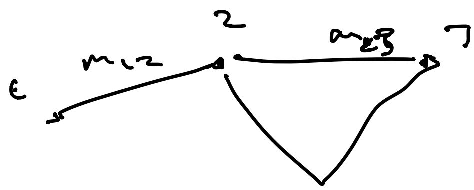
Coxeter gps & Artin gps

L' simple graph on $\{1, \dots, n\}$
labels on edges m_{ij}



s_i, $i \in \mathrm{Vert}(L') = I$

$S = \{s_i\}$

$$W = \underbrace{\langle S \mid s_i^2 \ \forall i \in I, \ (s_i s_j)^{m_{ij}} \ \forall \text{ edges } \{i,j\}}_{R}$$

$(s_i s_j)^{m_{ij}} = 1$ $\qquad s_i = s_i^{-1}$

$(s_j s_i)^{m_{ij}} = ((s_i s_j)^{-1})^{m_{ij}}$

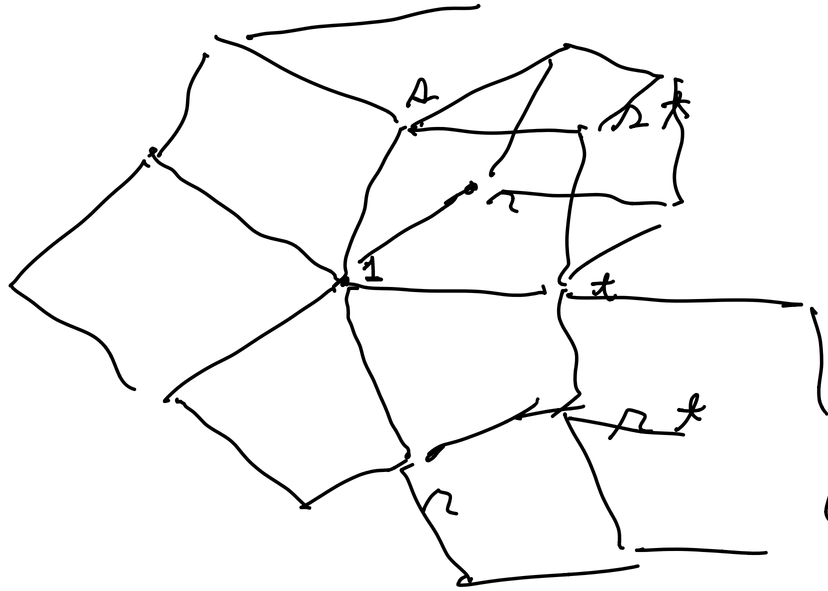
$A =$ ARTIN GA

$\hat{S} = \{a_i\}_{i \in I}$

$A_{\Gamma} = A_{(W, S)}$

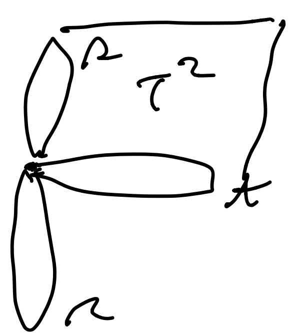
$$= \langle S \mid \underbrace{a_i a_j \cdots}_{m_{ij}} = \underbrace{a_j a_i \cdots}_{m_{ij}}$$

RACG $\quad \tilde{P}_L = \Sigma(W, S)$



Cayley Graph of W

For Artin gps $\qquad$ presentation ~~comp~~ complex



$m_{st} = 2$

L^1 = defining graph

$\mathcal{S}(W,S) = \{ T \subset S \mid W_T = \langle T \rangle \text{ is finite} \}$

$L(W,S) =$ simplicial cx

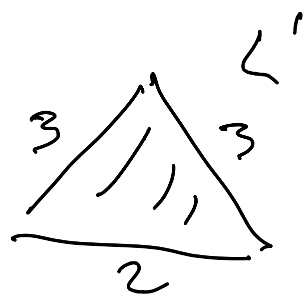
$L'(W,S) = L'$

$T \subset S$ spans a simplex σ_T

$\iff W_T$ is finite

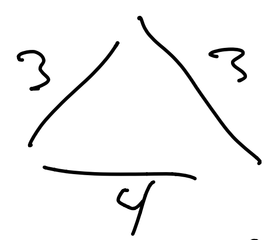
$\iff$

$\mathcal{S}(W,S) =$ set of simplices in $L(W,S)$



$W =$ finite

$W =$ symmetric



$= L$

W infinite

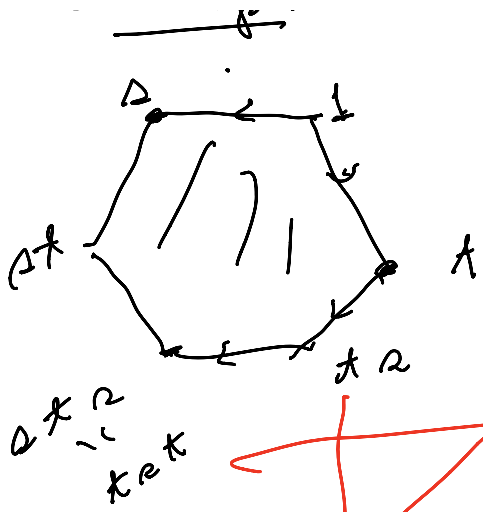
Define <u>Coxeter Zonotope</u>

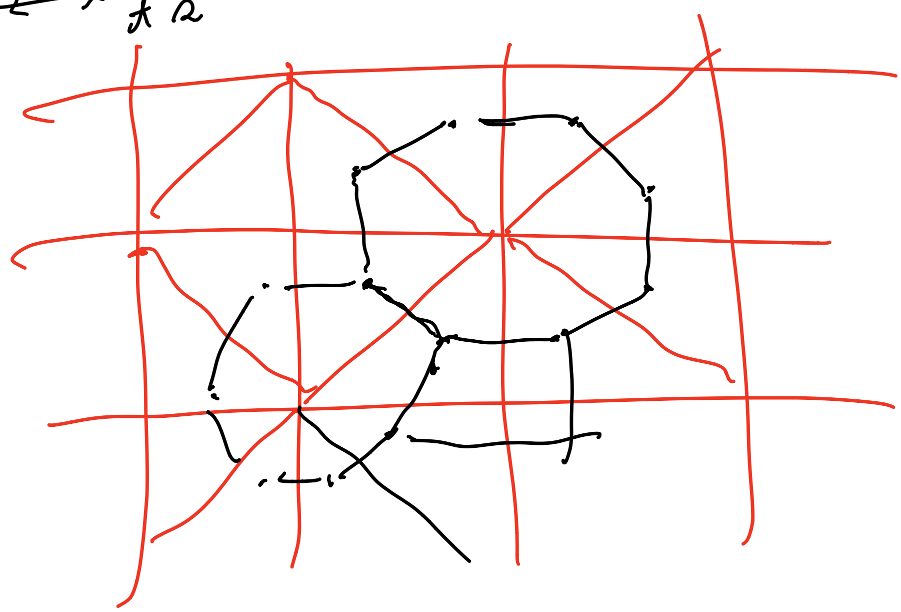
of finite gp W.

Example $W =$ dihedral gp

of order $2m$

$m = 3$





Start with $\mathrm{Cay}(W, S)$

fill in "Coxeter zonotopes"

$I = \{0, \dots n\}$

<u>Fact</u> W is finite

$\iff$ W is sp generated reflection across face spherical simplex where dihedral angles π/m_{ij}

matrix

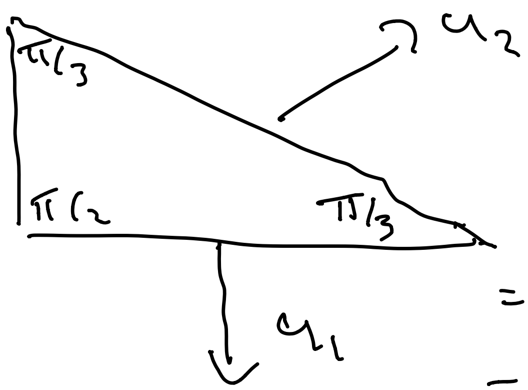
$\Longleftrightarrow$ ~~Matrix~~ Cosine matrix c_{ij}

$$c_{ij} = -\cos(\pi / m_{ij})$$

&

$$c_{ij} = u_i \, u_j$$

u_i = unit normal to i^{th} face.



$$u_1 \cdot u_2 = \cos(\pi - \pi/3) = -\cos \pi/3$$

$\boxed{c_{ij} \text{ is positive def'n}}$

<u>Why</u>

$\exists$ Spherical simplex σ_I exterior angles are $\pi - \pi/m_{ij}$ $\Longrightarrow$ (c_{ij}) " >0

& conversely

- $C > 0 \Rightarrow \exists$ basis $(u_0, \dots u_n)$

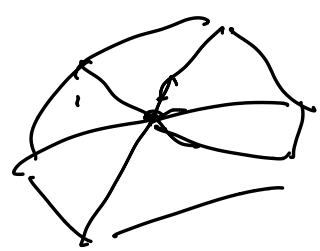
$$u_i^T u_j = C_{ij}$$

$u_i \cdot u_j = C_{ij}$. $\square$

- Given simplex σ you get reflection $S\rho$ on S^n

$$D(W, \sigma) = (W \times \sigma)/\sim$$

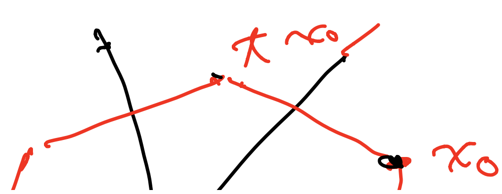
$$D(W,\sigma) \xrightarrow{\cong} S^n$$



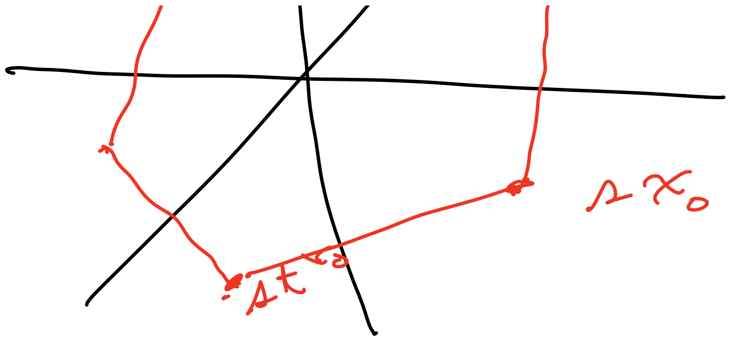
Def^n of Coxeter Zonotop.

$$W \curvearrowright S^{n-1}$$

and on $\mathbb{R}^n$



$W x_0$



Choose x_0 in interior of cone on simplex

Def'n $Z(W,S)$ = Convex hull of $W x_0$

1-skeleton of Z = Cayley graph of (W,S)

$\mathcal{S} = \{ T < S \mid W_T$ is finite $\}$

$$\boxed{\text{Poset of cells of} = \coprod_{\sigma \in \mathcal{S}} W / W_\sigma}$$

Feb 24

Coxeter zonotopes

Background of finite Coxeter gps

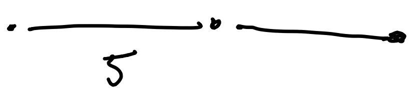
Almost All come from regular polytopes as a from root system

Regular polytopes: Coxeter gps have diagram line segment

m-gon

= graph A_n

B_n " cube

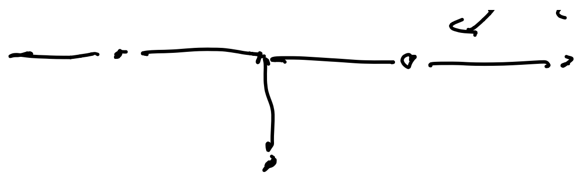


dodecahedr.

120-cell

24-cell

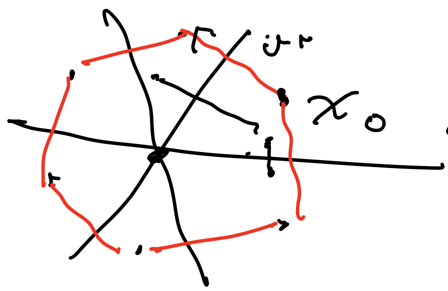
3 others E_6 E_7 E_8



Zonotopes W = finite Coxeter gp

$S = \{$reflections across faces of fundamental domain$\}$

~~Def~~



$x_0 \leftarrow C$ = fund cone

$x_0 \in C$

$Z = Z(W,S)$ = Convex hull of Wx_0

$\mathrm{Face}(C) \subset \mathrm{Fix}(s_i)$

u_i = unit normal to this face

Choose x_0 s.t.

$$(x_0 \cdot u_i)_{i=0}^{n-1} \in [0, \infty)^n$$

Assume that $(x_0 \cdot u_i) = 1$

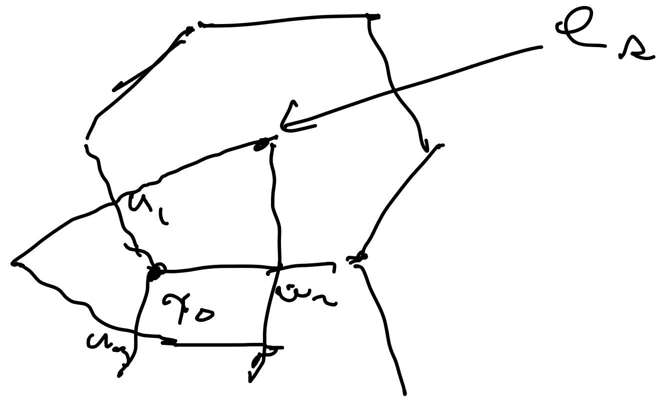
<u>Lemma</u> $Z = Z(W, S)$

$\mathrm{Face}(Z)$ = poset of faces

$$\mathrm{Face}(Z) = \coprod_{\emptyset \subset T \subset S} W/W_T$$

~~cells of dim i~~

$\dim(\text{cell}) = |T|$



e_s = ray opposite to face fixed by s

Claim

$T = S - \{s\}$

$W_T x_0 = \{$ vertices of codim 1 face of zonotope $\}$

linear form

$$v \xrightarrow{\varphi} v \cdot e_s$$

$$v = x_0 \qquad x_0 \cdot e_s = c$$

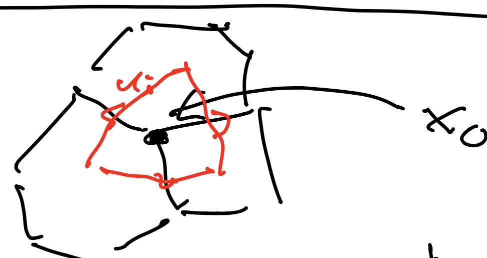
$$\varphi(x) = c = \text{supporting hyperplane}$$

$$(w x_0) \cdot e_s \le x_0 \cdot e_s$$

equality $\Leftrightarrow$ $w \in W_T$

blah blah

Proves lemma $\square$



$$Lk(x_0, Z) = \text{spherical simplex} \subset S^{n-1}$$

$\shortparallel$

σ

edges are parall
to outward normals u_i

$u_i \cdot u_j = \cos(\pi - \pi/m_{ij})$

$= -\cos(\pi/m_{ij})$

vertices of σ
$= \{u_i\}$

length of arc from u_i to u_j
is $\pi - \pi/m_{ij}$

$-\cos \pi/m_{ij} \to -\cos \pi/m_{ij}$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

This is pos. definite

$\cancel{[\ (W, S)\ ]}$

W arbitrary

$S =$ generators

$$\text{Coxeter gp} \leftarrow \begin{array}{l} s_i^2 = 1 \\ (s_i s_j)^{m_{ij}} = 1 \end{array}$$

$$\mathcal{S} = \mathcal{S}(W,S) = \{ T < S \mid \underset{\text{is finite}}{W_T} \underset{=}{=} \langle T \rangle \}$$

$L(W,S) =$ abstract simplicial cx with vertex set simplices the nonempty subsets of S.

Precearse spherical cx each edge has length $\pi - \pi/m_{ij}$

<u>Define</u> a cx $\Sigma(W,S)$

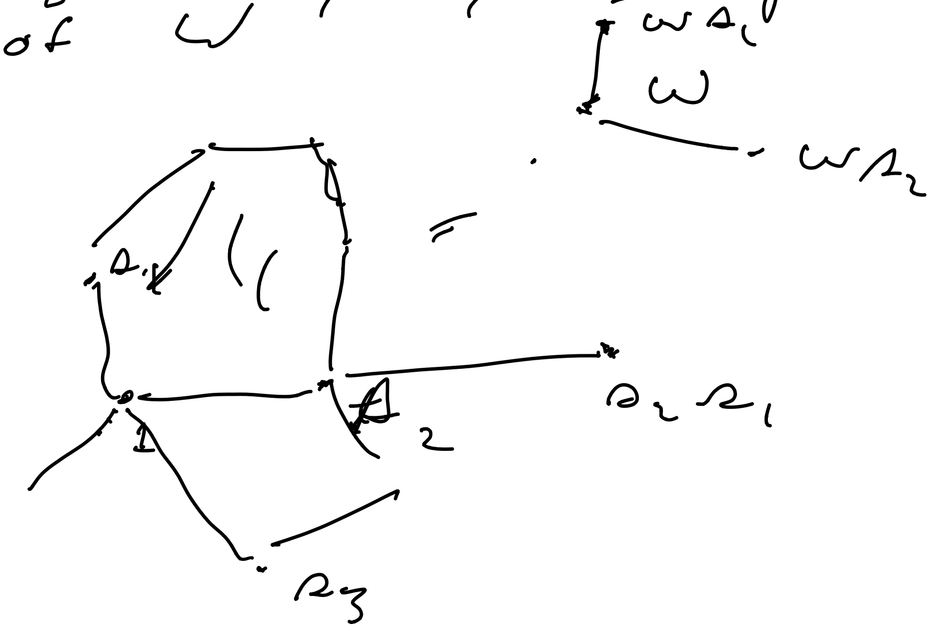
with poset of cells

$$\coprod_{T \in \mathcal{S}} W/W_T$$

$w W_T$ to $\ldots w$

Generalized Cayley graph of W

$T = \emptyset$



Make this into a CW

For each cell $w W_T$

I fill a zonotope $w Z(W_T)$ whose vertices are $w W_T$

Moussong: $Z(W, S)$

is CAT(0)

Feb 26

2-dim examples of $\Sigma(W,S)$

2-cell = Coxeter zonotope corresponding to $D_{m} =$ dihedral gp

= $2m$-gon.

L^1 = simpl. graph

$S =$ Vert(L^1)

an edge e is $\{s,t\}$, $s,t \in S$

(L^1, m) $\quad m$: Edges $\longrightarrow \{2,3,\dots\}$

$e = \{s,t\}$

$m(e)$ or $m(s,t)$

This defines presentation for (W,S):

$s^2 = 1$, $(st)^{m(s,t)} = 1$

When is $|W| < \infty$

$W \curvearrowright \mathbb{R}^n$ or S^{n-1}

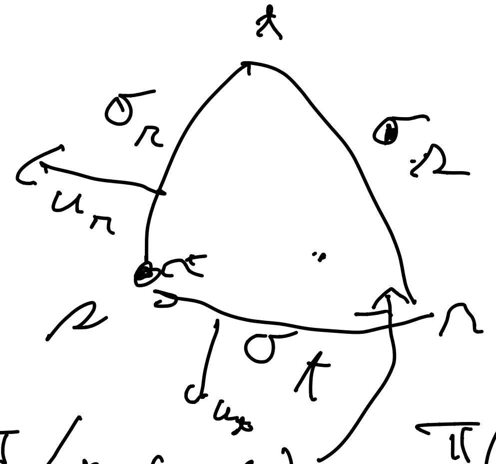
$n = \cancel{|S|}$ card(S)

~~σ not (W)~~ $L'(W) =$ complete graph on S

fund domain S^{n-1}

on $\Sigma(W)$

$\sigma(W) =$



$\pi / m(s,t) = \pi / m(e)$

$\sigma^*(W) =$ dual simplex

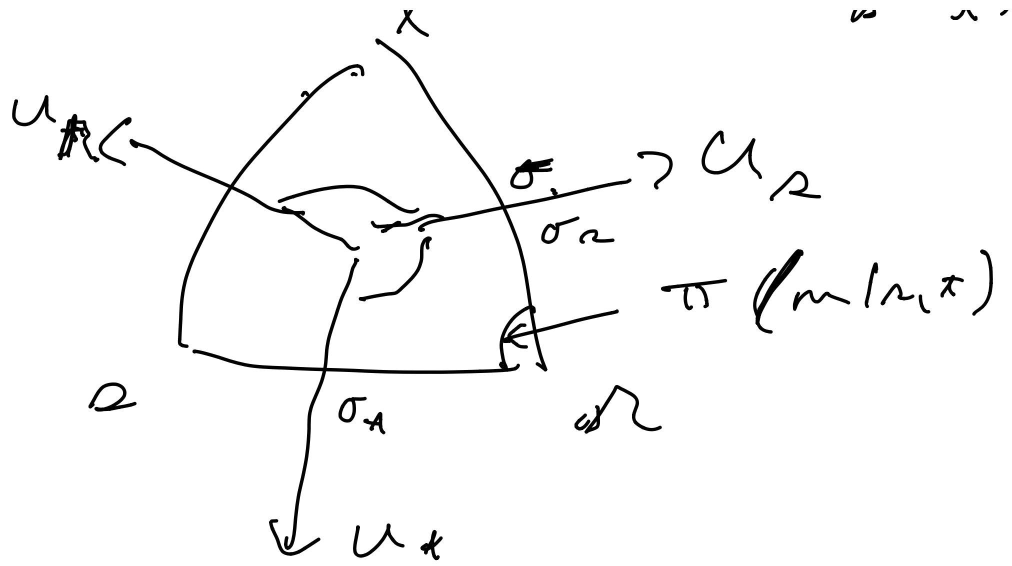
$u_s =$ unit normals

σ^* is spanned $\{u_s\} = \{S\}$

~~and co~~

length$(\{s,t\}) = \cancel{\cos \cos}$

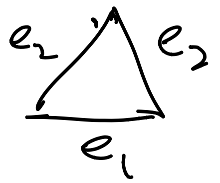
$= \cos^{-1}(u_s \cdot u_t)$



$$\tilde{\ell}(\{s,t\}) = \cos^{-1}(u_s \cdot u_t) = \pi - \frac{\pi}{m(s,t)}$$

Condition for ~~the~~ complex to be 2-dim

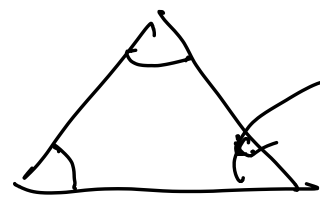
And $L^1(W,S)$ has no circuits ~~of~~ of length 3 ~~labels~~ spherical labels



$\subset L'$ 3-circuit

$$\sum \frac{1}{m(e)} > 1$$

σ



(*) $\boxed{\sum \frac{\pi}{m(s,t)} > \pi}$

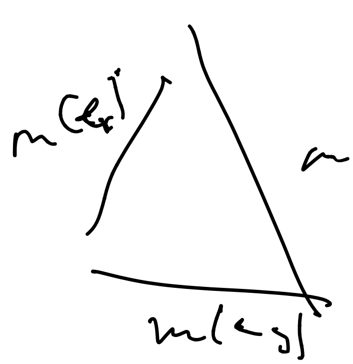
$\Rightarrow \exists$ spherical with those angles

Possibilities for $m(s,t)$

$(m(e_1), m(e_2), m(e_3))$ ~~$\{s,t,r\}$ =~~ $= (2, 2, n)$

or

$\boxed{(2, 3, p) \quad p = 3, 4, 5}$



No spherical triangles $\Leftrightarrow$

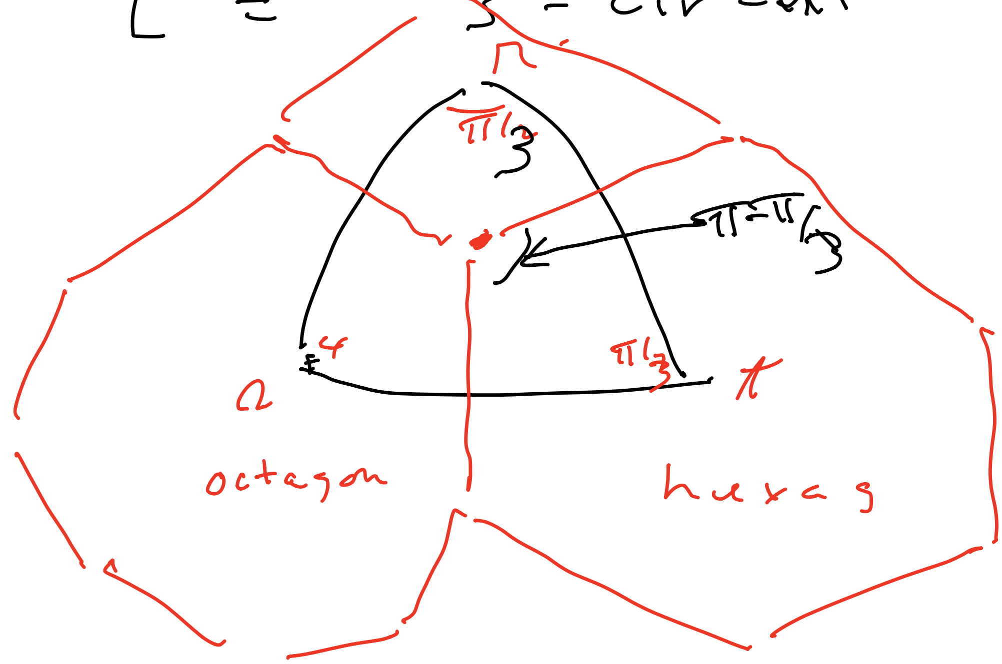
~~* is equivalent~~

$$\sum_{\text{circuit}} \pi - \pi/m(e) \geq 3\pi - \sum \frac{\pi}{m(e)}$$

$$\geq 3\pi - \pi$$

$$= 2\pi$$

$L' =$ 3 - circuit



Get 2-dim cell complex $\Sigma(W, S)$

~~that~~ No $L(W, S)$

has no spherical

3-circuits

Then $L(W, S) = L'(W, S)$

edge lengths of L'

$\pi - \pi / m(e)$

<u>Thm</u> $L'(W, S)$ graph edges = spherical arcs

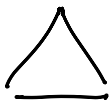
$L'(W, S)$ is $CAT(1)$

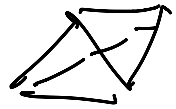
iff no 3-circuits of length $< 2\pi$

<u>Cor</u> Condition $\Rightarrow$ $\Sigma(W, S)$ is NPC and since simply connected $\Rightarrow$ $CAT(0)$.

Gromov Polyhedra

$L' =$ complete graph





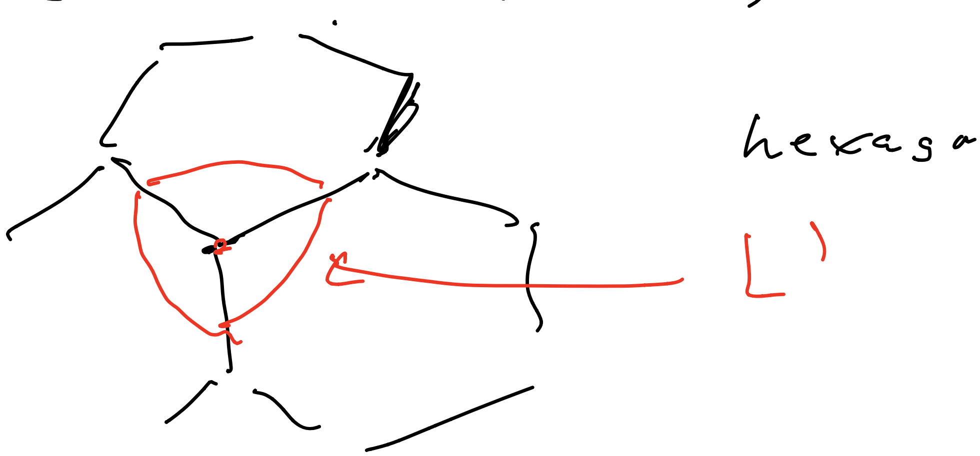
label each edge by ~~3~~ m,

If $m \geq 3$, Σ

$=$ 2dim cx where each 2-cell is $2m$-gon

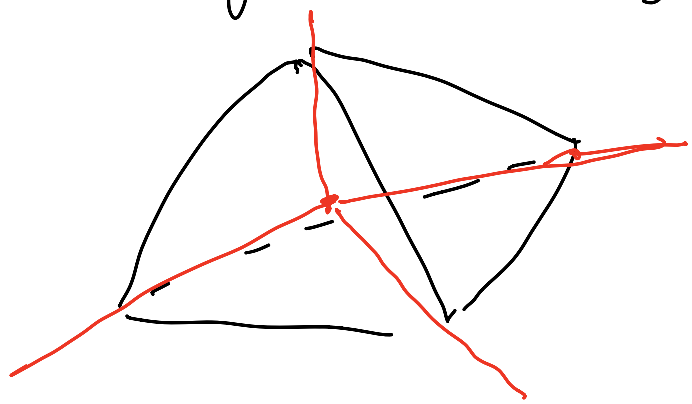
link of each vertex is complete.

& Σ is CAT(0)



$L' =$ 

hexagonal cx where
each vertex is
a complete graph



<u>Thm</u> (Gromov) For any $k \geq 3$

There exists a simply connected 2-complex $X_{k,l}$

each 2-cell is k-gon

& link of each vertex
~~is~~ is <u>complete graph</u> l-vertices

& is gp of symmetries
which is transitive
on triples (pt < edge < face) $\underline{l-1}$

$k = 3$ = 2-skeleton Δ

$k = 4$ 2-skeleton of cube

$k = 5$ 2-skeleton of dodecahedron $l = 4$

...

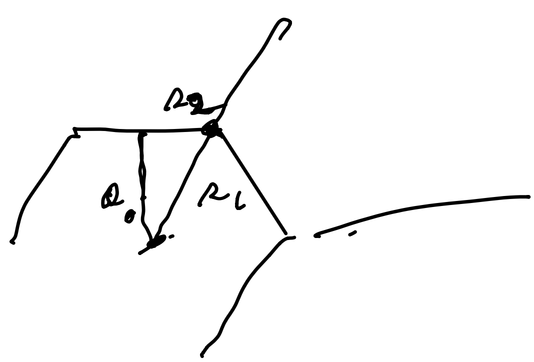
of 120-cell if $l = 5$

$k \geq 6$ Symmetry gp is ∞

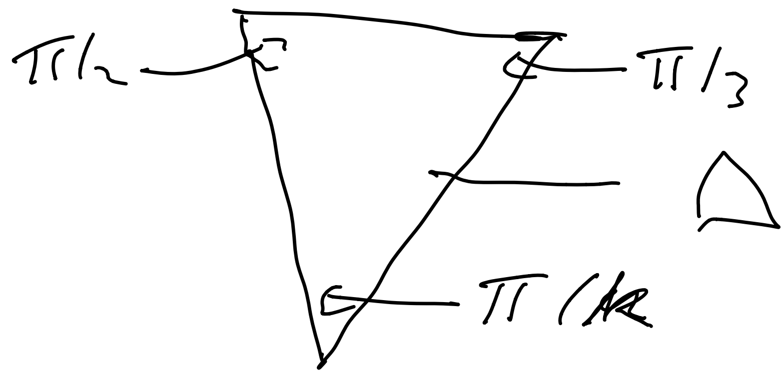
<u>Idea</u>: Symmetry gp with diagram

$$\underset{s_0}{\bullet} \overset{k}{\text{———}} \underset{s_1}{\bullet} \overset{3}{\text{———}} \underset{s_2}{\bullet} \;\cdots\; \text{———} \; s_{l-1}$$

W = Coxeter gp



Fund domain Euclidean triangle



$W =$ Coxeter

Form $D(W, \Delta)$

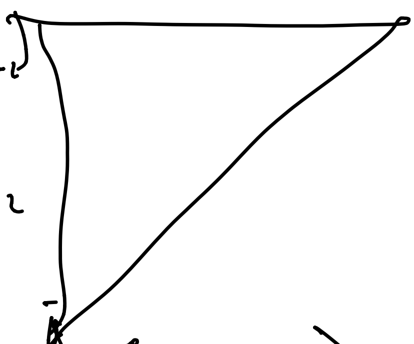
$= (W \times \Delta)/\sim$

$\langle s_0 \rangle \times \langle s_2, \ldots, s_{l-1} \rangle$

$C_2 \times \Sigma_{l-2}$

$\langle s_1, \ldots, s_{l-1} \rangle$

$= \Sigma_{l-1}$



$\langle s_0, s_1 \rangle \times \langle s_3, \ldots, s_{l-1} \rangle$

$\langle s_0, s_1 \rangle = D_k$

$$s_0 \overset{k}{\text{———}} s_1 \text{ —— } s \quad \cdots \quad s_{l-1}$$

Check this is

Haglund's X_{R}

---

coxeter gp Feb 28

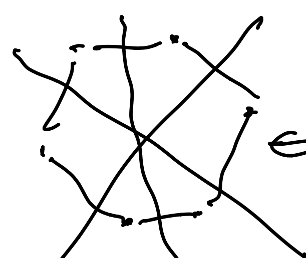
(W, S)

$\Sigma(W, S) =$ CAT(0) cell cx

$W \curvearrowright \Sigma$.

$Z(W_T, T) =$ Zonotope for Cox. gp W_T

$=$ cell of $\Sigma(W, S)$


$\leftarrow Z$

$T \subset S$

$\mathcal{S} = \{T \subset S \mid |W_T| < \infty\}$

$\{\text{cells of } \Sigma\} = \coprod_{T \in \mathcal{S}} W / W_T$

$\Sigma = \left(\coprod_{w \in W/W_T} w\, Z(W_T, T) \right) / \sim$

$L \omega \in W/W_T$ —

$L(W,S) =$ simplicial cx with ~~PS~~ piecewise structur

Simplices $\cong \mathcal{S}_{>\emptyset}$

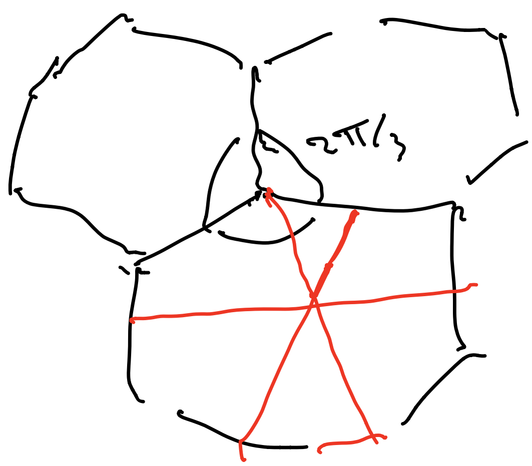
simplex $\sigma^*(W_T, T) \longleftrightarrow T$

ℓ_{st} = edge length of 1-simplex between s & t

$= \cos^{-1}(\pi - 2\pi/m_{st})$

Ex $L^1 =$ complete graph on S

label each edge $m \geq 3$

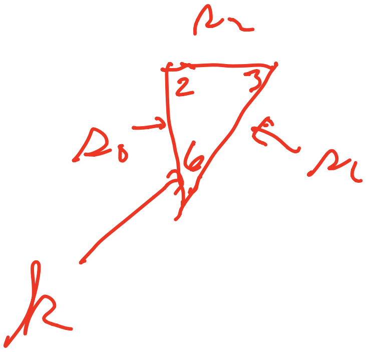


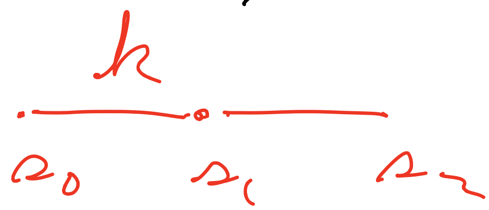
$L^2 = \Delta$

$Z(W_T, T)$

$=$ 2m-gons

Def our Gromov polyhedra.

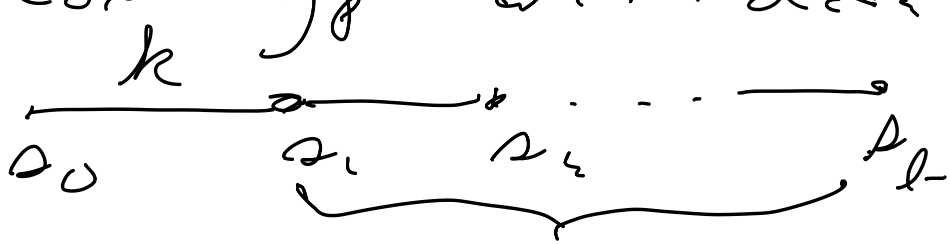




Haglund's thesis

Thm (Coxeter Polyhedra) For $k \geq 3$, $\exists$ cell complex of k-gons where link of each vertex is complete graph. (CAT(0) for $k \geq 6$)

$W' =$ Cox gp with diag.

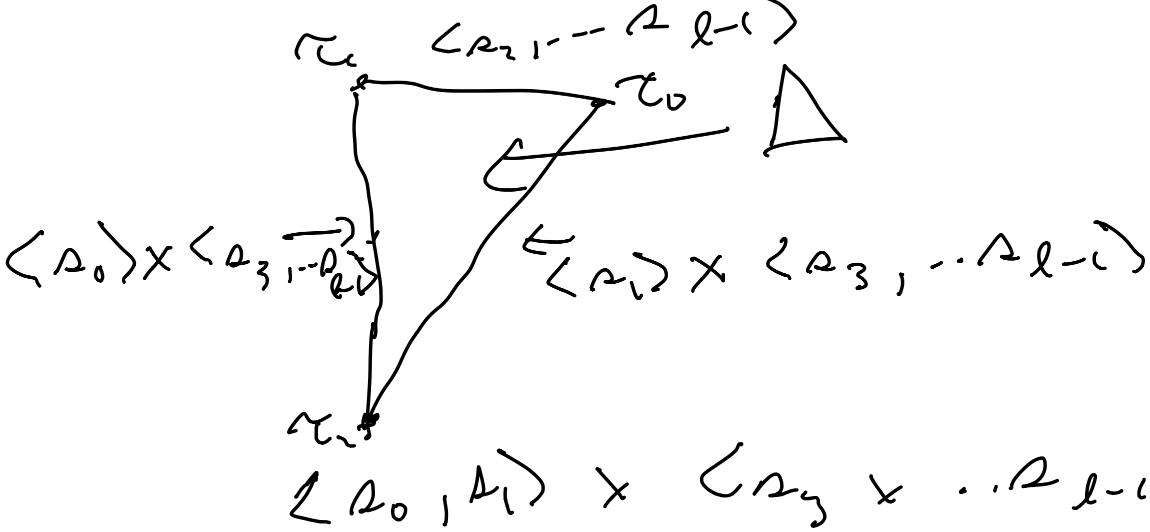


S_l = symmetric on $l-1$

"

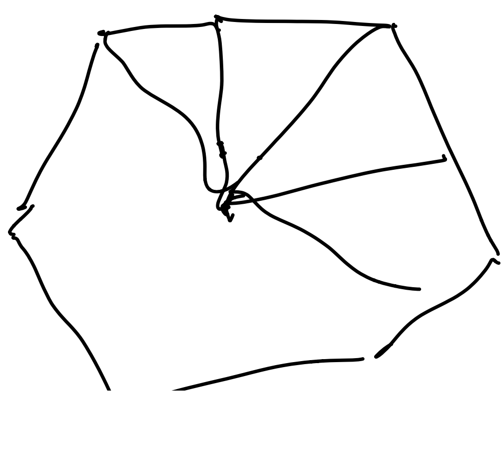
symmetries of complex

Strict fund. domain construct.



$$D(W'; \Delta) = \underline{B(W' \times \Delta)} / \sim$$

This gives



~~show $L(W, S)$ is CAT(1)~~

$L(W, S)$

has following properties:

$L =$ simplicial cx

$S =$ vert(L)

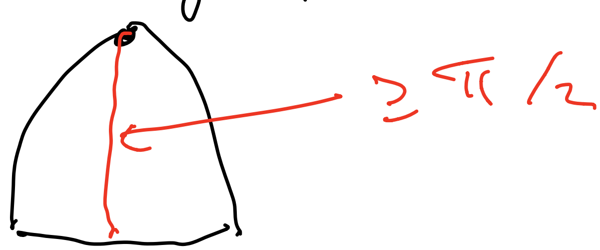
PS-structure

$\underline{l_{st} = \text{edge}}$ length of st

~~Def L is a metr~~

Suppose each $l_{st} \in \{\pi/2, \pi\}$

~~Suppose each~~ L has "size $\geq \pi/2$" if each simplex has prop



Def L is metric flag cx $\Longleftrightarrow$

each possible spherical simplex is filled in

($T \subset \mathrm{Vert}(L)$
~~all~~ $\{\ell_{st}\}_{s,t \in T}$

= {edge lengths of σ}

then $\sigma \in L$.

ℓ_{st} $\quad (= \cos^{-1}(u_s \cdot u_t))$

$$\left(\cos(\ell_{st})\right) = c_{st}$$

Should be pos. def.

Moussong Lemma If L has size $\geq \pi/2$ is PS simpl. cx

Then L is CAT(1)

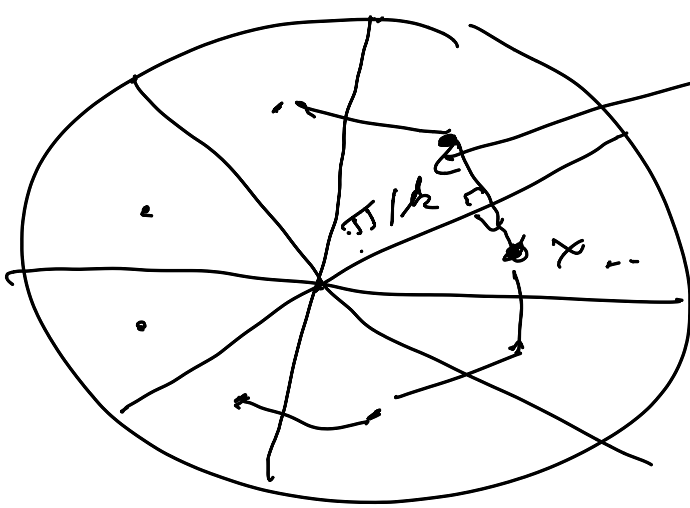
$\Longleftrightarrow$ it is a metric flag cx.

Cor (Moussong) $L(W,S)$ is CAT(1)

$\therefore$ $\Sigma(W,S)$ is CAT(0)

Give $\Sigma(W,S)$ a piecewise hyperbolic hyperbolic

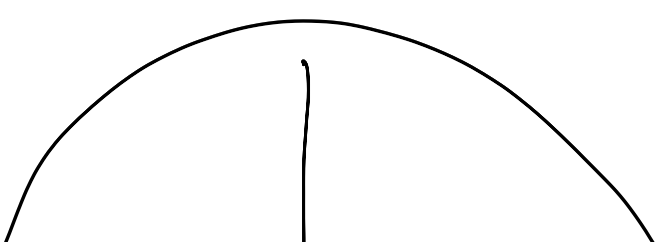
$\Sigma^h(W_T)$ = defined in $\mathbb{H}^n$



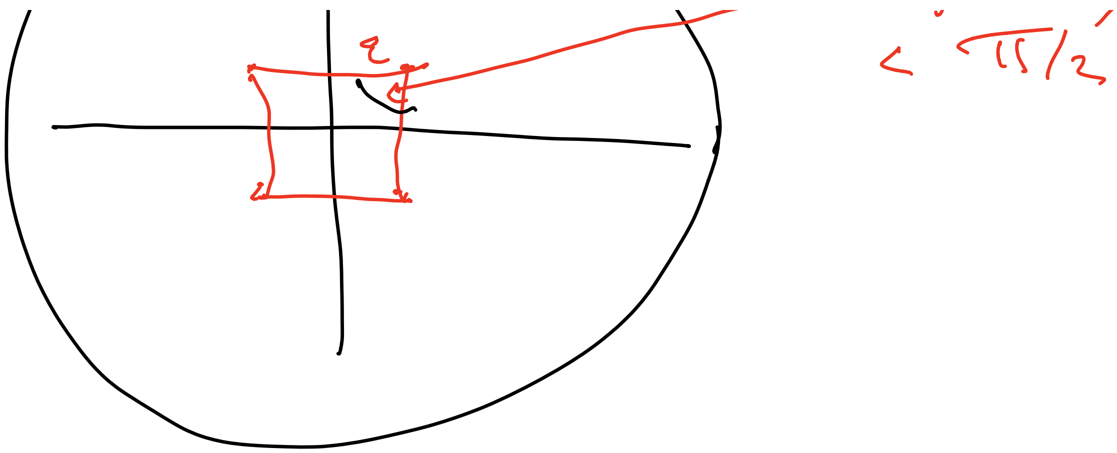
angle smaller than $\pi - \pi/k$

$\mathbb{H}_2$

orbit of x_0 under W_T



— slightly



Replace each euclidean
$Z(W_T)$ by hyperbolic
$Z^h(W_T)$

new edge lengths in
new $L^h(W,S)$ slightly
smaller

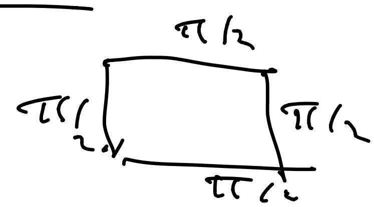
Q: When is $L^h(W,S)$
is CAT(-1)?

Ans $\Leftrightarrow$ $L(W,S)$ has
no closed geodesic ~~of~~ with $\ell = 2\pi$

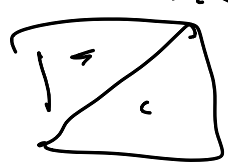
Def L is extra large satisfies $\ell(\text{closed geod}) > 2\pi$
Same in each link.

Gromov All right flag cx L is extra large
$\Longleftrightarrow$ satisfies no empty $\square$-cond

(i.e



exists $\Longleftrightarrow$ no diagonal



Cor (W, S) is RACG
Then $\Sigma(W, S)$ can be given a $CAT(-1)$ structure

$\Longleftrightarrow$ $\mathcal{L}(W,S)$ has no squares

$\; ''$

$\; \mathcal{L}$

$\Longleftrightarrow$ W is hyperbolic

gp.