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Morse theory and finiteness properties of groups

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Abstract. We examine the finiteness properties of certain subgroups of "right angled" Artin groups. In particular, we find an example of a group that is of type FP($\mathbb{Z}$) but is not finitely presented.

1. Introduction

The two fundamental finiteness properties in group theory are the properties of being finitely generated and of being finitely presented. There are many examples of finitely generated groups that are not finitely presented. Indeed, there are uncountably many finitely generated groups, but only countably many finitely presented ones [Ne]. A remarkable example of one such group was constructed by Stallings in [St]. It can be described as the kernel of the homomorphism of the direct product $F_2 \times F_2$ of free groups of rank 2 to the infinite cyclic group $\mathbb{Z}$ that sends both basis elements of both factors to $1 \in \mathbb{Z}$.

More general finiteness properties were introduced by C.T.C. Wall in [Wa 1]. A group H is said to be of type F_n if it has an Eilenberg-Mac Lane complex $\mathrm{K}(H,1)$ with finite n-skeleton. Equivalently, a group is of type F_n if it acts freely, faithfully, properly, cellularly, and cocompactly on an $(n-1)$-connected cell complex. Clearly, a group is finitely generated if and only if it is of type F_1, and is finitely presented if and only if it is of type F_2.

One can generalize this notion further by replacing the phrase "$(n-1)$-connected" in the latter definition by the phrase "homologically $(n-1)$-

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connected (with respect to a given coefficient ring R)''. This gives a weaker finiteness property, which we call $\mathrm{FH}_n(R)$. It has been an open question whether $\mathrm{FH}_n(\mathbb{Z})$ is equivalent to F_n for $n > 1$. It is known that $\mathrm{FH}_1(R)$ for any nontrivial R is equivalent to finite generation of the group, and hence to F_1.

The finiteness properties $\mathrm{FP}_n(R)$ used in the literature were introduced by Bieri [Bi 2] and are slightly weaker than the properties $\mathrm{FH}_n(R)$ above. Precise definitions and the relationship between these properties are given in Sect. 3. The main point is that we use the topologically defined $\mathrm{FH}_n(R)$ to detect if our groups satisfy the finiteness conditions $\mathrm{FP}_n(R)$.

The first example of a group which is finitely presented but not of type $\mathrm{FP}_3(\mathbb{Z})$ was given by Stallings in [St]. It may be described as the kernel of the homomorphism $F_2 \times F_2 \times F_2 \to \mathbb{Z}$ that sends all basis elements to $1 \in \mathbb{Z}$. This was generalized in [Bi 3] to produce examples of groups which were of type $\mathrm{FP}_n(\mathbb{Z})$ but not of type $\mathrm{FP}_{n+1}(\mathbb{Z})$ for all $n \geq 1$. Other examples of groups of type F_n but not of type $\mathrm{FP}_{n+1}(\mathbb{Z})$ for all $n \geq 1$ were discovered later by Stuhler [Stu] and by Abels and Brown [AB].

In this paper, we give a general construction which provides examples of groups having one type of finiteness property but not another. See Section 6 for specific examples. In particular, we produce groups of type $\mathrm{FP}(\mathbb{Z})$ which are not finitely presented.

Our construction begins with the class of ''right angled Artin groups''. These are also known as graph groups. We associate a right angled Artin group G_L to each finite flag complex L (a flag complex is a simplicial complex that is determined by its 1-skeleton). For example, if L is the n-sphere triangulated as the $(n+1)$-fold join of 0-spheres, then G_L is the $(n+1)$-fold direct product of free groups F_2 considered in the Stallings-Bieri examples above.

Next we examine the finiteness properties of the kernel H_L of the homomorphism $G_L \to \mathbb{Z}$ that ''sends all basis elements to 1''. Now, each Artin group G_L acts in a nice fashion on a CAT(0) space X_L (analogous to the action of a free group on a tree). Our method consists of constructing a map $f : X_L \to \mathbb{R}$ which is equivariant with respect to the homomorphism $G_L \to \mathbb{Z}$ and viewing it as a Morse function on X_L. This allows us to obtain information about the homotopy type of the point preimages or ''level sets'' of f. The key point is that H_L acts cocompactly on each level set, so this translates into information about finiteness properties of H_L.

This Morse theory point of view can be found in [Bi 1] and in [BNS]. The Bieri-Neumann-Strebel ''geometric invariants'' of [BNS] are related to homotopy types of halfspaces $f^{-1}[t, \infty)$ (or more precisely to the homotopy type of the pro-system $f^{-1}[t, \infty)$ as $t \to \infty$).

The main theorem relates finiteness properties of H_L to the topology of L.

Main Theorem. *Let L be a finite flag complex. Let $G = G_L$ be the associated right angled Artin group, and $H = H_L$ the subgroup of G_L described above. We compute homology with coefficients in a ring R with $0 \neq 1$.*

(1) $H \in \mathrm{FP}_{n+1}(R)$ *if and only if L is homologically n-connected.*
(2) $H \in \mathrm{FP}(R)$ *if and only if L is acyclic.*
(3) *H is finitely presented if and only if L is simply connected.*

The subgroups H_L are interesting test cases for other questions in group theory. The equivalence of finite generation of H_L and connectivity of L was already proven in [MV]. Meinert [Me] has related connectivity properties of L to finite generation of H_L in the more general setting of graph products of groups.

The outline of this paper is as follows. In Sects. 2, 3 and 4 we give the definitions and properties of Morse functions in the category of affine cell complexes, and we discuss finiteness properties of groups. In Sect. 5 we describe the Morse theory view of the homomorphism from the right angled Artin groups above to $\mathbb{Z}$. Theorem 4.1 can now be interpreted as proving the finiteness properties of H_L in the Main Theorem from the topological properties of L.

We define the notion of *sheets* in Sect. 6. These are used in Sects. 7 and 8 when we prove the reverse implications of the Main Theorem. Examples and applications of the Main Theorem are provided in Sect. 6.

It is worth noting that all implications in the Main Theorem follow directly from the homotopy description of the sublevel sets furnished in Theorem 8.6. However, we provide separate proofs of certain implications in Theorem 4.1, and Theorem 7.1. We feel that these demonstrate the power of the Morse theory and homology arguments alone. Another application of these ideas can be found in [B].

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2. Morse theory on affine cell complexes

In this section we develop Morse theory for affine cell complexes. This has some very interesting applications in geometric group theory. For example, it was the motivation for our construction in Sect. 5 below. In [B] it is used to construct an example of a hyperbolic group containing a finitely presented subgroup which is not hyperbolic.

Definition 2.1. *Let X be a CW complex. We say that X is an* ***affine cell-complex*** *if it is equipped with the following structure. An integer m is given, and for each cell e of X we are given a convex polyhedral cell* $C_e \subset \mathbb{R}^m$ *and a characteristic function* $\chi_e : C_e \to e$ *such that the restriction of* χ_e *to any face of* C_e *is a characteristic function of another cell, possibly precomposed by a partial affine homeomorphism* (= *restriction of an affine homeomorphism*) *of* $\mathbb{R}^m$. *An admissible characteristic function for e is any function obtained from* χ_e *by precomposing with a partial affine homeomorphism of* $\mathbb{R}^m$.

Thus the restriction of an admissible characteristic function to a face is an admissible characteristic function (of another cell).

Definition 2.2. *A map $f : X \to \mathbb{R}$ defined on an affine cell-complex X is a* ***Morse function*** *if*

- *for every cell e of X $f\chi_e : C_e \to \mathbb{R}$ extends to an affine map $\mathbb{R}^m \to \mathbb{R}$, and $f\chi_e$ is constant only when $\dim e = 0$, and*
- *the image of the 0-skeleton is discrete in $\mathbb{R}$.*

The main concern of this paper is to determine the topology of the f preimages of closed intervals in $\mathbb{R}$. We use the following notation and terminology. For a nonempty closed subset $J \subset \mathbb{R}$ denote by X_J the set $f^{-1}(J)$, and also let $X_t = X_{\{t\}}$. We call the sets X_t *level sets* of the Morse function f. We refer to the set $X_{(-\infty,t]}$ as the *sublevel set* corresponding to X_t.

Lemma 2.3. *If $J \subset J' \subset \mathbb{R}$ are connected and $X_{J'} \setminus X_J$ contains no vertices of X, then $X_J \hookrightarrow X_{J'}$ is a homotopy equivalence.*

Proof. For each cell e of X and each admissible characteristic function $\chi_e : C_e \to X$ we construct a strong deformation retraction $H_t^{\chi_e}$ of $C_e \cap (f\chi_e)^{-1}(J')$ to $C_e \cap (f\chi_e)^{-1}(J)$ so that

- if χ_e is precomposed by a partial affine homeomorphism h, then $H_t^{\chi_e}$ is conjugated by h, i.e. $H_t^{\chi_e h} = h^{-1}H_t^{\chi_e}h$ and
- the restriction of $H_t^{\chi_e}$ to a face of C_e is the strong deformation retraction associated to that face.

The construction is by induction on $\dim e$. The inductive step amounts to the following observation. If C is a convex cell in some Euclidean space and F and G are two disjoint faces with F top dimensional and G either top dimensional or just a vertex, then any strong deformation retraction from $\overline{\partial C \setminus F}$ to G extends to a strong deformation retraction from C to G.

The deformation retractions constructed above induce a strong deformation retraction from $X_{J'}$ to X_J. □

Ascending and descending links

Any affine cell complex can be equipped with a natural PL structure. This is done inductively over skeleta while at the same time proving that all admissible characteristic functions are PL. For the inductive step, first observe that every admissible attaching map $\partial C \to X^{(i)}$ is PL, since the restriction to each face of C is PL by the inductive hypothesis. Thus $X^{(i+1)}$ can be given the PL structure as the adjunction space of $X^{(i)}$ and the attaching maps defined on the boundary of convex $(i+1)$-cells in Euclidean space.

When $\alpha : A \to B$ is a PL map between polyhedra and $a \in A$ is an isolated point of $\alpha^{-1}\alpha(a)$, then α induces a map $Lk(a, A) \to Lk(\alpha(a), B)$ between links of a and $\alpha(a)$. We shall denote this map by α_* (the point a is understood from the context).

In particular, when $\chi_e : C_e \to X$ is an admissible characteristic function of a cell e of an affine cell complex X and w is a vertex of C_e, we have a well-defined map $\chi_{e*} : Lk(w, C_e) \to Lk(\chi_e(w), X)$. We may then identify the link $Lk(v, X)$ of a vertex v of X with

$$Lk(v,X) \;=\; \bigcup\{\, \chi_{e*}(Lk(w,C_e)) \;:\; \chi_e(w) = v\,\}.$$

Definition 2.4. *The* ***ascending link*** *(the* $\uparrow$*-link) is*

$$\begin{aligned} Lk_{\uparrow}(v,X) \;=\; \bigcup\{\, \chi_{e*}(Lk(w,C_e)) \;:\; \\ \chi_e(w) = v \text{ and } f\chi_e \text{ has a minimum at } w\,\} \;\subset\; Lk(v,X) \end{aligned}$$

and the ***descending link*** (the $\downarrow$-link) is

$$\begin{aligned} Lk_{\downarrow}(v,X) \;=\; \bigcup\{\, \chi_{e*}(Lk(w,C_e)) \;:\; \\ \chi_e(w) = v \text{ and } f\chi_e \text{ has a maximum at } w\} \;\subset\; Lk(v,X). \end{aligned}$$

Lemma 2.5. *Let* $f : X \to \mathbb{R}$ *be a Morse function on an affine cell complex as above. Suppose* $J \subset J' \subset \mathbb{R}$ *are closed and connected,* $\inf J = \inf J'$*, and* $J' \setminus J$ *contains only one point* r *of* f(0-cells)*. Then* $X_{J'}$ *is homotopy equivalent to* X_J *with the copies of* $Lk_{\downarrow}(v,X)$ *(*v *a vertex with* $f(v) = r$*) coned off.*

A similar statement holds when $\inf J = \inf J'$ is replaced by $\sup J = \sup J'$ and $Lk_{\downarrow}(v,X)$ by $Lk_{\uparrow}(v,X)$.
Proof. The argument is similar to that of Lemma 2.3. Since $X_{J' \cap (-\infty, r]} \hookrightarrow X_{J'}$ is a homotopy equivalence, we may assume by Lemma 2.3 that $\sup J' = r$. Let $r - \epsilon = \sup J$. If a cell e of X has the property that $\min f|e > r$ then e is disjoint from $X_{J'}$. For any admissible characteristic function $\chi_e : C_e \to X$ of any other cell e we construct, inductively on $\dim e$, a strong deformation retraction of $(f\chi_e)^{-1}(-\infty, r]$ onto the subset

$$(f\chi_e)^{-1}(-\infty, r-\epsilon] \;\cup\; \bigcup\{F \;:\; F \text{ is a face of } C_e \text{ with } f\chi_e(F) \subseteq (-\infty, r]\}$$

satisfying both naturality properties from the proof of Lemma 2.3.

These strong deformation retractions induce a strong deformation retraction of $X_{J'}$ onto X_J with the cones attached as stated in the Lemma. $\square$

Unless otherwise indicated, all homology groups are taken with coefficients in a fixed commutative ring R with $0 \neq 1$.

Corollary 2.6. *Let* $f : X \to \mathbb{R}$ *be a Morse function on an affine cell complex as above. Suppose that* $J \subset J' \subset \mathbb{R}$ *are nonempty and connected.*

(1) *If each* $\uparrow$*-link and each* $\downarrow$*-link is homologically* n*-connected, then inclusion* $X_J \hookrightarrow X_{J'}$ *induces isomorphism* in $\tilde{H}_i$ *for* $i \leq n$ *and epimorphism* in $\tilde{H}_{n+1}$.

(2) *If each* $\uparrow$*-link and each* $\downarrow$*-link is simply connected, then inclusion* $X_J \hookrightarrow X_{J'}$ *induces isomorphism* in π_1.

(3) *If each* $\uparrow$*-link and each* $\downarrow$*-link is connected, then the inclusion* $X_J \hookrightarrow X_{J'}$ *induces an epimorphism* in π_1.

Proof. Since $f(X^{(0)})$ is discrete, the proof follows by induction from Lemma 2.3, Lemma 2.5, the Mayer-Vietoris and the Seifert-Van Kampen Theorems. □

3. Finiteness properties of groups

In this section H is a discrete group, and R is a commutative ring with $1 \neq 0$ which we consider as a trivial RH-module.

Definition 3.1. *A group* H *is said to be of type* $\mathrm{FP}_n(R)$ *if there exists a resolution* (*exact sequence*)

$$P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

of the trivial RH*-module* R *by finitely generated projective* RH*-modules* P_i.

Definition 3.2. *A group* H *is said to be of type* $\mathrm{FP}(R)$ *if there exists a finite resolution*

$$0 \to P_n \to \cdots \to P_0 \to R \to 0$$

of R by finitely generated projectives over RH.

Definition 3.3. *A group* H *is of type* F_n *if there exists a* $\mathrm{K}(H,1)$ *with finite* n*-skeleton. One often writes* $H \in \mathrm{FP}_n(R)$ *to mean* H *is of type* $FP_n(R)$.

An application of Schanuel's Lemma from homological algebra gives us the following method of showing that a group H is of type $\mathrm{FP}_n(R)$ but not of type $\mathrm{FP}_{n+1}(R)$. A proof of this can be found in K. Brown's book [Br 1] (pp. 192–193) for example.

Proposition 3.4. *If there exists a resolution*

$$0 \to Z_n \to P_n \to \cdots \to P_0 \to R \to 0$$

where the P_i *are all finitely generated, projective over* RH *and* Z_n *is not finitely generated over* RH, *then* H *is of type* $FP_n(R)$ *but not of type* $FP_{n+1}(R)$. □

The most important way of producing projective resolutions (actually free resolutions) of R over RH comes from algebraic topology. The following definitions appeared in the introduction as natural generalizations of Wall's finiteness conditions F_n.

Definition 3.5. *A group H is said to be of type* $\mathrm{FH}_n(R)$ *if it acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on a cell complex X such that $\tilde{\mathrm{H}}_i(X,R) = 0$ for all $i \leq n-1$.*

Definition 3.6. *A group H is said to be of type* $\mathrm{FH}(R)$ *if it acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on an R-acyclic cell complex X.*

The following lemma relates the topological definitions FH to the classical finiteness conditions FP.

Lemma 3.7. *The following implications hold for a group H, and ring R with* $1 \neq 0$.

(1) *If $H \in FH_n(R)$ then $H \in FP_n(R)$.*

(2) *If $H \in FH(R)$ then $H \in FP(R)$.*

(3) *If H acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on an $(n-1)$-connected cell complex X, then H is of type F_n.*

Proof. From Definitions 3.5 and 3.6 we see that in parts 1 and 2 the group H acts nicely on a cell complex X which satisfies certain homological properties. All one has to do is write out the reduced chain complex of X with coefficients in R. Note that the vanishing of reduced homology groups implies that this is a resolution of R. The RH-module structure comes from the action of H on X, so compactness of the quotient X/H implies finite generation of the corresponding chain groups as RH-modules.

For part 3 note that X/H can be made into a $\mathrm{K}(H,1)$ by adding cells of dimension $n+1$ and higher. □

In a similar fashion, Proposition 3.4 yields the following useful topological criterion which tells when a group is of type $\mathrm{FP}_n(R)$ but not of type $\mathrm{FP}_{n+1}(R)$.

Lemma 3.8. *Suppose that a group H acts freely, faithfully, properly, cellularly, and cocompactly on a complex X which satisfies*

- $\tilde{H}_i(X,R) = 0$ for $0 \leq i \leq n-1$
- $\tilde{H}_n(X,R)$ is not finitely generated as an RH-module.

Then H is of type $\mathrm{FH}_n(R)$ but not of type $\mathrm{FP}_{n+1}(R)$. □

There are other finiteness properties in the literature. Finiteness properties of groups defined in terms of free resolutions were introduced by Serre [Se]. One says that a group is of type $\mathrm{FL}_n(R)$ (respectively $\mathrm{FL}(R)$) if it satisfies the conditions of Definition 3.1 (respectively Definition 3.2) with all occurrences of the word "projective" replaced by the word "free". A group is said to be of *finite type*, denoted F, if it has a finite Eilenberg-Mac Lane space.

The following set of implications shows the relationship between our properties FH and the classical finiteness conditions.

$$\mathrm{F}_n \rightarrow \mathrm{FH}_n \rightarrow \mathrm{FL}_n \rightarrow \mathrm{FP}_n$$

$$\mathrm{F} \rightarrow \mathrm{FH} \rightarrow \mathrm{FL} \rightarrow \mathrm{FP}$$

Example 6.3 (3) below shows that the implications $\mathrm{F} \rightarrow \mathrm{FH}$ and $\mathrm{F}_n \rightarrow \mathrm{FH}_n$ ($n \geq 2$) cannot be reversed. Clearly, $\mathrm{FH}_1 \rightarrow \mathrm{F}_1$. In [Se] Serre asked if $\mathrm{FP} \rightarrow \mathrm{FL}$ holds. It is still not known if this holds or if the implication $\mathrm{FL} \rightarrow \mathrm{FH}$ holds. The conditions FP_n and FL_n are known to be equivalent. One simply takes direct sums of the projectives in the resolution with complements to obtain a resolution which is free up to dimension n. These ideas were first observed by Wall [Wa 1]. The condition FL_n is equivalent to FH_n for $n = 1, 2$. A group of type FL_1 is, in particular, finitely generated, and is seen to be of type FH_1 by considering its Cayley complex with respect to a given presentation (with finite generating set). If the group is of type FL_2 then the first homology of the quotient of this Cayley complex is finitely generated. Thus we see that the group is of type FH_2 by equivariantly adding 2-cells to the Cayley complex, and killing the first homology of the complex. Finally, it is not known if the implication $\mathrm{FL}_n \rightarrow \mathrm{FH}_n$ holds for $n \geq 3$.

The next result says that if a finitely presented group H acts freely, faithfully, properly, cocompactly and cellularly on a cell complex Y then $\pi_1(Y)$ is finitely generated modulo the H action. It is used in Sect. 8 to conclude that certain groups are not finitely presentable.

Proposition 3.9. *Let H be a finitely presented group. Suppose that H acts freely, faithfully, properly discontinuously, cocompactly, and cellularly on a connected cell complex Y. Then it is possible to attach to Y finitely many H-orbits of 2-cells so that the resulting complex is simply connected.*

Proof. Since H is finitely presented, H acts freely, faithfully, properly discontinuously, cocompactly, and cellularly on a simply connected cell complex Z. Construct H-equivariant cellular maps $\alpha\colon Y^{(2)} \rightarrow Z$ and $\beta\colon Z^{(1)} \rightarrow Y^{(1)} \subset Y$. By attaching finitely many orbits of 2-cells to Y (one for every orbit of 2-cells in Z) we may assume that β extends to an equivariant cellular map $\tilde{\beta}: Z^{(2)} \rightarrow Y^{(2)}$. The map $Y^{(1)} \times \{0,1\} \rightarrow Y$ defined by $(y,0) \mapsto y$, $(y,1) \mapsto \beta\alpha(y)$ extends to an equivariant cellular map $F\colon Y^{(1)} \times \{0,1\} \cup Y^{(0)} \times [0,1] \rightarrow Y$ by connectivity of Y. After attaching finitely many orbits of 2-cells to Y (one for every orbit of 1-cells in Y) we may assume that F extends to an equivariant cellular map $\tilde{F}\colon Y^{(1)} \times [0,1] \rightarrow Y$.

We now claim that at this stage Y is simply connected. Indeed, let ℓ be a loop in $Y^{(1)}$. Then $\tilde{F}$ provides a homotopy between ℓ and $\beta\alpha(\ell)$. The loop $\alpha(\ell)$ is nullhomotopic in $Z^{(2)}$, and thus so is $\tilde{\beta}\alpha(\ell) = \beta\alpha(\ell)$. $\square$

4. Kernels of homomorphisms to $\mathbb{Z}$

In this section we assume that a group G acts freely, faithfully, properly, cocompactly, and cellularly on a contractible affine cell complex X and that the action preserves the affine structure. More precisely, if e is a cell of X with an admissible characteristic function χ_e and $g \in G$, then $g\chi_e$ is an admissible characteristic function for the cell $g(e)$. We also assume that $\phi\colon G \to \mathbb{Z}$ is an epimorphism, and that $f : X \to \mathbb{R}$ is a ϕ-equivariant Morse function (where $\mathbb{Z}$ acts on $\mathbb{R}$ by translations in the usual fashion). Let H denote the kernel of ϕ and observe that H acts freely and cocompactly on level sets X_t. We now investigate the relationship between finiteness properties of H and the homotopy types of ascending and descending links.

Theorem 4.1. *Let $f : X \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = Ker(\phi)$ be as above.*

(1) *Suppose that each $\uparrow$-link and each $\downarrow$-link is homologically n-connected. Then $H \in FH_{n+1}(R)$.*

(2) *If all $\uparrow$-links and $\downarrow$-links are R-acyclic, then $H \in \mathrm{FH}(R)$.*

(3) *If all $\uparrow$-links and $\downarrow$-links are simply connected, then H is finitely presented.* □

Proof. (1) By Corollary 2.6, for $t < s$ the inclusion $X_{(-\infty,t]} \hookrightarrow X_{(-\infty,s]}$ induces isomorphisms in $\tilde{H}_i$ for $i \le n$ and an epimorphism in $\tilde{H}_{n+1}$. Since $X = \cup_{r\in\mathbb{Z}} X_{(-\infty,r]}$ is acyclic, it follows that $\tilde{H}_i(X_{(-\infty,t]}) = 0$ for $i \le n$ and all t. Similarly, $\tilde{H}_i(X_{[t,\infty)}) = 0$ for $i \le n$ and all t. Since $X = X_{(-\infty,t]} \cup X_{[t,\infty)}$ is acyclic, it follows from Mayer-Vietoris that the intersection X_t is homologically n-connected. Thus H acts freely and cocompactly on a homologically n-connected complex and so is of type $\mathrm{FH}_{n+1}(R)$.

(2) It follows from the proof of (1) that X_t is acyclic.

(3) By (1) each X_t is connected. By Corollary 2.6 inclusion $X_t \hookrightarrow X$ induces an isomorphism in fundamental groups. Hence X_t is simply connected and H is finitely presented. □

5. Right-angled Artin groups

We define right angled Artin groups and their associated piecewise euclidean cubical complexes. The construction of the Eilenberg–Mac Lane spaces for the right angled Artin groups is not new. See for example [MV] and [CD].

Definition 5.1. *A simplicial complex L is said to be a flag complex if every finite collection of vertices of L which are pairwise adjacent spans a simplex in L. We see that such complexes L are completely determined by their* 1-*skeleton.*

It is worth noting that this definition does not impose any restrictions on the topology of L, since the first barycentric subdivision of a simplicial complex is a flag complex. For a proof see [Br 2], page 28.

Definition 5.2. *The* ***right angled Artin group*** G_L *associated to the finite flag complex* L *has finite generating set* $\{ g_1, \dots, g_N \}$ *in one to one correspondence with the vertex set* $L^{(0)} = \{ v_1, \dots, v_N \}$ *and has finite presentation*

$$G_L = \langle g_1, \dots, g_N \colon [g_i, g_j] = 1 \text{ for all edges } \{v_i, v_j\} \text{ in } L^{(1)} \rangle$$

Note that this is a subpresentation of the standard presentation of $\mathbb{Z}^N$, so that there is a natural epimorphism $G_L \to \mathbb{Z}^N$ taking the generators g_i to the standard basis elements of $\mathbb{Z}^N$. Composing this with the epimorphism

$$\mathbb{Z}^N \to \mathbb{Z} \colon (x_1, \dots, x_N) \mapsto \Sigma_i x_i$$

gives an epimorphism $\phi \colon G_L \to \mathbb{Z}$ which sends all the generators g_i to $1 \in \mathbb{Z}$.

Thus there is a short exact sequence of groups

$$1 \to H_L \to G_L \to \mathbb{Z} \to 1$$

and we are interested in finiteness properties of the kernel H_L.

We obtain information about H_L by constructing an Eilenberg-Mac Lane space Q_L for G_L which is a compact, piecewise euclidean cubical complex of nonpositive curvature, and a continuous map $Q_L \to S^1$ which induces the homomorphism $G_L \to \mathbb{Z}$. The universal cover X of Q_L is a CAT(0) metric space on which G_L acts cocompactly by deck transformations. The map $Q_L \to S^1$ lifts to a ϕ-equivariant Morse function

$$f \colon X \to \mathbb{R}$$

with the $\uparrow$-links and $\downarrow$-links all isomorphic to L, so the results of Section 4 apply to give one set of implications in the Main Theorem.

We begin with some definitions. Denote by $\square^n$ the *regular* n*-cube* in $\mathbb{R}^n$ with vertex at the origin, and edges defined by the unit basis vectors.

Definition 5.3. A *piecewise euclidean cubical complex is a cell complex constructed from a finite, disjoint collection of regular cubes, by glueing their faces via isometries. We shall refer to these as PE cubical complexes.*

Let us look at the local geometry of a PE cubical complex.

Definition 5.4. *Let* v *be a vertex of a regular cube* $\square^n$ *in* $\mathbb{R}^n$*. The euclidean metric on* $\mathbb{R}^n$ *gives an inner product on the tangent space* $T_v(\mathbb{R}^n)$*. We define the link of* v *in* $\square^n$ *(denoted by* $Lk(v, \square^n)$*) to be the set of unit tangent vectors* *at v which point into $\square^n$. It is a subset of the unit sphere S^{n-1} which is homeomorphic to an $(n-1)$-simplex. It has a natural spherical metric in which all dihedral angles are right angles. Such spherical simplices are termed all right simplices.*

Thus the link of a vertex in a PE cubical complex is a piecewise spherical cell complex, all of whose cells are all right simplices. The next definition gives a combinatorial version of nonpositive curvature for PE cubical complexes.

Definition 5.5. *A PE cubical complex is said to be nonpositively curved if the link of each vertex is a flag complex.*

The following lemma of Gromov [Gr] (p. 122) (see also [Da] Lemma 9.3) tells when a piecewise spherical cell complex, whose cells are all right simplices, satisfies the CAT(1) inequality. (Gromov's statement uses the "no triangle condition" instead of the "flag complex" condition below).

Lemma 5.6 (Gromov). *Let L be an all right, piecewise spherical complex. Then L is a $CAT(1)$ space if and only if it is a flag complex.* $\square$

A PE cubical complex whose links are all CAT(1) spaces satisfies a local CAT(0) inequality (see [Gr] p. 120 or [GH] p. 197), and so its universal cover is a CAT(0) metric space (and hence is contractible) — see [Gr] p. 119 or [GH] p. 193. There is a comprehensive survey of these results in section I of [Da].

The following definition will be useful in describing the link structure in our PE cubical complexes.

Definition 5.7. *Let L be a simplicial complex, and let*

$$\{ S^0_\alpha \}_{\alpha \in L^{(0)}}$$

be a collection of 0*-spheres indexed by the set of vertices in L. The spherical complex associated to L is denoted by* $\mathrm{S}(L)$ *and is defined to be the union*

$$\mathrm{S}(L) \;=\; \bigcup \{ S^0_{\alpha_0} * \cdots * S^0_{\alpha_m} : < \alpha_0, \ldots, \alpha_m > \text{ an } m\text{-simplex of } L \}$$

Lemma 5.8. *Let L be a flag complex. Then the associated spherical complex* $\mathrm{S}(K)$ *is also a flag complex.*

Proof. Let $\pi : \mathrm{S}(L)^{(0)} \to L^{(0)}$ be the 2-to-1 map which takes the two vertices of S^0_α to α for all $\alpha \in L^{(0)}$.

It is clear from the definition of $\mathrm{S}(L)$ that the pair of vertices in each 0-sphere S^0_α are not adjacent. Therefore, given a collection $\{v_0, \ldots, v_k\}$ of pairwise adjacent vertices in $\mathrm{S}(L)$ the set $\{\pi(v_0), \ldots, \pi(v_k)\}$ is a collection of distinct, pairwise adjacent vertices in L. These vertices span a simplex in L

(since L is a flag complex), and the original vertices span a simplex in the corresponding k-sphere in $\mathrm{S}(L)$. □

The following result about flag complexes will be used in local computations in Section 8. First we define the subcomplexes $St(\sigma, L)$ and $St'(\sigma, L)$ associated to a simplex σ in a simplicial complex L.

Definition 5.9. *Let L be a simplicial complex, and σ a simplex of L. We denote by $St(\sigma, L)$ the subcomplex of L consisting of all simplices which contain σ. This is called the closed star of σ in the literature.*

We denote by $St'(\sigma, L)$ the subcomplex of L consisting of all simplices which contain a face of σ.

Proposition 5.10. *Let L be a flag complex equipped with the all right $CAT(1)$ metric and let σ be a simplex of L.*

(1) *$St'(\sigma, L)$ is contractible.*

(2) *Let τ be another simplex of L and assume that $d(a, b) < \pi/2$ for a point $a \in \tau$ and a point b* of *σ. Then $\sigma \cap \tau \neq \emptyset$.*

Proof. We see that $St'(\sigma, L)$ is contractible as follows. For each simplex ρ in L, the closed star $St(\rho, L)$ is contractible as it is a union of simplices, any subcollection of which intersects in a common face containing ρ.

Given a collection $\{v_i, \ldots, v_k\}$ of vertices of σ let $\langle v_1, \ldots, v_k \rangle$ denote the face of σ which they span. Since L is a flag complex, we see that

$$St(v_1, L) \; \cap \; \cdots \; \cap St(v_k, L) \; = \; St(\langle v_1, \ldots, v_k \rangle, L)$$

is contractible. Thus the union

$$St'(\sigma, L) \; = \; \bigcup \{ \, St(v, L) : \; v \text{ a vertex of } \sigma \, \}$$

is also contractible.

For (2) note that the simplicial map from $St'(\sigma, L)$ to a 1-simplex (of length $\pi/2$), which maps σ to one vertex and the frontier $Fr(St'(\sigma, L))$ to the other vertex, is distance nonincreasing in the spherical metric. We see this by noting that the map restricts to a distance nonincreasing simplicial map on each simplex of $St'(\sigma, L)$.

Now suppose that $a \in \tau$ and $b \in \sigma$ are such that $d(a, b) < \pi/2$, and that $\sigma \cap \tau = \emptyset$. Then a geodesic γ from a to b intersects $Fr(St'(\sigma, L))$, and so $\gamma \cap St'(\sigma, L)$ maps onto the 1-simplex. But this implies that $d(a, b) \geq \pi/2$, a contradiction. □

The following lemma also concerns the spherical geometry of all-right piecewise spherical simplicial complexes. It is due to Gromov (see [Gr] p. 122 or [Da] Sublemma 3.13), and is the spherical geometry component of the proof of Lemma 5.6 above (Gromov's Lemma). We shall need this result in the proof of Lemma 8.3.

Lemma 5.11. *Let v be a vertex in an all-right, piecewise spherical simplicial complex, and let B be the ball (closed star) of radius $\pi/2$ about v. Let $x, y \in \partial B$ (the sphere of radius $\pi/2$ about v), and let γ be a geodesic from x to y which intersects the interior of B. Then the length of γ is at least π.* $\square$

The cubical complex Q_L

We define and describe properties of the PE cubical complex Q_L associated to a finite flag complex L. Specifically, we have.

Theorem 5.12. *Let L be a finite flag complex, G_L the associated right angled Artin group, and $\phi\colon G_L \to \mathbb{Z}$ the epimorphism defined above. Then there exists a nonpositively curved PE cubical complex Q_L and a map $l\colon Q_L \to S^1$ satisfying the following conditions*

(1) *l induces the homomorphism $\phi\colon G_L \to \mathbb{Z}$.*

(2) *Q_L has one vertex and the link of this vertex is isomorphic* to $\mathrm{S}(L)$.

(3) *The lift of l to the universal covers is a ϕ-equivariant Morse function $f : X \to \mathbb{R}$.*

(4) *All $\uparrow$- and $\downarrow$-links of X (with respect to the Morse function f) are isomorphic* to L.

Note that since the PE cubical complex Q_L is nonpositively curved, it is a $K(G_L, 1)$.

Proof. First construct a geometric realization of L in $\mathbb{R}^N$ (N equals the number of vertices of L) in the usual way; mapping each vertex v_i of L to the (endpoint of the) basis vector e_i and mapping an m-simplex

$$\sigma = \{ v_{i_0}, \ldots, v_{i_m} \}$$

to the convex hull in $\mathbb{R}^N$ of the images of the v_{i_j}. For each such σ define $\square_\sigma$ to be the regular $(m+1)$-cube based at the origin in $\mathbb{R}^N$ with edges defined by the basis vectors $\{ e_{i_j} \colon \; v_{i_j} \in \sigma \}$.

Now define Q_L to be the image of the union of cubes

$$\bigcup \{ \square_\sigma \;\; : \;\; \sigma \text{ a simplex of } L \}$$

under the projection $\mathbb{R}^N \to T^N = \mathbb{R}^N/\mathbb{Z}^N$. Here $\mathbb{Z}^N$ acts on $\mathbb{R}^N$ by translations in the usual fashion, so T^N is just the standard N-torus.

Clearly, Q_L is a PE cubical complex. Since the vertices of each $\square_\sigma$ have integer coordinates, the complex Q_L has just one vertex. Each m-simplex σ in

L corresponds to an $(m+1)$-cube $\square_\sigma$ in $\mathbb{R}^N$ which descends to a torus in the quotient Q_L. This torus contributes an m-sphere (triangulated as the $(m+1)$-fold join of 0-spheres) to the link of the single vertex in Q_L. Thus the link of this vertex in Q_L is precisely the complex $\mathrm{S}(L)$. This proves property (2).

Since L is a flag complex, Lemma 5.6 ensures that the link $\mathrm{S}(L)$ in Q_L is also flag, and so Q_L is nonpositively curved.

The linear map

$$l\colon \mathbb{R}^N \to \mathbb{R}\colon (x_1, \dots, x_N) \mapsto x_1 + \cdots + x_N$$

descends to a continuous map (fibration) $\mathbb{R}^N/\mathbb{Z}^N \to S^1$. Restricting to the subcomplex Q_L of the N-torus, gives a continuous map $Q_L \to S^1$ which we also denote by l.

The fundamental group of Q_L is easily computed from the 2-skeleton and is the Artin group G_L. The map l sends each basis vector in $\mathbb{R}^N$ to the unit interval in $\mathbb{R}$, and so l_* maps generators of $\pi_1(Q_L)$ to $1 \in \mathbb{Z}$. Thus l_* is the homomorphism $\phi\colon G_L \to \mathbb{Z}$, and (1) holds.

Lifting l to the universal covers gives a continuous map

$$f\colon X \to \mathbb{R}$$

which is clearly ϕ-equivariant. In order to establish (3), it remains to prove that f is a Morse function on X.

From the definition of f we see that for each m-cell e of X

$$f\chi_e = \tau l$$

where the attaching map χ_e may need to be precomposed with an isometry $\square^m \to \square^m$, and where l: $\square^m \to \mathbb{R}$ is the restriction of the linear map

$$\mathbb{R}^m \to \mathbb{R}\colon (x_1, \dots, x_N) \mapsto x_1 + \cdots + x_N$$

and $\tau\colon \mathbb{R} \to \mathbb{R}$ is an integer translation. Note that $f\chi_e$ is constant only when e is a vertex of X. Also, the f image of the 0-skeleton $X^{(0)}$ is just $\mathbb{Z} \subset \mathbb{R}$. Thus f is a Morse function on X.

Finally, note that the local picture of X at a vertex can be embedded in $\mathbb{R}^N$ by mapping the vertex to the origin. The local picture of f at this vertex is then given by the linear map $l\colon \mathbb{R}^N \to \mathbb{R}$ defined above. Thus one sees that the $\uparrow$- and $\downarrow$-links are isomorphic to L. $\square$

6. Sheets

In this section we define and record properties of *sheets*. These are special flats in the universal cover X of Q_L, which we view as building blocks. This point of view enables us to determine the homology and homotopy type of (sub)level sets in Sections 7 and 8. We also state the Main Theorem and give some examples.

Recall that Q_L is the union of tori, one for every simplex of L.

Definition 6.1. *A sheet in X is a flat which is a subcomplex of the preimage in X of one of the tori in Q_L.*

Note that in general there are flats in X that are not sheets. Indeed, when L consists of 2 points, X is a tree, and there are countably many sheets, but uncountably many biinfinite geodesics.

Alternatively, we can describe sheets as follows. Any simplicial map $L' \to L$ induces a map $Q_{L'} \to Q_L$. When L' is a single simplex of L, this map can be regarded as inclusion to a subcomplex (with underlying space a torus). A *sheet* is a component of the preimage in X of such a subcomplex as L' ranges over all simplices in L.

Given a vertex $v \in X$ and $A \subset X$ a union of sheets, we define $St(v, A)$ to be the cone on $Lk(v, A)$, and consider it as a small neighborhood of v in A. Similarly, $St_\downarrow(v, A)$ is just the cone on $Lk_\downarrow(v, A)$. We also use St to denote the closed star of a simplex in a simplicial complex as in Proposition 5.10. It should be clear from the context what the notation refers to.

There is a natural retraction $r_v\colon Lk(v, X) \to Lk_\downarrow(v, X)$. This map takes vertices in $Lk_\uparrow(v, X)$ to corresponding vertices in $Lk_\downarrow(v, X)$ by a central symmetry in v. It fixes the vertices of $Lk_\downarrow(v, X)$, and extends simplicially to give a map: $Lk(v, X) \to Lk_\downarrow(v, X)$. The map r_v can be coned off to give a retraction of neighborhoods $St(v, X) \to St_\downarrow(v, X)$ also denoted by r_v.

We record the following properties of sheets.

Proposition 6.2. (1) *X is covered by the sheets.*

(2) *The intersection of any collection of sheets is either empty, a vertex, or a sheet.*

(3) *All (sub)level sets of f restricted to a sheet are contractible. Moreover, all $\uparrow$- and $\downarrow$-links of the restriction are single simplices.*

(4) *The retraction r_v preserves sheets through v. More precisely, if $A \subseteq X$ is a union of sheets and $v \in A$, then the restriction of r_v induces retractions $Lk(v, A) \to Lk_\downarrow(v, A)$ and $St(v, A) \to St_\downarrow(v, A)$ and these are also denoted r_v. Moreover, $r_v^{-1}(Lk_\downarrow(v, A)) = Lk(v, A)$.*

Proof. Property (1) follows immediately from the definition of sheets and the fact that Q_L is a union of tori.

We see that property (2) holds for the intersection of any two sheets, and proceed by induction.

The restriction of f to a k-dimensional sheet is given by the linear map

$$l : \mathbb{R}^k \to \mathbb{R} : (x_1, \ldots, x_k) \mapsto x_1 + \cdots + x_k$$

where $\mathbb{R}^k$ has the usual cubing. So property (3) clearly holds.

Property (4) is immediate from the definitions of sheet and of the map r_v. $\square$

Here is the Main Theorem and some examples.

Main Theorem. *Let L be a finite flag complex. Let $G = G_L$ be the associated right angled Artin group, and $\phi: G \to \mathbb{Z}$ the homomorphism with kernel $H = H_L$ as in Theorem* 5.12. *We compute homology with coefficients in a ring R with* $0 \neq 1$.

(1) $H \in \mathrm{FP}_{n+1}(R)$ *if and only if L is homologically n-connected.*

(2) $H \in \mathrm{FP}(R)$ *if and only if L is acyclic.*

(3) *H is finitely presented if and only if L is simply connected.*

Note that all 3 implications $\Leftarrow$ follow from Theorem 5.12 and Theorem 4.1. In the next two sections we shall develop an understanding of (sub)level sets which is sufficient to prove the three implications $\Rightarrow$.

Examples 6.3.

(1) For $L = S^n$ and any R, H_L is of type FP_n but not of type FP_{n+1}. Triangulating S^n as the $(n+1)$-fold join of 0-spheres, the group G_L is F_2^{n+1}, the $(n+1)$-fold product of free groups of rank 2, and the homomorphism to $\mathbb{Z}$ sends each basis element of each coordinate F_2 to $1 \in \mathbb{Z}$. For $n = 1, 2$ this is the Stallings' example [St], and for all n this is due to Bieri [Bi 3].

(2) Let p be a prime and let L be the (appropriately triangulated) Moore space obtained from the n-sphere ($n \geq 1$) by attaching an $(n+1)$-cell via a degree p attaching map. Then L is acyclic over any field of characteristic $\neq p$, but over fields of characteristic p it is only $(n-1)$-connected. Thus H_L is of type FP over fields of characteristic $\neq p$. Over a field of characteristic p H_L is of type FP_n but not FP_{n+1}. For example, for $p = 3$ and $n = 1$ H_L is an example of a group that is not finitely presented (and not even of type $\mathrm{FP}_2(\mathbb{Z}/3)$) but is of type $\mathrm{FP}(R)$ whenever $3 \in R$ is invertible.

Bieri and Strebel [BS] have constructed a group that is of type $\mathrm{FP}_2(F)$ for any field F, but is not of type $\mathrm{FP}(\mathbb{Z})$.

(3) Let L be an acyclic nonsimply-connected finite flag complex, say of dimension 2. Then H_L is FP (over all rings) but it is not finitely presented. This example answers in the negative a long standing question concerning finiteness properties of groups which first appeared in [Bi 2] Chapter I (2).

Furthermore, the cohomological dimension of H_L is 2 since the level sets are 2-dimensional and acyclic. It is natural to ask if such groups H_L have 2-dimensional Eilenberg-Mac Lane spaces. Thus one obtains a family of potential counterexamples to the Eilenberg-Ganea conjecture.

If L is a flag triangulation of a spine of the Poincare homology sphere, then in Theorem 8.7 we show that H_L is either a counterexample to the Eilenberg-Ganea conjecture or that there is a counterexample to the Whitehead conjecture.

Recall that the Eilenberg-Ganea conjecture states that if a group H has cohomological dimension 2, then it has a 2-dimensional Eilenberg-Mac Lane space $K(H,1)$. The Whitehead conjecture states that every connected subcomplex of an aspherical 2-complex is aspherical.

7. Homology of (sub)level sets

We now fix a finite flag complex L and the associated right angled Artin group $G = G_L$. Let $X = \widetilde{Q}_L$, $\phi\colon G \to \mathbb{Z}$, $H = H_L$, and $f\colon X \to \mathbb{R}$ be as in Theorem 5.12. The goal of this section is to compute homology groups of sets X_J for a nonempty closed connected $J \subseteq \mathbb{R}$.

Theorem 7.1. *Let A be a* (*possibly infinite*) *union of sheets and vertices in X, and denote by A_J the intersection of A and the set X_J. Then*

$$H_*(A, A_J) \cong \bigoplus_{v \notin A_J} H_*(St_\downarrow(v, A), Lk_\downarrow(v, A))$$

where the sum ranges over the vertices in A that are not contained in A_J.
Since $H_*(\emptyset, \emptyset) = 0$ we could be summing over all vertices in X not contained in X_J.

Proof. We first construct a homomorphism

$$\Psi_A\colon H_*(A, A_J) \to \bigoplus_{v \notin A_J} H_*(St_\downarrow(v, A), Lk_\downarrow(v, A))$$

and then show it is an isomorphism.

The coordinate of Ψ_A corresponding to $v \in A \setminus A_J$ is defined as the composition

$$H_*(A, A_J) \to H_*(A, A \setminus Int\ St(v, A)) \cong H_*(St(v, A), Lk(v, A)) \xrightarrow{r_{A*}} H_*(St_\downarrow(v, A), Lk_\downarrow(v, A)).$$

An element x of $H_*(A, A_J)$ can be represented by a relative cycle supported in a finite subcomplex of A, and then the coordinates of $\Psi_A(x)$ corresponding to the vertices outside the support are 0. Therefore, Ψ_A is a well-defined homomorphism into the direct sum.

The homomorphism Ψ_A is natural, i.e. if $A \subset A'$ and $A'_J = A' \cap X_J$, then the diagram

$$\begin{array}{ccc} H_*(A, A_J) & \xrightarrow{\Psi_A} & \bigoplus H_*(St_\downarrow(v, A), Lk_\downarrow(v, A)) \\ \downarrow & & \downarrow \\ H_*(A', A'_J) & \xrightarrow{\Psi_{A'}} & \bigoplus H_*(St_\downarrow(v, A'), Lk_\downarrow(v, A')) \end{array}$$

commutes (the vertical arrows are induced by inclusion).

We now prove that Ψ_A is an isomorphism. Since homology commutes with direct limits, it suffices to prove this when A is a finite union of sheets and vertices of X. For such A we proceed by induction on the number of sheets and vertices in the union. When A is empty, a single sheet, or a vertex in A_J then both groups are zero. When A is a vertex in $A \backslash A_J$, then the only nonvanishing groups are in dimension 0, they are both isomorphic to the coefficient group, and Ψ_A is an isomorphism.

Now suppose that Ψ_A is an isomorphism whenever A is a union of $\leq k$ sheets and vertices, and let A be a union of $k+1$ sheets and vertices. Write $A = A' \cup S$ where S is a sheet or a vertex, and A' is a union of k sheets and vertices. Note that $A' \cap S$ is a union of $\leq k$ sheets and vertices (by Proposition 6.2(2)) and so $\Psi_{A' \cap S}$, $\Psi_{A'}$, and Ψ_S are isomorphisms. That Ψ_A is an isomorphism now follows from the Mayer-Vietoris sequence, the naturality of Ψ and the 5-lemma. $\square$

As a corollary, we have the following description of the homology of level and sublevel sets.

Corollary 7.2. *There are isomorphisms of* RH*-modules*

(1) $\tilde{H}_*(X_{(-\infty,t]}) \cong \bigoplus_{v \in X_{(t,\infty)}} \tilde{H}_*(L)$.

(2) $\tilde{H}_*(X_t) \cong \bigoplus_{v \notin X_t} \tilde{H}_*(L)$. $\square$

We can now prove two more implications in the Main Theorem.

Proof of (1) *and* (2) *of Main Theorem*. Let n be the smallest integer such that $\tilde{H}_n(L) \neq 0$. Then $\tilde{H}_n(X_t)$ is not a finitely generated RH-module, so the claim follows from Lemma 3.8.

8. Homotopy type of (sub)level sets

In this section we see how to construct X using sheets; starting with sheets through a given vertex, and then adding sheets which pass through neighboring vertices. This view of X enables us to determine the homotopy type of the (sub)level sets in Theorem 8.6 below, and hence to prove part (3) of the Main Theorem.

We begin with some lemmas describing the intersections of certain collections of sheets and (sub)level sets.

Lemma 8.1. *Let w be a vertex of X and let K be the union of a collection of sheets containing w. Let J be a closed and connected interval in* $\mathbb{R}$. *Then*

(1) *K is contractible.*

(2) *All $\uparrow$- and $\downarrow$-links of K are contractible except possibly at w, and at w the two are naturally isomorphic.*

(3) *$K_J = K \cap X_J$ is homotopy equivalent to $Lk_{\downarrow}(w, K)$ when $w \notin X_J$ and it is contractible if $w \in X_J$.*

Proof. K is a cone with conepoint w, hence contractible. We prove (2) for $\downarrow$-links, the other case being similar. Let x be any vertex of K different from w. There is a sheet $S \subseteq K$ that contains w and x and is minimal with this property. Thus $Lk_{\downarrow}(x, K)$ is the union of the simplex $\sigma = Lk_{\downarrow}(x, S)$ and the simplices of the form $Lk_{\downarrow}(x, S')$ where S' ranges over sheets in K that pass through x and w. All these simplices contain σ, so the union is contractible.

There is a natural central symmetry defined on each sheet S through w. This maps $Lk_{\uparrow}(w, S)$ bijectively to $Lk_{\downarrow}(w, S)$, and these combine to give the isomorphism between $Lk_{\uparrow}(w, K)$ and $Lk_{\downarrow}(w, K)$.

We now prove (3). First suppose that $w \in X_J$. Then it follows from Lemma 2.5 that the inclusion $K_J \hookrightarrow K$ is a homotopy equivalence, and thus K_J is contractible. If $w \notin X_J$, assume for concreteness that w is *above* X_J, i.e. that $f(w) > t$ for all $t \in J$. Choose $\epsilon \in (0, 1)$ such that $f(w) - t > \epsilon$ for all $t \in J$. Then again from Lemma 2.3 it follows that inclusions

$$K_J \hookrightarrow K_{(-\infty, f(w)-\epsilon]} \hookleftarrow K_{f(w)-\epsilon}$$

are homotopy equivalences. Finally, to show that $K_{f(w)-\epsilon}$ is homotopy equivalent to $Lk_{\downarrow}(w, K)$ observe that the correspondence

$$Lk_{\downarrow}(w, S) \leftrightarrow S \cap X_{f(w)-\epsilon}$$

is a 1-1 order preserving correspondence between elements of closed covers of $Lk_{\downarrow}(w, K)$ and $K_{f(w)-\epsilon}$ both of which are closed under intersections and have contractible elements. Hence $Lk_{\downarrow}(w, K)$ and $K_{f(w)-\epsilon}$ are homotopy equivalent (see [BoS] Sect. 8.2). □

The next lemma shows that the minimum distance from a vertex in X to a cube in X is attained at a vertex of the cube. It is used in Lemmas 8.4 and 8.5 below.

Lemma 8.2. *Let X be a CAT (0) PE cubical complex, $Q \subset X$ a cube, and $c \in X$ a vertex. Then the minimal distance from c to Q is attained at a unique point of Q. Moreover, this point is a vertex of Q.*

Proof. Suppose there are two points x and y of Q which realize this minimum distance. Then the geodesic $[x, y]$ lies in Q, since cubes are convex in X. Applying the CAT(0) inequality to the geodesic triangle xyc we see that all interior points of $[x, y]$ are closer to c than either of the endpoints. This contradicts the choice of x and y.

Let γ be a distance minimizing geodesic from c to Q, and suppose that γ does not end at a vertex of Q. Then γ meets Q in the interior of some face. Let e denote an edge of this face.

For each point $p \in \gamma$ choose the minimal cube Q_p in X which contains p. Consider the collection of all such cubes Q_p for $p \in \gamma$.

We claim that there is a cellular map from the union of such cubes to a 1-simplex which takes γ to an interior point of the simplex. Hence, c could not be a vertex, and the Lemma is proven.

Let Q_0 be the cube which contains the edge e and the initial segment of the geodesic γ, and is minimal among cubes with this property. Here we parameterize γ so that it starts at Q and ends at c. This cube contains the face of Q at which γ originates, and so has a metric product structure, defined as the product of a codimension one face times the edge e. Since γ minimizes the distance from c to Q we see that $Q_0 \cap \gamma$ is perpendicular to the e-coordinate.

The metric product structure induces a codimension-1 metric product structure on the face of intersection of Q_0 and the next cube Q_1 of the collection along the geodesic γ. This extends uniquely to a codimension-1 product structure of Q_1 and so on.

Finally, we map the e-coordinate onto a 1-simplex, collapsing the other coordinates to a point. The geodesic γ remains perpendicular to the e-coordinate throughout, and so maps to an interior point of the 1-simplex. □

Now we begin to build up X as a union of sheets. Choose a base vertex v in X and an ordering $v = v_1, v_2, v_3, \cdots$ of all vertices in X such that $d(v, v_i) \leq d(v, v_j)$ whenever $i \leq j$. Denote by K_i the union of *all* sheets through v_i. We will study the homotopy type of $K_1 \cup K_2 \cup \cdots \cup K_n$ by induction on n. To that end, we fix $n > 1$, let $w = v_n$, and let $K = K_n \cap (K_1 \cup K_2 \cup \cdots K_{n-1})$.

In Lemma 8.4 below, we give a description of K as the union of a collection of sheets containing v_n. This together with Lemma 8.1 ensures that K is contractible. This will be important in proving Theorem 8.6. First, we need to prove that sets like the K_i are convex in X.

Lemma 8.3. *Let $x \in X$ be a vertex, and let K_x denote the union of all sheets through x. Then K_x is a convex subset of X.*

Proof. We break the proof into a number of steps. The first step gives a local formulation of the problem in terms of convexity of subcomplexes of all-right piecewise spherical simplicial complexes. In the second step we show that the purely combinatorial condition of "fullness" ensures the convexity of a subcomplex of an all-right piecewise spherical simplicial complex. Steps 3 and 4 reduce the problem to that of checking that

$$Lk_{\downarrow}(y, K_x) \ \subset \ Lk_{\downarrow}(y, X) \ = \ L$$

is a full subcomplex, for all vertices $y \in K_x$. Finally, in step 5 we calculate these $\downarrow$-links, and complete the proof.

Step 1. (Local formulation of convexity) We show that K_x is convex in X by checking that each geodesic in K_x is also a geodesic in X.

Note that a segment of a geodesic which lies inside a cube in K_x will also be locally a geodesic in X. So we only need check the points where the geodesic in K_x passes through vertices of X, or passes from one cube to another through a common face.

Suppose a geodesic in K_x passes through a vertex y. In order for it to be a local geodesic in X we need that the distance in $Lk(y, X)$ between the two directions defined by this path should be at least π. This will hold if

$$Lk(y, K_x) \subset Lk(y, X) \qquad \text{is convex} \quad \text{(A)}$$

where we say that a subcomplex M of an all-right piecewise spherical simplicial complex N is *convex* if it satisfies the following: For all points $a, b \in M$ such that $d_N(a, b) < \pi$, the geodesic $[a, b]$ is contained in M.

Similarly, if a geodesic in K_x intersects a cube Q in a single interior point, then it will be a local geodesic in X provided that the link of Q in K_x is a convex subcomplex of the link of Q in X. Let y be any vertex of Q, and let σ_Q be the simplex of $Lk(y, X)$ determined by Q. There is a natural identification of the link of Q in K_x with $Lk(\sigma_Q, Lk(y, K_x))$. Likewise the link of Q in X can be identified with $Lk(\sigma_Q, Lk(y, X))$, and so the convexity condition becomes

$$Lk(\sigma_Q, Lk(y, K_x)) \subset Lk(\sigma_Q, Lk(y, X)) \qquad \text{is convex} \quad \text{(B)}$$

for all cubes Q in K_x and vertices $y \in Q$.

So we only have to verify conditions (A) and (B) above. In the following steps we reduce this to verifying a simple combinatorial condition at $\downarrow$-links.

Step 2. (Combinatorial condition for convexity in all-right piecewise spherical complexes) In this step we prove the following. Let $M \subset N$ be a full subcomplex of an all-right piecewise spherical simplicial complex N. Then M is convex in N.

Recall that a subcomplex $M \subset N$ is said to be *full* if it satisfies the following: If a set of vertices of M spans a simplex $\tau \subset N$, then $\tau \subset M$.

Suppose that $M \subset N$ is full, and that $a, b \in M$ satisfy $d_N(a, b) < \pi$. We have to show that the geodesic $[a, b]$ is contained in M. Let σ_a (respectively σ_b) denote the minimal simplex of N which contains a (respectively b). Note that σ_a and σ_b are contained in M.

Now either $a = b$ and the result is trivial, or $a \neq b$ and so the collection of all simplices σ which intersect $[a, b]$ nontrivially in their interiors is not empty. We claim that the vertices of such σ are contained in the union of the set of vertices of σ_a and the set of vertices of σ_b. Hence, $\sigma \subset M$ by fullness. But $[a, b]$ is contained in the union of such σ, and so $[a, b] \subset M$ as required.

Suppose σ is a simplex of N whose interior intersects $[a, b]$ nontrivially, and let $v \in \sigma$ be a vertex. The path $[a, b]$ intersects the open star about v (=open ball in N of radius $\pi/2$ about v). Now if both $d_N(v, a)$ and $d_N(v, b)$ are at least $\pi/2$ Lemma 5.11 implies that the length of $[a, b]$ is at least π contradicting the assumption $d_N(a, b) < \pi$. Thus one of $d_N(a, v)$ and $d_N(b, v)$ is strictly less than $\pi/2$. Suppose $d_N(a, v) < \pi/2$. Then either $a = v$ or a lies in the interior of a simplex with vertex v. In either case, v is a vertex of σ_a, and we are done.

Step 3. (Reduction to vertices of K_x) Step 2 implies that, in order to check that properties (A) and (B) hold, we need only verify that

$$Lk(y, K_x) \text{ is a full subcomplex of } Lk(y, X) \qquad (\mathrm{A}')$$

for all vertices $y \in K_x$ and

$$Lk(\sigma_Q, Lk(y, K_x)) \text{ is a full subcomplex of } Lk(\sigma_Q, Lk(y, X)) \qquad (\mathrm{B}')$$

for all cubes $Q \subset K_x$ and vertices $y \in Q$.

But property (B$'$) follows from property (A$'$) by the following observation. Let M be a full subcomplex of the simplicial complex N, and let $\sigma \subset M$ be a simplex. Then $Lk(\sigma, M)$ is a full subcomplex of $Lk(\sigma, N)$.

Step 4. (Reduction to $\downarrow$-links) Property (A$'$) reduces to verifying the following statement.

$$Lk_{\downarrow}(y, K_x) \text{ is a full subcomplex of } Lk_{\downarrow}(y, X) = L \qquad (\mathrm{A}'')$$

for each vertex $y \in K_x$.

This reduction follows from the following simple observation. Let M be a full subcomplex of the simplicial complex N. Then the associated spherical complex S(M) is a full subcomplex of the associated spherical complex S(N).

Step 5. ($\downarrow$-link computations) Finally, we verify property (A$''$). Let $y \in K_x$ be a vertex. If $y = x$ then $Lk_{\downarrow}(y, K_x) = Lk_{\downarrow}(y, X)$ is clearly convex. So we assume that $y \neq x$. Then the geodesic $[x, y]$ defines a unique point of $Lk(y, X)$. Taking the r_y image gives a point $p \in L = Lk_{\downarrow}(y, X)$. Since K_x is a geodesic cone on x we see that $p \in Lk_{\downarrow}(y, K_x)$, and that in fact $Lk_{\downarrow}(y, K_x)$ is just the union U of all simplices $\sigma \subset L$ which contain p. Note that simplicial structure on U can be described by: τ is a simplex of U if and only if there exists a simplex σ containing both τ and p.

This is seen to be full in L as follows. Let σ_p denote the minimal simplex of L which contains p. Then σ_p is a face of each simplex containing p. In particular, if $v \in U$ is a vertex, then either $v \in \sigma_p$ or $\{v\} \cup \sigma_p$ spans a simplex. Hence, if $\{v_1, \dots, v_j\} \subset U$ is a collection of vertices which span a simplex $\sigma \subset L$, then $\{v_1, \dots, v_j\} \cup \sigma_p$ spans a simplex of U. Thus $\sigma \subset U$, and so U is a full subcomplex of L. □

Lemma 8.4. *Let $K = K_n \cap (K_1 \cup \cdots \cup K_{n-1})$ be as above. Then K is the union of those sheets that contain w and at least one v_j with $j < n$.*

Proof. This union is contained in K, so we need only prove the reverse inclusion. That is, given $x \in K$, there exists a sheet $S \subset X$ through v_n which contains some v_j ($j < n$) and contains x.

First we prove this for vertices $v \in K$. Now $v \in K$ implies that $v \in K_n \cap K_i$ for some $i < n$. Thus $v_i, v_n \in K_v$ where K_v denotes the union of all sheets through v.

By Lemma 8.3 K_v is convex, and so contains the geodesic $[v_i, v_n]$. Let Q denote the minimal cube containing v_n and the end segment of the geodesic $[v_i, v_n]$. By minimality, $Q \subset K_v$ and so Q lies in a sheet S through v. We claim that one of the vertices of Q is in the set $\{v_1, \dots, v_{n-1}\}$. Thus the sheet S contains v, v_n and some v_j ($j < n$) as required.

To see this claim, note that the ordering on the v_i implies that the geodesic $[v_1, v_i]$ is not longer than $[v_1, v_n]$, and so the CAT(0) inequality implies that the distance from v_1 to any interior point of $[v_i, v_n]$ is strictly smaller than $d(v_1, v_n)$. In particular this is true for the points of Q which lie in the interior of $[v_i, v_n]$. Thus v_n is not the closest point of Q to v_1, and so Lemma 8.2 ensures that one of the other vertices of Q is v_j for some $j < n$.

Finally, suppose that $x \in K$ is not a vertex. Then $x \in S' \subset K_n \cap K_i$ for some sheet S' and some $i < n$. We know that each vertex of S' is contained in a sheet through v_n and some v_j ($j < n$). Since there are only finitely many sheets through v_n, then one such sheet S must contain a maximal, general position subset of vertices of S', and hence contains all of S' as required. $\square$

Next we investigate the link of $w = v_n$ in K.

Lemma 8.5. *$Lk_{\downarrow}(w, K) \cong Lk_{\uparrow}(w, K)$ is contractible.*

Proof. Denote by a the point in $Lk(w, X)$ determined by the geodesic $[w, v]$ and let $b = r(a) \in Lk_{\downarrow}(w, X)$ be the image of a under the retraction r_w (see section 6). Let σ denote the smallest simplex of $Lk_{\downarrow}(w, X)$ that contains b.

Let $\mathscr{T}$ denote the collection of simplices τ in $L = Lk_{\downarrow}(w, X)$ such that the sheet through w corresponding to τ contains one of the v_i $(i \leq n-1)$. By Lemma 8.4 $Lk_{\downarrow}(w, K)$ and $Lk_{\uparrow}(w, K)$ are isomorphic to the simplicial complex determined by $\mathscr{T}$. Claim 2 below implies that this complex is actually just $St'(\sigma, L)$ which is contractible by Proposition 5.10 (1).

Claim 1. Every face of σ is in the collection $\mathscr{T}$.

Indeed, let S_0 be the sheet through w such that $Lk_{\downarrow}(w, S_0) = \sigma$. We know from Proposition 6.2 (4) that $Lk(w, S_0)$ contains a. Let Q be the smallest cube that contains w and such that $Lk(w, Q)$ contains a. Thus $Q \subset S_0$ and $\sigma = r_w(Lk(w, Q))$. We shall show that each vertex of Q which is distance one from w is in the collection $\{v_1, \ldots, v_{n-1}\}$. Hence, σ and every face of σ is in $\mathscr{T}$.

Let v' be a vertex of Q which is distance one from w. Then the angle at w defined by $[w, v]$ and $[w, v']$ is less than $\pi/2$. Thus the distance from v to a point on the edge $[w, v']$ which is close to w is less than $d(v, w)$. Lemma 8.2 implies that

$$d(v, v') = d(v, [w, v']) < d(v, w)$$

and so $v' \in \{v_1, \ldots, v_{n-1}\}$.

Claim 2. $\mathscr{T} = \{\, \tau \text{ a simplex of } L\colon\ \tau \cap \sigma \neq \emptyset \,\}$

For the inclusion $\subset$, let τ be an element of $\mathscr{T}$, and let S be the corresponding sheet in K. Then S contains v_i for some $i < n$. In the triangle $\Delta(w, v, v_i)$ we have $d(v, w) \geq d(v, v_i)$ and thus the angle at w is $< \pi/2$. It follows that $Lk(w, S)$ contains a point within distance $< \pi/2$ from the point a defined above, and hence $\tau = Lk_{\downarrow}(w, S) = r_w(Lk(w, S))$ contains a point within distance $< \pi/2$ from $b = r_w(a)$. Thus it must intersect σ by Proposition 5.10 (2).

Now for the reverse inclusion. It is clear that if $S_1 \subset S_2$ are sheets and $Lk_{\downarrow}(w, S_1)$ is in $\mathscr{T}$, then $Lk_{\downarrow}(w, S_2)$ is also in $\mathscr{T}$. In particular, if a vertex of a simplex τ of L is in $\mathscr{T}$, then τ is in $\mathscr{T}$. If $\tau \cap \sigma \neq \emptyset$ then τ contains a vertex of σ. But we've seen that all vertices of σ are in $\mathscr{T}$ in Claim 1. Hence $\tau \in \mathscr{T}$. $\square$

Now we are in a position to determine the homotopy type of sublevel sets X_J.

Theorem 8.6. *X_J is homotopy equivalent to the wedge of L's, one for every vertex of X not in X_J.*

Note: When L is connected, the choice of basepoints is irrelevant. When L is disconnected, different copies of L in the above wedge might have basepoints in different components.

Proof. We write X_J as the increasing union of sets of the form

$$X(n) = X_J \cap (K_1 \cup K_2 \cup \cdots \cup K_n)\,.$$

We may express $X(n)$ as the union $X(n) = X(n-1) \cup (X_J \cap K_n)$. Note that the intersection

$$X(n-1) \cap (X_J \cap K_n) \;=\; X_J \cap (K_n \cap (K_1 \cup K_2 \cup \cdots \cup K_{n-1}))$$

is contractible. This follows from Lemma 8.1 (3) (setting $K_w = K_n \cap (K_1 \cup K_2 \cup \cdots \cup K_{n-1})$) and Lemma 8.5.

Now Lemma 8.1 (3) implies that $X_J \cap K_n$ is either contractible or homotopy equivalent to L depending on whether or not $v_n \in X_J$. Thus, it follows inductively that $X(n)$ is homotopy equivalent to the wedge of L's, one for every v_i, $i \leq n$ not contained in X_J. Further, inclusions $X(n-1) \hookrightarrow X(n)$ respect the wedge structure, so the theorem follows. $\square$

We can now complete the proof of the Main Theorem.

Proof of (3) of Main Theorem. If L is not connected then part (1) of the Main Theorem implies that H will not be finitely generated, and hence not finitely presented.

If L is connected but not simply-connected, then $\pi_1(X_t)$ is the free product of $\pi_1(L)$'s, one for every vertex not in X_t. If H were finitely presented, then Proposition 3.9 would imply that $\pi_1(X_t)$ is generated by the H translates of finitely many loops. Since X is contractible, each of these finitely many loops is null homotopic in X, and hence is null homotopic in some block $X_{[t-T,t+T]}$. Since H acts by "horizontal" translations, all the H-orbits of these loops will be null homotopic in $X_{[t-T,t+T]}$. This implies that the inclusion $X_t \hookrightarrow X_{[t-T,t+T]}$ induces the trivial map on π_1. But Corollary 2.6 (3) says that this inclusion induces an epimorphism in π_1, and thus $X_{[t-T,t+T]}$ is simply connected, contradicting Theorem 8.6 above. $\square$

We end this section with an examination of the level sets in the case when L is a flag triangulation of a spine of the Poincaré homology sphere. We have the following result which shows that at least one of the Eilenberg-Ganea and Whitehead conjectures must be false.

Theorem 8.7. *Let L be a flag triangulation of a spine of the Poincare homology sphere. Then either H_L is a counterexample to the Eilenberg-Ganea conjecture or there is a counterexample to the Whitehead conjecture.*

We begin by observing that there are 'arbitrarily large' copies of L which are quasi-isometrically embedded in the level set X_t. Given a vertex $v \in X$ we identify the complex L with the descending link $Lk_{\downarrow}(v, X)$. Thus each point $x \in L$ determines a unique geodesic $g_x \subset K_v$ through v.

Definition 8.8. *Let $t \in \mathbb{R}$ and $v \in X$ be a vertex such that $f(v) > t$, and $M \subset L$ be a subcomplex of $L = Lk_{\downarrow}(v, X)$. Define the shadow of M on X_t to be the set*

$$S_{v,M} \;=\; \bigcup \{\, g_x : \; x \in M \,\} \;\cap\; X_t$$

Note that $S_{v,L}$ is homeomorphic to L, by the map which sends each simplex $\sigma \subset L$ to $S_{v,\sigma} \subset S_{v,L}$. Give L a metric by requiring that each 2-simplex is an equilateral triangle with sidelengths equal to $|f(v) - t|$ and then taking the induced path metric. The homeomorphism above becomes a quasi-isometry, with constants which are independent of $|f(v) - t|$. To see this note that this is obviously true if $S_{v,L}$ inherited its metric from K_v (the quasi-isometry constants would be bounded by a multiple of the cardinality of the 1-skeleton of L). But K_v is a convex subset of X by Lemma 8.3. Hence we obtain the same quasi-isometry inequality for $S_{v,L}$ as a subset of X.

Proof of Theorem 8.7. If H_L does not have geometric dimension 2 we are done. Otherwise, let Y be a contractible 2-complex on which H_L acts freely, faithfully, properly and cellularly. Since Y is contractible we can define an H_L-equivariant PL map $\phi : X_t \to Y$.

Now X_t and $\phi(X_t)$ are quasi-isometric to each other (if both are metrized by H_L-equivariant path-metrics). Thus point preimages of ϕ will have diameters bounded by the quasi-isometry constants. We denote the restriction $\phi|_{S_{v,L}}$ by ϕ_v.

By the observation following Definition 8.8 above, we may choose a vertex $v \in X$ so that $|f(v) - t|$ is large in comparison with the quasi-isometry constants. For each point $x \in S_{v,L}$ the geodesic $[x, v]$ determines a unique minimal simplex $\sigma \subset L = Lk_{\downarrow}(v, X)$. The preimage $\phi_v^{-1}(\phi_v(x))$ will be contained in the contractible (by Proposition 5.10 (1)) subcomplex $S_{v,St'(\sigma,L)}$. For vertices v with $|f(v) - t|$ large enough one can define a left homotopy inverse to ϕ_v by taking each vertex of $\phi_v(S_{v,L})$ to a point of its ϕ_v-preimage, and extending over skeleta.

Thus the shadow $S_{v,L}$ is a homotopy retract of $\phi_v(S_{v,L})$, and so we have

$$\pi_2(\phi_v(S_{v,L})) \;\supset\; \pi_2(S_{v,L}) \;=\; \pi_2(L) \;=\; \pi_2(\widetilde{L}) \;=\; H_2(S^3 - \{120 \text{ points}\}) \;\neq\; 0$$

The last equality above follows from the fact that L is a spine of the Poincare homology 3-sphere. In conclusion $\phi_r(S_{v.L}) \subset Y$ is a connected subcomplex of the contractible 2-complex Y which is not aspherical, and so gives a counterexample to the Whitehead conjecture. □

Remark. Fix a metric on L. We conjecture that there is $\varepsilon > 0$ such that if $g : L \to K$ is a surjective PL ε-map, then K is homotopy equivalent to L with 1- and 2-cells attached. This conjecture implies that the geometric dimension of H_L is 3.

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Note added in proof. Meier, Meinert, and Van Wyk [MMV] have generalized the Main Theorem to all (including irrational!) characters $\mathrm{G_L} \to \mathbb{R}$.