_*^{RJ} by

$$x * y := s_0(xy).$$

We also define

$$\partial := i_{SR} \circ d_1 : \Omega_*^{RJ} \to \Omega_*^{RJ}.$$

Since $d_1 \circ i_{SR} = 0$ in the exact sequence (8.7.6), we have $\partial^2 = 0$.

For any class $a \in \Omega_*^{RJ}$, $\mu(a)$ is in the image of $\Omega_*^J(P^1)$, so can be written $\mu(a) = \alpha x_0 + \alpha' x_1$ with $\alpha,\ \alpha' \in \Omega_*^J$. Then $a = \varepsilon\mu(a) = \alpha + \alpha'\varepsilon(x_1)$ and we have $\partial a = \varepsilon\Delta\mu a = \varepsilon(\alpha' x_0) = \alpha'\varepsilon(x_0) = \alpha'$.

Lemma 8.7.7 *(i)* $a * b = a.b + 2w_2.\partial a.\partial b$.
(ii) $\partial(a.b) = a.\partial b + \partial a.b - \varepsilon(x_1).\partial a.\partial b$.
(iii) $\partial(x.\partial(y)) = \partial x.\partial y$.
(iv) $\partial(a * b) = a.\partial b + \partial a.b + w_1.\partial a.\partial b$.

Proof (i) Let $a,\ b \in \Omega_*^{RJ}$, and write $\mu(a) = \alpha x_0 + \alpha' x_1$, $\mu(b) = \beta x_0 + \beta' x_1$, so $s_0(a.b) = \varepsilon\Delta(\alpha\beta x_1 + (\alpha\beta' + \alpha'\beta)x_1^2 + \alpha'\beta' x_1^3)$. Calculations give $\varepsilon\Delta(x_1) = \varepsilon(x_0)$, $\varepsilon\Delta(x_1^2) = \varepsilon(x_1)$ and $\varepsilon\Delta(x_1^3) = 3\varepsilon(x_1^2) - 2\varepsilon(x_2)$; thus

$$\begin{aligned} s_0(a.b) &= \alpha\beta\varepsilon(x_0) + (\alpha\beta' + \alpha'\beta)\varepsilon(x_1) + \alpha'\beta'(3\varepsilon(x_1^2) - 2\varepsilon(x_2)) \\ &= (\alpha + \alpha'\varepsilon(x_1))(\beta + \beta'\varepsilon(x_1)) + 2\alpha'\beta'(\varepsilon(x_1^2) - \varepsilon(x_2)) \\ &= a.b + 2\partial a.\partial b.(\varepsilon(x_1^2) - \varepsilon(x_2)). \end{aligned}$$

(ii) With $a,\ b$ as above, we have

$$\begin{aligned} \partial(a.b) &= \varepsilon\Delta\mu(a.b) \\ &= \varepsilon\Delta(\alpha\beta + (\alpha\beta' + \alpha'\beta)x_1 + \alpha'\beta' x_1^2) \\ &= \varepsilon((\alpha\beta' + \alpha'\beta)x_0 + \alpha'\beta' x_1) \\ &= (\alpha + \alpha'\varepsilon(x_1))\beta' + \alpha'(\beta + \beta'\varepsilon(x_1)) - \alpha'\beta'\varepsilon(x_1) \\ &= a.\partial b + \partial a.b - (\varepsilon(x_1)).\partial a.\partial b. \end{aligned}$$

(iii) follows from (ii) since $\partial^2 = 0$. (iv) now follows from (i)–(iii). □

Here w_2 is given by $\varepsilon(x_1^2) - \varepsilon(x_2)$, and $w_1 = 2\partial(w_2) - \varepsilon(x_1)$. If $d = 2$ we have $w_2 := [(P^1)^2] - [P^2]$.

Since $\partial^2 = 0$, $(\Omega_*^{RJ}, \partial)$ defines a chain complex: denote its homology by H_*^J. Explicitly,

$$H_n^J := \frac{\mathrm{Ker}(\partial : \Omega_n^{RJ} \to \Omega_{n-d}^{RJ})}{\mathrm{Im}(\partial : \Omega_{n+d}^{RJ} \to \Omega_n^{RJ})}.$$

Although, in the case $d = 2$, ∂ is not a derivation, it follows from Lemma 8.7.7 that Ker ∂ is a subring of Ω_*^{RU} and that *Im* ∂ is an ideal in it, so that the quotient H_*^J is a ring.

We next want an exact sequence derived from (8.7.6). The general procedure, due to Massey [87], is as follows. Suppose given an exact sequence

$$\dots P \xrightarrow{a} P \xrightarrow{b} Q \xrightarrow{c} P \xrightarrow{a} P \dots$$

(he calls this an exact couple). Then $d := b \circ c$ has $d^2 = 0$ since $c \circ b = 0$, so we can form the homology H of Q with respect to d. Set $A := a(P) \subset P$.

Lemma 8.7.8 *There is an exact sequence*

$$\dots A \xrightarrow{a_1} A \xrightarrow{b_1} H \xrightarrow{c_1} A \dots$$

If a, b, c have respective degrees d_a, d_b, d_c then a_1, b_1, c_1 have degrees $d_a, d_b - d_a, d_c$.

Observe that this gives another exact couple, called the derived couple.

Proof Define a_1 as the restriction of a. For $y = a(x) \in A$ define $b_1(y)$ as the class of $b(x)$: we have $db(x) = bcb(x) = 0$ so do get a class in H. And for an element $\zeta \in H$ represented by $z \in Q$ with $bc(z) = 0$ define $c_1(\zeta) = c(z)$: this is indeed in $\mathrm{Ker}(b) = \mathrm{Im}(a)$. Any other representative is of form $z + bc(w)$ and $c(z + bc(w)) = c(z)$.

Composites vanish since $b_1(a_1(ax))$ is the class of $b(a(x)) = 0$; $c_1(b_1(ax))$ is represented by $c(b(x)) = 0$; and $a_1(c_1(\zeta)) = a(c(z)) = 0$.

If $y = a(w)$ and $b_1(y) = 0$, then $b(w) = d(x) = b(c(x))$ for some x so $w - c(x) \in \mathrm{Ker}(b) = \mathrm{Im}(a)$; $w = c(x) + a(v)$ so $y = a^2(v) \in \mathrm{Im}(a_1)$. If $c_1(\zeta) = 0$, then $z \in \mathrm{Ker}(c) = \mathrm{Im}(b)$: set $z = b(y)$: then $\zeta = b_1(ay)$. If $a_1(x) = 0$, then $x \in \mathrm{Ker}(a) = \mathrm{Im}(c)$. This proves exactness. □

Write A_n^J for the image of $\theta : \Omega_{n-d+1}^{SJ} \to \Omega_n^{SJ}$. Applying Lemma 8.7.8 to (8.7.6) gives the exact sequence

$$\dots \to A_n^J \to A_{n+d-1}^J \to H_n^J \to A_{n-d}^J \to A_{n-1}^J \to \dots \qquad (8.7.9)$$

As in the preceding section, the completion of the calculation of the cobordism rings depends on exhibiting particular examples. We will give these, but omit the detailed calculations, some of which yield

Lemma 8.7.10 *We have* $s_{n+2}(s_2(M^{d(n+2)})) \equiv (n+1)c_1^n[M^{dn}]$ *(mod 2)* $c_1^{n+2}[s_2(M^{dn})] = -c_1^n[M^{dn}]$.

As in the proof of Theorem 8.6.9, for n not a power of 2, choose integers $r,\ s$ with $r+s=n$ and $\binom{n}{r}$ odd: for example, write $n=2^p(2q+1)$ (so $q\geq 1$) and set $r=2^{p+1}q$, $s=2^p$. Define cobordism classes by
$z_{2dn} := \rho(P^{2r}\times P^{2s})\in\Omega_{2dn}^{RJ}$,
$z_{2dn-d} := d_1(P^{2r}\times P^{2s})\in\Omega_{2dn-d}^{SJ}$.
It follows using Lemma 8.7.10 that $s_m(z_{dm})$ is odd in both cases, and from the formulae relating the maps that $d_1 z_{2dn}=z_{2dn-d}$.

In case $n=2^j>1$ is a power of 2, first set $z_{2dn}:=P^n\times P^n$. In the case $d=1$, since $P^n(\mathbb{R})\times P^n(\mathbb{R})\sim P^n(\mathbb{C})$, which is orientable, this gives $d_1 z_{2n}=0$. If $d=2$ we set $z_{2dn-d}:=d_1(P^n\times P^n)$.

In the preceding section we defined Poincaré series and calculated the series for Ω_*^U and Ω_*^{SO} over $\mathbb{Q}$ and for Ω_*^O over $\mathbb{Z}_2$. It now follows from the exact sequence (8.7.5) that

$$P(\Omega_*^{RO};\mathbb{Z}_2)(t)=(1-t^2)\prod_{i\ not\ of\ form\ 2^n-1}(1-t^i)^{-1}.$$

Theorem 8.7.11 *(i) Ω_*^{SO}/Tors is a polynomial ring.*

(ii) All torsion in Ω_*^{SO} has order 2.

(iii) Classes in Ω_*^{SO} are detected by Stiefel–Whitney numbers and Pontrjagin numbers.

(iv) The image of Ω_*^{SO} in $\Omega_*^{RO}\subset\Omega_*^O$ is Ker ∂; the image of Tors Ω_*^{SO} is Im ∂.

A presentation of Ω_*^{SO} by generators and relations is not convenient: (iv) gives a better description.

Proof (i) By Proposition 8.6.4, $\Omega_*^{SO}\otimes\mathbb{Q}$ is a polynomial ring, with one generator in each dimension divisible by 4. It follows (again from [96] or [113]) that $\Omega_*^{SO}\otimes\mathbb{Z}[\frac{1}{2}]$ is a polynomial ring. We next claim that Ω_*^{SO}/Tors is a polynomial ring: it suffices to observe that since by Lemma 8.6.5 $s_n(p)[P^{2n}(\mathbb{C})]=2n+1\neq 0$, so the classes of the $P^{2n}(\mathbb{C})$ are polynomial generators of $\Omega_*^{SO}\otimes\mathbb{Q}$ and since also these numbers are odd, their images in Ω_*^O generate a polynomial algebra.

Next observe that Ω_*^{RO} is a subring of Ω_*^O. This follows from Lemma 8.7.7 (i) and the fact that Ω_*^O has exponent 2, or more simply from the fact that $P^1=S^1$ can be regarded as a subgroup of P in this case. Next, the map ∂ is a derivation: this follows from (ii) of the same Lemma, and the fact that $\epsilon(x_1)$ is a class of dimension 1, hence is zero as $\Omega_1^O=0$.

We defined classes $z_n\in\Omega_n^{RO}$ above, for $n,\ n+1$ not powers of 2, and showed that in each case, $s_n(z_n)=1$. If $n=2^j\geq 2$, the class x_n of $P^n(\mathbb{R})$ has $s_n(x_n)=1$ by Proposition 8.6.3 above. Hence Ω_*^O is the polynomial ring in these generators. Now $P^{2^i}(\mathbb{R})^2$ is cobordant to $P^{2^i}(\mathbb{C})$, which is orientable. The classes z_n

and x_n^2 thus all belong to Ω_*^{RO} and generate a polynomial ring. Since this subring has the same Poincaré polynomial as Ω_*^{RO}, it is the whole ring.

We have now calculated the derivation ∂ on all generators of Ω_*^{RO}, for as $P^{2^i}(\mathbb{C})$ is orientable, $\partial(x_n^2) = 0$. Since we can regard Ω_*^{RO} as the tensor product of the algebras $\mathbb{Z}_2[z_m, z_{m+1}]$ with $m+1 = 2^p(2q+1)$, $p \geq 1$, $q \geq 1$ and $\mathbb{Z}_2[x_n^2]$, the ring $H_*(\Omega_*^{RO}; \partial)$ is the tensor product of the homologies of these subalgebras, which is the polynomial algebra in the z_{m+1}^2 and x_n^2. Thus $H_*^O := H_*(\Omega_*^{RO}; \partial)$ is a polynomial ring over $\mathbb{Z}_2$, with one generator in each dimension divisible by 4.

We now recall the exact sequence given by (8.7.9) with $J = O$:

$$\ldots \to A_n^O \xrightarrow{a_1} A_n^O \xrightarrow{b_1} H_n^O \xrightarrow{c_1} A_{n-1}^O \to A_{n-1}^O \to \ldots$$

Since a_1 is induced from α it is multiplication by 2. Thus $A_n^O = 2\Omega_n^{SO}$. The torsion-free rank of A_{4n}^O is equal to the number $p(n)$ of partitions of n, so the image of b_1, isomorphic to $A_{4n}^O/2A_{4n}^O$ has rank over $\mathbb{Z}_2$ at least this. Hence b_1 is surjective in these, hence in all degrees. By exactness, the kernel of multiplication by 2 on A_n^O vanishes. Thus A_n^O is torsion-free. This proves (ii).

By (8.7.6) the kernel of $\Omega^{SO} \to \Omega^{RO} \subset \Omega^O$ is the image of multiplication by 2, so is torsion-free. An element on which all Stiefel–Whitney numbers vanish is in this kernel, hence of infinite order, hence by Proposition 8.6.4 is detected by Pontrjagin numbers; thus (iii) holds.

The first assertion of (iv) follows from Proposition 8.7.4 (ii); the second now follows from the above calculation that $\Omega_*^{SO}/\text{Tors}$ and $\text{Ker}\,\partial/Im\,\partial$ have the same Poincaré polynomial. □

Calculations in homology lead to the further results, completing the above.

Lemma 8.7.12 *(i) As $\mathcal{S}_2$-module, $H^*(\mathbb{TSO}; \mathbb{Z}_2)$ is the direct sum of a free module and copies of $\mathcal{S}_2/Sq^1.\mathcal{S}_2$.*

(ii) The spectrum $\mathbb{TSO}$ is homotopy equivalent to a wedge of spectra $\mathbb{K}(\mathbb{Z}_2, n)$ and $\mathbb{K}(\mathbb{Z}, n)$.

(iii) An element of Ω_*^O is in the image of Ω_*^{SO} (resp. of Ω_*^{RO}) if and only if all Stiefel–Whitney numbers with w_1 (resp. w_1^2) as a factor vanish.

We turn to Ω_*^{SU}. It again follows from Proposition B.4.1 that

$$P(\Omega_*^{RU}; \mathbb{Q})(t) = P(H_*(B(RU) : \mathbb{Q}))(t) = (1-t^2)^{-1}\prod_{i=3}^{\infty}(1-t^{2i})^{-1},$$

$$P(\Omega_*^{SU}; \mathbb{Q})(t) = P(H_*(B(SU) : \mathbb{Q}))(t) = \prod_{i=2}^{\infty}(1-t^{2i})^{-1}.$$

Since Ω_*^{RU} is a direct summand of Ω_*^U, it is torsion-free. Write $\overline{\Omega}_*^{SU}$ for the pure subgroup $(\Omega_*^{SU} \otimes \mathbb{Q}) \cap \Omega_*^U$ generated by Ω_*^{SU}. It follows by comparing Poincaré series that an element of Ω_*^U is in the image of $\overline{\Omega}_*^{SU}$ (resp. of Ω_*^{RU}) if and only if all Chern numbers with c_1 (resp. c_1^2) as a factor vanish.

Calculations at odd primes were made by Novikov [113]. The following is analogous to Theorem 8.6.11 for Ω_*^U.

Theorem 8.7.13 *The ring $\Omega_*^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra with one generator in each even dimension $\neq 2$. The class of a manifold M^{2m} is determined by Chern numbers, and M^{2m} qualifies as a generator if and only if $s_m(c)[M]$ is $\pm r_m r_{m+1}$ times a power of 2.*

The structure at the prime 2 is not simple: a precise description of the torsion-free quotient is given by [147, p. 265]. The torsion subgroup is described by

Theorem 8.7.14 *(i) All torsion in Ω_*^{SU} has order 2. We have*

$$P(Tors\ \Omega_*^{SU}; \mathbb{Z}_2)(t) = (t + t^2) \prod_{i=1}^{\infty} (1 - t^{8i})^{-1}.$$

(ii) The image of $\Omega_{2j}^{SU} \to \Omega_{2j}^{RU}$ is Ker ∂ if $2j \not\equiv 4 \pmod 8$ and is Im ∂ if $2j \equiv 4 \pmod 8$.

Proof We outline the main arguments involved in the proof. First recall that by Proposition 8.7.4, Ω_*^{RU} maps injectively to Ω_*^U, so is torsion-free.

Since $2\alpha = 0$, in the sequence $\Omega_*^{SU} \xrightarrow{\alpha} \Omega_*^{SU} \to \Omega_*^{RU}$ the image of the first map has exponent 2; the quotient by it embeds in a free group, so there is no further torsion. More precisely, as Ω_*^{RU} vanishes in odd dimensions, the sequence (8.7.6) reduces to

$$0 \to \Omega_{2k-1}^{SU} \xrightarrow{\alpha} \Omega_{2k}^{SU} \to \Omega_{2k}^{RU} \to \Omega_{2k-2}^{SU} \xrightarrow{\alpha} \Omega_{2k-1}^{SU} \to 0. \qquad (8.7.15)$$

To calculate the 2-torsion, we again use the derived couple (8.7.9): here $d = 2$, $J = U$, so we have

$$\ldots \to A_n^U \to A_{n+1}^U \to H_n^U \to A_{n-2}^U \to A_{n-1}^U \to \ldots$$

Also, $A_n^U := \theta(\Omega_{n-1}^{SU}) \subset \Omega_n^{SU}$ has exponent 2. Since Ω_*^{RU} vanishes in odd dimensions, so does H_*^U: it follows that the map $\alpha : A_{2k-1}^U \to A_{2k}^U$ is an isomorphism. It thus follows from (8.7.15) that $A_{2k}^U = \mathrm{Tors}(\Omega_{2k}^{SU})$ and $A_{2k-1}^U \cong \Omega_{2k-1}^{SU}$.

We now claim that the map $\alpha : A_{2k-2}^U \to A_{2k-1}^U$ vanishes. For $A_{2k-2}^U = \theta(\Omega_{2k-3}^{SU})$ and Ω_{2k-3}^{SU} is in the image of the map θ given by multiplication by η_2. Thus the image of α is contained in the image of θ^3. However by §B.3(x), we have $\eta_2^3 = 0$.

The sequence (8.7.9) thus reduces to split short exact sequences

$$0 \to A^U_{2k+1} \to H^U_{2k} \to A^U_{2k-2} \to 0. \tag{8.7.16}$$

To calculate H^U_* we use another sequence. Since Ω^{RU}_* is torsion-free we have a short exact sequence $0 \to \Omega^{RU}_* \xrightarrow{2} \Omega^{RU}_* \to \Omega^{RU}_* \otimes \mathbb{Z}_2 \to 0$, which we regard as an exact sequence of chain complexes with ∂ as differential. There is thus an exact homology sequence, which we denote

$$\ldots \to H^U_* \xrightarrow{2} H^U_* \to H^V_* \xrightarrow{\partial} H^U_* \to \ldots.$$

Now the groups H^U_n have exponent 2 since, for each RU-manifold M, we have $\partial[P^1(\mathbb{C}) \times M] = 2[M]$. Thus the map $2 : H^U_* \to H^U_*$ is zero. The groups H^U_n and H^V_n vanish in odd degrees, and ∂ has degree -2, so the sequence reduces to

$$0 \to H^U_{2k} \to H^V_{2k} \to H^U_{2k-2} \to 0. \tag{8.7.17}$$

We next compute H^V_*. We think of Ω^{RU}_* as a polynomial subring of Ω^U_*, and ∂ as a derivation: a correct formulation is given in Lemma 8.7.7.

We have defined elements $z_{2n} \in \Omega^{RU}_{2n}$ for $n,\ n+1$ not powers of 2 such that $s_{2n}(z_{2n})$ is odd. For $m = 2^j \geq 2$, define

$x_{4m} := \rho(P^m(\mathbb{C}) \times P^m(\mathbb{C}))$;

$x_{4m-2} := d_1(P^m(\mathbb{C}) \times P^m(\mathbb{C}))$.

Calculations similar to those in the preceding case yield

$s_{m,m}(c)(x_{4m}) \equiv 1 \pmod 2$ (the class s_{2m} does not suffice here), and

$s_{2m-1}(c)(x_{4m-2}) \equiv 2 \pmod 4$.

It follows that the z_{2n}, x_{4m-2} and x_{4m} give polynomial generators of $\Omega^U_* \otimes \mathbb{Z}[\frac{1}{2}]$ and $\Omega^U_* \otimes \mathbb{Z}_2$ in all dimensions except 2 and 4. Since all are in Ω^{RU}_*, we only need to add the class x_2 of $P^1(\mathbb{C})$ to obtain a complete set of generators of $\Omega^{RU}_* \otimes \mathbb{Z}_2$.

We have $\partial z_{4n} = z_{4n-2}$, $\partial x_{4m} = x_{4m-2}$ and $\partial x_2 = 0$, so if ∂ were a true derivation, we would have H^V_* polynomial with generators x_2 and z^2_{4n} in each dimension divisible by 8. In fact it follows from Lemma 8.7.7 that the elements $h_2 = z_2$ and $h_{8n} := z^2_{4n} + z_2 z_{4n-2} z_{4n}$ $(n \geq 2)$ are cycles. It follows that H^V_* is a polynomial algebra with their classes as generators.

Further calculations using Lemma 8.7.7 exhibit elements of H^U_* mapping to h_2^2 and the h_{8k}. Thus $H^U_{4n} \to H^V_{4n}$ is surjective, so by the exact sequence (8.7.17), H^U_{4n-2} vanishes and the maps $H^V_{4n+2} \to H^U_{4n} \to H^V_{4n}$ are isomorphisms; moreover, H^U_* is a polynomial algebra with the classes of h_2^2 and the h_{8k} as generators. The Poincaré series of H^U_* is thus given by $P(H^U_*; t) = (1 - t^4)^{-1} \prod_{k=2}^{\infty} (1 - t^{8k})^{-1}$.

Since H_n^U vanishes unless n is divisible by 4, $A_n^U = 0$ unless $n \equiv 1$ or $n \equiv 2$ (mod 4). It now follows from the exact sequence (8.7.16) and the isomorphism $A_{2k-1}^U \to A_{2k}^U$ that if $PB(t)$ denotes the Poincaré series of the even part of A_*^U, then $P(H_*^U; t) = (t^2 + t^{-2})PB(t)$; thus $PB(t) = t^2 \prod_{k=1}^{\infty}(1 - t^{8k})^{-1}$. Hence $P(A_*^U; t) = (t + t^2) \prod_{k=1}^{\infty}(1 - t^{8k})^{-1}$,and the rank of Tors Ω_n^{SU} is as stated. □

Further calculations yield more detailed results.

Theorem 8.7.18 [9] *(i) Write* $\mathcal{S}_2^* := \mathcal{S}_2/\langle Sq^1\rangle$. *Then as* $\mathcal{S}_2$*-module,* $H^*(\mathbb{TSU}; \mathbb{Z}_2)$ *is a sum of copies of* $\mathcal{S}_2^*$ *and* $\mathcal{S}_2^*/\mathcal{S}_2^*.Sq^2$.

(ii) The spectrum $\mathbb{TSU}$ *is homotopy equivalent to a wedge of copies of spectra* $\mathbb{K}(\mathbb{Z}, n)$ *and spectra* $\mathbb{BO}\langle k\rangle$.

(iii) An SU-manifold bounds in Ω_*^{SU} *if and only if all its Chern numbers and KO characteristic numbers vanish.*

8.8 Groups of knots and homotopy spheres

We first consider k-connected cobordism, where the manifolds M and cobordisms W are to be k-connected for some integer $k \geqq 1$. In this case, M is orientable: we make the further convention that M is oriented.

Since the set of k-connected manifolds is not closed under disjoint union, we define an addition on the set of cobordism classes using connected sum. We remark that in general, the disjoint union and connected sum of two manifolds are cobordant: a cobordism of $M \cup M'$ to $M \sharp M'$ is given by taking $(M \times I) \cup (M' \times I)$ and attaching a 1-handle to join $M \times 1$ and $M' \times 1$.

Lemma 8.8.1 *Connected sum of k-connected manifolds of a given dimension* $n > 2$ *is a commutative associative operation with unit, compatible with cobordism. The set of equivalence classes thus acquires the structure of an abelian group* $\Omega_n\langle k\rangle$.

Proof The operation is well-defined by Theorem 2.7.4 (with the remark following dealing with orientation); by Proposition 2.7.6, it is commutative and associative, and the sphere S^n acts as unit. That the connected sum $M \sharp M'$ is k-connected if M and M' are follows if $k = 0$ from the definition, if $k = 1$ from the fact that for $n > 2$ removing a point does not introduce a fundamental group, and if $k > 1$ from the fact that removing a point does not change homology in dimension $< n$.

We must next check that the operation is compatible with cobordism. Let V and W be connected cobordisms, of dimension $n + 1$, and $f_- : D^n \to \partial_- V$, $f_+ : D^n \to \partial_+ V$, $g_- : D^n \to \partial_- V$, and $g_+ : D^n \to \partial_+ V$ be used to define the connected sums $\partial_- V \sharp \partial_- W$ and $\partial_+ V \sharp \partial_+ W$. Join $f_-(0)$ to $f_+(0)$ by an arc α in V: a tubular neighbourhood of the arc gives an imbedding $F : D^n \times I \to V$ with $f_- = F|D^n \times 0$ and $f_+ = F|D^n \times 1$. Similarly define $G : D^n \times I \to W$. Now delete the interiors of the images of F and G and glue the boundaries, and we have a cobordism of $\partial_- V \sharp \partial_- W$ to $\partial_+ V \sharp \partial_+ W$.

The inverse $\overline{M}$ of M is as usual obtained by change of orientation. We can regard $M \times I$ as a cobordism of $M \cup \overline{M}$ to the empty set. Attaching $D^n \times I$ with one end in M and one in $\overline{M}$ gives W with $\partial W = M \sharp \overline{M}$. Now remove a disc from the interior of W to obtain a cobordism of $M \sharp \overline{M}$ to S^m. □

For 0-connected cobordism (where we do not assume M oriented), we noted above that disjoint union is cobordant to connected sum, so that the map $\Omega_n\langle 0\rangle \to \Omega_n^O$ is surjective for $n \geqq 1$; it is easily seen to be bijective.

For k-connected cobordism, we need the connective covers of groups and classifying spaces. For any X we denote by $X^{\langle k\rangle}$ the $(k-1)$-connected cover of X: thus the map $\pi_r(X^{\langle k\rangle}) \to \pi_r(X)$ is zero for $r < k$ and an isomorphism for $r \geq k$. Observe that $B(J^{\langle k-1\rangle}) = (B(J))^{\langle k\rangle}$: we will write $BJ^{\langle k\rangle}$ for $B(J^{\langle k\rangle})$, which is k-connected.

The classifying map $\tau_M : M \to B(O)$ of its normal bundle lifts to a map $\tau_M^k : M \to BO^{\langle k\rangle}$ if M is k-connected, and the lift is unique up to homotopy if M is $(k+1)$-connected. We now claim

Theorem 8.8.2 *If $m > 2k+2$, there is a natural isomorphism $\Omega_m\langle k\rangle \to \pi_m^S(T(O^{\langle k\rangle}))$.*

Proof It follows from the remark preceding the theorem that there is a natural map $\psi_m^k : \Omega_m\langle k\rangle \to \pi_m^S(T(O^{\langle k\rangle}))$.

By Theorem 8.1.3 the Thom construction induces a bijection from the set of cobordism classes of m-manifolds whose stable normal bundle is induced from $BO^{\langle k\rangle}$ with the set $\pi_m^S(T(O^{\langle k\rangle}))$.

By Theorem 7.2.1, if X is a finite CW-complex and $m \geq 2r$, any normal map $(f : M \to X, \nu, T)$ is normally cobordant to a normal map $(f' : M' \to X, \nu, T')$ such that f' is r-connected. Applying this with X (a high enough skeleton of) $BO^{\langle k\rangle}$ and $r = k+1$, we see that if $m > 2k+2$ any element of $\pi_m^S(T(O^{\langle k\rangle}))$ is represented by a manifold M^m with $f : M \to BO^{\langle k\rangle}$ $(k+1)$-connected. It thus follows from the exact sequence

$$\pi_{k+1}(M) \to \pi_{k+1}(BO^{\langle k\rangle}) \to \pi_{k+1}(f) \to \pi_k(M) \to \pi_k(BO^{\langle k\rangle}) \to \dots$$

that M is k-connected, so the map ψ_m^k is surjective.

Similarly, given a cobordism W between two k-connected M, M' defining the same element of $\Omega_m\langle k\rangle$, provided $m+1 > 2k+2$ we can perform surgery on W, eventually making $W \to BO^{\langle k\rangle}$ $(k+1)$-connected and hence W k-connected. Hence ψ^k_m is injective. $\square$

Corollary 8.8.3 *Excluding small m, we have isomorphisms* $\Omega_m\langle 1\rangle \cong \Omega^{SO}_m$, $\Omega_m\langle 2\rangle \cong \Omega_m\langle 3\rangle \cong \Omega^{Spin}_m$.

For we have $BO^{\langle 1\rangle} = B(SO)$, $BO^{\langle 2\rangle} = BO^{\langle 3\rangle} = B(Spin)$.

The above argument deals with the cases $m > 2k+2$ using surgery below the middle dimension. The cases $m = 2k$ and $m = 2k+1$ are of special interest. The case $m = 2k$ was discussed in Theorem 5.6.12.

If $m < 2k$ a k-connected m-manifold is a homotopy sphere (terminology introduced in §5.6), and the value of k is irrelevant to further study. From now on we focus on homotopy spheres. We begin our treatment by deriving a number of exact sequences: in all cases exactness will follow from the general principle of Lemma 8.3.1. First, however, we list the types of cobordism to be considered. In each case there is a natural definition of addition by connected sum, which gives the set a group structure. This is treated in Lemma 8.8.1 for k-connected cobordism and is similar in other cases. At the centre of our interest are groups of homotopy spheres:

(θ) Submanifolds $\Sigma^m \subset S^{m+k}$ with a homotopy equivalence $\Sigma^m \to S^m$. We denote the set of cobordism classes by Θ^k_m. We also consider

($f\theta$) Submanifolds $\Sigma^m \subset S^{m+k}$ with a homotopy equivalence $\Sigma^m \to S^m$ and a framing of the normal bundle (with compatible orientation class). Here we denote the set of cobordism classes by $F\Theta^k_m$. In parallel with these we consider

(so) The standard submanifold $S^m \subset S^{m+k}$ and a framing of the normal bundle (with compatible orientation class). Framings are classified up to homotopy by $\pi_m(SO_k)$, and we can identify this with the cobordism group.

(sph) Submanifolds $M^m \subset S^{m+k}$ with a framing of the normal bundle.

For a cobordism we must have a submanifold W^{m+1} of $S^{m+k} \times I$ (equal in case (so) to $S^m \times I$) together in cases (θ) and ($f\theta$) with a homotopy equivalence $W^{m+1} \to S^m$, in cases (so), ($f\theta$) and (sph) with a framing of the normal bundle of W^{m+1} in $S^{m+k} \times I$; in each case inducing the given structures on $\partial_- W$ and $\partial_+ W$.

A structure of type (so) is stronger than one of type ($f\theta$) which in turn is stronger than one of type (sph). Each of these three inclusions induces by Lemma 8.3.1 an exact sequence, and by the Corollary to that Lemma we also have an exact sequence of the three relative groups. We now reinterpret these.

As in §7.8, write $B(G_n)$ for the classifying space for spherical fibrations with fibre S^{n-1} and G_n for the monoid of maps of S^{n-1} to itself of degree ± 1, with multiplication given by composition of maps. Fixing the orientation gives a submonoid SG_n and a classifying space $B(SG_n)$. We write $F_n \subset G_{n+1}$ for the set of base-point preserving maps $S^n \to S^n$ of degree ± 1, and SF_n for those of degree $+1$. The suspension of a self-map of S^{n-1} is a self-map of the same degree of S^n which fixes a base point; thus we have an inclusion $G_n \subset F_n$. There are corresponding classifying spaces $B(F_n)$ and $B(SF_n)$. Since all components of $\Omega^n S^n$, including SF_n, are homotopy equivalent, we have $\pi_r(F_n) \cong \pi_{r+n}(S^n)$. Further discussion is given in §B.2.

Now (sph) is the cobordism group of submanifolds $M^m \subset S^{m+k}$ with a framing of the normal bundle; by Proposition 8.1.4 the group of cobordism classes is identified with $\pi_{m+k}(S^k) = \pi_m(\Omega^k S^k) = \pi_m(SF_k)$. We now see that the (so, sph) sequence can be identified with the exact homotopy sequence of (SF_k, SO_k).

The relative term for the $(so, f\theta)$ sequence is represented by manifolds W with boundary, with an assigned embedding $W^{m+1} \subset D^{m+k+1}$, with $\partial W = S^m \subset S^{m+k}$, a framing of the normal bundle of W^{m+1} in D^{m+k+1}, and a homotopy equivalence of W with a point. Since W is contractible, the framing of its normal bundle is unique (up to homotopy) and can be ignored. We regard D^{m+k+1} as the upper hemisphere of S^{m+k+1} and complete W to a closed manifold $\overline{W} \subset S^{m+k+1}$ by attaching the standard disc $D^{m+1} \subset D^{m+k+1}$ in the lower hemisphere and rounding the corner. There is a natural homotopy equivalence of $\overline{W}$ with S^{m+1}. Conversely, given a homotopy sphere $\Sigma^{m+1} \subset S^{m+k+1}$ we have (by the Disc Theorem 2.5.6) an essentially unique embedding $D^{m+1} \to \Sigma^{m+1}$; its neighbourhood in S^{m+k+1} may be identified with a disc D^{m+k+1}, and the whole construction can be reversed. The relative group is thus identified with Θ^k_{m+1}.

The relative term for the $(f\theta, sph)$ sequence is represented by manifolds W with boundary, with $W \subset D^{m+k+1}$, framed normal bundle, and a homotopy equivalence $\partial W \to S^m$. We denote the corresponding group of cobordism classes by P^{*k}_{m+1}.

By Lemma 8.3.2 we now have

Proposition 8.8.4 *We have a commutative braid of long exact sequences.*

$$
\begin{tikzcd}[column sep=small]
\pi_n(SO_k) \arrow[dr] \arrow[rr, bend left] & & \pi_n(SF_k) \arrow[dr] \arrow[rr, bend left] & & P^{*k}_n \arrow[dr] \arrow[rr, bend left] & & \Theta^k_{n-1} \\
& F\Theta^k_n \arrow[ur] \arrow[dr] & & \pi_n(SG_k, SO_k) \arrow[ur] \arrow[dr] & & F\Theta^k_{n-1} \arrow[ur] \arrow[dr] & \\
P^{*k}_{n+1} \arrow[ur] \arrow[rr, bend right] & & \Theta^k_n \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(SO_k) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(SG_k)
\end{tikzcd}
$$

For each of the above 6 sequences of groups, the natural inclusions $S^{n+k} \subset S^{n+k+1}$ and $D^{n+k+1} \subset D^{n+k+2}$ induce maps which increase k by 1. In each case we see that for k large enough ($k > n+1$ suffices), these maps are isomorphisms and the groups stabilise. We denote the limiting groups by omitting k from the notation (and also the asterisk from P^*). All sequences of Proposition 8.8.4 thus remain exact when we omit the affix k. We may identify $\pi_n(SG)$ with the stable homotopy group π_n^S and the map $\pi_n(SO) \to \pi_n(SG)$ with the classical J-homomorphism $J_n : \pi_n(SO) \to \pi_n^S$.

We now give calculations for the stabilised groups. By §B.3(xi) $\pi_n(SO)$ is isomorphic to $\mathbb{Z}$ for $n \equiv -1 \pmod 4$, to $\mathbb{Z}_2$ for $n \equiv 0$ or $n \equiv 1 \pmod 8$, and is trivial otherwise. We proved in Proposition 7.8.4 by surgery that P_n is isomorphic to $\mathbb{Z}$ for $n \equiv 0 \pmod 4$, to $\mathbb{Z}_2$ for $n \equiv 2 \pmod 4$, and is trivial otherwise (provided $n > 5$). It follows from the stabilised braid (8.8.4) that the groups Θ_n are closely related to the stable homotopy groups π_n^S. A first deduction is

Proposition 8.8.5 *All the groups in the* stabilised *diagram of (8.8.4) with $n \geq 5$ are finitely generated abelian groups, and all are finite with the exceptions of $\pi_{4r-1}(SO)$, $\pi_{4r}(SF, SO)$, P_{4r} and $F\Theta_{4r-1}$, which have rank 1.*

For the case $n = 4s - 1$ we first consider an element y of the group $\pi_{4s}(SF, SO)$ of cobordism classes of framed manifolds N with boundary diffeomorphic to S^{4s-1}. The boundary $x \in \pi_{4s-1}(SO)$ induces an orthogonal bundle $\xi(x)$ over S^{4s}: we can then form the Pontrjagin class $p_s(\xi(x))$ and evaluate on the fundamental class $[S^{4s}]$ giving an integer $p_s(x)$, say. Additivity properties of bundles and classes show that we have a homomorphism $p_s : \pi_{4s-1}(SO) \to \mathbb{Z}$. According to [22], if x_0 generates $\pi_{4s-1}(SO)$ then (up to sign) $p_s(x_0) = a_s(2s-1)!$, where we set $a_s = 2$ if s is odd and $a_s = 1$ if s is even. Thus the image of y in $\pi_{4s-1}(SO)$ is $p_s(x)/a_s(2s-1)!$ times a generator.

On the other hand, attaching a disc to the boundary of N yields a closed manifold M. The normal bundle of M is trivial except on the disc, so is induced from a bundle over S^{4s} which we can identify with the above bundle $\xi(x)$. According to the signature theorem 8.6.7, the signature of M is given by $L_s(\nu_M)[M]$. Since all the intermediate Pontrjagin classes of ν_M vanish, it follows by Lemma 8.6.8 that

$$L_s(\nu_M) = 2^{2s}(2^{2s-1} - 1)B_s p_s(x)/(2s)!,$$

so the image of y in P_{4s} is $2^{2s-3}(2^{2s-1} - 1)B_s p_s(x)/(2s)!$ times a generator.

Thus in some sense the generators in $\pi_{4s-1}(SO)$ and P_{4s} differ by a factor $a_s 2^{2s-2}(2^{2s-1} - 1)B_s/4s$. It now follows from exactness of the braid that

Proposition 8.8.6 *We have* $|\Theta_{4s-1}| = a_s 2^{2s-2}(2^{2s-1}-1)B_s|\pi^S_{4s-1}|/4s$.

More precisely, according to Adams [5] (see also §B.3(xviii)), $\mathrm{Ker} J_{4s-1}$ is a subgroup of $\pi_{4s-1}(SO)$ of index $\mathrm{den}(B_s/4s)$. Here, if $z \in \mathbb{Q}$ is expressed as a fraction p/q with $p, q \in \mathbb{Z}$ as small as possible, we write $p := \mathrm{num}(z)$ and $q := \mathrm{den}(z)$ for the numerator and denominator of z. Thus $|\pi^S_{4s-1}| = \mathrm{den}(B_s/4s)|\mathrm{Coker}\, J_{4s-1}|$. It also follows that $p_s(\mathrm{Ker}\, J_{4s-1}) = a_s(2s-1)!\mathrm{den}(B_s/4s)$, hence the signatures of the manifolds M obtained by closing elements of $\pi_{4s}(SF, SO)$ form the group of multiples of
$a_s 2^{2s+1}(2^{2s-1}-1)\mathrm{num}(B_s/4s)$.

The integer $m(2s) := \mathrm{den}(B_s/4s)$ is given by the following formula (due to Milnor and Kervaire [102], see also Adams [4]). For n an integer and p a prime, denote by $\nu_p(n)$ the greatest integer r such that p^r divides n. Then

For p odd, $\nu_p(m(t)) = 1 + \nu_p(t)$ if $t \equiv 0 \pmod{(p-1)}$, and $= 0$ if not.

For $p = 2$, $\nu_p(m(t)) = 2 + \nu_2(t)$ if t is even, and $= 1$ if t is odd.

Since $P_{4s} \cong \mathbb{Z}$, with the isomorphism given by $\sigma/8$, it follows that the image of $\pi_{4s}(SF, SO)$ in P_{4s}, which is the kernel of $P_{4s} \to \Theta_{4s-1}$ is a subgroup of index $a_s 2^{2s-2}(2^{2s-1}-1)\mathrm{num}(B_s/4s)$, so this number is the order of the group traditionally denoted bP_{4s}, which is the kernel of the epimorphism $\Theta_{4s-1} \to \pi_{4s-1}(SF, SO)$, the latter group having order $|\mathrm{Coker}\, J_{4s-1}|$.

For other values of n, we compare Θ_n with π^S_n via the intermediary $\pi_n(SF, SO)$ (or, if $n \equiv 0 \pmod 4$, its torsion subgroup). It was shown by Adams [5] that J_n is a (split) monomorphism if $n \equiv 0$ or $1 \pmod 8$. Thus if $n \not\equiv -1 \pmod 4$ $\mathrm{Tors}\, \pi_n(SF, SO)$ is the cokernel of J_n. Moreover Θ_n maps onto $\mathrm{Tors}\, \pi_n(SF, SO)$ except perhaps when $n \equiv 2 \pmod 4$. In this last case, π^S_n maps onto $\pi_n(SF, SO)$ and we have the map $K_n : \pi^S_n \to P_n \cong \mathbb{Z}_2$, defining the Kervaire invariant of framed manifolds. Thus if $n \equiv 0 \pmod 4$, Θ_n is isomorphic to $\mathrm{Tors}\, \pi_n(SF, SO)$; if either $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and K_n vanishes, Θ_n maps isomorphically to $\pi_n(SF, SO)$.

The delicate question of deciding for which values of n K_n is zero, is known as the 'Kervaire invariant problem'. It was shown by Browder [30] that K_n vanishes unless $n + 2 = 2^{k+2}$ is a power of 2. There are simple constructions showing K_n non-zero if $k = 0$, 1 or 2 (the classical framings of the tangent bundles of S^1, S^3 and S^7 induce framings of the projective spaces, and one uses $P^1(\mathbb{R}) \times P^1(\mathbb{R})$, $P^3(\mathbb{R}) \times P^3(\mathbb{R})$, and $P^7(\mathbb{R}) \times P^7(\mathbb{R})$); there is a somewhat less simple example for $k = 3$, and a proof by strenuous calculations [17] if $k = 4$. Recently it was shown by Hill, Hopkins, and Ravenel [69] that K_n vanishes for all $k \geq 6$, leaving only the case $k = 5$ (dimension 126) open.

A modified version of the braid (8.8.4) turns out to have better properties: we will replace the term SF_k by SG_k. Now $\pi_n(SG_k)$ is the group of homotopy classes of maps $S^n \times S^{k-1} \to S^{k-1}$ with the restriction to S^{k-1} homotopic to the identity. By the Thom construction, we can identify this with cobordism classes of framed manifolds $M^n \subset S^n \times S^{k-1}$ such that M^n has intersection number 1 with $* \times S^{k-1}$, or equivalently, the projection of M^n on S^n has degree 1.

Correspondingly, we can interpret $\pi_n(SG_k, SO_k)$ as the group of cobordism classes of framed manifolds $(M^n, \partial M) \subset (D^n, \partial D^n) \times S^{k-1}$ such that the projection $\partial M \subset S^{n-1} \times S^{k-1} \to S^{n-1}$ is a diffeomorphism and the projection $S^{n-1} \to S^{k-1}$ is induced by the framing. We have thus interpreted the exact homotopy sequence of (SG_k, SO_k) as an exact cobordism sequence.

We now need a replacement for the group denoted P_n^{*k} above. Write P_n^k for the group of cobordism classes of framed manifolds $(M^n, \partial M) \subset (D^n, \partial D^n) \times S^{k-1}$ such that the projection $\partial M \subset S^{n-1} \times S^{k-1} \to S^{n-1}$ is a homotopy equivalence and the projection $S^{n-1} \to S^{k-1}$ is induced by the framing. We now claim

Proposition 8.8.7 *We have a commutative braid of long exact sequences.*

$$\begin{tikzcd}
\pi_n(SO_k) \arrow[rr, bend left] \arrow[dr] & & \pi_n(SG_k) \arrow[rr, bend left] \arrow[dr] & & P_n^k \arrow[rr, bend left] \arrow[dr] & & \Theta_{n-1}^k \\
& F\Theta_n^k \arrow[ur] \arrow[dr] & & \pi_n(SG_k, SO_k) \arrow[ur] \arrow[dr] & & F\Theta_{n-1}^k \arrow[ur] \arrow[dr] & \\
P_{n+1}^k \arrow[ur] \arrow[rr, bend right] & & \Theta_n^k \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(SO_k) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(SF_k)
\end{tikzcd}$$

Proof The exact homotopy sequence of (SG_k, SO_k) was described above, and the exact sequence $\pi_n(SO_k) \to F\Theta_n^k \to \Theta_n^k$ is as before. We next describe the remaining maps.

First consider $F\Theta_n^k \to \pi_n(SG_k)$. Given a framed homotopy sphere $\Sigma_n \subset S^{n+k}$, take a tubular neighbourhood T. By Proposition 5.6.6, there is a diffeomorphism $h : \partial T \to S^n \times \partial D^k$: choose h such that the standard framing of $S^n \times \partial D^k$ pulls back to the framing of ∂T induced from that on S^{n+k}. Now the first vector of the normal framing of Σ^n induces a map $f : \Sigma^n \to \partial T$, and we take $h(f(\Sigma^n)) \subset S^n \times S^{k-1}$, with the framing induced by the remaining vectors of the normal framing of Σ^n.

The map $\Theta_n^k \to \pi_n(SG_k, SO_k)$ is defined similarly. We identify Θ_n^k with the group of framed homotopy discs in D^{n+k} with boundary the standard S^{n-1}. Now follow through the same steps as above.

The map $\pi_n(SG_k, SO_k) \to P_n^k$ is a forgetful map, defined by weakening the structure.

To define $P_{n+1}^k \to F\Theta_n^k$, we start with a framed manifold $(M^{n+1}, \partial M) \subset (D^{n+1}, \partial D^{n+1}) \times S^{k-1}$ such that the projection $\partial M \subset S^n \times S^{k-1} \to S^n$ is a homotopy equivalence. Map this to $\partial M \subset S^n \times S^{k-1} \subset S^{n+k}$ with the given framing extended by the normal vector to $S^n \times S^{k-1}$ in S^{n+k}.

The maps so far defined form a commutative diagram, and we define the remaining maps $P_{n+1}^k \to \Theta_n^k$ and $\pi_n(SG_k) \to P_n^k$ as the composites in the diagram. It follow easily that all four sequences have order 2. The exactness of the two remaining sequences follows again (with a little care) from Lemma 8.3.1. □

Since $G_n \subset F_n \subset G_{n+1}$, the stabilisations as $n \to \infty$ have the same homotopy groups; it follows that the same goes for the diagrams (8.8.4) and (8.8.7).

The reason why the second braid is an improvement on the first is the following.

Proposition 8.8.8 *The natural map $P_m^k \to P_m$ is surjective for $k \geq 2$ and an isomorphism for $k \geq 3$.*

Proof Recall that P_m^k is the group of cobordism classes of framed manifolds $(M^m, \partial M) \subset (D^m, \partial D^m) \times S^{k-1}$ such that the projection $\partial M \subset S^{m-1} \times S^{k-1} \to S^{m-1}$ is a homotopy equivalence and the projection $\partial M \to S^{k-1}$ is induced by the framing.

For surjectivity, since $P_{2n+1} = 0$, it suffices to consider the case $m = 2n$ even. By Proposition 7.8.3, generators of P_{2n} are represented by framed manifolds M constructed by attaching n-handles to D^{2n}. Since changing orientation and forming boundary sums respect this description, it follows that all elements of P_{2n} are so represented.

Write $e_i : S^{n-1} \times D^n \to S^{2n-1}$ for the attaching maps of the handles. Since all embeddings of S^{n-1} in S^{2n-1} are isotopic, there is a diffeomorphism of the image of e_i to the submanifold obtained from $\partial D^n \times D^n \subset \partial(D^n \times D^n)$ by rounding the corner. Thus e_i extends to an embedding $f_i : (D^n, \partial D^n) \times D^n \to (D^{2n}, \partial D^{2n})$ diffeomorphic to that induced by the map $D^n \times D^n \to D^{2n}$ rounding the corner.

We seek to construct a smooth embedding $F : (M, \partial M) \to (D^{2n}, \partial D^{2n}) \times S^1$ such that the trivial normal bundle agrees with the given stable framing; in fact we replace S^1 by I and then wrap round by $t \to e^{2\pi i t}$. Choose distinct points $t_i \in I$ and a smooth map $\phi : S^{2n-1} \to I$ such that the image of $\phi \circ e_i$ is the point

t_i. We first define a continuous map F_0: it is given on D^{2n} by $F_0(z) = (z, \phi(z))$ and on the handle h_i (identified with $D^n \times D^n$) by $F_0(x, y) = (f_i(x, y), t_i)$.

The map F_0 is not injective: each handle overlaps the core. Write $\nu : (D^n, \partial D^n) \to ([0, 1], 0)$ for the map given by $(1 - \|x\|^2)$, and deform the map of the handle to $F_1(x, y) = (f_i(x, y), t_i + \epsilon\nu(x))$, where ϵ is small enough that the handles remain disjoint. Then F_1 is injective, but has a corner along each copy of $S^{n-1} \times S^{n-1}$. We define a map F_2 by rounding these corners. This has the desired effect of deforming the interior part $\mathring{D}^n \times D^n$ of each handle into the interior of $D^{2n} \times I$, and gives the desired smooth embedding in $(D^{2n}, \partial D^{2n}) \times S^1$.

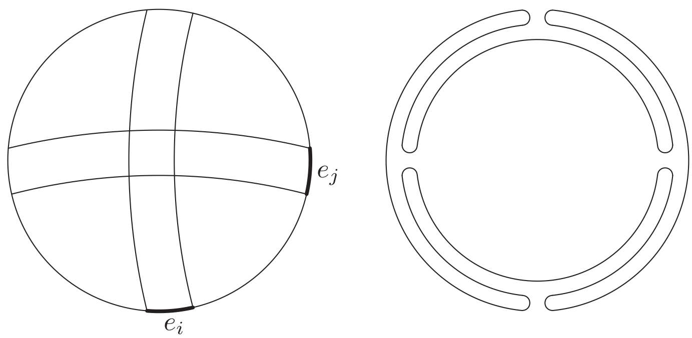

Figure 8.2 Embedding a plumbed manifold

We attempt to illustrate this in Figure 8.2: here the first figure represents a disc with two handles, pictured as a basket suspended by a couple of handles; the second figure indicates how these fit at the boundary. To prove injectivity, suppose given a framed manifold $(M^m, \partial M) \subset (V, \partial V)$ with $(V, \partial V) = (D^m, \partial D^m) \times S^{k-1}$ and $\partial M \subset S^{m-1} \times S^{k-1} \to S^{m-1}$ a homotopy equivalence, such that M represents 0 in the stabilised group P_m. Then there is a cobordism W of M to a disc: we seek to extend the embedding of $M = \partial_- W$ to $(W, \partial_c W) \to (V, \partial V) \times I$, ideally such that on $\partial_c W$ we have a product embedding. It is enough to consider a single r-handle attached to (the interior of) M. In view of the clause in Theorems 7.5.2, 7.5.4 (m even), and 7.6.1 (m odd) stating that for (simply-connected) surgery on manifolds of dimension $2n$ or $2n + 1$ it is sufficient to perform surgery on spheres S^r with $r \leq n$, we may suppose here that $2r \leq m$, and that M is $(r - 1)$-connected.

Using the first vector of the framing, we extend the a-sphere of the handle to an embedding $\phi : S^r \times I \to V$ such that $\phi(S^r \times \{0\})$ is the a-sphere and the rest of the image is disjoint from M. We next show that $\phi(S^r \times \{1\})$ is nullhomotopic in the complement $V \setminus M$ of M. Since M has codimension greater than 2, the complement is 1-connected.

We now use the hypothesis that $H_m(M, \partial M) \to H_m(V, \partial V)$ is surjective. It follows that $H_i(V, M \cup \partial V) = 0$ with possible exceptions $i = k$, $i = m + k - 1$, $r + 1 \le i \le m + 1 - r$. By the universal coefficient theorem, the same holds for $H^i(V, M \cup \partial V)$. By duality, $H_i(V \setminus M) = 0$ except perhaps for $i = m - 1$, $i = 0$, $k + r - 2 \le i \le m + k - r - 1$. Since $k \ge 2$, $V \setminus M$ is r-connected, so our r-sphere is indeed nullhomotopic in $V \setminus M$.

We can thus extend the map ϕ on a collar neighbourhood of the boundary to a map $\psi : (D^{r+1}, \partial D^{r+1}) \to (V, M)$ with $\psi^{-1}(M) = \partial D^{r+1}$.

The map ψ is covered by a stable normal framing of the handle. As in the proof of Theorem 7.1.1, this framing determines a regular homotopy class of immersions $D^{r+1} \times D^{m+k-r-2} \to V$. We wish the immersion to restrict to the given embedding $S^r \times D^{m-r} \to M$. Since $m \ge 2r$ and $k \ge 3$, we have $m + k - r - 2 \ge r + 1$. Thus $\pi_r(SO_{m+k-r-2})$ maps onto $\pi_r(SO)$, so the stable framing induces a normal framing. It follows by Theorem 6.2.1 that the map ψ is homotopic (relative to its boundary) to an immersion.

If $m + k - 1 > 2(r + 1)$ putting this map in general position makes it an embedding; in the critical case $m = 2r$ and $k = 3$, we can use the Whitney trick (see Theorem 6.3.4 but allow boundaries) to obtain an embedding. Now using the normal framing on this handle allows us to extend the embedding of M to the desired embedding of M with the handle. □

Inserting this result in the braid diagram (8.8.7) together with results in § B.3 (ix) on homotopy groups of spheres, (xiv) on homotopy groups of orthogonal groups and (xix) on $\pi_r(SO_k) \to \pi_r(SG_k)$, it follows that

Theorem 8.8.9 *All groups in the diagram*

$$\begin{tikzcd}
 & \pi_n(SO_k) \arrow[dr] \arrow[rr, bend left] & & \pi_n(SG_k) \arrow[dr] & \\
\pi_{n+1}(SG_k, SO_k) \arrow[ur] \arrow[dr] & & F\Theta^k_n \arrow[ur] \arrow[dr] & & \pi_n(SG_k, SO_k) \\
 & P^k_{n+1} \arrow[ur] \arrow[rr, bend right] & & \Theta^k_n \arrow[ur] &
\end{tikzcd}$$

for $k \geq 3$ *are finite* except *for*
$(A) : n = 4s + 1,\ k = 4s + 2$*:* $\pi(SO) \to F\Theta \to \pi(SG)$*, of rank 1,*

$(S) : n = 4s - 1,\ k > 2s + 1$*:* $\begin{array}{ccc} \pi(SG, SO) & \to & \pi(SO) \\ \downarrow & & \downarrow \\ P & \to & F\Theta \end{array}$ *of rank 1,*

$(O) : n = 4s - 1,\ k \leq 2s$*:* $P \to F\Theta \to \Theta$*, of rank 1,*
$(B) : n = 4s - 1,\ k = 4s$*: 'the direct sum of the diagrams (A) and (S)',*
$(C) : n = 4s - 1,\ k = 2s + 1$*: 'the direct sum of the diagrams (A) and (O)'.*

In particular, Θ^k_{4s-1} has rank 1 if $k \leq 2s + 1$, and otherwise Θ^k_n is finite.

We can use the above results to investigate groups of embeddings of spheres in spheres. Denote by Σ^k_m the set of diffeotopy classes of embeddings $S^m \to S^{m+k}$. Since by Lemma 2.5.11 orientation-preserving embeddings $(D^{m+k}, D^m) \to (S^{m+k}, S^m)$ are unique up to diffeotopy, we can define a connected sum of two embeddings by removing an embedded disc-pair from each, and glueing along the boundary (with an orientation reversal). It follows that Σ^k_m acquires the structure of a group.

Since diffeotopic embeddings are cobordant, there is a natural forgetful map $\sigma : \Sigma^k_m \to \Theta^k_m$. By Lemma 8.3.1 the map σ lies in an exact sequence $\ldots \to R^k_{m+1} \to \Sigma^k_m \to \Theta^k_m \to R^k_m \to \ldots$ The relative term R^k_{m+1} is the set of cobordism classes of homotopy discs $\Delta^{m+1} \subset D^{m+k+1}$ together with a diffeomorphism $S^m \to \partial\Delta^{m+1}$. It follows from Corollary 5.6.3 that for $m \geq 5$ Δ^{m+1} is diffeomorphic to D^{m+1}. If also $k \geq 3$, it now follows from Theorem 5.6.7 (i) that $\Delta^{m+1} \subset D^{m+k+1}$ is diffeomorphic to the standard pair. Thus R^k_{m+1} is the cobordism group of standard pairs together with a diffeomorphism of S^m on the boundary. The embedding now plays no part, thus for $m \geq 5$, $k \geq 3$ the map $R^k_{m+1} \to R_{m+1}$ is an isomorphism. Hence $R^k_{m+1} \cong R_{m+1} \cong \Theta_{m+1}$. This proves

Proposition 8.8.10 *There is an exact sequence* $\ldots \to \Theta_{m+1} \to \Sigma^k_m \to \Theta^k_m \to \Theta_m \to \ldots$

Since the groups Θ_m are all finite, it follows that the rank of Σ^k_m is the same as that of Θ^k_m: thus is 1 if $m = 4s - 1$ and $k \leq 2s + 1$, and zero otherwise.

It follows from the Whitney embedding theorem that $\Sigma^k_m = 0$ for k large. More precisely, by Theorem 6.4.11, any two embeddings of S^m in S^{m+k} are isotopic (and hence Σ^k_m vanishes) provided $2k > m + 3$. However in the limiting case $2k = m + 3$ the group does not vanish: if also k is odd, it is infinite by the above; more precisely, by [61], we have $\Sigma^{2s+1}_{4s-1} \cong \mathbb{Z}$. It was shown in [64] that in the other critical case, $\Sigma^{2s}_{4s-3} \cong \mathbb{Z}_2$.

8.9 Notes on Chapter 8

§8.1 The first result in this area is due to Pontrjagin [123], who succeeded in relating framed bordism to homotopy groups of spheres. Thom's paper [150], as well as formally introducing the construction, obtained a transversality theorem.

§8.2 I also believe that at least part of his motivation was the problem of representing homology classes by embedded submanifolds.

§8.3 I do not know where it was first observed that the definition of bordism naturally leads to exact sequences. The second technique was formally introduced in [158].

§8.4 In his paper [12], Atiyah introduced bordism as a homology theory, showed that smooth oriented manifolds are also orientable for this theory, and made applications to bordism groups.

There are other abstract structures using bordism. Graeme Segal defined in [134] axioms for quantum field theory, which we can summarise as follows. A cobordism category is a category with objects (diffeomorphism classes of) closed manifolds (of a given dimension) and morphisms (diffeotopy classes of) bordisms: to obtain interesting examples one usually imposes extra structure: for example, an embedded submanifold of codimension 2.

A 'topological field theory' is then a functor ϕ from such a category to, for example, the category of vector spaces over $\mathbb{C}$ and maps: it is required also to take disjoint unions to tensor products. Since the empty manifold is mapped to $\mathbb{C}$, if M is a closed manifold, and so a cobordism from the empty set to itself, $\phi(M)$ is a linear map $\mathbb{C} \to \mathbb{C}$: multiplication by a number, giving an invariant $\alpha(M) \in \mathbb{C}$. Non-trivial examples are not easy to construct.

§8.5 The main reference for this section is the book [38], which has a wealth of information about actions of finite cyclic groups. Chapter IV of that book contains the calculation of equivariant bordism groups of $\mathbb{Z}_2$-actions. The $\mathbb{Z}_p$-actions are discussed in Chapter VII: the results are, of course, not complete. However many geometrical consequences of their calculations are given throughout the book.

§8.6 In his original 1954 paper [150], as well as introducing transversality and using it to reduce the calculation of cobordism groups to a homotopy problem, Thom was able to give the full calculation of Ω_*^O, using Serre's calculation [135] of cohomology of Eilenberg–MacLane spaces, and to calculate $\Omega_*^{SO} \otimes \mathbb{Q}$ using Proposition B.4.1. Milnor's paper [96] followed in 1960 and Novikov's [113] appeared in 1962. Milnor's book [103] gives an alternative introduction to characteristic classes, the calculation of the cohomology of classifying spaces, cobordism and the calculation of cobordism rings, including the Hirzebruch signature theorem.

Unitary bordism has more structure than the calculation in Theorem 8.6.11 shows. One aspect of this is:

Theorem 8.9.1 *There is an isomorphism of the universal formal group over* $\mathbb{Z}$ *on* Ω_*^U.

We explain this statement. If T is a connected 1-dimensional analytic Lie group with multiplication $\mu : T \times T \to T$, and x a local coordinate at the unit, we can expand $\mu \circ x$ as a power series $F(x, y)$ with $F \in \mathbb{R}[[x, y]]$. The group properties are reflected in the identities

$$F(x, 0) = F(0, x) = x, \quad F(x, y) = F(y, x), \quad F(F(x, y), z) = F(x, F(y, z)).$$

One thus defines a formal group over a ring R as a formal power series in 2 variables $F(x, y)$, with constant term 0, satisfying these rules. The simplest examples are $F_r(x, y) = x + y + rxy$ for $r \in R$.

Now consider $P := P^\infty(\mathbb{C}) \cong B(U_1)$. There is a multiplication map $\mu : P \times P \to P$ induced, for example, by tensor product of line bundles.

Since P has a cell structure with one cell in each even dimension we can identify $\Omega_*^U(P)$ with $\Omega_*^U[[z]]$, with a generator $z \in \Omega_2^U(B(U_1))$ which can be taken as defined by the inclusion $P^1(\mathbb{C}) \subset P^\infty(\mathbb{C})$. Now $\mu^*(z) \in \Omega_*^U[[x, y]]$ defines a formal group.

For the proof of Theorem 8.9.1 we refer to Quillen [127]. This result is the jumping off point for the use of complex cobordism theory as a tool for elaborate calculations in homotopy theory. It is used to set up the so-called Adams–Novikov spectral sequence. One can localise Ω_*^U homology theory at a prime p; it then splits into the so-called BP-theories with much smaller coefficient group (polynomial with generators only in dimensions $p^r(2p - 2)$). We refer to [129] for an introduction to this area.

§8.7 Certain exact sequences were devised by the author [157] to relate $\Omega_*^{\mathbf{O}}$ and $\Omega_*^{\mathbf{SO}}$, as a means of calculating the latter. A more abstract proof was found by Atiyah [12] (who invented bordism theory for the purpose). My original insight was that the apparently complicated structure of Ω_*^{SO} might be the similar to the structure of $H^*(X; \mathbb{Z})$ for a space X such that each of $H^*(X : \mathbb{Q})$ and $H^*(X; \mathbb{Z}_2)$ is a polynomial ring.

The original exact sequences were extended by Conner and Floyd to the case of $\Omega_*^{\mathbf{U}}$ and $\Omega_*^{\mathbf{SU}}$, and used in the calculations of the latter, with details in [39].

Further calculations of Ω^{SU} were obtained by Anderson, Brown, and Peterson [9].

The groups Ω^{Spin} were calculated, also by Anderson, Brown, and Peterson, in [10]. They first determine the structure of $H^*(\mathbb{TS}pin; \mathbb{Z}_2)$ as a module over the Steenrod algebra: it is a sum of copies of $\mathcal{S}_2$, $\mathcal{S}_2/\mathcal{S}_2(Sq^3)$ and $\mathcal{S}_2/\mathcal{S}_2(Sq^1, Sq^2)$. They deduce that the Thom spectrum is homotopy equivalent to a wedge of spectra of type $\mathbb{K}(\mathbb{Z}_2, n)$ and $\mathbb{BO}\langle n\rangle$; and thence that cobordism class in Ω_*^{Spin} is determines by Stiefel–Whitney and *KO*-characteristic numbers.

Complete results are also available for $Spin^c$: here cobordism class is determined by Stiefel–Whitney numbers and characteristic numbers in $\mathbb{Q}$: calculations for this case can be reduced to those for *Spin* in view of the isomorphism $\Omega_n^{Spin^c} \cong \tilde{\Omega}_{n-2}^{Spin}(P^\infty(\mathbb{C}))$.

In addition to the original references, Stong's book [147] aims to give complete details of all the calculations involved in determining the cobordism groups mentioned above, and their interrelations with each other and with framed bordism.

For Ω_*^{Sp}, it was again shown in [113] that the tensor product by $\mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra. Extensive calculations have been made by Kochman [80].

§8.8 The sequences 8.8.4 were extracted from the methods introduced by Milnor and Kervaire [79] for calculating the groups Θ_n. Our account follows the presentation by Levine [85], which in turn combined the earlier work of Milnor and Kervaire (see, for example, [79]) with ideas of Haefliger [61].

Milnor's discovery [92] of non-diffeomorphic differential structures on the topological manifold S^7 was a great surprise: up to then, though smooth and piecewise linear (PL) structures were used, the philosophy was that one was really studying problems in pure topology. Likewise the existence of non-trivial embeddings of spheres in spheres contrasts with the theorem of Stallings [143] (in the topological category) and Zeeman [183] (in the piecewise linear category) that embeddings of spheres in spheres, in codimension at least 3, are topologically unknotted. It is thus possible to regard all the results about embeddings of spheres in spheres as a manifestation of smoothing theory.

Explicit results of this kind were obtained by Rourke and Sanderson. In the first of the three papers [130] they set out to construct a theory of neighbourhoods of locally flat submanifolds of PL manifolds to play the role in PL topology of the tubular neighbourhoods in differential topology. By introducing a notion of 'block bundles' they constructed a (simplicial) space $B\widetilde{PL}_k$ such that for any PL manifold M^m the set of isomorphism classes of regular neighbourhoods of M embedded locally flatly in PL $(m+k)$-manifolds maps bijectively to the homotopy set $[M : B\widetilde{PL}_k]$.

In the third paper, after defining various simplicial spaces, in particular a piecewise differentiable version $B\widetilde{PD}_k$ of $B\widetilde{PL}_k$ which is homotopy equivalent to it, they interpret the braid (8.8.7) as the homotopy braid coming from the inclusions $B(SO_k) \subset B\widetilde{SPL}_k \subset B(SG_k)$.

In the subsequent paper [131], Rourke and Sanderson construct a theory of neighbourhoods of locally flat submanifolds of topological manifolds. A starting point is the notion of microbundle introduced by Milnor [99]. Following a subsequent idea of Haefliger, they consider a microbundle with fibre dimension $(n+k)$ together with a submicrobundle with fibre dimension n. From these they form a (simplicial) classifying space $BTop^n_{n+k}$ and establish the existence of a (Kan) fibration $Top^n_{n+k} \to Top_n$, whose fibre is denoted $Top_{n+k,n}$. An $(n+k)$ dimensional neighbourhood of a manifold N^n induces a lift of $N \to BTop_n$ to a cross-section of the induced fibration.

They then establish that if $i \le k$ and either $n \le 2$ or $n+k \ge 5$ the map $\pi_i(Top_{n+k,k}) \to \pi_i(Top_n)$ is an isomorphism. It follows that with this dimension restriction, neighbourhoods of N are classified by maps $N \to BTop_k$. This leads to obstruction theories to the existence of normal microbundles or block bundles with fibre D^k or $\mathbb{R}^k$.

It also follows that the above results in the PL case carry over to the Top case. Thus one can identify $F\Theta^k_n$ with $\pi_n(STop_k)$, Θ^k_n with $\pi_n(STop_k, SO_k)$, and P^k_n with $\pi_n(SG_k, STop_k)$. Thus the stability theorem Proposition 8.8.8 establishes a homotopy pullback diagram

$$\begin{array}{ccc} STop_k & \to & SG_k \\ \downarrow & & \downarrow \\ STop & \to & SG \end{array},$$

and the exact sequence of Proposition 8.8.10 interprets Σ^k_n as the homotopy group of the diagram

$$\begin{array}{ccc} SO_k & \to & STop_k \\ \downarrow & & \downarrow \\ SO & \to & STop \end{array},$$

and hence of the diagram

$$\begin{array}{ccc} SO_k & \to & SG_k \\ \downarrow & & \downarrow \\ SO & \to & SG \end{array}.$$

This final result had been obtained by Haefliger in [64].