Feb 14

1) Reflection Group Trick

2) Poincaré duality groups

3) Modified Trick + Bestvina Brady Construction

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1) R. Gp. Trick

Given a group π of type F, (i.e, $B\pi$ is finite cw complex)

Goal: Produce a closed aspherical mfld M s.t. M retracts onto $B\pi$. In other words,

$$\pi_1(M) \xrightarrow{r_*} \pi \to 1$$

is a retraction.

Main Use: Many conjectures for π_1(aspherical mfld) are equivalent to knowing conjecture for all gps of type F.

Trick.

- $X = B\pi$. "Thicken" X to a compact mfld N with ∂. $(X \sim N)$
- Triangulate ∂N as flag simplicial complex L

Use L to produce a RACG W_L.

- Also use dual cells to simplices of L

to cellulate ∂N. If $v \in \operatorname{Vert} L$

$N_v = \operatorname{Star}(v, bL)$

- Use N as fund domain for W_L-action

- $D = D(W_L, N)$

$= (W_L \times N)/\sim$

- Let Γ = torsion-free subgp of finite index in W_L

Say $\Gamma = \ker(W_L \to C_2^N)$

$\shortparallel$

$\pi_1(P_L)$

$$\boxed{M = D/\Gamma}$$

Show M is aspherical.

"Thicken" X" i.e

simplicial cx. $X \overset{\text{embed}}{\hookrightarrow} \mathbb{R}^N$

as subcx.

Take a regular nbhd N.

Isotropy subsp of

$N_i = \langle s_i \rangle$

of $N_i \cap N_j = \langle s_i, s_j \rangle$

Use N as Fund. domain

$M = D/\Gamma$ is mfld

Prop M is aspherical

$\tilde{M}$ = contractible

M retracts onto N

$\tilde{N} \to N$ universal

$\partial \tilde{N} = \tilde{L}$ = infinite flag cx

$\pi_1 \curvearrowright \tilde{L}$

through simplicial maps

$W_{\tilde{L}}$ = RACG corresponding $\tilde{L}$

Acts on $\tilde{M}$

$$\tilde{M} = D(W_{\tilde{L}}, \tilde{N})$$

$$= \cancel{\tilde{B}}(W_{\tilde{L}} \times \tilde{N})/\sim$$

look at $\boxed{W_{\tilde{L}} \rtimes \cancel{\pi_1(L)}\ G}$

$W_{\tilde L} \rtimes \cancel{\pi_1(L)}^{=0} \curvearrowright \tilde M$

$G = \pi_1 N$

quotient space is N

Γ = torsion free subg of W_L

~~$\tilde\Gamma$~~ $= W_{\tilde L} \xrightarrow{p} W_L$

$p^{-1}(\Gamma) = \tilde\Gamma$ = torsion free

$$\boxed{M = \tilde M / \tilde\Gamma}$$

<u>Claim</u> $\tilde M$ is contractible

Uses earlier techn

~~$W_{\tilde L}$~~ acts $P_{\tilde L}$ < cube

is NPC

$$W_{\tilde{L}} \curvearrowright P_{\tilde{L}}$$

$$\begin{matrix} & \| \end{matrix}$$

CAT(0) ∴ contract.

<u>Claim</u> $\tilde{M}$ is h.e to $\tilde{P}_L$. ∴ contractible

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<u>Poincaré duality gp</u>

M^n closed mfld

$[M] \in H_n(M^n; \mathbb{Z})$

<u>Poincaré du</u>

$$\cap [M] : H^k_c(M) \xrightarrow[\cong]{} H_{n-k}(M)$$

Suppose M is aspheri-

$\tilde{M}$ = Contractible n-mfld

$$\begin{array}{c} H^k_c(\tilde M) \\ \wr\wr \\ H^k_c(\mathbb{R}^n) \end{array} = \begin{cases} \mathbb{Z} & k=n \\ 0 & k\neq n \end{cases}$$

$$H^k_c(\tilde M) = H_{n-k}(\tilde M)$$

$$G = \pi_1(M) \curvearrowright \tilde M$$

Note $H^k_c(\tilde M) = H^k(BG; \mathbb{Z}G)$

Def'n G is a

PD^n-gp ~~T-F~~ if

① G is type FP

(2)

$$H^{k}(BG;\mathbb{Z}G) = \begin{cases} \mathbb{Z} & k=n \\ 0 & k\neq n \end{cases}$$

Question: Is every PD^n-gp $G \cong \pi_1(M^n)$, M^n aspherical mfld

Ans Use BB constr to get G **not** finitely presented

$G = \pi_1($~~fin~~ finite aspherical $BG)$

$$\begin{tikzcd} \Gamma \arrow[r, bend left, "r"] & G \arrow[l, bend left, "i"] \arrow[r] & 1 \end{tikzcd}$$

$r \circ i = \mathrm{id}$

Feb 17

Reflection Gp Trick

Thm (1st version) G is gp type F

Then G is retract of

Γ = fund gp of closed aspherical mfld $\sim B\Gamma$

2nd version

Thm (2nd version) G is type FH then

G is retract of PD^n-gp

Γ.

(type FH $\qquad G \curvearrowright X$ = acyclic cx

X/G = finite cx.

Def PD^n gp Γ means

- Γ is type FP ($\Leftarrow$ FH)

i.e.

$$0 \to C_n \to C_{n-1} \quad \to C_0 \to \mathbb{Z} \to 0$$

(of projective $\mathbb{Z}\Gamma$ modules)

- $$H^*(\Gamma; \mathbb{Z}\Gamma) = \begin{cases} \mathbb{Z} & * = n \\ 0 & * \neq n \end{cases}$$

X/Γ compact $\qquad \overset{\parallel}{H^*_c(X; \mathbb{Z})}$

Construction in RCT:

Step 1 Thicken finite cx $Y = BG$ to mfld with ∂, say N.

($N \sim BG$)

Step 2 Triangulate ∂N as flag simplicial cx, L.

Form RACC

$W_L \leftarrow$ generators = (vertices of L)

Step 3 Make N into reflecti.. ~~?~~ orbifold"

Take dual cells to the ~~sta~~ simplices of L are faces of N

$$N_i \subset \partial N \ , \quad \langle s_i \rangle \subset (C_2)^I$$

Form ~~Spa~~ mfld

$$\underset{\text{-rc}}{D}(W_L, N) = (W_L \times N)/\sim$$

W_L acts on this strict fund. domain.

Γ' = torsion free subgp subgp of finite index in W_L

$$\left(W_L \longrightarrow C_2^{\,I} \right)$$

Γ' = kernel]

$$M = ~~D(W~~ \; D(W_L, N)/\Gamma'$$

M retracts on N

Show M ~~sch~~ aspherical

& determine $\pi_1(M) = \Gamma$

Step 2/ Show $\tilde{M}$ = contractible

$\tilde{L} = p^{-1}(L)$

$\tilde{N}$ = universal cover $\supset p^{-1}(L) \subset \partial\tilde{W}$

$\downarrow p$

$G = \pi_1(W)$ acts on $\tilde{L}$

$N = \partial W = L$

$W_{\tilde{L}}$ = RACC for $\tilde{L}$

$D(W_{\tilde{L}}, \tilde{N}) = \tilde{M}$

$\tilde{N}$ ← contractible

$\tilde{M}$ is contractible.

$$\begin{tikzcd}
 & \Gamma' \arrow[r, phantom, "\subset"] & W_L \\
p^{-1}(\Gamma') = & \tilde{\Gamma} \arrow[u] \arrow[r, phantom, "\subset"] & W_{\tilde{L}} \arrow[u, "p"]
\end{tikzcd}$$

Original is $\boxed{M = \tilde{M} / \Gamma'}$

$\pi_1 M = \Gamma$.

2^{nd} version G is type

$G \curvearrowright X$ = acyclic

$Y = X/G$ = finite cx

Y has cover space which: no homology

* ~~[illegible]~~

Step 1 Thicken Y to N

Step 2 L = triangulation of ∂N

$D = D(W_L, N)$

$\Gamma' < W_L$.

D/Γ = mfld M.

Step 3 M is covered by acyclic mfld $\tilde{M}$

$\tilde{D} = D(W_{\tilde{L}}, \tilde{N})$

↑ acyclic

is a acyclic mfld

$\Gamma = p^{-1}(\Gamma^{\ell})$ is same as before

Pf of 2nd Thm

$\tilde M$ is acyclic &

$\tilde M / \Gamma = M$ compact.

$\Rightarrow$ Γ is ~~FH~~. FH

retraction $M \to N$

induces $r : \Gamma \to G$.

Finally

Need to show Γ is PD^n.

$$C_i = C_i(\tilde M) = \cancel{C_i(\mathbb{Z}}$$

$$= C_i(\Gamma, \mathbb{Z}G)$$

~~$\tilde M$ acyclic~~

$$H^*(\Gamma; \mathbb{Z}\Gamma) = H^*_c(\tilde M, \mathbb{Z})$$

Poincare duality $= H_{n-*}(\tilde M)$

$$= \begin{cases} \mathbb{Z} & * = n \end{cases}$$

$\} \quad \sigma \qquad x \in h.$

Cor (Using BB)

$G = BB_L \subset A_L$

If L is acyclic but not simply connected

then G is type FH

Finitely presented

(So G is retract of P)

which is not finitely presented

Finally counterexamples

Conj if Γ PD^n gp

then $\Gamma = \pi_1$(closed aspherical mfld)

Th

~~Thm~~ **Thm** $\exists\ \Gamma = PD^n$ gp which not finitely present i.e. $\Gamma \neq$ (fund closed mfld)

Feb 19

- Jim Fowler's thesis
- Branched covers of CAT(0) cube complexes
- Higman gps
- Leary's construction

Last time: $PD^n(\mathbb{Z})$

(G is type FP (or FH)

& $H^*(G, \mathbb{Z}G) = \begin{cases} \mathbb{Z} \\ 0 \end{cases}$)

Replace $\mathbb{Z}$ by R ($R = \mathbb{Q}$ or $\mathbb{Z}[\frac{1}{2}]$

$PD^n(R)$ - gp = type $FH(R)$

$$H^*(G; RG) = \begin{cases} R & * = n \\ 0 & * \neq n. \end{cases}$$

<u>Thm</u> (Fowler): $\exists$ fin presented $PD^n(R)$ G, $R \neq \mathbb{Z}$, $R \subset \mathbb{Q}$

which not type F.

<u>Pf</u> $\exists$ e gp H which is type $FH(R)$ but not type $FH(\mathbb{Z}$

(Do BD with L = R-acyclic but not $\mathbb{Z}$-acyclic.

(Reflection gp. gives G retracts onto H)

--- as before. $\square$

Branched Covers

$X = \text{CAT}(0)$ cube Cx. ~~polyhedron~~

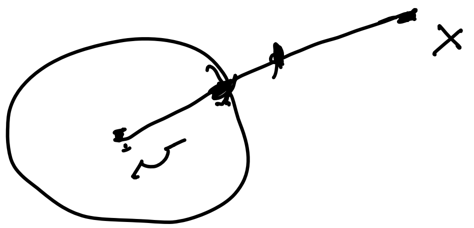
$v \in X$ a vertex

$L = Lk(v, X)$

$L \hookrightarrow X - v$

Geodesic retraction

$$X - v \xrightarrow{r} L$$



$\exists$ unique γ geodesic from γ to v

$L =$ sphere of radius ε

$$\gamma_x : [0,1] \to X$$

$$x \to \gamma_x(t)$$

Conclusion L is a retract of $X - v$

$\pi_1(L)$ is retract of $\pi_1(X - v)$

$\pi_1(L) \longrightarrow \pi_1(X - v)$ is injective.

X = CAT(0) cube cx

V ~~Φ~~ = set of vertices.

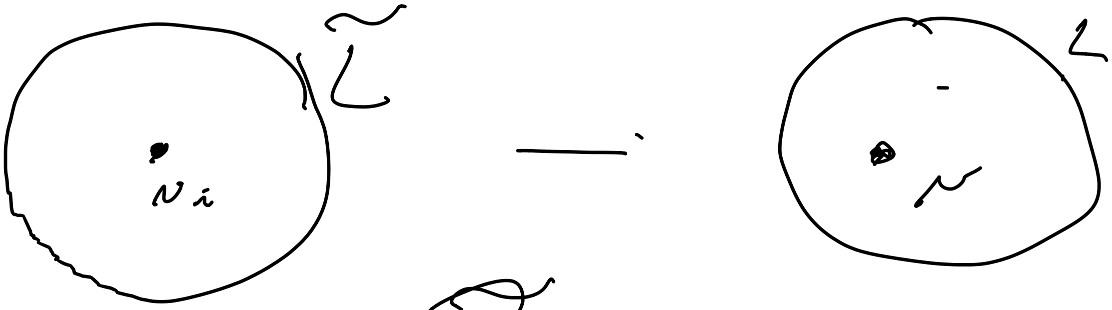
$X - V$

Take universal cover

$\widetilde{X - V}$, universal branched of X along V

is metric completion, i.e.

locally in $X - V$



Locally ~~$X - V$~~

$\tilde{X}$ = universal branched cover

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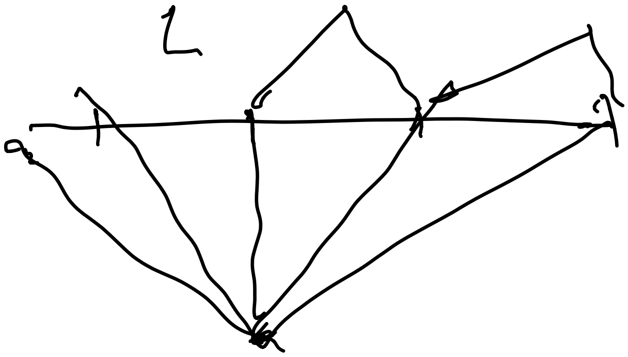

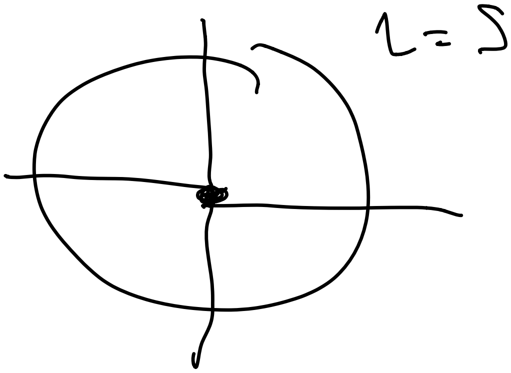

Thm (Leary) ∃ uncountably groups ($G_L(S)$) which type FH but not finitely presented.

Remark ∃ uncountably many finitely many fin gen. But only countabl ~~finitely~~ many finitely presented

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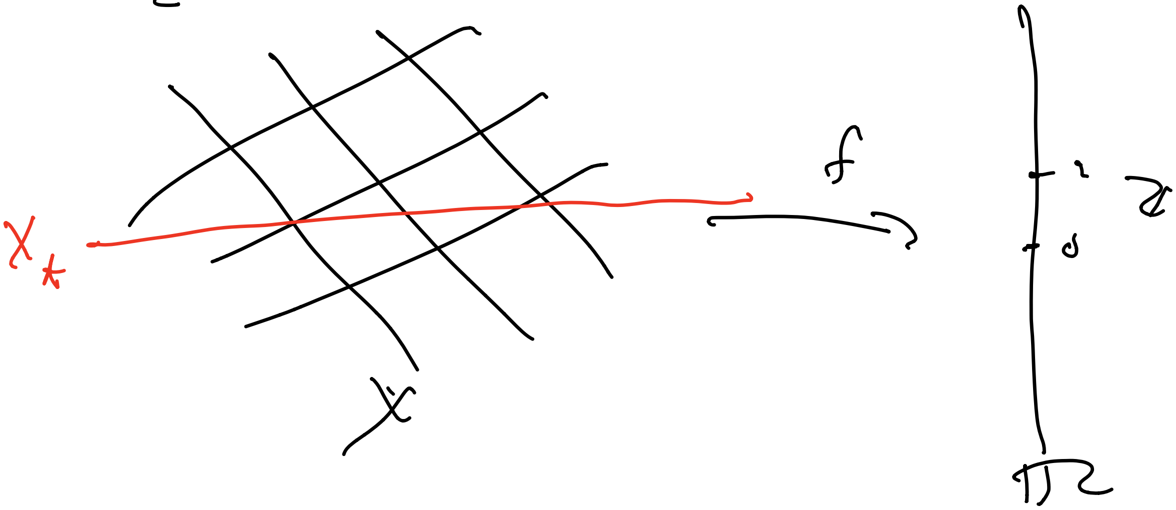
Uses Best - Brady

$$\begin{tikzcd} 0 \arrow[r] & BB_L \arrow[r] \arrow[d, equal] & A_L \arrow[r, "\ell"] & \mathbb{Z} \arrow[r] & 0 \\ & H & & & \end{tikzcd}$$

$X = X_L =$ CAT(0) cube for A_L



Morse function linear
each cube

X_t / BB_L = Counterexam

Leary's choice for

$L =$ acyclic but not
1-connect-

want $\tilde{L}$ = acyclic (∴ contractible

Let $S \subset \mathbb{Z}$

$X^{(S)}$ = universal branched of X along all vertice v $f(v) \in \mathbb{Z} - S$.

$$\begin{tikzcd} X^{(S)} \arrow[r] & X \arrow[d] \\ & X/BB_L \end{tikzcd}$$

$G_L(S)$ = gp of deck transformations of $X^{(S)} \to X/BB_L$

* $f(x) \notin \mathbb{Z}$

* ~~A~~ Also gp of deck

$$X_t^{(S)} \longrightarrow X_t/BB_L$$

$$\begin{tikzcd} X^{(S)} \arrow[r] \arrow[rr, bend right] & X \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

g = Morse

$$Lk_{\downarrow}(v, X^{(S)}) = \; = Lk$$ [illegible]

$$= \begin{cases} L & \text{if } f(v) \in S \\ \tilde{L} & \text{if } f(v) \in \mathbb{Z} - S \end{cases}$$

<u>Conclusion</u>

X_π is always acyclic.

~~$G_L(S)$ is~~

$\pi_1(X_\pi)$ = free product of copies $\pi_1(L)$
