Morse Theory on Cell complexes

Bestvina-Brady is on course page

X convex cell complex

Def^n $f : X \to \mathbb{R}$ is Morse function, if

1) $f|_C : C \to \mathbb{R}$ is
   cell a ~~nonconstant~~
   linear, nonconstant
   if $C \neq$ vertex

2) $f(X^0) =$ discrete

~~Lemma 2.1~~ J = interval in $\mathbb{R}$.

$X_J = f^{-1}(J)$ (sublevel ...)

set

Lemma If $J \subset J'$ and $\nexists$ vertex in $X_{J'} - X_J$

Then $X_{J'}$ is h.e. to X_J

Pf Construct deformation retract of $X_{J'}$ onto X_J

Descending & Ascending links

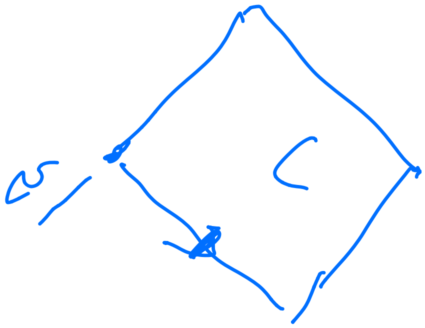
$v \in C$ a vertex

$Lk_{\downarrow}(v, C)$

$\cup \; F \subset C$ faces $\ni v$ if $f|_F$ have max at v.

$$Lk_{\downarrow}(v, C) = \bigcup_{\substack{F \leq C \\ v \in F}} Lk(v, F)$$

v is max $f|$, is ma



$= Lk_{\downarrow}(v, C)$

$= Lk(v, \pi$

$$Lk_{\downarrow}(v, X) = \bigcup_{C,\ v \in C} Lk_{\downarrow}(v, C)$$

Similarly $Lk_{\uparrow}(v, X)$ = ascending Lk

<u>Lemma</u> 2.5 $J \subset J' \subset \Pi$

There is only one $r \in J' - J$

Then $$X_{J'} \simeq X_J \cup \bigcup_{v \in f^{-1}(r)} \mathrm{Cone}(Lk_{\downarrow}(v))$$

~~Assume $[\inf J' = \inf J]$~~

Consider one cell at time.

$X_{J'} \longrightarrow X_J \cup \mathrm{Cone}(L, L)$

replace

$X_{J'} \cap f^{-1}(-\infty, r])$

<u>Cor</u> 2.6 Suppose $J \subset J'$

non empty connected intervals

Assume each $v' \in X_{J'} - X_J$

$Lk_{\downarrow}(v)$ & $Lk_{\uparrow}(v)$ are

both homologically n-connected

(i.e. $\tilde{H}_i(L) = 0$ $i \leq n$)

~~Then~~

(1) If ~~the~~ both links

are n-connected

then $X_J \to X_{J'}$ is

iso on H_i for $i \leq n$

onto for $i = n+1$.

② Similar statement for $Lh_{\downarrow}$ & $Lh_{\uparrow}$ being simply connected

The $\pi_1(X_J) \to \pi_1(X_{J'})$ is iso.

③ . . . –

Pf Assume $\inf J = \inf J'$

$$X_{J'} = X_J \cup \text{Cone } L$$

$L = Lh_{\downarrow}$ or more csp.

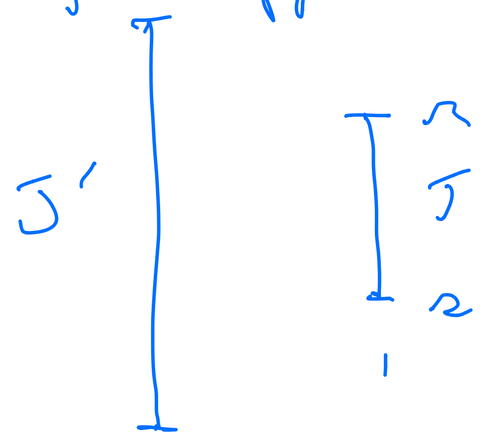
Exact sequence $(X_{J'}, X_J)$

$$H_i(\text{Cone } L, L) \hookrightarrow H_{\pi}(X_J) \to H_i(X_{J'}) \to H_i(X_{J'}, X_J)$$

$$H_i(X_{J'}, X_J) = H_i(\text{Cone } L, L) = \tilde{H}_{i-1}(L)$$

iso for $i \leq n$.

$\therefore$ ~~…~~ J Suppose $\inf J = \inf J'$



$X_{J'} \cap [b, \infty) \sim X_J \cup$ ~~…~~

Case 1 h_1

$X_{J'} \cap [-\infty, b] \sim X_J \cap [-\infty, b)$

Mayer Vietoris is

$J' = J' \cap [b, \infty) = J_1$

$\cup\ J' \cap [-\infty, b] = J_2$

$J_0 = f^{-1}(b)$

$X_{J'} = X_{J_1} \cup X_{J_2}$

$\uparrow$ X_{J_0}

MV & Five Lemma gives result

(2) $\pi_1(X_J) \ni \pi_1(X_{J'})$

$\inf J = \inf J'$

$X_{J'} \simeq X_J \cup_{L_i} \mathrm{Cone}(L_i)$

v.k Th

$\pi_1(X_{J'}) = \pi_1(X_J) *_{\pi_1(L)} (e)$

$= \pi_1(X_J) / \langle \pi_1(L) \rangle$

$\pi_1(L) = 1$

$= \pi_1(X_J)$

∎