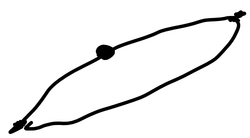
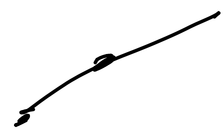
Thm. Geodesics are unique in a CAT(0)-space

~~a~~ That is, no digon



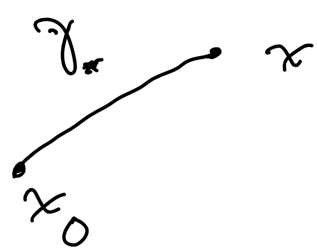
NOT allowed



comparison triang.

Thm If X is complete geodesic space then CAT(0) $\Longrightarrow$ Contractible

Pf Geodesic contraction



$\gamma_x : [0, d] \longrightarrow X$

$\gamma_x(0) = x_0$

$(x, t) \longrightarrow \cancel{\gamma(t)}$ $\gamma_x(tx)$

Show continuous $\square$

Show distance function is

convex

$$\gamma_1, \gamma_2 : [c_i, d_i] \longrightarrow X$$

$$d(\gamma_1(s), \gamma_2(t)) : [c_1, d_1] \times [c_2, d_2] \longrightarrow \mathbb{R}$$

is convex function

$\Longrightarrow$ continuity

Def X is NPC ~~for~~ if it is locally CAT(0)

Also Define CAT(K) for any $K \in \mathbb{R}$.

Use comparison triangles in either hyperbolic plane or S^2 (radius $1/\sqrt{K}$)

Main Case $K > 0$, $K = +1$

perimeter of Triangle $\leq 2\pi$

Def X is CAT(1), comparison holds for all triangles of perimeter $\leq 2\pi$

Thm (CARTAN-Hadamard)
X is complete geodesic locally CAT(K).

1) If $K \leq 0$, then $\pi_1(X) = 0$
$\Longrightarrow$ X is CAT(K) (CAT(0))

2) If $K > 0$, then X has

X '' is CAT(K^{+1})
$\Longleftarrow$ No short closed geodesics (length $\frac{2\pi}{\sqrt{K}}$)

Cor X is NPC ~~simply~~
Then $\tilde{X}$ is CAT(0).

Cor If X is NPC, then X is aspherical.

Link Condition: A cell cx X

is locally $CAT(\kappa)$

$\iff$ $\forall x \in X$, $Lk(x, X)$

is $CAT(1)$

$Lk(x, X)$ is spherical cell cx

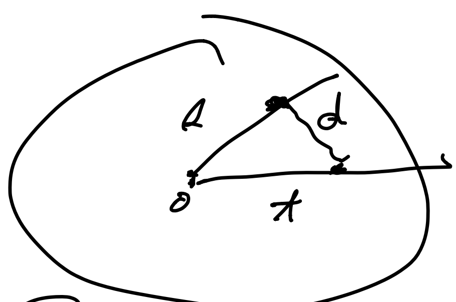
all cells = convex cells in S^n

X = cell cx

X locally isometric to

$Cone^{\kappa}_{\varepsilon}(Lk(x, X))$

L is piecewise spherical Metric on convex $Cone^{0}(L)$



Law cosines in $\mathbb{R}^2$

$$d^2 = s^2 + t^2 - 2st\cos\theta$$

defines distance in $\mathrm{Cone}^0(L)$

Similarly κ-distance in $\mathrm{Cone}^\kappa(L)$.

<u>Remark</u> locally CAT(0) at x
$\iff$ nbhd isometric to $\mathrm{Cone}^0_\varepsilon(L)$

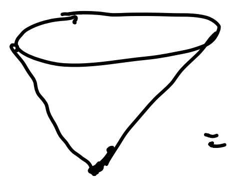
<u>Thm</u> (Link condition)

X is cell cx, cells of constant curvature κ, then

X is CAT(κ) $\iff$

each ~~link~~ $Lk(x, X)$ is CAT(1) $\forall x$.

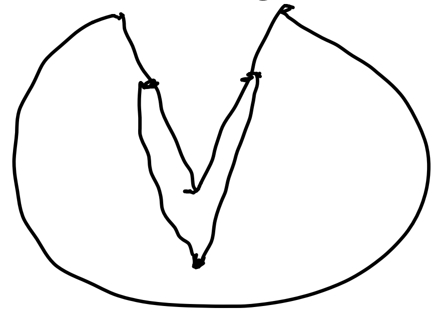
(Eventually, using induction on dim, check for vertices x.)



$\leftarrow S^1$ $\ell < 2\pi \implies$ non unique geodesics

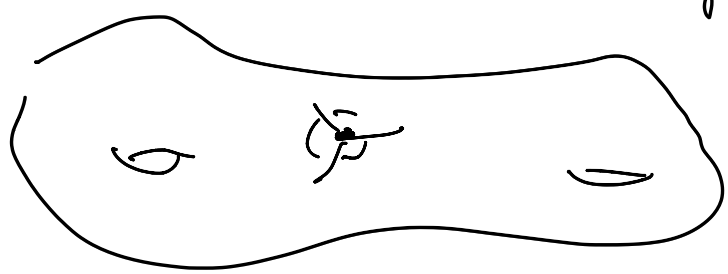
= cone

cut op



Pf of Link Condition is more or less above picture

Remark X = surface (made of polygons)

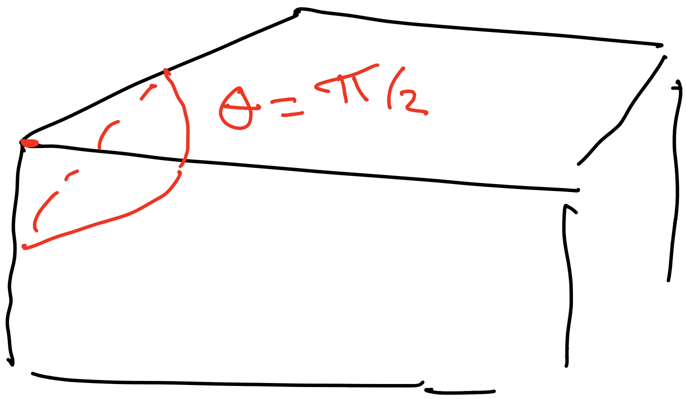


X^2 is NPC $\iff$ at

each cone pt

the cone angle $\geq 2\pi$.

Links in cube cx



Def A spherical simplex is all right if each length $= \pi/2$ (each dihedral angle $= \pi/2$

Conclusion If X is cube cx then $Lk(x, X)$ is an all right Δ-complex

$x \in X$
$x =$ vert