Introduction

Group theory $\subset$ topology

via fundamental gp

$X =$ CW cx

$G = \pi_1(X)$

$\tilde{X} =$ univ. cover

$G \curvearrowright \tilde{X}$ via deck transformation

<u>Def</u> X is <u>aspherical</u>

if $\tilde{X}$ is contractible

equiv (since $\pi_i(\tilde{X}) = \pi_i(X)$, $i > 1$)

$\pi_i(X) = 0 \quad \forall\, i \geq 2.$

$\underbrace{\text{aspherical}}_{}$

<u>Fact</u>: $\forall\, G$, $\exists$ CW cx

X s.t. $\pi_1(X) = G$.

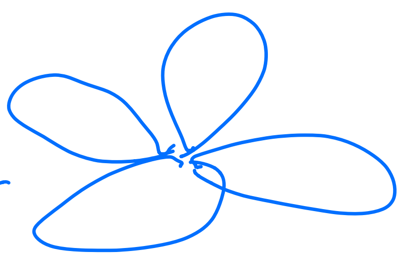
X is unique up to homotopy

$X = BG$ classifying space

$\exists\, X^2$, 2-dim

with $\pi_1(X^2) = G$.

<u>Pf</u> $G = \langle S \mid R \rangle$

$X^2 =$  $\cup_{w \in R} \bigvee D^2$

$\simeq \vee S^1$

$w: \partial D^2 \longrightarrow \vee S^1$

<u>Remark</u>) If G has

torsion, BG ∞-dim'l

2) G acts freely on EG
$= \{\widetilde{BG}\}$

<u>Def'n</u> $G \curvearrowright X$ is <u>proper</u>

<u>means</u> 1) X/G is Hausdorff

2) Each isotropy subsp is finite, $G_x = \{g \in G \mid gx = x\}$

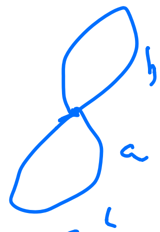
<u>Examples</u> of aspherical CW complexes:

1) a) $X = S^1$, $\tilde{X} = \mathbb{R}$

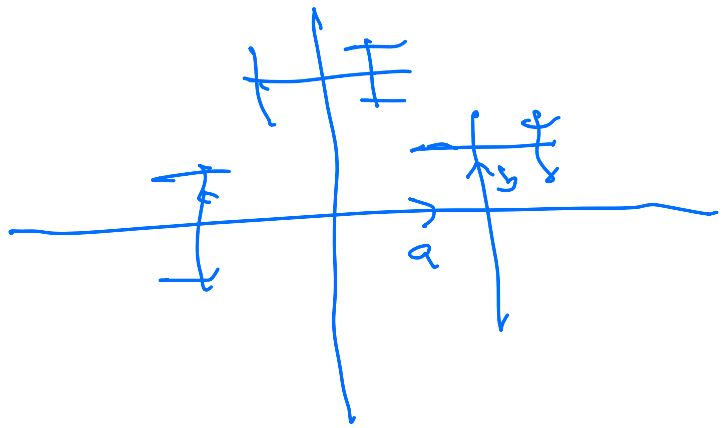
$G = \pi_1(X) = \mathbb{Z}$

b) X = a connected graph

$\pi_1(X) =$ free gp $= C$



$X = S^1$



c) Product of graphs

$X = X_1 \times X_2$

$\dim X = 2$

cells = {squares, edges or vertices}

2) $\dim X = 2$

a) $X =$ surface $= M^2$

X is aspherical $\iff$ $X \neq S^2$ or $\neq \mathbb{RP}^2$

e.g. $X = S^1 \times S^1 = T^2$

$\cancel{\partial} \; T^2 = B(\mathbb{Z} \times \mathbb{Z})$

b) Many 2-dim X^2 (presentation complexes are aspherical)

1-relator gps

$X^2 = \vee S^1 \cup_f D^2$

$f : \partial D^2 \to \vee S^1$

If $f \neq g^m$ for $m \geq 1$

$g \in \pi_1(\vee S^1)$

~~Thm~~ (Lyndon) Then X^2 is aspherical

Small cancellation theory:

<u>Thm</u> ~~Small~~ G = small cancellation gp. Then X^2 is aspherical.

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3) 3-manifolds are usually aspherical

<u>Thm</u> (Sphere Thm. Papakyriakopoulos)

$\pi_2(M^3) \neq 0$

$\Rightarrow \exists$ essential embedded $S^2 \subset M^3$

$\Rightarrow$ either $M^3 = M_1 \# M_2$

or $M^3 = S^2 \times S^1$

or $M^3 = (S^2 \times S^1)/\mathbb{Z}_2$

Thm If M^3 is closed 3 mfld, which is prime (not connected sum) & $\pi_1(M^3) = \infty$

Then M^3 is aspherical.

4) Nonpositively curved Riemannian

M^n $K(M^n) =$ Sectional curv.

Thm If M^n is closed, $K(M^n) \leq 0$

Then M^n is aspherical

T^n $K(T^n) = 0$

$\tilde{T}^n = \mathbb{R}^n$

$\tilde{M}^n = \mathbb{H}^n =$ hyperbolic space

Thm (Cartan-Hadamard) $K(M^n) \leq 0$

$T_x M^n \xrightarrow{\exp} M^n$ is a covering map

$\cong$

$\mathbb{R}^n$

Standard Example of NPC cube complexes

Input = simplicial cx $= L$

$\text{Vert}(L) = I$

Define a subcomplex

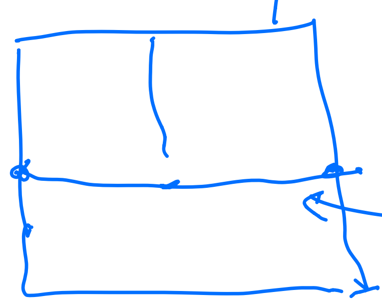
$$P_L \subset [-1,1]^I = \prod_I [-1,1]$$

$$\sigma \in L$$

$$\square^\sigma \subset \sigma \times [-1,1]^{\text{Vect}\,\sigma}$$

= cube determ...

$$I = \{1, 2\}$$



$$\sigma = \{1\}$$

$$\square^\sigma$$

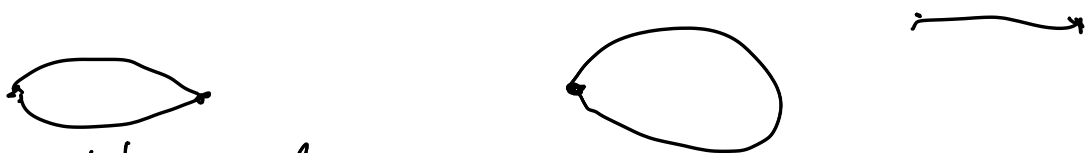
$$P_L = \bigcup_{\sigma \in L} \text{faces } [-1,1]^I \text{ which parallel } \square^\sigma$$

Jan 8

Classical,

cell complex is finite collection of convex cells in some Euclidean space s.t $\cap$ of two cells $= \emptyset$ or common face.

Usually, cell cx X = top space (the union) formed by gluing convex cells via isometries of faces



allowed

Standard examples of cell

simplex $= \Delta^n =$ affine span of $n+1$ pts

cube $= \square$

simplicial cx = each cell = simplex classical cell cx

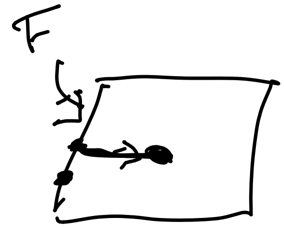
Δ-complex = "drop classical

Links

$F < P$ = convex cell

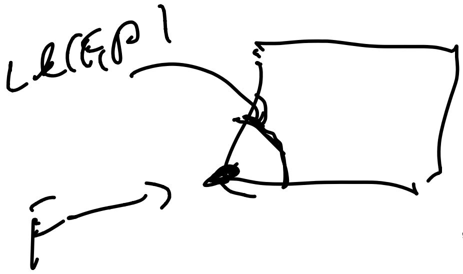
" face

$x \in \operatorname{Int} F$



$N(F, P)$ = inward pointing tangent vectors which are $\perp$ to

$Lk(F, P)$ = unit vectors in $N(F, P)$

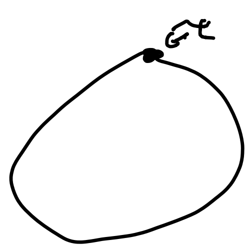


X = cell cx

$\tau \in X$ is a cell

$$Lk(\tau, X) = \bigcup_{\sigma > \tau} Lk(\tau, \sigma)$$

wrong if faces are glued to them.



$Lk(\tau, X) = 2 \text{ pts} = S^0$

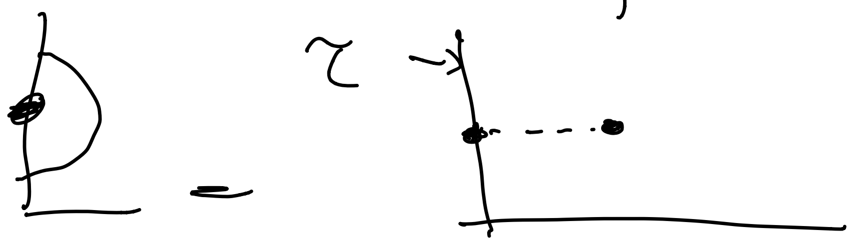
ε- neighborhoods

$x = 0\text{-cell} \in \mathrm{Vert}\, X$

$N_\varepsilon(x) = \varepsilon$ - nbhd of x

$\partial N_\varepsilon(x) = \{ y \in N_\varepsilon \mid d(y,x) = \varepsilon$

$= Lk(x, X)$



$Lk(\tau, X) = \bigcup_{\sigma \supset \tau} N_\varepsilon(\tau, \sigma]$

$= \partial$ (Normal "link")

The local topology of X is determ by Link

<u>Thm</u> $x \in X$

$Lk(x, X) \cong \partial N_\varepsilon(x, X)$

i.e locally

$N_\varepsilon(x) \cong \mathrm{Cone}(Lk(x, X))$

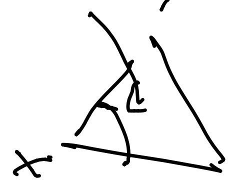
<u>Cor</u> $Lk(x, X) = S^{n-1}$

$\forall\, x \in \mathrm{Vert}(X) \implies$

X is n-dim. mfld

<u>Examples</u> $\qquad\qquad Lk(x, X)$

1) $X = \Delta^n$



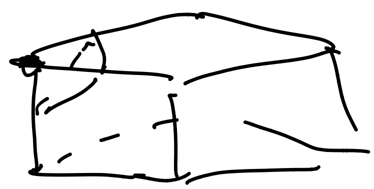
$\cancel{= \partial\Delta^n = S^{n-1}} \quad \cancel{Lk(x)}$

$Lk(x, \Delta^n) = \Delta^{n-1}$

2) $X = \partial\Delta^n = S^{n-1}$

$Lk(x, \partial\Delta^n) = S^{n-2}$

3)



$= \square^n$

$Lk(x, \square^n) = \Delta^{n-1}$

In cubical cx Lk ~~of~~ vertex = simplicial cx

Recall P_L $\qquad L =$ finite simpl. cx

$I = \mathrm{Vert}(L)$ ...

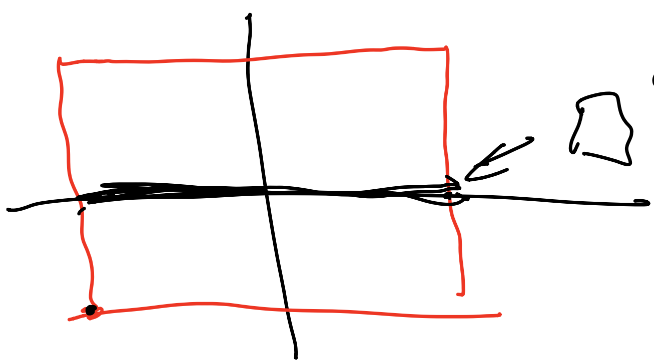
Goal: define $P_L \subset [-1,1]^I$

$$\mathrm{Vert}(P_L) = \mathrm{Vert}([-1,1]^I)$$
$$= \{ (\varepsilon_i)_I \mid \varepsilon_i \in \{\pm 1\} \}$$
$$= \{\pm 1\}^I$$

link vertex in $P_L = L$

$\sigma \in L$ be a simplex

$$\square^\sigma = [-1,1]^{\mathrm{Vert}\,\sigma} \subset \mathbb{R}^I$$



$\square^\sigma$ σ $\sigma =$ interva

$\partial\sigma =$ has 2 endpoin

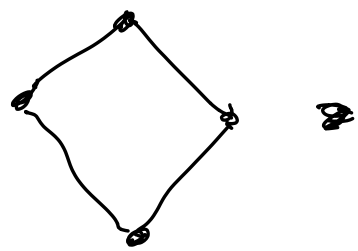
$$P_L = \bigcup_{\sigma \in L} \text{faces parallel to } \square^\sigma$$

<u>Ex</u> a) $L = \Delta^{n-1}$

$P_L = [-1,1]^n$

b) $L =$ m-gon $= S^1$ m vertices, m edges

$m=4$



$= T^2 \subset [-1,1]^4$

$L =$ m-gon $\qquad P_L = M^2$ $= 2$-dim mfld

<u>Remark</u>

m-edges

$P_L =$ 2-mfld with ∂

$\chi(P_L) = ?$

$\#$ vertices on P_L $= 2^m \qquad C_2 = \{\pm 1\}$

$\#$ ~~ed~~ edges orbits $= m \qquad C_2^m \curvearrowright P_L$

$\# \quad m(2^m/2) = 2^m\left(\frac{m}{2}\right)$

$\#$ squares $= 2^m\left(\frac{m}{4}\right)$

$\chi = 2^m\left(1 - \frac{m}{2} + \frac{m}{4}\right)$

vertex $\qquad$ square

$= 2^m\left(1 - \frac{m}{4}\right)$

$$= 2^{m-2}(4-m)$$

$$\chi = 2 - 2g \qquad g = \frac{2-\chi}{2}$$

$$\cancel{= 2^{m-2}}$$

$$g = \frac{2 - 2^{m-2}(4-m)}{2}$$

$$= 1 - 2^{m-3}(4-m)$$

M^2 = surface of genus g

$C_2 = \{\pm 1\} \qquad C_2^I$

acts on $[-1,1]^I$

generated by r_i

$$r_i : \begin{array}{l} x_i \longrightarrow -x_i \\ x_j \longrightarrow x_j \quad j \neq i \end{array}$$

$C_2^I \curvearrowright [-1,1]^I$ $\langle r_i \rangle$ generated reflect…

P_L is stable under this action

June 10

L = simplicial cx

P_L associated cube cx

$L =$ 

$\tilde{P}_L$ = univ. cover of P_L

$C_2 = \{\pm 1\}$

$(C_2)^I \curvearrowright [-1,1]^I$

i. on P_L.

$i \in I$, r_i = reflection across hyperplane $x_i = 0$

$(C_2)^I = \langle r_i \rangle$

$W_L =$ gp of all lifts of of element of C_2^I to $\widetilde{P}_L$.

Let s_i = lift of r_i which takes v to adjacent vertex along edge type i

<u>Prop</u> We short exact sequence

$$1 \to \pi \to W_L \to C_2^I \to 1$$
$$\pi = \pi_1(P_L)$$

<u>Facts</u> · $W_L \curvearrowright \widetilde{P}_L^1$ = 1-skeleton

acts simply transitively on

$\mathrm{Vert}(P_L)^{I}$

Because C_2 does on

$\mathrm{Vert}(P_L)$

<u>Remark</u> (G, S) ← generators

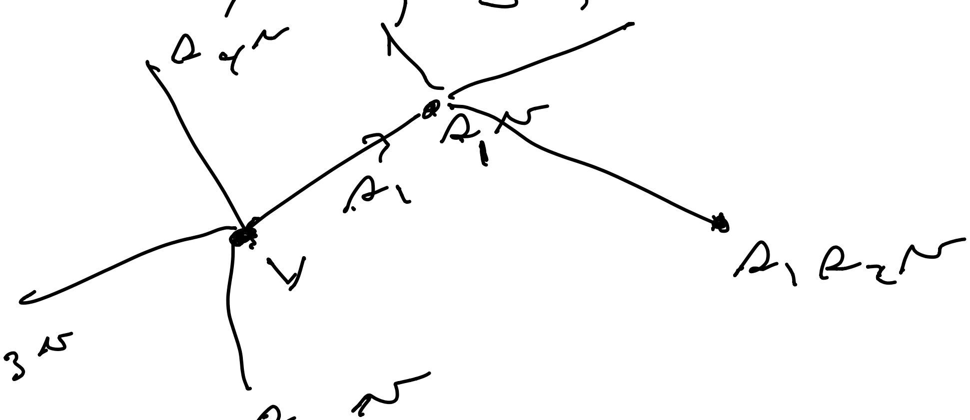
$\mathrm{Cay}(G, S)$ = graph

Vert set = G

$g \xrightarrow{s} gs$ Edge = $\{(g, gs)\}$

$G \curvearrowright \Omega$ = graph
simply transitive on $\mathrm{Vert}(\Omega)$

Then $\Omega \cong \mathrm{Cay}(G, S)$



s_i = left m

Then s_i ~~is fast~~ generator

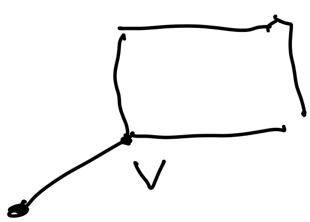
for $W_L = \langle s_i \rangle_{i \in I}$

Cayley 2-cx = simply 2 cx, $G \curvearrowright \mathrm{Vert}(\mathrm{Cay})$

1-skeleton = Cayley graph

~~2-skeletal~~

2-cells ~~are~~ give relations



Each square gives relation of form

$s_i s_j = s_j s_i$

$(s_i s_j)^2 = 1$

<u>Conclusion</u> W_L has presentation $\langle S \mid R \rangle$

$S = \{s_i\}$ $i \in \mathrm{Vert}(L)$

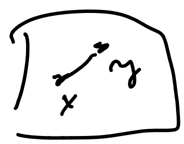
$R = \{ s_i^2 = 1, (s_i s_j)^2 = 1 \}$

$\{i,j\} \in \text{Edge } L.$

<u>Remark</u> W_L is "right-angled Coxeter gp corresponding L"

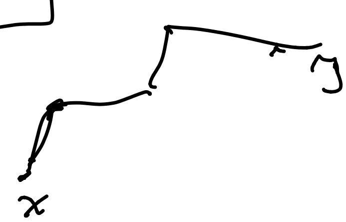
Discussion of CAT(0)-metric spa

$X =$ cube cx (cell cx)



$x, y \in \sigma$

$d(x,y) =$ usual distan



$$d(x,y) = \inf \left\{ \ell(\gamma) \text{ where } \gamma \text{ is broken geodesic} \right\}$$

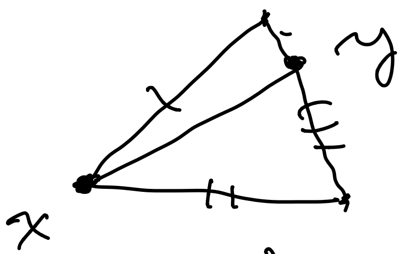
$X =$ geodesic space means $\exists$ path of length $d(x,y)$ between any 2 pts.

Suppose X = "geodesic space"

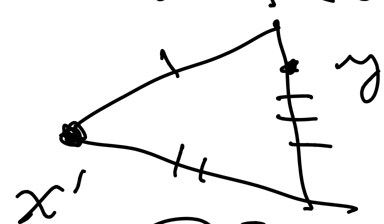
<u>Def</u> X is CAT(0)

Given any triangle in X

(3 vertices, 3 edges)



in X



$\mathbb{R}^2$ = Euclidean plane

$$\boxed{d(x,y) \le d(x',y')}$$

$\forall\, x,y$ in triangle

CAT(0)

curvature ≤ 0

Toponogov

Alexandrov

~~Car~~ Comparision