2

Geometrical tools

We can regard a compact smooth manifold as built up by glueing together smaller pieces, which are easier to analyse. In this chapter we begin the description of this process. After obtaining some basic results on Riemannian metrics, we study geodesics for such metrics. The key result is that any two nearby points are joined by a unique shortest geodesic. This leads us to study the way in which a closed submanifold lies in a manifold: we describe the structure of a neighbourhood of the submanifold as having the form of a tube.

A diffeotopy, or differentiable isotopy, can be considered either as deforming the embedding of one manifold in another or as an embedding of a product with I. If the deformation can be extended to the whole manifold, the two embeddings are equivalent. The diffeotopy extension theorem asserts that under certain conditions, this extension is possible; it may thus be looked on as a uniqueness theorem. We apply this result to obtain a uniqueness theorem for tubular neighbourhoods, which enables us to pass from knowledge of the structure of a compact submanifold M of a manifold N to knowledge of a neighbourhood of M: the only extra piece of information needed is the structure of the normal bundle $\mathbb{N}(N/M)$. This contributes to the general aim of building up global results from merely local ones.

We define inverse procedures for straightening a corner, to yield a manifold with boundary, and for introducing corners: it will be useful in Chapter 5 to be able to effectively ignore corners.

Finally we discuss glueing and the inverse process of cutting: these are simple geometrical constructions which, given some smooth manifolds (perhaps with boundaries and corners) and additional data where necessary, give rise to new manifolds. On account of their perspicuity, these methods are traditional in describing the topology of surfaces, and they remain a very powerful tool in higher dimensions.

2.1 Riemannian metrics

We recall that if M^m is a smooth manifold, the bundle over M associated to the tangent bundle and whose fibre over P is the set of all positive definite quadratic forms on T_PM is called the Riemann bundle, and any cross-section of it a Riemannian structure on M; in local coordinates this takes the form $\sum_{i,j=1}^m g_{i,j}(x)dx_i dx_j$.

We saw in Theorem 1.3.1 that every smooth manifold M^m has a Riemannian structure. Such a structure induces an inner product on each T_PM, which we use to introduce the notion of length of tangent vectors. A (smooth) *path* in M is a smooth map p to M with source $\mathbb{R}$ or an interval contained in $\mathbb{R}$. For a path p, we define the *length* of p between two of its points by

$$l(p) = \int_a^b \frac{ds}{dt} dt,$$

where $(ds/dt)^2 = \sum_{i,j} g_{i,j}(dx_i/dt)(dx_j/dt)^2$, the derivatives being taken along the path. We set

$$\rho(P, Q) = \inf\{l(p) : p \text{ a path joining } P \text{ to } Q\};$$

this is defined if and only if P, Q are in the same component of M. We could also, for example, define $\rho(P, Q) = 1$ whenever P and Q are in different components, but the case of interest is when M is connected.

We call ρ the *Riemannian metric*: we now show that it *is* a metric.

Theorem 2.1.1 *The function ρ defines a metric on M which induces the given topology on M.*

Proof The triangle inequality follows since, as in Lemma 1.1.8, we can (up to re-parametrising, which does not alter length) combine smooth paths from P to Q and from Q to R to give a smooth path from P to R. That $\rho(P, Q) = 0$ implies $P = Q$ follows from the argument below.

To show that the metric induces the given topology, we need to establish that, for any point $P \in M$,
(i) any neighbourhood of P in M contains $\{Q \in M \mid \rho(P, Q) < A\}$ for some A,
(ii) any such set is a neighbourhood of P.

Choose a coordinate neighbourhood $\varphi : U \to \mathbb{R}^m$ with $\varphi(P) = O$. By a linear change of coordinates in $\mathbb{R}^m$, we can reduce the matrix $\left(g_{i,j}(P)\right)$ to the identity, so at P the metric ds^2 agrees with the Euclidean metric $\sum_1^n dx_i^2$. Hence there is a neighbourhood of P on which the ratio is bounded:

$$\frac{1}{2}\sum_{1}^{n} dx_i^2 \leq \sum_{i,j} g_{i,j}(x)dx_i dx_j \leq 2\sum_{1}^{n} dx_i^2$$

for $\|x\| < A$, say.

Thus if p is a path in M with $\varphi(p) \subset \mathring{D}^n(A)$, and $l(\varphi(p))$ denotes the length of $\varphi(p)$ in the Euclidean metric, $\frac{1}{2}l(\varphi(p)) \leq l(p) \leq 2l(\varphi(p))$.

Now (ii) follows since, if $B \leq A$, then for any $Q = \varphi^{-1}(x)$ with $\|x\| < \frac{B}{2}$, taking the path p_3 such that $\varphi(p_3)$ is the straight segment from O to x gives

$$\rho(P, Q) \leq l(p_3) \leq 2l(\varphi(p_3)) < B,$$

so the set $\{Q \mid \rho(P, Q) < B\}$ contains the neighbourhood $\varphi^{-1}\{\mathring{D}^m(\frac{1}{2}B)\}$.

As to (i), first note that if $\varphi(Q) = x$ with $\|x\| < \frac{A}{2}$, and p_1 is a path from Q with $\varphi(p_1)$ leaving $\mathring{D}^m(A)$, then $l(\varphi(p_1)) \geq \frac{A}{2}$, hence $l(p_1) \geq \frac{A}{4}$. Thus for any path p_2 from P to Q with $\varphi(p_2)$ leaving $\mathring{D}^m(A)$, we have $l(p_2) \geq \frac{A}{2}$.

Now for any $B < \frac{A}{4}$, since any path p from P with $l(p) < B$ is contained in $\varphi^{-1}\{\mathring{D}^m(A)\}$, it follows that $D := \{Q \in M \mid \rho(P, Q) < B\}$ is also contained in this region; and now since we need only consider paths p in this region, and $l(\varphi(p)) < 2l(p)$, D is contained in $\varphi^{-1}\{\mathring{D}^m(2B)\}$. □

The basic results about Riemannian metrics: existence of a Riemannian structure, and the definition and properties of a metric: apply without essential change also to manifolds with boundary.

Next let V^v be a submanifold of a smooth manifold M^m. If $P \in V$, the inclusion $i : V \to M$ induces $di : T_P V \to T_P M$ of rank v, hence the dual map $di^* : T_P^\vee M \to T_P^\vee V$ also has rank v, and its kernel has rank $(m - v)$.

The kernel of $di^* : T_P^\vee M \to T_P^\vee V$ is called the *normal space* to V in M at P; we will denote it by $N_P(M/V)$. The union of these normal spaces is the *normal bundle* $\mathbb{N}(M/V)$ of V in M. We must check that the normal bundle is indeed a vector bundle over V. Let $\varphi : U \to \mathbb{R}^m$ be a coordinate neighbourhood of P in M with $U \cap V = \varphi^{-1}(\mathbb{R}^v)$; then in $U \cap V$ we may take $dx_{v+1}, \ldots, dx_m$ as a basis for the normal space. These give the local product maps φ_α required of a fibre bundle; as with the tangent bundle, the maps $g_{\alpha\beta}$ come from Jacobians on change of coordinates.

A Riemannian structure on M induces one on V. The distinction between $T_P^\vee M$ and $T_P M$ disappears, and in this case we can regard $\mathbb{N}(M/V)$ as a subbundle of the restriction $\mathbb{T}(M)|V$ of $\mathbb{T}(M)$ to V.

Proposition 2.1.2 *$\mathbb{T}(M)|V$ is the Whitney sum of $\mathbb{N}(M/V)$ and $\mathbb{T}(V)$,* *Proof* Since all the above bundles are defined, and the latter two are subbundles of the first, it is sufficient to verify that at each point the fibre of the first is the direct sum of the latter two. Since we have a positive definite inner product, it will be sufficient to verify that the fibre $N_p(M/V)$ of $\mathbb{N}(M/V)$ over P is the orthogonal complement of the fibre $T_P V$ of $\mathbb{T}(V)$ in the fibre $T_P M$ of $\mathbb{T}(M)$, or that it is the annihilator of $T_P V$ in $T_P^\vee M$. But since di^* is dual to di, the kernel of di^* is certainly the annihilator of the image of di. □

We say that a submanifold V of M meets ∂M *orthogonally* if the normal vectors to V and ∂M at each point of ∂V are perpendicular.

Lemma 2.1.3 *Let M be a manifold with boundary, V a submanifold. Then M has a Riemannian metric in which V meets ∂M orthogonally.*

Proof We construct a metric just as in Theorem 1.3.1; the only point to watch is that V meets ∂M orthogonally in each of the partial metrics to be fitted together. But since V is a submanifold, at a point of ∂V, there is a coordinate map of an open set of (M, V) to $(\mathbb{R}^n_+, \mathbb{R}^m_+)$, and the Euclidean metric will do. Now when we fit these together, V continues to meet ∂M orthogonally. □

2.2 Geodesics

For a connected manifold M^m with a Riemannian structure, we have already defined the length of a path and the distance function as the infimum of lengths of paths, and shown in Theorem 2.1.1 that the infimum $\rho(P, Q)$ of lengths of paths joining P to Q is a metric defining the topology on M.

We now focus attention on the paths minimising this distance. Recall that the length of a path $p : U \to M$ (U open in $\mathbb{R}$) between two of its points is defined by $l(p) := \int_a^b \frac{ds}{dt} dt$, where $(ds/dt)^2 = \sum_{i,j} g_{i,j}(dx_i/dt)(dx_j/dt)^2$, the derivatives being taken along the path. We now define the *energy* of p by

$$E(p) := (b-a)\int_a^b \left(\frac{ds}{dt}\right)^2 dt.$$

Then a *geodesic* is defined to be a smooth path $p : U \to M$ giving an extremal value to the energy between any two of its points.

By Schwarz' inequality,

$$l(p)^2 = \left(\int_a^b \frac{ds}{dt} dt\right)^2 \leq \int_a^b dt \int_a^b \left(\frac{ds}{dt}\right)^2 dt = (b-a)\int_a^b \left(\frac{ds}{dt}\right)^2 dt = E(p),$$

with equality if and only if ds/dt is constant, so that the curve is parametrised proportionately to arc length. Since any curve can be parametrised by arc length, the geodesic gives an extremal value also to the length of the path.

Proposition 2.2.1 *In local coordinates, geodesics are defined by equations*

$$\frac{d^2x_i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx_j}{dt}\frac{dx_k}{dt} = 0.$$

Proof Euler's equation for the variational problem of minimising the integral of $G := \sum_{j,k} g_{jk} \frac{dx_j}{dt}\frac{dx_k}{dt}$ is $\frac{\partial G}{\partial x_r} = \frac{d}{dt}\left(\frac{\partial G}{\partial y_r}\right)$, where $y_r = \frac{dx_r}{dt}$. This gives

$$\begin{aligned}\sum_{j,k} \frac{\partial g_{jk}}{\partial x_r}\frac{dx_j}{dt}\frac{dx_k}{dt} &= \frac{d}{dt}\left(2\sum_j g_{rj}\frac{dx_j}{dt}\right)\\ &= 2g_{rj}\frac{d^2x_j}{dt^2} + 2\frac{\partial g_{rj}}{\partial x_k}\frac{dx_j}{dt}\frac{dx_k}{dt}\\ &= 2g_{rj}\frac{d^2x_j}{dt^2} + \frac{dx_j}{dt}\frac{dx_k}{dt}\left(\frac{\partial g_{rj}}{\partial x_k} + \frac{\partial g_{rk}}{\partial x_j}\right),\end{aligned}$$

where in the last step we use symmetry under the interchange of j and k. If g^{ij} is the inverse matrix to $g_{i,j}$, multiply by g^{ir}, sum over r and simplify:

$$\frac{d^2x_i}{dt^2} + \frac{1}{2}\sum_r g^{ir}\left(\frac{\partial g_{rj}}{\partial x_k} + \frac{\partial g_{rk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_r}\right)\frac{dx_j}{dt}\frac{dx_k}{dt} = 0.$$

The coefficient of the last term is usually abbreviated to Γ^i_{jk}. □

Theorem 2.2.2 *For any point $Q \in M$, we can find a neighbourhood V of Q in M and an $\varepsilon > 0$ such that for any $P \in V$, and $\upsilon \in T_P(M)$ with $\|\upsilon\| < \varepsilon$, there is a unique geodesic $p(t)$ with*

$$p(0) = P, \qquad \left.\frac{d}{dt}p(t)\right|_{t=0} = \upsilon.$$

This is defined for $|t| < 2$, stays in V, and depends smoothly on p, υ, t.

Proof We take a coordinate neighbourhood φ on M at Q mapping onto $\mathring{D}^m(3)$ and apply the Existence Theorem for Ordinary Differential Equations (Theorem 1.4.1). Consider the system

$$\left.\begin{aligned} dx_i/dt &= y_i\\ dy_i/dt &= \Gamma^i_{jk}(x)y_jy_k\end{aligned}\right\},$$

where $x \in \mathring{D}^m(3)$, $\|y\| < 3$ corresponds to the U of that theorem, and $x \in D^m(2)$, $\|y\| \le 2$ to its K. Then for some $\varepsilon > 0$, we find a unique solution $x = f(x_0, y_0, t)$ for all $\|x_0\| \le 2$, $\|y_0\| < 2$, $|t| < \varepsilon$ depending smoothly on all its arguments, and lying in $\|x\| < 3$. Lifting to V by φ^{-1}, this gives a geodesic in M.

To deduce the theorem, we need only change parameter by $t' = \frac{2}{\varepsilon}t$; this has the effect of multiplying the initial $\frac{d}{dt}p(t)$ by the inverse factor, and so altering the condition $\|v\| \le 2$ to $\|v\| \le \varepsilon$. □

It is worth emphasising that though the argument involved defining a flow in the tangent bundle $\mathbb{T}(M)$, the geodesic itself is a path in M.

As for flows in general, the local existence and uniqueness of geodesics given by Theorem 2.2.2 does not imply global existence, but does imply uniqueness in the whole range of existence (by applying the result to a sequence of points along the geodesic), given the initial point and direction.

Let $P \in M$, $v \in T_PM$, and suppose that the geodesic with direction v at P can be defined for $|t| \le 1$. Then we write $\exp(P, v)$ for the point at $|t| = 1$ on the geodesic, and call exp the *exponential map*. We also define the map Exp from a subset of $\mathbb{T}(M)$ to $M \times M$ by $\mathrm{Exp}(P, v) = (P, \exp(P, v))$. We have shown that these maps are defined on a neighbourhood $\mathcal{V}$ of $\mathbb{T}^0(M)$ in $\mathbb{T}(M)$.

A submanifold $V \subset M$ is called *totally geodesic* if each geodesic in M tangent to V is contained in V. Thus a one-dimensional submanifold is totally geodesic if and only if it is a geodesic.

We now obtain further properties of the exponential map.

Proposition 2.2.3 *The Jacobian determinant of* Exp *is non-zero on* $\mathbb{T}^0(M)$.

Proof For $P \in M$, let $\varphi : U \to \mathbb{R}^m$ be a coordinate neighbourhood, and choose $x_1, \ldots, x_m$ as coordinates in M, $dx_1, \ldots, dx_m$ as coordinates in the fibres T_PM; write the latter as $v_1, \ldots, v_m$, and write coordinates in $M \times M$ as $x_1, \ldots, x_m$, $z_1, \ldots, z_m$. Then we have $\mathrm{Exp}(x, v) = (x, z)$, so it remains to compute the partial derivatives of the z_i at 0. Now z is the point at $t = 1$ on the solution of the equation $\frac{dz}{dt} = y$ with initial condition $z = x$, $y = v_0$, i.e. at the point t_0 on the solution with initial condition $z = x$, $y = v_0/t_0 = v$. Hence

$$z = x + t_0 v + \text{smaller terms, where } t_0 \text{ is small, } v \text{ fixed,}$$

and so to find $\frac{\partial z_i}{\partial v_j}$, set $(v_0)_i = t_0\delta_{ij}$; then

$$\frac{\partial z_i}{\partial v_j} = \left.\frac{\partial z_i(v_0)}{\partial t_0}\right|_{t_0=0} = \delta_{ij}.$$

This proves the result: for later reference note also that $\frac{\partial z_i}{\partial x_j} = \delta_{ij}$. □

It follows from Proposition 2.2.3 and the Inverse Function Theorem 1.2.5 that $\mathbb{T}^0(M)$ has a neighbourhood $\mathcal{V}'$ in $\mathbb{T}(M)$ on which Exp is defined, and is a local diffeomorphism. It now follows using Corollary A.2.6 that $\mathbb{T}^0(M)$ has a neighbourhood $\mathcal{V}''$ in $\mathbb{T}(M)$ on which Exp is defined, and is a diffeomorphism.

We have an even sharper statement.

Theorem 2.2.4 *There is a neighbourhood W of $\Delta(M)$ in $M \times M$ such that if $(x, y) \in W$, there is a unique geodesic from x to y of length $\rho(x, y)$. Hence* Exp *defines a diffeomorphism of* $\mathrm{Exp}^{-1}(W)$ *onto W.*

Proof For each $P \in M$, it follows from the above that we can find a neighbourhood V_P of P such that Exp^{-1} defines a diffeomorphism of $V_P \times V_P$ on a neighbourhood of $\mathbb{T}^0(V_P)$. Then if U_P is a sufficiently small neighbourhood of P, each pair of points in U_P is joined by a unique geodesic lying in U_P, and (as in the proof of Theorem 2.1.1) each geodesic going outside U_P is longer. Thus this geodesic gives a minimum length for curves in U_P joining the two points. (In the technical language of Calculus of Variations, the metric is positive definite, the problem is regular, and we have constructed a semi-field of extremals, passing through a point and covering a neighbourhood.)

The geodesic gives the global minimum, which we defined as the distance $\rho(x, y)$. Thus Exp^{-1} is a diffeomorphism on $U_P \times U_P$: we take W as the union of such neighbourhoods. □

This has the following useful application.

Corollary 2.2.5 *There exist a neighbourhood W of $\Delta(M)$ in $M \times M$ and a C^∞ map $H : W \times [0, 1] \to M$ such that for each $(P, Q) \in W$, $H(P, Q, 0) = P$ and $H(P, Q, 1 - t) = H(Q, P, t)$.*

Proof Take W as given by the theorem. Then for each $(P, Q) \in W$ there is a unique geodesic $g_{P,Q} : [0, \rho(P, Q)] \to M$ with $g_{P,Q}(0) = P$ and $g_{P,Q}(1) = Q$. We can thus take $H(P, Q, t) = g_{P,Q}(t.\rho(P, Q))$. □

We will need a variant of this below (for Proposition 6.4.4).

Proposition 2.2.6 *For M a smooth manifold, the map $e_M : \mathbb{T}(M) \to M \times M$ given by $e_M(\xi) = (exp(\xi), exp(-\xi))$ is a local diffeomorphism along $\Delta(M)$ and there exist neighbourhoods A_M of $\mathbb{T}^0(M)$ in $\mathbb{T}(M)$ and O_M of $\Delta(M)$ in $M \times M$ such that e_M gives a diffeomorphism of A_M on O_M.*

For it follows from the proof of Proposition 2.2.3 that, in the natural local coordinates, the differential of e_M takes the form $(x, v) \mapsto (x + v, x - v)$, so is an isomorphism. The conclusion now follows as above.

In the region where geodesics are unique, the distance function also has the expected properties.

Proposition 2.2.7 *On the set W of Theorem 2.2.4, the square $\rho(x, y)^2$ of the distance is a smooth function.*

Proof In view of Theorem 2.2.4, it suffices to show that taking the square of the length of the geodesic defines a smooth function on a neighbourhood of $\mathbb{T}^0(M)$ in $\mathbb{T}(M)$. But this function is just the square of the length of the tangent vector in question, so is a smooth function since the Riemannian structure is smooth. □

We recall that a metric space is complete if each Cauchy sequence of points converges to a limit point, or equivalently, if each bounded closed subset is compact. With this concept, we can give the global forms of the above theorems.

Theorem 2.2.8 *M is complete if and only if geodesics may be indefinitely produced, i.e. if* exp *and* Exp *are definable on $\mathbb{T}(M)$. Any two points in a complete manifold may be joined by geodesics: the length of at least one such is the distance between them.*

Proof Suppose first M is complete, and $p(t)$ a geodesic which exists only for $t < k$. Then the points $p(t - \frac{1}{n})$ form a Cauchy sequence: since M is complete, these have a limit point P. But by Theorem 2.2.2, P has a compact neighbourhood K such that any geodesic within K may be produced a distance ε. This gives a contradiction.

Now suppose exp globally definable, but that there are pairs of points (P, Q) not joined by a geodesic of length $\rho(P, Q)$. Let r be the greatest lower bound of the distance of such points Q from P (by Theorem 2.2.4, $r > 0$), let $K_1 = \{v \in T_P M \mid \|v\| \leq r\}$, and let $K = \exp(K_1)$. Then K_1 is compact, hence so is K, by definition of r, K contains all points at distance less than r from P. Choose $2\varepsilon < r$ as the number ε in Theorem 2.2.2, and choose Q such that $\rho(P, Q) = r_0 < r + \varepsilon$, but P and Q are not joined by a geodesic of length $\rho(P, Q)$. Now let P_i be a smooth path from P to Q of length at most $r_0 + 1/i$, and let R_i be the point on it at distance $r - \varepsilon$ from P. The R_i lie in the compact set K; let R be a cluster point. Then

$$\rho(P, R) \leq \limsup \rho(P, R_i) = r - \varepsilon,$$
$$\rho(R, Q) \leq \limsup \rho(R_i, Q) = r_0 - r + \varepsilon,$$

so by the triangle inequality we have

$$\rho(P, R) = r - \varepsilon, \quad \rho(R, Q) = r_0 - r + \varepsilon.$$

By the definition of r, ε; P can be joined to R by a geodesic of length $r - \varepsilon$; R to Q by on of length $r_0 - r + \varepsilon$. If these met at an angle at Q, we could construct a shorter path by rounding the corner in a neighbourhood of Q. Hence

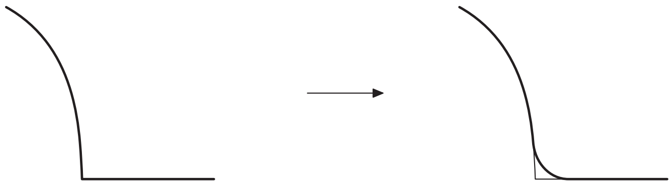

Figure 2.1 Rounding the corner of a path

they have the same direction at Q, so by the uniqueness theorem form part of the same geodesic. Thus P is joined to Q by a geodesic of length $\rho(R, Q)$: a contradiction. The idea of this proof is sketched in Figure 2.1.

Finally, suppose $\exp(T_P M) = M$. Then a bounded set lies within a finite distance from P, so is contained in the image of a closed and bounded, hence compact, subset of $T_P M$. But the image of this set is also compact, so it follows that M is complete. □

Theorem 2.2.9 *Any connected manifold has a Riemannian metric in which it is complete.*

Proof We make a slight refinement of the proof of Theorem 1.3.1, asserting the existence of Riemannian structures. Let $\varphi_\alpha : U_\alpha \to \mathring{D}^m(3)$ be the coordinate neighbourhoods constructed in Theorem 1.1.4, and define $\Phi_\alpha \in \mathcal{F}_i$ by

$$\Phi_\alpha(P) = \begin{cases} Bp(2\frac{1}{2} - \|x\|) & \text{if } P \in U_\alpha, \varphi_\alpha(P) = x \\ 0 & \text{if } P \notin U_\alpha. \end{cases}$$

Then write $ds^2 = \sum \Phi_\alpha(\sum dx_i^2) \circ \varphi_\alpha$. As in the earlier proof, we see that this is a metric. In $\varphi_\alpha^{-1}(\mathring{D}^m(1\frac{1}{2}))$, it is greater than or equal to the Euclidean metric, so the set of points at distance $\leq \frac{1}{3}$ from $\varphi_\alpha^{-1}(D^m)$ is a closed subset of $\varphi_\alpha^{-1}(D^m(2))$, so is compact. As in Theorem 2.2.8, it follows that all geodesics from a point of $\varphi_\alpha^{-1}(D^m)$, and hence from any point of M, may be produced a distance at least $\frac{1}{3}$ from any point. Thus they can all be produced indefinitely. □

Corollary 2.2.10 *(i) For any smooth manifold V, there is a proper map $V \to \mathbb{R}_+$.*

(ii) If M is non-compact, there is a proper map $V \to M$.

Proof (i) Choose a complete Riemannian metric on V; then for any $P_0 \in V$, the distance from P_0 is a proper map $\rho(P_0, -) : V \to \mathbb{R}_+$. For we saw above that the preimage of any set $[0, K]$ is compact. The square $\rho(P_0, -)^2$ is also proper, and is smooth.

Since the composite of two proper maps is proper, (ii) will follow if we can construct a proper map $\mathbb{R}_+ \to M$. Choose a non-compact component M_0 of

M and a point $Q_0 \in M_0$. Suppose inductively chosen $Q_i \in M_i$: then remove $\{P \in M_i \mid \rho(P, Q_0) < i\}$ from M_i, let M_{i+1} be a non-compact component of the complement, and choose any $Q_{i+1} \in M_{i+1}$.

Since $Q_i,\ Q_{i+1}$ lie in the connected set M_i, they can be joined by a path $[i, i+1] \to M_i$. Joining all these paths gives a map $\varphi : \mathbb{R}_+ = [0, \infty) \to M$. Since, for any $P \in M_i$, $\rho(P, Q_0) \geq i - 1$, the map φ is proper. □

2.3 Tubular neighbourhoods

We will now apply the results of §2.2 in the context of a submanifold V^v of M^m. Then we proceed to consider boundaries.

Proposition 2.3.1 *The Jacobian determinant of* $\exp : \mathbb{N}(M/V) \to M$ *on* $\mathbb{T}^0(V)$ *is non-zero.*

Proof Let $P \in V$, and let $\varphi : U \to \mathbb{R}^n$ be a coordinate neighbourhood of P in M such that $U \cap V = \varphi^{-1}(\mathbb{R}^m)$. Then if $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$, we can take as local coordinates in $\mathbb{N}(M/V)$ $x_1, \ldots, x_m$ (coordinates in V) and $v_{m+1}, \ldots, v_n$ (coordinates in the fibre) where $v_i = dx_i$. Now refer back to Proposition 2.2.3, where we showed that if $\exp(x, v) = z$, then $\frac{\partial z_i}{\partial x_j} = \frac{\partial z_i}{\partial v_j} = \delta_{ij}$ so that with respect to our coordinates, the Jacobian matrix is the unit matrix, so its determinant is non-zero. □

Theorem 2.3.2 *Let V be a submanifold of M. Then*

(i) the map $\exp : \mathbb{N}(M/V) \to M$ *is a local diffeomorphism at* $\mathbb{T}^0(V)$*,*

(ii) there is a neighbourhood of $\mathbb{T}^0(V)$ *in* $\mathbb{N}(M/V)$ *on which* exp *is a diffeomorphism to a neighbourhood U of V in M,*

(iii) V has a neighbourhood U' in M such that each point P of U is joined to V by a unique geodesic of length $\rho(P, V)$; this meets V orthogonally.

Proof (i) follows from Proposition 2.3.1 and the Inverse Function Theorem 1.2.5.

(ii) follows from this by applying Corollary A.2.6.

(iii) Let $Q \in V$, and let $U_1 \subset U_0$ be neighbourhoods of Q in M as in the proof of Theorem 2.2.4: any two points in U_0 are joined by a unique geodesic of minimal length, and the minimal geodesic joining two points of U_1 lies in U_0. We may suppose $\bar{U}_0$ compact.

For $P \in U_1$, let r_P be the greatest lower bound of distances of P from points of V. If we have points $Q_i \in V$ with $\rho(P, Q_i) < \frac{1}{i}$, then for $i > D^{-1}$ we have $Q_i \in U_0$, and since $\bar{U}_0$ is compact, the points Q_i have a cluster point Q; since V is closed, we have $Q \in V$, and now $\rho(P, Q) = r_P$. By the above choice of U_0, P and Q are joined by a unique geodesic of minimal length. This meets V orthogonally for if not, by a small modification near Q, we could make it shorter (take a path orthogonal to V, and smooth off the corner), giving a shorter path from P to V. Hence there is a point R' of $\mathbb{N}(M/V)$ lying over Q with $\exp(R') = P$.

We may now take U' as the union of the U_1. □

Taking the intersection $U \cap U'$ gives a neighbourhood of V on which both exp is a diffeomorphism and the geodesics give shortest distances from V.

For V^v a closed submanifold of a smooth manifold M^m, a *tubular neighbourhood* of V in M consists of a bundle B over V with fibre the disc D^{n-m} and an embedding $\psi : B \to M$ (as submanifold with boundary) extending the map taking the centre of each disc to the corresponding point of V.

As with coordinate neighbourhoods, the actual neighbourhood $\psi(B)$ is the more geometrical concept; but the mapping ψ is more convenient to work with. A tubular neighbourhood is pictured in Figure 2.2.

For any tubular neighbourhood, the map ψ induces an isomorphism of the normal bundle of V in M with that in B, and hence with the vector bundle associated to B. If M^m has a Riemannian structure, the normal bundle $\mathbb{N}(M/V)$ has group O_{m-v}. We may then take B as the associated disc bundle, consisting of vectors of $\mathbb{N}(M/V)$ of at most unit length.

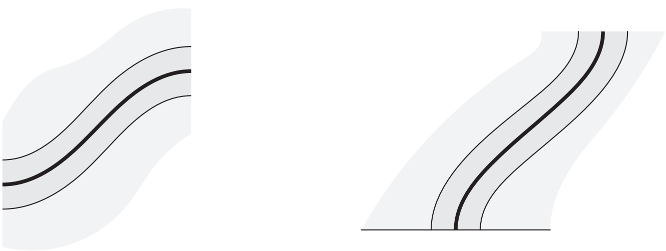

Figure 2.2 Tubular neighbourhood in a manifold and in one with boundary

Theorem 2.3.3 *For any submanifold V of a smooth manifold M, there exists a tubular neighbourhood of V in M.*

Proof Choose a Riemannian metric on M. Let W be a neighbourhood of $\mathbb{T}^0(V)$ in $\mathbb{N}(M/V)$ mapped diffeomorphically by exp: the existence of such W is guaranteed by Theorem 2.3.2. Let f be a positive continuous function on V such that vectors in $N_P(M/V)$ of length less than $f(P)$, are contained in W: the existence of such f follows from Lemma A.2.4 (i). By Proposition 1.1.7, we can find a positive smooth function g on V such that $0 < g(P) < f(P)$ for all $P \in V$. We now define a diffeomorphism ψ. For each $P \in V$, $v \in N_P(M/V)$, set

$$\psi(P, V) = \exp(P, g(P)v).$$

Multiplication by $g(P)$ in the fibre is possible since $g(P) \neq 0$, and for $\|v\| \leq 1$ we have $|g(P)v| \leq g(P) < f(P)$, so $(P, g(P)v) \in W$. $\square$

We will extend this result to the case of manifolds with boundary, but need first to develop further ideas.

We now combine Whitney's embedding theorem with the existence of tubular neighbourhoods to give a general method of constructing maps into smooth manifolds. We illustrate by showing the existence of smooth approximations, extending Lemma 1.1.7.

Let V be a compact manifold. By Theorem 1.2.11, there exists a smooth embedding $i : V \to \mathbb{R}^N$ for some N. By Theorem 2.3.3 there exist a disc bundle $\pi : W^N \to V$ and a smooth embedding $\psi : W \to \mathbb{R}^N$, extending i, and whose image is a neighbourhood U of $i(V)$. Further, we can choose the discs to have radius ε; U is then a ε-neighbourhood of $i(V)$. We have a retraction $\phi := \pi \circ \psi^{-1} : U \to V$; for each $x \in V$, the preimage $\phi^{-1}(x)$ is a disc of radius ε.

Proposition 2.3.4 *Let M and V be smooth manifolds with $V \subset \mathbb{R}^N$ compact.*

(i) For any continuous $f : M \to V$ and any $\varepsilon > 0$ there exists a smooth $h : M \to V$ with $\|h(x) - f(x)\| < \varepsilon$ for every $x \in M$.

(ii) If moreover F is a closed subset of M such that f is smooth on some open set $U \supset F$, we can find h such that also $h = f$ on a neighbourhood of F.

Proof Choose a tubular neighbourhood of V in $\mathbb{R}^N$ as above. Applying Proposition 1.1.7 to each component of $M \xrightarrow{f} V \subset \mathbb{R}^N$ gives a smooth map $h : M \to \mathbb{R}^N$ within distance ε of f, and hence with image contained in U. Thus $\phi \circ h$ gives a map $M \to V$, and since ϕ moves each point within a disc of radius $< \varepsilon$, h is within ε of f.

The same argument, but using (iii) of Proposition 1.1.7, gives (ii). $\square$

For N a smooth manifold with boundary, the discussion of geodesics at non-boundary points is the same as before. At a boundary point P, we see from the differential equations that local geodesics can be constructed for all inward-pointing tangent vectors and for no outward-pointing ones. There are several possibilities for those tangent to the boundary; as examples, the reader may consider D^2 and the closure of $\mathbb{R}^2 \setminus D^2$, each with the metric induced from $\mathbb{R}^2$.

A Riemannian metric on M is *adapted to the boundary* if ∂M is totally geodesic.

Lemma 2.3.5 *Let M^m have a Riemannian metric. Then the product metric for $M \times \mathbb{R}^1_+$ is adapted to the boundary.*

Proof Let $x_1, \ldots, x_m$ be local coordinates in M, and x_0 the coordinate in $\mathbb{R}^1_+$. Then for the metric $g_{i,j}$ we have $g_{0j} = \delta_{0j}$. Hence one of the defining equations for geodesics is simply $d^2x_0/dt^2 = 0$. Thus if initially $x_0 = dx_0/dt = 0$, we have $x_0 = 0$ all along the geodesic, which thus stays in $M \times \{0\}$. □

A similar argument gives the following.

Lemma 2.3.6 *If $V \subset M$ is a submanifold whose normal bundle is trivial, then M has a Riemannian metric in which the submanifold V is totally geodesic.*

Proof It follows from Theorem 2.3.3 that V has a neighbourhood in M diffeomorphic to $V \times \mathbb{R}^c$, where c is the codimension of V in M. We may choose any metric on V and then take the product metric on $V \times \mathbb{R}^c$: in any coordinate neighbourhood of V with metric $ds^2 = \sum g_{i,j} dx_i dx_j$ this is given by $ds'^2 = \sum g_{i,j} dx_i dx_j + \sum dy_k^2$. A short calculation shows that any geodesic initially tangent to $V \times \{0\}$ remains in this submanifold.

As in the proof of Theorem 1.3.1, we can now construct a metric on M which agrees with this metric on some neighbourhood of V in M. The result follows. □

Proposition 2.3.7 *(i) Every manifold M^m with boundary has a Riemannian metric adapted to the boundary.*

(ii) Given a submanifold V^v of M^m, there is a metric on M such that V meets ∂M orthogonally, and the restriction of the metric to V is adapted to the boundary.

Proof (i) By Theorem 1.5.5, ∂M has a collar neighbourhood $\psi : \partial M \times I \to M$. Let φ be a metric on M, φ' the product of some metric on ∂M with the standard metric of I. We define a metric φ'' by

$$\varphi'' = \begin{cases} \varphi & \text{outside the image of } \psi \\ \varphi' + (\varphi - \varphi')Bp(3t-1) & \text{at } \psi(P, t). \end{cases}$$

The latter agrees with φ in a neighbourhood of $t = 1$, so is smooth everywhere; it is a Riemannian structure, as a positive linear combination of positive form is another, and it agrees with φ' near $t = 0$, so by Lemma 2.3.5, it is adapted to ∂M.

(ii) By Proposition 1.5.6(ii), we may suppose that the restriction to $\partial V \times I$ of the collar neighbourhood of ∂M gives a collar neighbourhood for ∂V. Then the metric constructed above has both the desired properties. □

The definition of tubular neighbourhood of a closed submanifold V^v of a manifold M^m with boundary is the same as before: we require a bundle B over

V with fibre the disc D^{n-m} and an embedding $\psi : B \to M$ (as submanifold with boundary) extending the map taking the centre of each disc to the corresponding point of V.

If $\pi : B \to V$ is the projection of a disc bundle, Σ the boundary sphere-bundle of B, and $C = \pi^{-1}(\partial V)$, then B has the structure of a smooth manifold with corner, and $\angle B = \Sigma \cap C$ separates ∂B into two parts, with closures Σ and C. It follows that if (B, ψ) is a tubular neighbourhood of V, $\psi(C) = \partial M \cap \psi(B)$.

Theorem 2.3.8 *If M is a manifold with boundary, V a submanifold, then there exists a tubular neighbourhood of V in M.*

Proof By Proposition 2.3.7 (ii), we can choose a Riemannian metric for M, adapted to the boundary, in which V meets ∂M orthogonally. As in the proof of Theorem 2.3.3, we consider the exponential map of the normal bundle $\mathbb{N}(M/V)$. We need to show that this is well defined. The crucial point is that since the metric is adapted to the boundary, and the vectors in C are normal to V and hence tangent to ∂M, integrating them gives curves in ∂M and hence, at least locally, a map $C \to \partial M$. The previous argument shows that this map is a local diffeomorphism.

The arguments needed to go from having a local diffeomorphism to the result are the same as those for Theorem 2.3.3. □

2.4 Diffeotopy extension theorems

Let V^v, M^m be smooth manifolds, possibly with boundary. A *diffeotopy* of V in M is an embedding $h : V \times I \to M \times I$ which is *level-preserving*, i.e. we can write

$$h(x, t) = (h_t(x), t) \qquad m \in V, t \in I.$$

It follows that each h_t is also an embedding. We also say that h is a diffeotopy between h_0 and h_1.

h is called *normalised* if it extends to a level-preserving embedding $h : V \times \mathbb{R} \to M \times \mathbb{R}$ such that $h_t = h_0$ when $t \leq 0$, and $h_t = h_1$ when $t \geq 1$. If h is any diffeotopy, the map $H : V \times \mathbb{R} \to M \times \mathbb{R}$ given by $H(m, t) = (h_{Bp(t)}(m), t)$ is a normalised diffeotopy between h_0 and h_1.

A *diffeotopy* of M is a diffeomorphism k of $M \times I$ which is level-preserving, thus in particular it is a diffeotopy of M in M. The diffeotopy k of M *covers* the diffeotopy h of V in M if, for all $x \in V, t \in I$, $k_t(h_0(x)) = h_t(x)$. A diffeotopy covered by a diffeotopy of M is called an *ambient diffeotopy*.

Lemma 2.4.1 *Diffeotopy is an equivalence relation.*

Proof The definition $h(x, t) = (h_0(x), t)$ gives a diffeotopy between h_0 and itself. If h' gives one between h_0 and h_1, then h'', where $h''(x, t) = h'(x, 1 - t)$ gives a diffeotopy between h_1 and h_0. Finally, let h', h'' be normalised diffeotopies between h_0 and h_1 and h_1 and h_2. Then set

$$h_t'''(x) = \begin{cases} h'_{3t}(x) & \text{if } t \leq 1/2 \\ h''_{3t-2}(x) & \text{if } t \geq 1/2; \end{cases}$$

this is a smooth embedding, since h' and h'' are so, and we have $h_t''' = h_1$ for $\frac{1}{3} \leq t \leq \frac{2}{3}$, so that the two parts of the definition fit smoothly. □

One of the basic problems in differential topology is to determine the set of equivalence classes. We will accomplish this in some cases in Chapter 6.

The *support* of a diffeomorphism h of a smooth manifold M is the closure of the set of points P with $h(P) \neq P$.

The support of a diffeotopy h of V in M is the closure of the set of points $P \in V$ such that $h_t(P)$ is not independent of t.

Theorem 2.4.2 (Diffeotopy Extension Theorem) *Let V, M be smooth manifolds, perhaps with boundary, and let $h : V \times \mathbb{R} \to M \times \mathbb{R}$ be a diffeotopy of V in M, whose support K is compact, and contained in $\mathring{M}$. Then there is a diffeotopy k of M, whose support is compact and contained in $\mathring{M}$, and which covers h; in particular, h is ambient.*

Proof Since K is contained in $\mathring{M}$, we can ignore the boundary of M, and suppose simply that M is a smooth manifold, for if the result is proved in this case, the diffeotopy k of M which we obtain, having compact support, equals the identity on a neighbourhood of $\partial M \times \mathbb{R}$, and can therefore be extended to the boundary as the identity.

Let k be a diffeotopy of $M \times \mathbb{R}$. Then k defines a vector field on $M \times \mathbb{R}$ as follows. Write ∂_t for the vector field which projects to 0 on M and to $\partial/\partial t$ on $\mathbb{R}$, and define a vector field on $M \times \mathbb{R}$ by $\xi_k := dk(\partial_t)$. Since k is level-preserving, the projection of ξ_k on the second factor is still $\partial/\partial t$. Also, if k has compact support, $\xi_k = \partial_t$ except at some points of a compact set.

Conversely, suppose given a vector field ξ whose projection on $\mathbb{R}$ is $\partial/\partial t$. If ξ is complete, it gives rise to a 1-parameter group (φ_t) of diffeomorphisms of $M \times \mathbb{R}$, and hence to the diffeotopy given by $k(P, t) = (\varphi_t(P), t)$. Moreover, the local uniqueness clause in Theorem 1.4.2 implies that if k gives rise to ξ, then we recover the original k.

Since by Proposition 1.4.4 (ii) and (iii), the vector field ∂_t on $M \times \mathbb{R}$ is complete, it follows by (iv) that if $\xi = \partial_t$ except on a compact set, then ξ is complete.

We conclude that to construct the diffeotopy, it is sufficient to construct the vector field ξ. By the above argument, we see that the necessary and sufficient condition that k covers h is that on $h(V \times \mathbb{R})$, we have $\xi = dh(\partial/\partial t)$. Thus the problem is reduced to the construction of a vector field ξ on $M \times \mathbb{R}$ satisfying:

(i) $\xi = \partial_t$ outside a compact set,

(ii) the projection of ξ on $\mathbb{R}$ is everywhere $\partial/\partial t$,

(iii) on $h(V \times \mathbb{R})$, $\xi = dh(\partial/\partial t)$.

We assert that if we can do this in a neighbourhood of each point of $h(V \times \mathbb{R})$, ξ can be constructed. For such neighbourhoods, together with the complement U_0 of $h(V \times \mathbb{R})$, form an open covering of $M \times \mathbb{R}$. By Theorem 1.1.5, there is a smooth partition $\{\Psi_\alpha\}$ of unity strictly subordinate to this covering. If ξ_α is a function on the support U_α of Ψ_α which satisfies conditions (i) – (iii), the function $\xi := \sum_\alpha \xi_\alpha \Psi_\alpha$ (where $\xi_0 := \partial_t$) will satisfy all the conditions.

Now $h(V \times \mathbb{R})$ is a submanifold of $M \times \mathbb{R}$, hence in a neighbourhood of any point of it we can find a coordinate neighbourhood $\varphi : U \to \mathbb{R}^{m+1}$ with $U \cap \mathrm{Im} h = \varphi^{-1}(\mathbb{R}^{v+1})$; say for simplicity that the image of U is $\mathring{D}^{m+1}$. Then $d\varphi(dh(\partial/\partial t)) = \sum a_i \partial/\partial x_i$ in $\mathring{D}^{v+1}$; we define ξ by taking the same formula in $\mathring{D}^{m+1}$ (i.e. by taking the a_i independent of the last $m - v$ coordinates).

In the case of boundaries, the a_i are only defined on $\mathring{D}^{v+1}_+$. But by Whitney's Extension Theorem 1.5.1, they can be extended to smooth functions on $\mathring{D}^{v+1}$, and then extended to $\mathring{D}^{m+1}$ as above. This completes the proof of the result. □

Corollary 2.4.3 *If M is a smooth manifold, V a compact submanifold (perhaps with boundary), then any diffeotopy of the inclusion $i : V \subset M$ is an ambient diffeotopy.*

Corollary 2.4.4 *If M is a smooth manifold with boundary, any diffeotopy of a compact submanifold (perhaps with boundary) of $\mathring{M}$ is covered by a diffeotopy of M.*

Proof By the theorem, it is covered by a diffeotopy of $\mathring{M}$ with compact support. Thus ∂M has a neighbourhood in $\mathring{M}$ fixed by the diffeotopy, which can thus be extended to M, defining it to be fixed on ∂M. □

Proposition 2.4.5 *Any diffeotopy of ∂M is covered by a diffeotopy of M.*

Proof We shall suppose the diffeotopy h_t of ∂M normalised so that $h_t = 1$ for $t \leq \frac{1}{3}$ and $h_t = h_1$ for $t \geq \frac{2}{3}$. Let $\psi : \partial M \times I \to M$ be a collar neighbourhood of ∂M in M (such exist by Theorem 1.5.5). Then we define a covering diffeotopy

k_t of M by

$$k_t(Q) = \begin{cases} Q & \text{if } Q \notin \mathrm{Im}(\psi), \\ Q & \text{if } Q = \psi(P, s),\ s \geq t, \\ \psi(h_{t-s}(P), s) & \text{if } Q = \psi(P, s),\ s \leq t. \end{cases}$$

Thus for $s = 0$, k_t agrees with h_t, and for $s \geq \frac{2}{3}$, $k_t(P) = P$, so that k is everywhere smooth, and does cover h. □

Theorem 2.4.6 *Let M be a manifold with boundary, V a submanifold (perhaps with boundary). Any diffeotopy of V in M with compact support is covered by a diffeotopy of M with compact support.*

Proof Following the proof of Theorem 2.4.2, we see that it only remains to show that we can construct ξ in a neighbourhood of each point of $h(V \times \mathbb{R})$. In this case, in a neighbourhood of any point of $h(V \times \mathbb{R})$ we can find a coordinate neighbourhood $\varphi : U \to \mathbb{R}^{m+1}$ with $U \cap \mathrm{Im} h = \varphi^{-1}(\mathbb{R}_+^{v+1})$. By Theorem 1.5.1 we can write $d\varphi(dh(\partial/\partial t)) = \sum a_i \partial/\partial x_i$ in $\mathring{D}^{v+1}$ with the a_i smooth in $\mathbb{R}^{v+1}$ and define ξ by taking the same formula in $\mathbb{R}^{m+1}$. □

We shall need one or two further kinds of diffeotopy extension, when we come to consider corners, but feel that by now proofs may be left to the reader. We mention one immediate application of our results.

Proposition 2.4.7 *Let M^m be a manifold (perhaps with boundary), V^v a compact submanifold with boundary. Then there is a submanifold U^v of M^m containing V^v.*

Proof First suppose that M has no boundary. Let $\varphi : \partial V \times I \to V$ be a tubular neighbourhood of ∂V in V. We define a diffeotopy of V by

$$\begin{cases} h_t(P) = P & P \notin \mathrm{Im}\,\varphi \\ h_t\varphi(P, u) = \varphi(P, f(t, u)) & \end{cases},$$

where f is chosen with

$f(t, u) = u$ for $u > 1 - \varepsilon$,

$f(0, u) = u$,

$f(t, 0) > 0$ for $0 < \varepsilon$,

and $\partial f/\partial u > 0$ everywhere; so that the diffeotopy 'pushes' the boundary a little way into V: for example, we can take $f(t, u) = u + Bp(t - u)$ provided $t \leq k$, where in this range $Bp'(t) < 1$. Now h_t is a diffeotopy, hence (V being compact) is ambient, and so covered by H_t, say, $h_k(V) \subset \mathring{V}$. We can thus take $U = H_k^{-1}(\mathring{V})$.

If M is bounded, we argue similarly, using that part of the boundary of V not contained in ∂V. □

This result has the effect that to describe a neighbourhood of V in M, we can use tubular neighbourhoods of U; tubes round V do not give neighbourhoods.

2.5 Tubular neighbourhood theorem

We shall now use our results on diffeotopy extension to complete the discussion in §2.3 of tubular neighbourhoods by showing that these are, essentially, unique.

We recall the definition. If B is an $(m - v)$-disc bundle over V, with group O_{m-v}, and central cross-section B_0, then a tubular neighbourhood of V in M is an embedding $\varphi : B \to M$, as submanifold with boundary, extending the projection of B_0 on V.

We say that two tubular neighbourhoods $\varphi : B \to M$ and $\varphi' : B' \to M$ are *equivalent* if there is a bundle map $\chi : B \to B'$ over the identity map of V, and an ambient diffeotopy of φ on $\varphi'_0\chi$ which is fixed on B_0.

Our object is to show that any two tubular neighbourhoods are equivalent. Since we shall use the result of §2.4 we shall have to assume that V is compact. One might expect that this assumption was unnecessary; however, it cannot be omitted, as the example of Figure 2.3 illustrates.

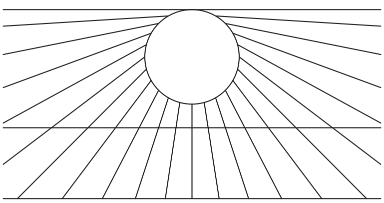

Figure 2.3 Example of a bad tubular neighbourhood

In the figure, T is the set defined by $-3 \leq y < 3$ and $x^2 + (y - 2)^2 \geq 1$, and the projection of T on $\mathbb{R}^1$ is defined by straight lines through (0, 3). This gives a tubular neighbourhood of $\mathbb{R}^1$ in $\mathbb{R}^2_+$, which is not a closed subset, so is not equivalent to a standard one.

The same example thus also shows the necessity of the compactness hypothesis in Theorem 2.4.2.

Let $\varphi : B \to M$ be a tubular neighbourhood for V in M. We consider the bundle E associated to B but with fibre $\mathbb{R}^{m-v}$. Then B is a submanifold with boundary of E. For the tubular neighbourhoods of §2.3, E is simply the normal bundle $\mathbb{N}(M/V)$.

We say that an embedding $\bar{\varphi} : E \to M$ as open submanifold, extending the projection of E_0 on V, is a *open tubular neighbourhood* of V in M.

Lemma 2.5.1 *Any tubular neighbourhood $\varphi : B \to M$ can be extended to an open tubular neighbourhood $\bar{\varphi} : E \to M$.*

Remember that we are assuming that V is compact. We use the same idea as for Proposition 2.4.7.

Proof We can define a diffeotopy of φ as follows. Recall that over each neighbourhood U in V, B is a product of U with a vector space; in the sequel, we permit ourselves to form sums and products by scalars in these vector spaces, using the standard notation. Then our diffeotopy is $\varphi_t(x, v) = \varphi(x, tv)$ for $\frac{1}{2} \le t \le 1$ (where $x \in V$, $v \in D^{m-v}$). Since V, and so also B, is compact, the diffeotopy is ambient: say it is covered by the diffeotopy k_t of M. But $\varphi_{1/2}$ can be extended to a open tubular neighbourhood, for example, by the map

$$\bar{\bar{\varphi}}(x, v) = \varphi\left(x, \frac{\gamma(\|v\|)}{\|v\|} \cdot v\right),$$

where γ is smooth, $\gamma(t) = \frac{1}{2}t$ for $0 \le t \le 1$, $\gamma'(t) > 0$, and $\gamma(t) < 1$. We can now define $\bar{\varphi} = k_{1/2}^{-1} \circ \bar{\bar{\varphi}}$.

A suitable γ can be constructed by using bump functions, for example, we may take

$$\gamma(t) = \frac{1}{3}\int_0^t \{1 + (e^{-x} - 1)Bp(x-1)\}dx.$$

□

Lemma 2.5.2 *Let $\bar{\varphi} : E \to M$, $\bar{\varphi}' : E' \to M$ be open tubular neighbourhoods of V in M such that $\operatorname{Im}\bar{\varphi} \subset \operatorname{Im}\bar{\varphi}'$. Then for some bundle map $\bar{\chi} : E \to E'$, there is a diffeotopy of $\bar{\varphi}$ on $\bar{\varphi}' \circ \bar{\chi}$ which is fixed on B_0.*

Proof Let $j = \bar{\varphi}'^{-1} \circ \bar{\varphi} : E \to E'$, then j is an embedding. Consider the mappings j_t given by $j_t(e) = t^{-1}j(te)$ for $0 < t \le 1$, $e \in E$; where the multiplications by t^{-1}, t are again scalar multiplications in the fibre. Clearly $j_1 = j$; we shall show that the definition of j_t can be extended to $t = 0$, and that j_0 can be taken as $\bar{\chi}$: $\bar{\varphi}' \circ j_t$ will then give the required diffeotopy of $\bar{\varphi} = \bar{\varphi}' \circ j$ on $\bar{\varphi}'\bar{\chi}$; it is fixed on B_0.

Take local coordinates $x = (x_1, \ldots, x_v)$ in V, and let y, z be Euclidean coordinates in the fibres of E, E'. Then setting $j(x, y) = (\alpha(x, y), \beta(x, y))$ we have

$$j_t(x, y) = (\alpha(x, ty), t^{-1}\beta(x, ty)).$$

But j carries the zero cross-section of E onto that of E', so

$$\alpha(x,0)=x, \qquad \beta(x,0)=0.$$

Now by Lemma 1.2.3, applied to β (regarded as a function of y with x as a parameter), there are smooth functions β_i with $\beta(x,y)=\sum y_i\beta_i(x,y)$. Then $t^{-1}\beta(x,ty)=\sum y_i\beta_i(x,ty)$, so we can write j_t in the form

$$j_t(x,y)=(\alpha(x,ty),\sum y_i\beta_i(x,ty)),$$

where the left-hand side is a smooth function also at $t=0$. This shows that we have a smooth map $J: E\times I\to E'\times I$ defined by the j_t; to have a diffeotopy, we must check that the Jacobian is everywhere non-zero. This follows for $t\neq 0$, since j is a diffeomorphic embedding, and multiplication by t or t^{-1} gives a diffeomorphism. Now

$$j_0(x,y)=\left(x,\sum y_i\beta_i(x,0)\right)=\left(x,\sum y_i\left.\frac{\partial\beta}{\partial y_i}\right|_{y=0}\right)$$

induces a linear map of each fibre, with matrix $(\partial\beta_j/\partial y_i)=(\partial z_j/\partial y_i)$ which is also the matrix of partial derivatives of j on B_0. Since j_0 is an embedding, this is non-zero. It follows that j_0 is a GL_{m-v}-bundle map, hence a diffeomorphism. We can thus take $\bar{\chi}=j_0$. We have also verified by the same token that J is a diffeotopy. □

Corollary 2.5.3 *The result holds also without the assumption* $\operatorname{Im}\bar{\varphi}\subset\operatorname{Im}\bar{\varphi}'$.

Proof For $\operatorname{Im}\bar{\varphi}\cap\operatorname{Im}\bar{\varphi}'$ is a neighbourhood of V, which thus has a tubular neighbourhood, hence also a open one $\bar{\varphi}''$, with $\operatorname{Im}\bar{\varphi}''\subset\operatorname{Im}\bar{\varphi}\cap\operatorname{Im}\bar{\varphi}'$. Then there are bundle maps modulo which $\bar{\varphi}''$ is diffeotopic both to $\bar{\varphi}$ and $\bar{\varphi}'$, whence the result follows. □

Lemma 2.5.4 *Let $\bar{\varphi}: E\to M$, $\bar{\varphi}': E'\to M$ be open tubular neighbourhoods of V in M where the bundles E, E' have group O_{m-v}. Then the conclusion of Lemma 2.5.2 holds, with $\bar{\chi}$ an O_{m-v}-bundle map.*

Proof It suffices to show that any $\psi: E\to E'$ which is a $GL_{m-v}(\mathbb{R})$-bundle map is diffeotopic to an O_{m-v}-bundle map. As above, in coordinates, ψ is given by

$$\psi(x,y)=(x,z) \quad\text{where}\quad z_i=\sum a_{ij}(x)y_j.$$

Now since the group is the orthogonal group, we can speak of the length of a vector in the fibre (compare §1.2). By the Gram–Schmidt orthogonalisation process, take the vectors b_i with components a_{ij}, and write $b_i=\sum_{j=1}^{i}\lambda_{ij}e_j$, where the e_i are orthonormal, and each $\lambda_{ij} > 0$. If e_i has components e_{ij}, consider now the diffeotopy

$$k_t(x, y) = (x, z_t), \quad \text{where} \quad (z_t)_i = \sum_{j,k} (t\lambda_{ij} + (1-t)\delta_{ij}) e_{jk} y_k.$$

That this is a diffeotopy follows as no matrix $(t\lambda_{ij} + (1-t)\delta_{ij})$ is singular (for the matrix is triangular, with non-zero diagonal terms); k_1 is the given map ψ, and k_0 takes one orthonormal base to another, so is an O_{m-v}-bundle map. □

Theorem 2.5.5 (Tubular Neighbourhood Theorem) *Let M^m be a smooth manifold and V^v a compact submanifold. Then any two tubular neighbourhoods of V in M are equivalent.*

Proof Let $\varphi : B \to M$, $\varphi' : B' \to M$ be tubular neighbourhoods of V in M. By Lemma 2.5.1, φ and φ' extend to open tubular neighbourhoods $\bar{\varphi}$, $\bar{\varphi}'$. By Corollary 2.5.3, there is a bundle map $\bar{\chi} : E \to E'$ such that there is a diffeotopy of $\bar{\varphi}$ on $\bar{\varphi}' \circ \bar{\chi}$, fixed on B_0. By Lemma 2.5.4, we may take $\bar{\chi}$ as an O_{m-v}-bundle map. Then $\bar{\chi}$ maps B into B', and so we can take χ as its restriction. It follows that χ is a bundle isomorphism. Also, by Theorem 2.4.2, the diffeotopy we have constructed is in fact ambient. □

As a first corollary, we obtain a useful little result.

Theorem 2.5.6 (Disc Theorem) *Let M be a connected manifold (perhaps with boundary), $f_1, f_2 : D^m \to M^m$ embeddings as submanifold with boundary. Then f_1 and f_2 are ambient diffeotopic unless M is oriented and f_1, f_2 have opposite orientations.*

Proof Let $P_i = f_i(O) \quad (i = 1, 2)$. Since $\mathring{M}$ is connected, there is a smooth path connecting P_1 and P_2 in $\mathring{M}$, i.e. a diffeotopy of P_1 and P_2, considered as submanifolds of zero dimension. By the diffeotopy extension theorem, there is an ambient diffeotopy. Hence we may suppose $P_1 = P_2 = P$. Now f_1, f_2 are tubular neighbourhoods of P, so by Theorem 2.5.5, there is an orthogonal transformation χ of D^m, such that f_1 and $f_2 \circ \chi$ are ambient diffeotopic.

Now if $\chi \in SO_m$, then f_2 is diffeotopic, so also ambient diffeotopic to $f_2 \circ \chi$, so the result follows. If not, and M is orientable, we have the case excluded by the theorem. If M is non-orientable, there is an orientation-reversing smooth path (see the discussion after the definition of orientability), and if we take P on an ambient diffeotopy round such a path, the sign of the determinant of χ will change. □

We shall use numerous extensions of Theorem 2.5.5 in the sequel; let us indicate one or two briefly here. The definition of equivalence remains the same.

Proposition 2.5.7 *Any two collar neighbourhoods of ∂M in M are equivalent if ∂M is compact.*

Proof The proof follows the same pattern. The analogues of Lemma 2.5.1 and Lemma 2.5.2 follow as before. In Lemma 2.5.4, note only that our group is not $GL_1(\mathbb{R})$ or O_1, but simply $GL_1^+(\mathbb{R})$ or SO_1 – the trivial group. This makes for a slight simplification in the argument. □

Proposition 2.5.8 *The result of Theorem 2.5.5 holds also if M has a boundary.*

We note that in proving uniqueness of tubular neighbourhoods, in contrast to the case where we had to prove existence, no extra difficulties arise in the case where we have boundaries.

We now present an alternative approach to the existence of tubular neighbourhoods which, while less immediate than the use of the exponential map, is more flexible for generalisations.

We begin with notation. For $\pi : E \to B$ the projection map of a vector bundle, we identify B with the zero cross-section (the zero vectors in the fibres). The map π induces $\pi_* : \mathbb{T}(E) \to \mathbb{T}(B)$, hence for each $e \in E$ a linear map $T_eE \to T_eB$. Vectors in the kernel are called *vertical tangent vectors* of E, and we write $\mathbb{T}^v(E)$ for the bundle of vertical tangent vectors.

Define a *partial tubular neighbourhood* of a submanifold V of M to consist of a neighbourhood U of V in the normal bundle $\mathbb{N}(M/V)$ together with a map $\psi : U \to M$ such that, for each $v \in V$, $\psi(v) = v$ and the composite

$$N_v(M/V) = T_v^v(\mathbb{N}(M/V)) \xrightarrow{d\psi} T_v(M) \to N_v(M/V)$$

is the identity. We will construct partial tubular neighbourhoods by piecing together ones constructed over coordinate neighbourhoods in V. The definition implies that at each $v \in V$ the map $d\psi : T_vU \to T_uM$ is the identity on the common subspace T_vV and an isomorphism on the quotient, hence by the Inverse Function Theorem 1.2.5 that ψ is a local diffeomorphism. Thus if V is a closed submanifold with a partial tubular neighbourhood in M, it follows from Corollary A.2.6 that V has a neighbourhood U' in U such that $\psi \mid U'$ is an embedding; and so, by the same argument as in Theorem 2.3.3, that V has a tubular neighbourhood in M.

Proposition 2.5.9 *Any submanifold V of a manifold M has a partial tubular neighbourhood.*

Proof Since V is a submanifold, at any point $P \in V$ there is a chart $\phi : (U_P, U_P \cap V) \to (\mathbb{R}^m, \mathbb{R}^v)$. Identifying $\mathbb{R}^m \cong \mathbb{R}^v \times \mathbb{R}^{m-v}$ gives a partial tubular neighbourhood for $U_P \cap V$.

These open sets $U_P \cap V$ form an open cover $\{V_a\}$ of V; by Theorem 1.1.5, there is a smooth partition $\{\eta_a\}$ of unity strictly subordinate to it. Write W_a for the closure of the support of η_a: thus W_a is closed, $W_a \subset V_a$, and the W_a cover V. We will construct a partial tubular neighbourhood over V by extending over a neighbourhood of one set W_a at a time. Arguing as in Proposition 1.1.7, we can construct a smooth function ε_a on V, vanishing outside V_a, taking the value 2 on a neighbourhood of W_a, and with all values in [0,2].

First consider two open subsets V_a, V_b of V and partial tubular neighbourhoods $\psi_a : U_a \to M$, $\psi_b : U_b \to M$. Since each of the images is a neighbourhood of $V_a \cap V_b$ in M, the composite $\phi_{ab} := \psi_b^{-1} \circ \psi_a$ is defined on a neighbourhood X of the zero cross-section of $V_a \cap V_b$ in $\mathbb{N}(M/V)$. Using a trivialisation of $\mathbb{N}(M/V)$ over V_b, we can write ϕ_{ab}, which is a partial map of $\mathbb{R}^k \times (V_a \cap V_b)$ to itself, as $\phi_{ab}(x, y) = (\{f_i(x, y)\}, g(x, y))$ in a neighbourhood of $\{0\} \times (V_a \cap V_b)$. Since ϕ_{ab} preserves the zero cross-section, each $f_i(x, y)$ vanishes when $x = 0$. Hence by Lemma 1.2.3, we can write $f_i(x, y) = \sum_k x_k f_{ik}(x, y)$. Define a deformation by

$$\Phi^t(x, y) := \left(\left\{\sum_k x_k f_{ik}(tx, y)\right\}, g(tx, y)\right).$$

As in the proof of Lemma 2.5.2, this is well defined and smooth for a range including $t = 0$. By definition, $\Phi^1 = \phi_{ab} = \psi_b^{-1} \circ \psi_a$. It follows from the definition of partial tubular neighbourhood that Φ^0 reduces to the identity.

Define $\epsilon : V_a \cap V_b \to I$ by $\epsilon(z) = Bp(\frac{1}{2}(1 + \varepsilon_a(z) - \varepsilon_b(z)))$; thus $\epsilon = 1$ if $\varepsilon_a - \varepsilon_b \geq 1$ and $\epsilon = 0$ where $\varepsilon_a - \varepsilon_b \leq -1$. Now define ψ_{ab} by

$$\psi_{ab}(z) = \begin{cases} \psi_a(z) & \text{if } z \in W_a' \setminus X_b, \\ \psi_b(\Phi^{\epsilon(z)}(z)) & \text{if } z \in V_a \cap V_b, \\ \psi_b(z) & \text{if} z \in W_b' \setminus X_a. \end{cases}$$

Each formula defines a smooth map on an open set.

On the overlap $W_a' \cap (V_b \setminus X_b)$ we have $\varepsilon_a(z) = 2,\ \varepsilon_b(z) \leq 1$, so $\epsilon(z) = 1$ and the first two formulae agree. Similarly on $W_b' \cap (V_a \setminus X_a)$ we have $\epsilon(z) = 0$, and the latter two formulae agree. Hence ψ_{ab} is defined and smooth on $W_a' \cup W_b'$.

It remains to check the derivative along the zero section. This reduces to checking the x-derivative at $x = 0$ of $\sum_k x_k f_{ik}(tx, y)$, which indeed reduces to the identity.

By Theorem 1.1.4 we may suppose the covering $\{V_a\}$ locally finite and hence countable, so label the pairs by $n \in \mathbb{N}$. We now construct a partial tubular neighbourhood over V by extending over one set at a time. Suppose a partial tubular neighbourhood constructed on a neighbourhood X of $\bigcup_{r=1}^{n-1} W_r$. Then the construction in the first part of the proof yields a partial tubular neighbourhood on a neighbourhood of $\bigcup_{r=1}^{n} W_r$; moreover, the alteration takes part only inside V_n. Since the covering is locally finite, each point of V has a neighbourhood which is only affected by a finite number of steps of the construction, so the sequence of maps converges, being ultimately constant on a neighbourhood of any given point. The limit gives the desired partial tubular neighbourhood of V. □

Proposition 2.5.10 *Given submanifolds $V \subset W \subset M$, there exists a partial tubular neighbourhood $\psi : U \to M$ of V in M such that the restriction of ψ to $U \cap \mathbb{N}(W/V)$ is a partial tubular neighbourhood of V in W. Hence if V is closed, there exists a tubular neighbourhood $\psi : \mathbb{N}(M/V) \to M$ of V in M whose restriction to $\mathbb{N}(W/V)$ is a tubular neighbourhood of V in W.*

Proof The proof of Proposition 2.5.9 goes through with the only change being the requirement on each of the partial tubular neighbourhoods of compatibility with W. As before, the existence of a tubular neighbourhood follows from that of a partial tubular neighbourhood. □

This rather weak relative form of Theorem 2.3.3 will be used in §6.3.

Clearly the argument adapts to further cases such as $V \subset W_1 \subset W_2 \subset M$ or to having two submanifolds W_1 and W_2 of M such that at each $v \in V$ there is a chart with each of the W_i mapping to a coordinate subspace of $\mathbb{R}^m$. Let us make one such result explicit.

Lemma 2.5.11 *Let $V^v \to M^m$ be an embedding of connected oriented manifolds. Then there exist orientation preserving embeddings $\phi : (D^m, D^v) \to (M, V)$, and any two such are isotopic.*

2.6 Corners and straightening

We recall that M^m is a manifold with corner if it has an atlas, with charts mapping to open sets in $\mathbb{R}^m_{++}$, and that the corner $\angle M$ is the set of points mapping to $\mathbb{R}^{m-2}$. At such a point $\angle M$ has two sides in ∂M: one corresponding locally to $x_1 = 0$, the other to $x_2 = 0$. Globally, the two sides define a double covering of $\angle M$, and we say that the corner is two-sided if this covering is trivial.

Lemma 2.6.1 *If $\angle M$ is two-sided, there is a smooth embedding $h : \angle M \times I^2 \to M$ with $h(x, 0, 0) = x$ for each $x \in \angle M$ and $h^{-1}(\partial M) = (I \times \{0\}) \cup (\{0\} \times I)$. Moreover, h is unique up to diffeotopy.*

Proof Since we are only interested in a neighbourhood of $\angle M$, we can delete from ∂M the complement of a neighbourhood of $\angle M$, and thus suppose that ∂M consists of such a neighbourhood and hence, since $\angle M$ is two-sided, can be split into two components, $\partial_1 M$ and $\partial_2 M$, each with boundary $\angle M$.

Let $h_1 : \angle M \times I \to \partial_1 M$ be a collar neighbourhood of $\angle M$ in $\partial_1 M$.

By Proposition 1.5.6, there is a smooth embedding $h_2 : (\partial_1 M) \times I \to M$ giving a neighbourhood of $\partial_1 M$. We may now set $h(x, t_1, t_2) = h_2(h_1(x, t_1), t_2)$ for $x \in \angle M$ and $t_1, t_2 \in I$. Uniqueness up to diffeotopy follows from the corresponding result for collars. $\square$

We can call the map we have constructed a *bicollar neighbourhood* of $\angle M$.

Define $\partial^s M$ by cutting ∂M along $\angle M$. By the arguments of Proposition 1.5.6, we can find a map $\partial^s M \times I \to M$ which is an embedding except that a bicollar neighbourhood is covered twice: call the image a *semicollar* of ∂M. Both a bicollar and a semicollar are pictured in Figure 2.4.

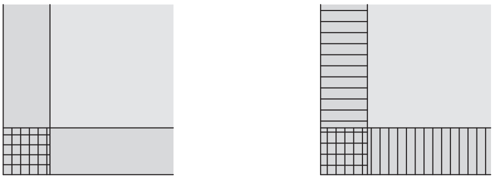

Figure 2.4 A bicollar and a semicollar

Proposition 2.6.2 *If M is a manifold with corner, there exist a manifold with boundary N and a homeomorphism $h : M \to N$ which is a diffeomorphism except on $\angle M$. Moreover, there is a construction of such an N which gives a result unique up to diffeomorphism.*

Proof Our construction is as follows. N will be M itself, with a different differential structure, defined by a new set of coordinate neighbourhoods. At points of $M \setminus \angle M$, the differential structure and coordinate neighbourhoods are unchanged. Let $h : \angle M \times I^2 \to M$ be a bicollar neighbourhood as above. Then a coordinate neighbourhood for $\angle M$, with coordinates $x_3, \ldots, x_m$ determines one for the neighbourhood with additional coordinates t_1, t_2.

We define N by the same mapping, but followed by taking the new coordinates as $(z_1, z_2) = (t_1^2 - t_2^2, 2t_1t_2)$. Since $t_1 + it_2$ lies in the first quadrant of the complex plane $\mathbb{C}$, $z_1 + iz_2 = (t_1 + it_2)^2$ lies in the upper half-plane $z_2 \geq 0$. We thus have the structure of smooth manifold with boundary. Uniqueness up to diffeomorphism follows from the uniqueness in Lemma 2.6.1. $\square$

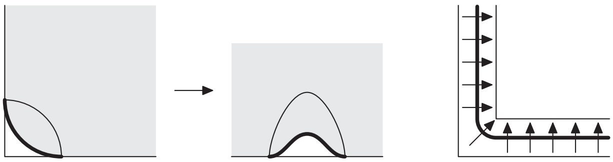

Figure 2.5 Rounding a corner

The resulting manifold N is said to be derived from M by *straightening the corner*.

We have discussed straightening corners, but may also consider the converse process, the introduction of corners. Given a manifold with boundary N, and a submanifold L of ∂N of codimension 1, we can construct a tubular neighbourhood of L in N, and redefine the differentiable structure, again using the change of coordinates $(z_1, z_2) = (t_1^2 - t_2^2, 2t_1t_2)$ in $\mathbb{R}^2$, to introduce a corner along L. The resulting M is unique up to diffeomorphism.

Since we have just reversed the above procedure, if we straighten the corner, we return to a manifold diffeomorphic to N. The procedure is roughly illustrated in Figure 2.5.

Lemma 2.6.3 *If L is a submanifold of ∂N of codimension 1, we can introduce a corner on L in an essentially unique way. If we straighten it, we recover L.*

While the above method of straightening is satisfactory, it is desirable to have alternative constructions, and be able to recognise when they give the same result.

We begin with the picture in the case when M has no corner. We can take a smooth vector field ξ on M, inward pointing at the boundary, and integrate to construct a collar neighbourhood $\varphi : \partial M \times I \to M$. A smooth submanifold $L \subset M$ of codimension 1, contained in the collar neighbourhood, and transverse everywhere to ξ, can be identified with the graph of a smooth map $\partial M \to I$. If L lies in the interior of the collar, it separates the collar into two pieces; it follows from Theorem 1.5.4 (taking the function $f = t$ as the projection on I and the vector field as $\partial/\partial t$) that each is diffeomorphic to $\partial M \times I$. Now L separates M into an outer part lying between L and ∂M, and hence diffeomorphic to $\partial M \times I$ and an inner part L^*; it follows from Lemma 2.7.2 below that the inner part is diffeomorphic to M.

We say that a smooth vector field $\sum_1^n a_i \frac{\partial}{\partial x_i}$ is inward pointing in $\mathbb{R}^n_{++}$ if $a_1 > 0$ on $x_1 = 0$ and $a_2 > 0$ on $x_2 = 0$. This definition is intrinsic, so passes to manifolds with corners.

Proposition 2.6.4 *Let ξ be a smooth vector field on M, inward pointing at boundaries and corners, $L \subset M$ a smooth submanifold of codimension 1, contained in a semicollar, and transverse everywhere to ξ. Then the inner region of L is diffeomorphic to the manifold N defined by straightening the corner.*

Proof The manifold N is obtained by applying the above change of coordinates $(z_1, z_2) = (t_1^2 - t_2^2, 2t_1t_2)$ at the corner. The image of the vector field ξ is not smooth at points corresponding to the corner, so we argue as follows.

In N we have a collar neighbourhood of $z_2 = 0$ given locally by (z_1, t). It contains a smooth submanifold L_ε given locally by $z_2 = \varepsilon$. The region between L_ε and ∂N is a collar, and the inner region for L_ε is diffeomorphic to N.

The region $0 \le z_2 \le \varepsilon$ in N becomes $0 \le t_1, t_2$ and $2t_1t_2 \le \varepsilon$ in M. The boundary L_ε given by $2t_1t_2 = \varepsilon$ is transverse to ξ, for we have $\sum_i a_i \frac{\partial}{\partial t_i}(2t_1t_2) = 2a_1t_2 + 2a_2t_1 > 0$ since, at least for ε small enough, we have $a_1, a_2, t_1, t_2 > 0$.

If L is any other submanifold transverse to ξ, there is an L' contained in the collar region, transverse to ξ, and disjoint from both L and L_ε. Hence the inner regions for L, L' and L_ε are all diffeomorphic. □

Once we have a semicollar, we can regard a neighbourhood of $\angle M$ as the product of $\angle M$ with the region $1 > y > |x|$ in $\mathbb{R}^2$ and then construct L as the graph of a function $\mu(x)$ defined by smoothing $|x|$. An example of such a function can be constructed as follows.

The function $\varepsilon^{-1}Bp(1 - \frac{|x|}{\varepsilon})$ is smooth, non-negative, vanishes unless $|x| < \varepsilon$, and has $\int_{-\infty}^{\infty} \delta_\varepsilon(x)dx = 1$. Now set $\mu(x) := \int_{-\infty}^{\infty} |y|\delta_\varepsilon(x-y)dy$. Then $\mu(x)$ is smooth, even, $\mu(x) = |x|$ if $|x| \ge \varepsilon$, and $\mu(x)$ is strictly increasing for $x > 0$.

Corollary 2.6.5 *D^{r+s} is derived from $D^r \times D^s$ by straightening the corner.*

Proof We can take the vector field in $D^r \times D^s$ to be the radial vector field $\sum_1^{r+s} x_i \frac{\partial}{\partial x_i}$, which is indeed inward pointing at the corner. We can then take S^{r+s-1} as the above L. □

We have given details for rounding corners in the simplest case. It is not possible to approximate any submanifold (not even any locally tame one) of a smooth manifold by a smooth submanifold, but the technique of rounding corners can be extended to the boundary of a submanifold of zero codimension: we have already mentioned the existence of smooth regular neighbourhoods.

2.7 Cutting and glueing

Let $M_i(i = 1, 2)$ be manifolds with boundary, $\partial M_i = Q_i$, and suppose given a diffeomorphism $h : Q_1 \to Q_2$. Take the disjoint union $M_1 \cup M_2$, and identify points corresponding under h to give a topological space N, and an identification map $\pi : M_1 \cup M_2 \to N$. Choose collar neighbourhoods $\varphi_i : Q_i \times I \to M_i$, and define a map $\varphi_i : Q_1 \times D^1 \to N$ by

$$\varphi(q, t) = \begin{cases} \pi\varphi_1(q, t) & \text{if } t \geq 0 \\ \pi\varphi_2(h(q), t) & \text{if } t \leq 0; \end{cases}$$

these agree on $t = 0$ since Q_1 and Q_2 were identified using h. Then φ is injective; in fact, it is an embedding. Define a function f on N to be smooth provided $f \circ \pi$ is a smooth function on $M_1 \cup M_2$ and $f \circ \varphi$ a smooth function on $Q_1 \times D^1$. The axioms defining a smooth manifold are now satisfied: coordinate neighbourhoods in M_1, $Q_1 \times D^1$, and in M_2 give rise to coordinate neighbourhoods in N, and where these overlap, they agree.

We have not made full use of the assumption $\partial M_i = Q_i$, and none of the above argument is affected if ∂M_i is the disjoint union of a certain set of components, and Q_i the union of a subset of these components. In this case, the remaining boundary components form the boundary of N.

More generally, suppose given manifolds M_1, M_2 with corner, smooth parts Q_i of ∂M_i, and a diffeomorphism $h : Q_1 \to Q_2$. Then by Proposition 1.5.6 (i) we have collar neighbourhoods of each Q_i, and the same definition now applies.

We say that N is obtained by *glueing* M_1 to M_2 by h (or, along Q_1).

Lemma 2.7.1 *The manifold defined by glueing M_1 to M_2 by h is determined up to diffeomorphism.*

Proof The only arbitrary element in the definition was the choice of collar neighbourhoods of the Q_i. The result follows since these are unique up to diffeotopy. □

The manifold obtained by glueing M to itself via the identity map $\partial M \to \partial M$ is said to be obtained by *doubling* M, and denoted $D(M)$.

Another simple but useful case is the following.

Lemma 2.7.2 *The result of glueing M to $\partial M \times I$ by the map $h : \partial M \to \partial M \times \{0\}$ given by $h(x) = (x, 0)$ is diffeomorphic to M.*

Proof Let $k : \partial M \times I \to M$ be a collar neighbourhood of ∂M. Define $p : M \cup (\partial M \times I) \to M$ by:

p is the identity on $M \setminus Im(k)$,

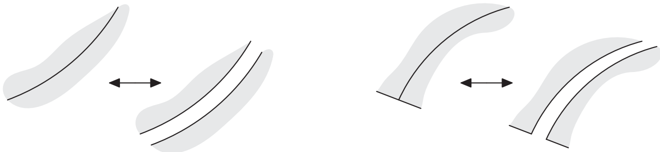

Figure 2.6 Cutting and glueing

$p(k(x,t)) = k(x, \alpha(t))$ for $x \in \partial M,\ t \in I$,
$p(x,t) = k(x, \frac{1}{2}(1-t))\ x \in \partial M,\ t \in I.$

This induces a bijection between the manifold obtained by glueing and M provided that $\alpha(t)$ increases from $\frac{1}{2}$ to 1 as t increases from 0 to 1. To make it a diffeomorphism it will suffice if also $\alpha(t) = \frac{1}{2}(1+t)$ for $t < \varepsilon$ and $\alpha(t) = t$ for $t > 1-\varepsilon$, for some small ε, for example, take $\alpha(t) = \frac{1}{2}\{(t+1) + (t-1)$ $Bp(3t-1)\}$. □

Glueing, and its inverse operation cutting, are both illustrated in Figure 2.6. Now let Q^{n-1} be a submanifold of N^n, with inclusion map $i : Q \to N$. For each point $P \in Q$, $di(T_PQ)$ is a subspace of T_PN of unit codimension, and so separates this real vector space into two components. We define a manifold M as follows. Its points are those of $N \setminus Q$, together with two points for each point P of Q, one associated with each complementary component of $di(T_PQ)$ in T_PN or, as we shall say, *side* of Q in N. There is thus a natural projection $\pi : M \to N$. We take for coordinate neighbourhoods in M those induced by π from coordinate neighbourhoods in $N \setminus Q$; in addition, for each coordinate neighbourhood $f : U \to \mathbb{R}^n$ with $f^{-1}(\mathbb{R}^{n-1}) = U \cap Q$ two coordinate neighbourhoods in M; induced by π from the restriction of f to the inverse images of $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ (in the latter case, we must change the sign of the first coordinate to obtain a coordinate neighbourhood of standard type). Here, of course, the points of N corresponding to a certain side of Q in N are mapped by the coordinate neighbourhood for the corresponding side of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$; since df is nonsingular, it preserves the distinction between sides.

We say that the resulting manifold M is obtained by *cutting* N along Q.

The same definition can be given more generally in the case when N has a boundary and Q is a submanifold of codimension 1 (so $\partial Q = Q \cap \partial N$): in this case the points corresponding to ∂Q form the corner $\angle M$; this divides ∂M into two parts: a part $\partial_1 M$ obtained by cutting ∂N along ∂Q and a part $\partial_2 M$ which is a double covering of Q. The double covering is given by the two sides of Q, or equivalently by the normal bundle, which we can take to have fibre D^1 with boundary giving the two points.

For example, if $N \setminus Q$ has just two components, with closures M_1 and M_2 so that $\partial M_1 = Q = \partial M_2$, then cutting N along Q yields the disjoint union of M_1 and M_2.

Proposition 2.7.3 *If N is defined by glueing M_1 to M_2 along Q_1, and we cut N along $\pi(Q_1)$, we recover M_1 and M_2. Conversely, if N^n and its submanifold Q^{n-1} are connected, Q separates N with parts M_1 and M_2 and we glue M_1 to M_2 along Q, we recover N.*

Proof The first part is immediate from the definition of glueing. For the converse, if the above conditions are satisfied, we obtain M_1 and M_2. Now if $\varphi : Q \times D^1 \to N$ is a tubular neighbourhood of Q in N, φ defines by restriction collar neighbourhoods of Q in M_1, M_2. If these are used in the glueing process, we recover N. The second part of the result now follows from Lemma 2.7.1. □

There are alternative definitions of cutting, which yield the same result up to diffeomorphism. One is to let ρ be a complete metric on N, and define M as the metric completion of $N \setminus Q$.

We can also define a manifold M' by deleting from N the interior of a tubular neighbourhood of Q. We see directly that this is obtained from the manifold M obtained by cutting N along Q by removing the interior of a collar neighbourhood of the boundary, hence by Lemma 2.7.2 is diffeomorphic to M.

We have seen that cutting and glueing are inverse operations, but cutting as defined above is more general than the inverse of glueing as it includes the case when the normal covering of Q in N is non-trivial. However we can also define glueing more generally: let Q_1, Q_2 be smooth parts of ∂M, not necessarily disjoint, and $h : Q_1 \to Q_2$ a diffeomorphism. The definition of glueing along h is now, as above, the quotient of M by identifying along h, with smooth structure defined using a choice of collar neighbourhoods of the Q_i. We see easily that this remains inverse to the cutting operation.

An important application of glueing is the following. Let M_1^m, M_2^m be connected smooth manifolds, $f_i : D^m \to M_i^m$ embeddings. Delete the interiors of the images of the f_i, and glue the result along the boundary $f_i(S^{m-1})$ by $f_2 f_1^{-1}$. Since removing a disc does not disconnect M if $m > 1$, the result is connected: it is called the *connected sum*, and written $M_1 \# M_2$. The construction is pictured in Figure 2.7.

Theorem 2.7.4 *$M_1 \# M_2$ is determined up to diffeomorphism by summands, unless these are both orientable, when there are two determinations.*

Proof By the Disc Theorem 2.5.6, the embeddings f_i are unique up to ambient diffeotopy and a possible change of orientation. By Lemma 2.7.1 the result

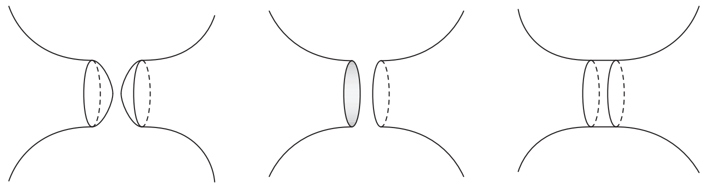

Figure 2.7 The connected sum

of glueing, given f_1 and f_2, is unique up to diffeomorphism. Hence the result follows, except for considerations of orientation. Now if f_1,f_2 are replaced by $f_1 \circ r$, $f_2 \circ r$, where r is a reflection, the connected sum is unaltered. If neither M_i is orientable, the result is trivial; if only M_2 is orientable, using the above possibility of simultaneous reversal, uniqueness again follows. If both are orientable, the result has two possible cases. □

To make the result precise in the orientable case, we suppose the M_i both oriented, and that one of the f_i preserves, the other reverses orientation. The result is then again unique, and has a canonical orientation inducing the given ones of the M_i.

The connected sum is also defined for manifolds with boundaries and corners; we simply suppose that the f_i map into the interior. However, in this case we also have a different sum operation. Let $f_i : D^{m-1} \to \partial M_i^m$ be an embedding. Introduce a corner along $f_i(S^{m-2})$. We may now glue the $f_i(D^{m-1})$ together by $f_2 f_1^{-1}$. The result is called a *boundary sum* $M_1 + M_2$ of M_1 and M_2.

Proposition 2.7.5 *If M_1^m, M_2^m are connected manifolds with connected boundaries, $M_1 + M_2$ is determined up to diffeomorphism by M_1 and M_2 unless ∂M_1 and ∂M_2 are both orientable, when there are two sums.*

Proof This follows by the Disc Theorem exactly as for Theorem 2.7.4. □

We conclude by summing up the simple properties of those operations.

Proposition 2.7.6 *Both operations are commutative and associative, with units: $M^m \# S^m \cong M^m$, $M^m + D^m \cong M^m$. We have $\partial(M_1 + M_2) = \partial M_1 \# \partial M_2$.*

Proof Commutativity and associativity are immediate. To form $M^m \# S^m$ we simply delete one disc from M^m, and replace it by another disc.

The second result may be seen as follows. D^m is obtained from $D^{m-1} \times I$ by straightening the corner. Derive N from M by introducing a corner along $f(S^{m-2})$ as above; then glueing on $D^{m-1} \times I$ does not affect N other than by a diffeomorphism by Lemma 2.7.2. The result follows by straightening the corner.

The last part is merely an observation of what happens to the boundary for the sum operation. □

2.8 Notes on Chapter 2

§2.1 I have proved a little more than I need at this point, but the existence of a neighbourhood of $\Delta(M)$ of pairs joined by minimal geodesics allows us to go further and define a continuous family of paths joining nearby points.

§2.2 These (classical) results on geodesics could be taken as the jumping off point for further results in differential geometry. Another treatment of this material is given in Milnor [98, II].

§2.3 It seems that tubular neighbourhoods, along with fibre bundles, were first introduced by Whitney [174].

§2.4 Our results are restricted to the case of diffeotopies of compact support. This restriction is necessary; otherwise we have counterexamples; but it may be possible to improve the result. The result was first proved by Thom [152], with a sharper version obtained independently by Cerf [36] and Palais [118].

§2.5 The tubular neighbourhood theorem was first proved by Milnor in lectures at Princeton University in 1961; an equivalent result was obtained in [36].

The construction of tubular neighbourhoods by local piecing together of partial tubular neighbourhoods is the method adopted by Cerf [36] and Lang [83]; it gives a proof of Theorem 2.5.10 without using the clumsy hypothesis that the normal bundle is trivial.

§2.6, §2.7, For a corner which is not two-sided, there is an analogue of a bicollar neighbourhood which is an embedding of a bundle over $\angle M$ with fibre $I \times I$ and group $\mathbb{Z}_2$ interchanging the components.

The disc theorem justifies the definition of connected sum. This seems to be due to Milnor, in the context of homotopy spheres.

Both these sections are designed for use in Chapter 5 for the theory of handle decompositions.