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Some groups of type VF

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Abstract. A group is of type *VF* if it has a finite-index subgroup which has a finite classifying space. We construct groups of type *VF* in which the centralizers of some elements of finite order are not of type *VF* and groups of type *VF* containing infinitely many conjugacy classes of finite subgroups. It follows that a group G of type *VF* need not admit a finite-type universal proper G-space. We construct groups G for which the minimal dimension of a universal proper G-space is strictly greater than the virtual cohomological dimension of G. Each of our groups embeds in $GL_m(\mathbb{Z})$ for sufficiently large m. Some applications to K-theory are also considered.

1. Introduction

In [6], M. Bestvina and N. Brady gave a method for constructing torsion-free groups with given homological finiteness properties. Examples that they constructed include: for each $n \geq 0$, groups of type F_n that are not of type F_{n+1}, groups that are of type *FP* over certain rings but not *FP* over others, and groups that are *FP* over the integers but that are not finitely presentable. (These finiteness conditions will be defined below.) We show how to construct virtually torsion-free groups with surprising homological properties as finite extensions of the groups considered by M. Bestvina and N. Brady. (By a finite extension of a group H we mean a group G containing H as a finite-index normal subgroup.)

The strongest of the homological finiteness conditions is *F*, where a group H is of type *F* if there is a finite Eilenberg-Mac Lane space $K(H, 1)$. Groups of type *F* are necessarily torsion-free. We exhibit groups that are virtually of type *F* or 'of type *VF*' that contain infinitely many conjugacy classes of finite subgroups, and groups of type *VF* in which the centralizers of some elements of finite order are not of type *VF*. The question of whether such examples can exist is implicit in [14,15,19,1], and was posed explicitly (by K.S. Brown amongst others) in the problem list from the 1977 Durham Symposium edited by C.T.C. Wall [64], and by W. Lück in [43]. One reason why these examples are of interest is that they show that the property of having a finite (resp. finitely dominated, finite type) classifying space for proper actions is not preserved under finite extensions. (The classifying space for proper actions of G is also known as the universal proper G-space, and usually denoted by $\underline{E}G$.) A result due to K.S. Brown ([16], IX.13.2) implies that any group of type *VF* contains only finitely many conjugacy classes of subgroups of prime power order. Our examples include groups of type *VF* containing infinitely many conjugacy classes of finite abelian subgroups.

A construction due to J.-P. Serre ([57], p. 96–97) shows that any group G of type *VF* admits a finite-dimensional model for its classifying space for proper actions, $\underline{E}G$. For each of the groups that we construct there is a 'nice' finite-dimensional (but not finite) model for $\underline{E}G$. In certain cases a study of this model gives rise to a lower bound for the dimension of any model which is strictly greater than the virtual cohomological dimension. In particular, for any $n \geq 1$ we exhibit a group G of type *VF* with virtual cohomological dimension $3n$ and such that any model for $\underline{E}G$ has dimension at least $4n$. These groups answer a question posed by K.S. Brown [15,43,49].

Using the results of [23], presentations may be given for the groups that we construct, and we give some examples including an explicit presentation for a group of type *VF* that contains infinitely many conjugacy classes of subgroups of order 120. Moreover, this group has virtual cohomological dimension three and any model for its classifying space for proper actions has dimension at least four. Each of the groups can be embedded in the integral general linear group $GL_n(\mathbb{Z})$ for sufficiently large n. For example, the group mentioned above embeds in $GL_{46}(\mathbb{Z})$. This group answers a question of M. Bridson [11].

Our method can also be used to construct groups satisfying weaker finiteness conditions than *VF*. For example, we exhibit a finitely presented subgroup of $GL_6(\mathbb{Z})$ containing infinitely many conjugacy classes of elements of order two, and for each $n \geq 4$ a finitely presented subgroup of $GL_{4n}(\mathbb{Z})$ containing infinitely many conjugacy classes of elements of order m, for each $m > 1$ dividing n. These groups answer a question posed by F. Grunewald and V. Platonov [29,54]. (This question has also been answered by M. Bridson in [11], and an answer was implicit in [27].)

Modulo torsion, the Farrell-Jones conjectures and the Baum-Connes conjecture relate the homology of centralizers of elements of finite order in the group G to the K-theory of various algebras associated to G [5,21,44]. The Baum-Connes conjecture may be shown to hold for the groups that we construct. Using this fact we exhibit groups of type *VF* for which the higher algebraic K-groups of certain group algebras and the topological K-groups of the reduced C^*-algebra have infinite rank.

The input to the Bestvina-Brady construction is a finite flag complex L, and the output is a torsion-free group H_L whose homological finiteness properties are controlled by the homotopy type of L. For example, H_L is *FP* over the integers but not finitely presentable if and only if L is acyclic but not contractible. The input for our generalization consists of a finite flag complex L, together with a group Q of automorphisms of L, and the output is a group $\widetilde{H} = H_L \rtimes Q$ whose homological properties are controlled by the homotopy type of L and of L^P, the P-fixed points in L, for each $P \leq Q$. For example, $\widetilde{H}$ is of type *VF* if and only if L is contractible, and $\widetilde{H}$ contains infinitely many conjugacy classes of finite subgroups if and only if Q acts on L without a global fixed point.

To construct examples, we rely on various results concerning actions of finite groups on simplicial complexes, including work of R. Oliver and earlier work that is surveyed by him in [52]. The Bestvina-Brady groups H_L are subgroups of right-angled Artin groups (or graph groups), and we also use a result of J.S. Crisp describing the fixed subgroup for a group of symmetries of an Artin group [20]. In fact, we only need a simple special case of Crisp's theorem, for which we provide a proof.

The remainder of the paper is organized as follows. Section 2 recalls the definitions and properties of the finiteness conditions that will be used. Section 3 contains a statement of the Bestvina-Brady theorem, together with some information concerning its proof which is required in Sects. 4–6. Section 4 describes our construction, and contains our main theorem, Theorem 3. Section 5 defines classifying spaces for proper actions, and explains their connection to our work. Section 6 compares various algebraic dimensions for a group with the dimension of its classifying space for proper actions. In Sect. 8 presentations and faithful integral linear representations for our groups are given. Sections 7 and 9 describe our examples and some of the questions that they resolve. The examples in Sect. 9 are rather more explicit than those in Sect. 7. Section 7 also discusses some related questions which our method fails to resolve, notably a conjecture of H. Bass concerning groups of type *FP* over the rationals [4]. Finally, in Sect. 10 partial information concerning the rational homology of Bestvina-Brady groups is obtained and used to construct group algebras whose higher algebraic K-groups have infinite rank.

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2. Finiteness conditions

Let G be a discrete group, let R be a commutative ring, and let RG denote the group algebra of G. The group G is of type F (resp. type F_n) if there is an Eilenberg-Mac Lane space $K(G, 1)$ which is finite (resp. has finite n-skeleton). It is easily seen that every group G is F_0, that G is F_1 if and only if G can be finitely generated, and that G is F_2 if and only if G can be finitely presented. The ring R may be made into a module for RG by letting every element of G act as the identity. This module is known as the trivial module for RG. The group G is *FP* over R if the trivial module admits a finite resolution by finitely generated projective RG-modules, i.e., for some n and finitely generated projective modules $P_0, \ldots, P_n$, there is an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to R \to 0.$$

If the P_i may be chosen to be finitely generated free RG-modules then G is *FL* over R.

If the trivial module admits a projective resolution (of any length) in which the first n terms are finitely generated, then G is of type FP_n over R. Equivalently, there are finitely generated projective modules $P_0, \ldots, P_n$ and an exact sequence

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to R \to 0.$$

G is of type F_∞ (resp. FP_∞ over R) if, for all $n \geq 0$, G is of type F_n (resp. FP_n over R).

If X is any of the above finiteness conditions, a group G is said to be of type VX or 'virtually of type X' if G has a finite-index subgroup H of type X. Many of the virtual notions are redundant, since for each n and R, VF_n is equivalent to F_n, and VFP_n over R is equivalent to FP_n over R. A theorem due to Serre shows that if G is *VFP* over R and contains no R-torsion, then G is *FP* over R [7,16,57]. In general, if G is *VF* (resp. *VFP* over R) then G is F_∞ (resp. FP_∞ over R). Note that the R. Thompson group is of type F_∞, but is not *VFP* over any non-trivial R, and therefore not of type *VF*, since it contains a free abelian group of infinite rank [17].

3. The Bestvina-Brady construction

Let L be a simplicial complex. L is a flag complex (or full simplicial complex) if there is no larger simplicial complex with the same 1-skeleton. Given a simplicial graph L^1, one can recover a flag complex L as the clique complex of L^1, i.e., the complex whose n-simplices are the complete subgraphs of L^1 on $(n+1)$ vertices. The barycentric subdivision of any simplicial complex is a flag complex.

Given a finite flag complex L, the right-angled Artin group (or graph group) G_L is a group whose generators are in 1-1 correspondence with the vertex set of L, subject only to the relations that the ends of every edge of L commute with each other. For example, in the case when L is the n-simplex, G_L is free abelian of rank $n+1$, and in the case when L consists of n vertices and no edges, G_L is free of rank n. Provided that L is not empty, there is a surjective homomorphism $\phi_L : G \to \mathbb{Z}$ that sends each of the given generators to 1. The group H_L is defined to be the kernel of ϕ_L. (For example, in the case when L is the n-simplex, H_L is free abelian of rank n, and in the case when L consists of $n > 1$ points, H_L is free of infinite rank.)

Before stating the Bestvina-Brady theorem, we recall some topological definitions. A topological space X is 0-connected if and only if X is non-empty and path connected. For $n \geq 1$, X is n-connected if and only if X is 0-connected and for each i with $1 \leq i \leq n$, the homotopy group $\pi_i(X)$ is trivial. X is R-homologically n-connected if $H_i(X; R)$ is isomorphic to the ith homology group of a point for all i with $0 \leq i \leq n$, and X is R-acyclic if X is R-homologically n-connected for all n.

Theorem 1 [6]. *Let L be a non-empty finite flag complex, let $n \geq 0$, and define $H = H_L$ as above.*

H is of type F if and only if L is contractible;
H is of type F_{n+1} if and only if L is n-connected;
H is of type FP over R if and only if L is R-acyclic;
H is of type FP_{n+1} over R if and only if L is R-homologically n-connected. □

The proof of this theorem uses nice models for an Eilenberg-Mac Lane space $K(G, 1)$ for $G = G_L$, and for its universal cover, EG. The model for $K(G, 1)$ has one 0-cell, and for $n > 0$ its n-cells are n-dimensional cubes in bijective correspondence with the $(n-1)$-simplices of L. The corresponding model for EG will be denoted by $X = X_L$. Thus X is a CW-complex whose n-cells are cubes indexed by pairs $(gx_0, \{v_1, \ldots, v_n\})$, where x_0 is some fixed vertex of X, $g \in G$ and $v_1, \ldots, v_n$ are 0-simplices of L that span an $(n-1)$-simplex (or equivalently, distinct elements of the given generating set for G that all commute). For each i, the $(n-1)$-cubes $(gx_0, \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots v_n\})$ and $(gv_ix_0, \{v_1, \ldots, v_{i-1}, v_{i+1} \ldots v_n\})$ are a pair of opposite faces of the n-cube given above. The action of G on X is given by $h.(gx_0, \sigma) = (hgx_0, \sigma)$.

There is a CAT(0)-metric on X in which each of the cubes is isometric to a standard Euclidean cube of side length 1. This implies that X is contractible, and gives one method for proving that X/G is a $K(G, 1)$. Make the real line $\mathbb{R}$ into a CW-complex with 0-cells the integers, let $\mathbb{Z}$ act on $\mathbb{R}$ by addition, and use $\phi_L : G_L \to \mathbb{Z}$ to define an action of G on $\mathbb{R}$ in which

H acts trivially. Then there is a G-equivariant map $f\colon X \to \mathbb{R}$ which is cellular, affine on each cube, and for each $g \in G$ and $v \in L^0$ sends the edge from gx_0 to gvx_0 to an interval $[m, m+1]$. It is possible to choose f so that $f(x_0) = 0$, in which case $m = \phi_L(g)$. For $t \in \mathbb{R}$, define $X_t = f^{-1}(t)$. The cellular structure on X gives rise to a cellular structure on X_t, but note that X_t intersects the cubes of X 'diagonally', so X_t is not a cube complex. For each t, H is the setwise stabilizer in G of X_t. Since X is a model for EG, it is also an infinite model for EH. On the other hand X_t is a finite H-CW-complex, but in general is not contractible. The proof of the theorem involves comparing the homotopy types of X and X_t using a 'cellular Morse-theory'. In this way it is shown that X is homotopy equivalent to a space made by attaching infinitely many cones on L to X_t.

4. Finite extensions

The map $L \mapsto G_L$ is a functor from flag complexes to groups, and in particular any automorphism of L induces an automorphism of G_L. Let Q be a group of automorphisms of L, and suppose that Q acts admissibly on L, in the sense that for every simplex σ of L, the setwise and pointwise stabilizers of σ are equal. ("Admissible" is a slightly weaker condition on an action than "regular" as in [10], p. 114–117.) For an arbitrary action of a finite group Q on a simplicial complex L, the induced action of Q on the barycentric subdivision L' is an admissible action on a flag complex, so insisting that the action should be admissible places no restriction on the equivariant homotopy type of L. The hypothesis that the action of Q on L be admissible ensures that the fixed points L^Q form a subcomplex of L. The following theorem is a special case of a theorem of J.S. Crisp [20].

Theorem 2 [20]. *Let L be a finite flag complex, and suppose that Q acts admissibly on L, with fixed points $M = L^Q$. Then the subgroup of G_L fixed by Q is isomorphic to G_M.*

Proof. Given a choice of base-vertex x_0 for X, the action of Q on L and on $G = G_L$ induces an action of Q by isometries on $X = X_L$, fixing x_0. The fact that X is CAT(0) implies that the fixed point subset X^Q is contractible ([9] or [12]). The fact that Q acts admissibly on L implies that the action of Q on X is cellular, and hence that X^Q is a subcomplex of X.

The vertices of X^Q are precisely the vertices gx_0 such that $g \in G^Q$. Since X^Q is a connected subcomplex of X, any two vertices of X^Q must be joined by a path in the 1-skeleton of X^Q. But the edges of X^Q are the edges (gx_0, gvx_0) such that $v \in M^0$ and $g \in G^Q$, and so the natural map from G_M to $(G_L)^Q$ is surjective. The kernel, K, of this map is trivial, since the map induces a homeomorphism from X_M/K to X^Q, and both X_M and X^Q are contractible. □

Remark. Crisp's theorem is far more general. As well as applying in the case when the action of Q on L is not admissible, it also applies to other

Artin groups (not right-angled) for which a suitable CAT(0) space exists [20].

For any finite group Q acting on a non-empty flag complex L, the homomorphism $\phi_L : G_L \to \mathbb{Z}$ is Q-equivariant (for the trivial action on $\mathbb{Z}$). It follows that Q acts on H_L, and we may form a group $\widetilde{H} = \widetilde{H}(L, Q) = H_L \rtimes Q$, the split extension with kernel H_L and quotient Q. The homomorphism ϕ_L extends uniquely to a homomorphism, which we shall also denote by ϕ_L, from $\widetilde{G} = G_L \rtimes Q$ to $\mathbb{Z}$, and $\widetilde{H}$ is equal to the kernel of this homomorphism. Since H_L is torsion-free and normal in $\widetilde{H}$, any finite subgroup of $\widetilde{H}$ will map isomorphically to a subgroup of Q.

Theorem 3. *Let L be a non-empty finite flag complex with an effective admissible action of a finite group Q, and define $\widetilde{H} = \widetilde{H}(L, Q)$ as above. Let P be a subgroup of Q, let $M = L^P$, and let $\mathcal{P}$ be the set of finite subgroups of $\widetilde{H}$ that map onto some conjugate of P in Q.*

1. *If $M = \emptyset$, then $\mathcal{P}$ contains infinitely many conjugacy classes of subgroups of $\widetilde{H}$, and for all $P' \in \mathcal{P}$, $C_{\widetilde{H}}(P') \cong C_Q(P)$.*
2. *If $M \neq \emptyset$, then all elements of $\mathcal{P}$ are conjugate in $\widetilde{H}$, and for any $P' \in \mathcal{P}$, $C_{\widetilde{H}}(P') \cong \widetilde{H}(M, C_Q(P)) = H_M \rtimes C_Q(P)$.*

Proof. A choice of base-vertex x_0 for X defines an action of Q on X, and the actions of G_L and Q on X together give a proper action of the split extension $\widetilde{G} = G_L \rtimes Q$. For this action the stabilizer of the vertex gx_0 is the subgroup gQg^{-1}, where by an abuse of notation we are using Q to denote the 'standard' copy of Q inside $\widetilde{G}$. The G-equivariant map $f : X \to \mathbb{R}$ defined during the description of the proof of Theorem 1 is also $\widetilde{G}$-equivariant, where $\widetilde{G}$ acts on $\mathbb{R}$ by the translation action of $\mathbb{Z} = \phi_L(\widetilde{G})$.

Since X is CAT(0) and $\widetilde{G}$ is acting by isometries, it follows that, for any finite subgroup P' of $\widetilde{G}$, the fixed points $X^{P'}$ form a contractible subcomplex of X. In particular, $X^{P'}$ contains some vertex gx_0, and so $P' \leq gQg^{-1}$. If P' and P'' are subgroups of $\widetilde{G}$ that both map isomorphically to $P \leq Q$, it follows that P' and P'' are conjugate in $\widetilde{G}$.

In the case when M is empty, $X^{P'}$ cannot contain any 1-cells, so must be a single point gx_0. In this case, $f(X^{P'})$ is the point $\phi_L(g) \in \mathbb{Z} \leq \mathbb{R}$. If $P'' = hP'h^{-1}$, then $f(X^{P''})$ is the point $\phi_L(hg)$. But if $h \in \widetilde{H}$, then $\phi_L(hg) = \phi_L(g)$, and so $\widetilde{H}$-conjugate subgroups isomorphic to P fix points of X of the same 'height'. It follows that in the case when M is empty, the subgroups $v^i P' v^{-i}$ for $i \in \mathbb{Z}$ all lie in distinct conjugacy classes in $\widetilde{H}$, where v is any of the standard generators for G_L.

In the case when M is not empty, let P' be a subgroup of $\widetilde{H}$ that maps isomorphically to P such that P' fixes x_0. If P'' is another subgroup of $\widetilde{H}$ that maps isomorphically to P, then there exists $g \in G_L$ such that P'' fixes the point gx_0. Since M is non-empty, there exists v, one of the generators for G_L, which commutes with P'. Then for some $i \in \mathbb{Z}$, the element $g' = gv^i$ lies in $\widetilde{H}$ and satisfies $g'P'g'^{-1} = gv^i P' v^{-i} g^{-1} = gP'g^{-1} = P''$.

The description of the centralizers in each case follows from Theorem 2. □

Remark (Inadmissible actions). A similar theorem can be proved in the more general case, in which Q is an arbitrary group of automorphisms of the flag complex L. The general version of Crisp's theorem (not Theorem 2) implies that the subgroup of G_L fixed by $P \leq Q$ is isomorphic to G_M, where M is a flag complex whose realization $|M|$ is homeomorphic to $|L|^P$, the P-fixed points in the realization of L. The vertices of M are the barycentres of simplices of L that are minimal amongst simplices that are preserved (setwise) by P, and simplices of M are sets of vertices that are contained within a single simplex of L. The inclusion of G_M in G_L sends the generator corresponding to the simplex $(v_1, \ldots, v_n)$ to the product $v_1 \cdots v_n$, and so the composite of the inclusion with ϕ_L is not equal to ϕ_M, but to a homomorphism $\phi' : G_M \to \mathbb{Z}$ that sends each generator to some strictly positive integer. J. Meier, H. Meinert and L. VanWyk have given a version of the Bestvina-Brady theorem that applies also to homomorphisms such as ϕ' [48].

In this way, one obtains the same statement as Theorem 3 for the case when Q does not act admissibly (with the new definition of M), except that if $P \leq Q$ fixes some point of $|L|$ but does not fix a vertex, the set $\mathcal{P}$ may contain more than one (but still only finitely many) conjugacy classes of subgroup of $\widetilde{H}$. In fact, the number of conjugacy classes in $\mathcal{P}$ is equal to the highest common factor of the numbers of vertices of the minimal simplices that are preserved by P, or equivalently the index of $\phi_L(G_M)$ in $\mathbb{Z}$. This phenomenon can be seen already in the simplest possible case, namely $Q = \Sigma_n$, the symmetric group of degree n, acting as the full automorphism group of the $(n-1)$-simplex L. In this case $\widetilde{G}$ is the wreath product $\mathbb{Z} \wr \Sigma_n$ and $\widetilde{H}$ is an extension with kernel $H_L = \mathbb{Z}^{n-1}$ and quotient Σ_n, which contains n conjugacy classes of subgroup isomorphic to Σ_n. In this special case, this can also be verified directly by computing $H^1(\Sigma_n; H_L)$.

5. Classifying spaces for proper actions

For G a discrete group, a proper G-CW-complex is one in which all cell stabilizers are finite. A classifying space for proper actions of G, denoted by $\underline{E}G$, is a proper G-CW-complex such that for each finite $P \leq G$, the P-fixed point set $(\underline{E}G)^P$ is contractible. Equivalently, $\underline{E}G$ is a terminal object in the homotopy category of proper G-CW-complexes. From the second definition, it follows that $\underline{E}G$ is unique up to equivariant homotopy. For more details, including a proof that these definitions are equivalent and a construction of $\underline{E}G$ for general G, see any of [62,63,38,49]. The definitions of $\underline{E}G$ given above may be compared with those for EG, which is defined as either the universal cover of a $K(G, 1)$, or as a contractible free G-CW-complex, or as a terminal object in the homotopy category of free

G-CW-complexes. In particular, if G is torsion-free, there is no distinction between free and proper actions, and so $\underline{E}G = EG$.

Just as the finiteness conditions F, F_n and F_∞ may be viewed as conditions on the number of G-orbits of cells in a model for EG, one may consider finiteness conditions for $\underline{E}G$. For example, every finite subgroup of G fixes a point of $\underline{E}G$, and so must be a subgroup of a 0-cell stabilizer. Hence the existence of an $\underline{E}G$ with only finitely many orbits of 0-cells implies that G contains only finitely many conjugacy classes of finite subgroups. If P is a finite subgroup of G with normalizer $N = N_G(P)$, then it may be shown that $(\underline{E}G)^P$ is a model for $\underline{E}N$, which is finite (resp. finitely dominated, resp. of finite type) if the original $\underline{E}G$ had this same property. In particular, if there is an $\underline{E}G$ that is finite (resp. finitely dominated, resp. of finite type), then any torsion-free finite-index subgroup of any $N_G(P)$ must be of type F (resp. FP, resp. F_∞). In fact, it may be shown that conditions on finite subgroups and their normalizers characterize those G for which $\underline{E}G$ is finitely dominated (resp. finite type). The following theorem is a combination of Theorem 1 of F. Connolly and T. Koźniewski's [19] and Theorem 4.2 of W. Lück's [43]. (Note however that we have modified Lück's statement, which uses $N_G(P)/P$ instead of $N_G(P)$.)

Theorem 4. *For any discrete group G, the following are equivalent:*

1. *There is an $\underline{E}G$ of finite type;*
2. *G contains finitely many conjugacy classes of finite subgroups, and for each such P, $N_G(P)$ is of type F_∞.*

If G is of finite virtual cohomological dimension and satisfies condition 1 or 2, then $\underline{E}G$ is finitely dominated. □

This result indicates that Theorem 3 may be used to construct groups of type VF that do not have finite type models for $\underline{E}G$. However, more can be said. Any proper G-CW-complex that is equipped with a G-equivariant CAT(0)-metric is a model for $\underline{E}G$ (see [12,9], but note that no explicit mention of $\underline{E}G$ is made in [12]). In particular, for any finite flag complex L and group Q of admissible automorphisms of L, the space X_L is a model for $\underline{E}\widetilde{G} = \underline{E}(G_L \rtimes Q)$. The extension of the Bestvina-Brady argument to this context gives:

Theorem 5. *With notation as in Theorem 3, the following are equivalent:*

1. *For each $P \leq Q$, L^P is contractible;*
2. *For any $t \in \mathbb{R}$, X_t is a finite model for $\underline{E}\widetilde{H}$;*
3. *There is a finite model for $\underline{E}\widetilde{H}$;*
4. *$\underline{E}\widetilde{H}$ is finitely dominated;*
5. *There is a finite type model for $\underline{E}\widetilde{H}$.*

Proof. $2 \Rightarrow 3$, $3 \Rightarrow 4$ and $3 \Rightarrow 5$ are clear. Bestvina and Brady prove $1 \Rightarrow 2$ in the case when Q is trivial using the following lemma. For any L, the space X_L (which is an infinite model for EH_L) is homotopy equivalent to a CW-complex made by attaching infinitely many copies of the cone on L to X_t. The same argument shows that for any finite $P' \leq \widetilde{G}$, either $(X_t)^{P'} = \emptyset$, or $X^{P'}$ is homotopy equivalent to $(X_t)^{P'}$ with infinitely many copies of the cone on $M = L^P$ attached. The case when $(X_t)^{P'} = \emptyset$ can only occur if $M = \emptyset$. In particular, if each such M is contractible, then each $(X_t)^{P'}$ is homotopy equivalent to $X^{P'}$, which is contractible, and so X_t is a finite model for $\underline{\mathrm{E}}\widetilde{H}$.

If 1 does not hold, then either (a) there exists $P \leq Q$ for which $L^P = \emptyset$, or (b) there exists $P \leq Q$ and $n \geq 0$ for which $M = L^P$ is $(n-1)$-connected but not n-connected. (Here we adopt the convention that '-1-connected' means 'non-empty'.) By Theorem 3, in case (a) $\widetilde{H}$ contains infinitely many conjugacy classes of finite subgroups and so any model for $\underline{\mathrm{E}}\widetilde{H}$ contains infinitely many orbits of 0-cells. In case (b), let $P' \leq \widetilde{H}$ be a subgroup that maps isomorphically to $P \leq Q$, and let $M = L^P$. By Theorem 3, the normalizer in $\widetilde{H}$ of P' contains the torsion-free group H_M as a finite-index subgroup. By Theorem 1, H_M is type F_n but not type F_{n+1}, and so any model for $\underline{\mathrm{E}}\widetilde{H}$ must contain infinitely many orbits of $(n+1)$-cells (whose stabilizers contain a conjugate of P'). In either case, if 1 does not hold, then neither can 4 or 5. □

Note that the use of an explicit model for $\underline{\mathrm{E}}\widetilde{H}$ allows us to avoid using the 'difficult' implication in Theorem 4 while proving Theorem 5. Note also that condition 1 in Theorem 5 is equivalent to the statement that L is a model for $\underline{\mathrm{E}}Q$.

The following theorem is not explicitly stated by F. Connolly and T. Koźniewski or by W. Lück [19,43], but may be proved in the same way as Theorem 4.

Theorem 4′. *For any discrete group G, and any $n \geq 0$, the following are equivalent:*

1. *There is an $\underline{\mathrm{E}}G$ with finite n-skeleton;*
2. *G contains finitely many conjugacy classes of finite subgroups, and for each such P, $N_G(P)$ is of type F_n.* □

The analogue of Theorem 5 is:

Theorem 5′. *With notation as in Theorem 3, for any $n \geq 0$, the following are equivalent:*

1. *For each $P \leq Q$, L^P is $(n-1)$-connected;*
2. *For any $t \in \mathbb{R}$, a model for $\underline{\mathrm{E}}\widetilde{H}$ may be made by attaching cells of dimension strictly greater than n to X_t;*
3. *There is a model for $\underline{\mathrm{E}}\widetilde{H}$ with finite n-skeleton.*

Proof. $2 \Rightarrow 3$ is clear, and the proof that $3 \Rightarrow 1$ is identical to an argument used in the proof of Theorem 5. To prove $1 \Rightarrow 2$, use the Bestvina-Brady argument to show that when 1 holds, $(X_t)^{P'}$ is $(n-1)$-connected for any finite $P' \leq \widetilde{H}$. The following standard argument shows that an $\underline{E}\widetilde{H}$ may be made by attaching cells of dimension greater than n to X_t.

Let $P_1, \ldots, P_r$ be a set of conjugacy class representatives for finite subgroups of G, so that $|P_i| \geq |P_j|$ whenever $i < j$. (Even in the case $n = 0$, condition 1 implies that there are only finitely many conjugacy classes of finite subgroups.) Define Z_n to be X_t, so that for each i, $(Z_n)^{P_i}$ is $(n-1)$-connected. Now let $Y_0 = Z_n$ and suppose that Y_j for $0 \leq j < r$ has been made by attaching orbits of cells of types $D^{n+1} \times G/P_i$ for $i < j$ to Z_n, and that $(Y_j)^{P_i}$ is n-connected for each $i < j$. Choose cellular maps from the n-sphere to $(Y_j)^{P_j}$ that represent generators for $\pi_n((Y_j)^{P_j})$, and extend equivariantly to maps from $S^n \times G/P_j$ to Y_j. Define Y_{j+1} by attaching copies of $D^{n+1} \times G/P_j$ to Y_j along these maps. Note that $(Y_{j+1})^{P_i} = (Y_j)^{P_i}$ for each $i < j$ since the choice of ordering implies that P_i cannot be conjugate to a subgroup of P_j. Now $Z_{n+1} = Y_r$ consists of X_t with some orbits of $(n+1)$-cells attached, and has the property that $(Z_{n+1})^{P'}$ is n-connected for each finite $P' \leq \widetilde{H}$. The claim follows by induction on n. □

6. Notions of dimension

The minimal dimension of a model for $\underline{E}G$, which we shall denote by $\underline{\mathrm{gd}}G$ and refer to as the 'proper geometric dimension of G', is a geometrically-defined invariant of G. The relationship between $\underline{\mathrm{gd}}G$ and various algebraically-defined quantities, including the virtual cohomological dimension of G or vcd$G$, has attracted some interest. In this section, we shall give some information concerning $\underline{\mathrm{gd}}G$ and these algebraic quantities in the case when $G = H_L \rtimes Q$. This will enable us to construct examples of groups (of type *VF*) for which $\underline{\mathrm{gd}}G$ is strictly greater than vcd$G$. First we recall the definitions of various algebraic quantities. For more details concerning these definitions we refer the reader to [9].

Recall that the cohomological dimension of a group H or cd$H$, is defined to be the projective dimension of $\mathbb{Z}$ as a $\mathbb{Z}H$-module. (Note that cd$H$, and each of the other notions of dimension that we shall discuss, may be either a positive integer or infinity.) Recall also that the virtual cohomological dimension of a group G is by definition vcd$G = $ cd$H$ if G contains a finite-index torsion-free subgroup H, and vcd$G = \infty$ if G contains no such subgroup. (A lemma due to Serre shows that this is independent of choice of H [16,57].) If H is a torsion-free subgroup of G, then any model for $\underline{E}G$ will serve as a model for EH, and so whenever G is virtually torsion-free, vcd$G \leq \underline{\mathrm{gd}}G$. Note that there are groups G which are not virtually torsion-free but admit finite-dimensional models for $\underline{E}G$ [9,56]. The question of whether vcd$G = \underline{\mathrm{gd}}G$ for all virtually torsion-free groups was raised by K.S. Brown in [15], p. 32, but see also [43], Problem 7.10 and [49].

Another algebraic dimension for a group G is $\mathcal{F}$-cd$G$, which is defined in terms of relative homological algebra, relative to the class $\mathcal{F}$ of all finite subgroups of G. More precisely, $\mathcal{F}$-cd$G$ is defined to be the minimal length n of an exact sequence of G-modules

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0,$$

with the properties that

(a) each C_i is a direct summand of a sum of modules induced up from finite subgroups of G;
(b) for each finite subgroup $H \leq G$, there is an H-equivariant splitting of the exact sequence.

A theorem due independently to S. Bouc and to P.H. Kropholler and C.T.C. Wall [8,37] implies that the augmented cellular chain complex for any acyclic proper G-CW-complex is such an exact sequence. In particular, for any G one has that $\mathcal{F}$-cd$G \leq \underline{\mathrm{gd}}G$.

The closest algebraic approximation to $\underline{\mathrm{gd}}G$ however is another quantity, $\underline{\mathrm{cd}}G$, defined in terms of the category of 'Bredon coefficient systems' or 'modules over the orbit category $\mathcal{O}(\mathcal{F}, G)$'. Work of M. Dunwoody (for the 1-dimensional case) and W. Lück (for the high-dimensional case) shows that $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G$, except that possibly there may exist a group for which $\underline{\mathrm{cd}}G = 2$ and $\underline{\mathrm{gd}}G = 3$ [25,41]. In [9] Noel Brady and the current authors exhibit groups for which strict inequality holds.

Briefly, the orbit category $\mathcal{O}(\mathcal{F}, G)$ has as objects the G-sets G/P for P a finite subgroup of G, and morphisms the G-maps. An $\mathcal{O}(\mathcal{F}, G)$-module or Bredon coefficient system for proper G-CW-complexes is a contravariant functor from $\mathcal{O}(\mathcal{F}, G)$ to abelian groups. Examples of $\mathcal{O}(\mathcal{F}, G)$-modules include a constant functor $\mathbf{Z}$ taking each object to $\mathbb{Z}$ and each morphism to the identity, and for each finite $Q \leq G$ a functor $P_{G/Q}$ such that $P_{G/Q}(G/P)$ is the free abelian group with basis the morphisms in $\mathcal{O}(\mathcal{F}, G)$ from G/P to G/Q. Each $P_{G/Q}$ is projective, and any $\mathcal{O}(\mathcal{F}, G)$-module is a homomorphic image of a direct sum of such modules. It follows that there are enough projectives in the category of $\mathcal{O}(\mathcal{F}, G)$-modules. Thus $\underline{\mathrm{cd}}G$ may be defined to be the projective dimension in this category of the object $\mathbf{Z}$. For any proper G-CW-complex X, the map $G/P \mapsto \mathrm{Map}_G(G/P, X)$ describes a functor from $\mathcal{O}(\mathcal{F}, G)$ to CW-complexes, and the composite of this with the cellular chain complex functor gives rise to a chain complex of projective $\mathcal{O}(\mathcal{F}, G)$-modules. In particular, any model for $\underline{E}G$ of dimension n gives rise to an $\mathcal{O}(\mathcal{F}, G)$-projective resolution of $\mathbf{Z}$ of length n, and so $\underline{\mathrm{cd}}G \leq \underline{\mathrm{gd}}G$.

For more details concerning any of these definitions see [9], where the following inequalities are also proved: for any group G, one has that

$$\mathcal{F}\text{-cd}G \leq \underline{\mathrm{cd}}G \leq \underline{\mathrm{gd}}G \leq \max\{\underline{\mathrm{cd}}G, 3\},$$

and that vcd$G \leq \mathcal{F}$-cd$G$ provided that G is virtually torsion-free. To establish the existence of virtually torsion-free groups G for which vcd$G = \mathcal{F}$-cd$G < \underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G$, we shall use the cohomology of $\underline{E}G$ relative to its singular set to provide a lower bound for $\underline{\mathrm{cd}}G$. If X is a proper G-CW-complex, denote by X^{sing} the subcomplex consisting of all cells whose stabilizer is non-trivial. (In [43], W. Lück uses the notation $\sigma_G(X)$ for X^{sing} to emphasize that X^{sing} depends on G as well as on X.) If Y is another proper G-CW-complex, then any equivariant map from X to Y sends X^{sing} to Y^{sing}, and any equivariant homotopy from $X \times I$ to Y restricts to an equivariant homotopy from $X^{\mathrm{sing}} \times I$ to Y^{sing}. In particular, the pair $(\underline{E}G, \underline{E}G^{\mathrm{sing}})$ is well-defined up to equivariant homotopy, and the equivariant cohomology groups $H^*_G(\underline{E}G, \underline{E}G^{\mathrm{sing}}; \mathbb{Z}G)$ depend only on G. These groups provide a lower bound on the dimension of $\underline{E}G$. Moreover, it may be checked that

$$H^*_G(\underline{E}G, \underline{E}G^{\mathrm{sing}}; \mathbb{Z}G) \cong \mathrm{Ext}^*_{\mathcal{O}(\mathcal{F},G)}(\mathbf{Z}, P_{G/1}),$$

and so these groups also provide a lower bound for $\underline{\mathrm{cd}}G$.

Now suppose that L is an n-dimensional flag complex with an admissible action of the finite group Q. In this case $G = H_L \rtimes Q$ admits an $n+1$-dimensional model for $\underline{E}G$, namely the complex X_L, and H_L contains free abelian subgroups of rank n (since G_L contains free abelian subgroups of rank $n+1$). It follows that each of the notions of dimension defined in this section is equal to either n or $n+1$ for any such G. Ideally, one would like a result determining these quantities for all pairs (L, Q). However, the following theorem will suffice for constructing examples.

Theorem 6. *Let L be an n-dimensional acyclic flag complex with an admissible action of a finite group Q. Suppose that there exists an n-simplex σ of L whose stabilizer in Q is trivial, but such that the stabilizer of each simplex in the boundary of σ is non-trivial. For any $m \geq 1$, let $G = G(m)$ be the direct product of m copies of the group $H_L \rtimes Q$. Then G is type VFP over $\mathbb{Z}$, and*

$$\mathrm{vcd}G = \mathcal{F}\text{-}cdG = m.n, \qquad \underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G = m.(n+1).$$

If L is contractible, then G is of type VF.

Proof. The facts that G is *VFP* over $\mathbb{Z}$ and is *VF* when L is contractible follow from Theorem 1. Since H_L contains a free abelian subgroup of rank n, it follows that G contains a free abelian subgroup of rank $m.n$, which gives the required lower bound for $\mathrm{vcd}G$. For any t, the level set X_t is an acyclic n-dimensional space admitting a proper cellular action of $H_L \rtimes Q$. Hence the direct product $(X_t)^m$ is an $m.n$ dimensional acyclic G-CW-complex, and so by a theorem of Bouc and Kropholler-Wall [8,37], $\mathcal{F}\text{-cd}G \leq m.n$.

The direct product $X^m = (X_L)^m$ is an $m.(n+1)$-dimensional model for $\underline{E}G$, and so $\underline{\mathrm{gd}}G \leq m.(n+1)$. Recall the explicit description of the cellular structure of $X = X_L$ given after the statement of Theorem 1, and note that since each cell of X is a cube, the product of cells gives rise to a cubical cell structure on X^m. For the action of $H_L \rtimes Q$ on X, the stabilizer of the k-dimensional cube $(gx_0, \{v_1, \ldots, v_k\})$ is a subgroup isomorphic to the stabilizer in Q of the $(k-1)$-simplex $(v_1, \ldots, v_k)$. Under the hypotheses on L and Q, there is an $(n+1)$-cube in X lying in a free $H_L \rtimes Q$-orbit, such that each of its proper faces lies in a non-free $H_L \rtimes Q$-orbit. Similarly, the direct product X^m contains an $m.(n+1)$-cube in a free G-orbit, each of whose proper faces lies in a non-free G-orbit. It follows that the relative cellular chain complex $C_*(X^m, (X^m)^{\text{sing}})$ contains as a direct summand a chain complex consisting of a single copy of $\mathbb{Z}G$ in degree $m(n+1)$, and hence that

$$H^*_G(\underline{E}G, \underline{E}G^{\text{sing}}; \mathbb{Z}G) = H^*\text{Hom}_G(C_*(X^m, (X^m)^{\text{sing}}), \mathbb{Z}G)$$

is non-zero in degree $m.(n+1)$, and so $\underline{\text{cd}}G \geq m.(n+1)$.

The inequalities proved so far, together with the inequalities $\text{vcd}G \leq \mathcal{F}\text{-cd}G \leq \underline{\text{cd}}G \leq \underline{\text{gd}}G$ (which hold for any virtually torsion-free group [9]), imply the claimed equations. □

Remark. As remarked earlier, [9] exhibits groups G for which

$$\text{vcd}G = \mathcal{F}\text{-cd}G = \underline{\text{cd}}G = 2, \qquad \underline{\text{gd}}G = 3.$$

Those examples were based around the fact that no algebraic notion of dimension is able to capture the distinction between acyclic spaces and contractible spaces, an Eilenberg-Ganea type phenomenon. In contrast, examples coming from Theorem 6 show that $\text{vcd}G$ is not the best algebraic approximation to $\underline{\text{gd}}G$.

7. Examples

Throughout this section, A_n and Σ_n will denote the alternating and symmetric groups of degree n respectively, Fr_n will denote the free group of rank n, and C_n will denote the cyclic group of order n.

Example 1. Our motivating example was the following inadmissible action. Let $Q = \Sigma_n$, let L be the join of n pairs of points (an n-dimensional analogue of the octahedron), and let Q act by permuting the n pairs. In this case G_L is a direct product of n copies of Fr_2, and $\widetilde{G}$ is the wreath product $\text{Fr}_2 \wr \Sigma_n$. The fact that for this L, H_L is of type F_{n-1} but not of type F_n was known long before the work of Bestvina and Brady. In the case $n \leq 3$ it is due to J. Stallings and for general n to R. Bieri [60,7]. In this case, the centralizer C of Σ_n in $\widetilde{G}$ is the diagonal copy of Fr_2, and $\phi_L(C)$ is the index n subgroup of $\phi_L(\widetilde{G}) \cong \mathbb{Z}$. It follows that $C \cap H_L$ is free of infinite rank, and hence that the property F_m is not inherited by centralizers of finite subgroups.

Example 2. One well-known example of a virtually torsion-free group containing infinitely many conjugacy classes of finite subgroups is the infinite free product $C_n * C_n * C_n * \cdots$, for $n > 1$. The kernel of the map to C_n which is the identity map on each factor is isomorphic to Fr_∞. This group fits into the framework of Theorem 3 as follows. Take L to be n points, and let $Q = C_n$ permute these points freely. Then G_L is isomorphic to Fr_n, and H_L is isomorphic to Fr_∞. The split extension $\widetilde{G} = G_L \rtimes C_n$ is isomorphic to the free product $\mathbb{Z} * C_n$, and $\phi : \mathbb{Z} * C_n \to \mathbb{Z}$ is the homomorphism defined by the identity map on $\mathbb{Z}$ and the trivial map on C_n. Hence $\widetilde{H}$ is isomorphic to an infinite free product of copies of C_n, as claimed.

Before describing our main examples, we recall some results concerning actions of finite groups on contractible complexes. Let $\mathbb{F}_p$ denote the field of p elements. P.A. Smith showed that the fixed point set for any action of a p-group on any finite-dimensional $\mathbb{F}_p$-acyclic complex must be $\mathbb{F}_p$-acyclic [59]. Conversely, L.E. Jones showed that any finite $\mathbb{F}_p$-acyclic complex arises as the fixed-point set for some action of C_p on a finite contractible complex [34]. In [52] R. Oliver showed that for any finite group Q that is not of prime-power order, there exists an integer $n_Q \geq 0$ such that a finite complex M is equal to the fixed point set for some action of Q on a finite contractible complex if and only if the Euler characteristic of M satisfies $\chi(M) \equiv 1$ modulo n_Q. In [52], the groups for which $n_Q = 0$ and $n_Q = 1$ were determined, together with a characterization of the primes that divide n_Q for any Q. In a later paper [53], Oliver was able to compute n_Q for all Q.

Example 3. The finite groups Q for which Oliver's integer n_Q is 1 are precisely the groups that are not of the form (p-group)-by-cyclic-by-(q-group), for primes p and q. The smallest such group is A_5, and the smallest such abelian group is $C_{30} \times C_{30}$. Any Q for which $n_Q = 1$ acts on some finite contractible complex L without a fixed point, and acts on some finite contractible complex L_M with any specified fixed point complex M. (Remark: the second statement may also be deduced directly from the first: take L_M to be the join of L and M, with the trivial Q-action on M.)

If Q is any finite group for which $n_Q = 1$, by suitable choice of L we obtain groups $\widetilde{H} = H_L \rtimes Q$ where H_L is of type F, but any one of the following holds:

(a) $\widetilde{H}$ contains infinitely many conjugacy classes of subgroup isomorphic to Q;
(b) for any $n \geq 0$, the centralizer in $\widetilde{H}$ of Q is type F_n but not F_{n+1};
(c) the centralizer of Q has any combination of homological finiteness properties that are enjoyed by some Bestvina-Brady group H_M.

These examples give negative answers to Problems 7.2, 7.4, 7.5 of [43], which concerned the question of whether the property of having a finite (or finite type) model for $\underline{\mathrm{E}}G$ passes to finite extensions. In the case when L is contractible and L^Q is empty, there is no $\underline{\mathrm{E}}\widetilde{H}$ with finite 0-skeleton. This case also gives a negative answer to the case 'finite subgroups' of Problem F7 of [64], which asked whether finite subgroups of a group of type *VFP* over $\mathbb{Z}$ or *FP* over $\mathbb{Q}$ fall into finitely many conjugacy classes. A related question was recently posed in [11] by M. Bridson: is there, for some n, a subgroup of type F_∞ of $SL_n(\mathbb{Z})$ containing infinitely many conjugacy classes of finite subgroups? In Corollary 8 it will be shown that for any L and Q, the group $\widetilde{H}$ embeds in $SL_n(\mathbb{Z})$ for n sufficiently large. From this it follows that these groups afford a positive answer to Bridson's question.

Note also that for groups $\widetilde{H}$ for which L is contractible but L^Q is not, the level set X_t is a finite proper $\widetilde{H}$-CW-complex that is contractible, but there is no finite (or finite type) model for $\underline{E}\widetilde{H}$. Hence these groups give a negative answer to two of the three cases of Problem 7.8 of [43]. This problem is also implicit in A. Adem's paper [1]. The main theorems of [1] are proved for groups G such that there is a finite $\underline{E}G$, although the (now seen to be weaker) hypothesis that there exists some finite proper contractible G-CW-complex is also mentioned.

Example 4. Suppose now that Q is a finite group that is not of prime power order, but that Oliver's $n_Q \neq 1$. The smallest such groups are C_6 and Σ_3. Such a Q cannot act without a global fixed point on any finite contractible complex L, but for any M with $\chi(M) = 1$ there is an action of Q on such an L with $L^Q = M$. For any $n \geq 0$, there is a finite complex M with $\chi(M) = 1$ such that M is $(n-1)$-connected but not n-connected. (Eg., for $n = 0$ take $M = S^0 \vee S^1$, and for $n \geq 1$ take $M = \Sigma^{n-1}\mathbb{R}P^2$.)

Thus by a suitable choice of L we obtain groups $\widetilde{H} = H_L \rtimes Q$ of type *VF* in which for any $n \geq 0$, the centralizer $C_{\widetilde{H}}(Q)$ is type F_n but not type F_{n+1}. Note however, that for these groups Q our methods cannot construct a group of type *VF* containing infinitely many conjugacy classes of subgroup isomorphic to Q.

These groups also provide a negative answer to Problems 7.2, 7.5 and some cases of 7.8 of [43]. In the case when Q is cyclic they provide a negative answer to Problem E11 of [64], which asked if the centralizers of elements of finite order in a group of type *VFP* over $\mathbb{Z}$ are necessarily of type *VFP*.

Example 5. Suppose now that $Q = C_p$ for p a prime. For any $n \geq 1$, there is a finite $\mathbb{F}_p$-acyclic complex which is $(n-1)$-connected but not n-connected. The result of L.E. Jones mentioned above implies that for any $n \geq 1$ there is a choice of L for which H_L is of type F, but the centralizer in $\widetilde{H} = H_L \rtimes C_p$ of an element of order p is type F_n but not type F_{n+1}. The difference between these examples and those discussed in Example 4 is that here we require $n \geq 1$. In other words, we are unable to find an element of prime power order in a group of type *VF* whose centralizer is not finitely generated. In terms of the classifying space for proper actions, for any $n \geq 1$ we obtain a group $\widetilde{H}$ containing a normal subgroup of index p of type F, such that there is no model for $\underline{E}\widetilde{H}$ with finitely many orbits of $(n+1)$-cells.

Example 6. Examples 3–5 show that the property of having a finite type model for $\underline{E}G$ is not inherited by finite extensions. There is another interesting question concerning extensions. In [42], W. Lück showed that for any extension $N \rightarrowtail G \twoheadrightarrow \overline{G}$ of discrete groups, if N and G are both of type F_∞, then $\overline{G}$ is also of type F_∞. Problem 7.7 of [43] asks if the same property holds for $\underline{\mathrm{E}}(-)$. The examples given above in 3–5 show that there exist pairs H_L, Q such that H_L is of type F, Q is finite, and there is no finite type $\underline{\mathrm{E}}(H_L \rtimes Q)$. Suppose we are given such a pair H_L, Q. Now let $N \rightarrowtail \mathrm{Fr} \twoheadrightarrow Q$ be a presentation for Q, where Fr is a finitely generated free group (and hence so is N). Let G be the split extension $H_L \rtimes \mathrm{Fr}$, where Fr acts via the action of $Q = \mathrm{Fr}/N$. Since H_L and Fr are both of type F, so is G. The standard copy of N in G centralizes H_L, so is normal in G, and $\overline{G} = G/N \cong H_L \rtimes Q$. Here there are finite models for $\underline{\mathrm{E}}N$ and $\underline{\mathrm{E}}G$, but no model for $\underline{\mathrm{E}}\overline{G}$ of finite type.

Example 7. In Lemma IX.13.2 of [16], K.S. Brown showed that any group G containing a finite-index torsion-free normal subgroup H such that $H_*(H;\mathbb{F}_p)$ is finite can contain only finitely many conjugacy classes of p-subgroups. This homological hypothesis on H is considerably weaker than insisting that H be of type FP over $\mathbb{F}_p$. (See also the appendix to [30], where certain homological conditions are shown to imply that a group contains only finitely many conjugacy classes of elementary abelian p-subgroups.)

From the point of view of our examples, the case when H_L is FP over $\mathbb{F}_p$ corresponds to the case in which L is $\mathbb{F}_p$-acyclic. P.A. Smith's theorem implies the following, for any finite Q. If the group H_L is FP over $\mathbb{F}_p$, then the group $H_L \rtimes Q$ contains only finitely many conjugacy classes of p-subgroups, and the centralizer of any p-subgroup is VFP over $\mathbb{F}_p$. This confirms a special case of K. S. Brown's result, and also suggests:

Question 1. If G is any group of type VFP over $\mathbb{F}_p$, and P is a p-subgroup of G, is the centralizer $C_G(P)$ necessarily VFP over $\mathbb{F}_p$?

It is possible that every group G of type VFP over $\mathbb{F}_p$ admits a finite, proper G-CW-complex X that is $\mathbb{F}_p$-acyclic. For any such G, and any p-subgroup P, X^P is a finite, proper $C_G(P)$-CW-complex, and is $\mathbb{F}_p$-acyclic by P.A. Smith's theorem. Thus in this case, $C_G(P)$ is necessarily VFP over $\mathbb{F}_p$.

Example 8. Consider the case when $Q = C_p$ for p a prime, acting freely on some flag triangulation L of the n-sphere (here n must be odd if $p \neq 2$). For each n, this gives rise to a group $\widetilde{H} = H_L \rtimes C_p$ which is of type F_n and contains infinitely many conjugacy classes of elements of order p. (Contrast this with K. S. Brown's result mentioned in Example 7.)

Example 9. The case when $\widetilde{H}$ is of type FP over $\mathbb{Q}$ corresponds to the case when L is $\mathbb{Q}$-acyclic. Since L is finite, L is $\mathbb{Q}$-acyclic if and only if L is $\mathbb{F}_p$-acyclic for some prime p, if and only if L is $\mathbb{F}_p$-acyclic for almost all primes p. From [52], any finite M with $\chi(M) = 1$ can occur as the fixed-point set for an action of C_p on some $\mathbb{F}_q$-acyclic L, for any distinct primes p and q. In particular, the fixed point sets need not be connected. Hence our construction gives rise to groups of type FP over $\mathbb{Q}$ in which the centralizers of some elements of prime order are not finitely generated. (Contrast this with Example 5.)

Example 10. In (7.6) of [4] (see also Problem E1 of [64], and [3]), H. Bass conjectured that any group of type *FP* over $\mathbb{Q}$ contains only finitely many conjugacy classes of elements of finite order. Our methods cannot produce a counterexample to this conjecture, since the fixed point set for any action of C_n on a finite $\mathbb{Q}$-acyclic complex L satisfies $\chi(L^{C_n}) = 1$ (see Lemma 1 of [52]). In fact, all groups of the form $\widetilde{H}$ that are *FP* over $\mathbb{Q}$ satisfy a stronger property, which we describe below.

Let k be a field of characteristic zero, let G be a (discrete) group, and let $\Phi = \Phi(G)$ be the Frobenius category of G, i.e., the category whose objects are the finite subgroups of G and whose morphisms are the group homomorphisms induced by conjugation and inclusion in G. Let Φ' be the full subcategory of Φ whose objects are the finite cyclic subgroups of G. For each such k and G, there is a sequence of homomorphisms as follows:

$$\lim_{C\in\Phi'} K_0(kC) \xrightarrow{i} \lim_{P\in\Phi} K_0(kP) \xrightarrow{\iota_G} K_0(kG). \qquad (1)$$

The Artin induction theorem for kP-modules ([58], 9.2 and 12.5), implies that the cokernel of i is a torsion group. It is a well-known conjecture that ι_G is always an isomorphism (see for example [44], where this conjecture is described as a case of the Farrell-Jones conjectures).

In the case when G contains only finitely many conjugacy classes of elements of finite order, the category $\Phi'(G)$ is skeletally finite, and so the left-hand term in (1) is a finitely generated abelian group, and the middle term is an abelian group of finite rank. In fact, the rank can be described as the number of 'k-conjugacy classes' of elements of finite order in G. For the definition of k-conjugacy classes, and the case of a finite group, see [58], 12.4. For a proof of this claim, see [44], Sect. 8, or [1] which considers only the case when $k = \mathbb{C}$, and uses stronger hypotheses. Hattori-Stallings traces can be used to show that the image of ι_G has this same rank (a similar argument, although expressed in terms of G_0 instead of K_0, also occurs in [13]).

Now suppose that Q is a finite group that admits an admissible action without a global fixed point on some finite $\mathbb{Q}$-acyclic L. In this case, the alternating sum of the modules in the chain complex for L gives an expression for k, the trivial module for kQ, as an alternating sum of non-trivial permutation modules. If M is any kQ-module, then applying $-\otimes_k M$ to the chain complex for L gives an expression for M, viewed as an element of $K_0(kQ)$, as an alternating sum of modules induced up from proper subgroups. Hence $K_0(kQ)$ is generated by the images of the $K_0(kP)$, where $P < Q$.

For any G, let Φ'' be the full subcategory of $\Phi(G)$ with objects the subgroups that admit no action without a global fixed point on any finite

$\mathbb{Q}$-acyclic L. By the remarks above, the natural map

$$\lim_{P\in\Phi''} K_0(kP) \longrightarrow \lim_{Q\in\Phi} K_0(kQ) \tag{2}$$

is surjective. In each group $\widetilde{H}$ of type *FP* over $\mathbb{Q}$, the category Φ'' is skeletally finite, and so the limit groups in (2) are finitely generated.

In an earlier version of this article we asked whether a similar result holds for every group of type *FP* over $\mathbb{Q}$. One of the authors has since constructed groups of type *VF* containing infinitely many conjugacy classes of elements of finite order [40].

It appears to be unknown in general whether there is a bound on the orders of finite subgroups of a group G of type *FP* over $\mathbb{Q}$. P.H. Kropholler has shown that there is such a bound if G is also assumed to be of type FP_∞ over $\mathbb{Z}$ ([36], Sect. 5, or [39]).

Example 11. One may define finiteness conditions using cohomology relative to all finite subgroups, as in the definition of the dimension $\mathcal{F}$-cd$G$. Say that a group G is of type $\mathcal{F}FP$ if there is an integer n and an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

of G-modules having the properties that

(a) each C_i is a direct summand of a sum of modules induced up from finite subgroups of G;
(b) for each finite subgroup $H \leq G$, there is an H-equivariant splitting of the exact sequence;
(c) each C_i is finitely generated.

Now suppose that Q is a finite group acting admissibly on contractible L such that L^Q is empty, and let $G = H_L \rtimes Q$. As in Sect. 6, it follows that the cellular chain complex for any level set X_t satisfies (a), (b) and (c). On the other hand, there is no model for $\underline{E}G$ having finite 0-skeleton. Thus we see that even strong finiteness conditions for cohomology relative to all finite subgroups do not imply finiteness conditions for $\underline{E}G$.

Example 12. To apply Theorem 6 and deduce that there are groups G for which the minimal dimension of $\underline{E}G$ is strictly larger than vcd$G$, we need to construct pairs (L, Q) such that

(i) L is acyclic and n-dimensional;
(ii) there is an n-simplex $\sigma \in L$ in a free Q-orbit;
(iii) every simplex in the boundary of σ has non-trivial stabilizer.

For $n = 2$, the acyclic A_5-complex described in Example 4 of Sect. 9 is such a complex. Other examples for $Q = A_5$ may be made by attaching extra orbits of simplices to this one. Higher dimensional examples can be obtained using a join construction. If (L, Q) is an example where L has dimension n, and Q' is any non-trivial finite group, then the join $L * Q'$, with

$Q \times Q'$ acting via the join of the given action on L and the multiplication action of Q' on itself, is an example of dimension $n+1$. Furthermore, $L * Q'$ is contractible even if L was not. The simplest case of this construction is also described in Example 4 of Sect. 9.

In the case when Q is the projective general linear group $PGL_2(q)$ over a finite field of order q, M. Aschbacher and Y. Segev construct 2-complexes L that satisfy (ii) and (iii), and almost satisfy (i), in the sense that their reduced homology groups are finite [2]. If (L_1, Q_1) and (L_2, Q_2) are two such examples for which the reduced homology groups of L_1 and L_2 have coprime orders, then $L = L_1 * L_2$ with $Q = Q_1 \times Q_2$ is a 5-dimensional contractible example satisfying (i)–(iii).

Since there are pairs L and Q satisfying (i)–(iii) for which L is 2-dimensional, and others for which L is 3-dimensional and contractible, it follows that for each m there are groups G of type VFP over $\mathbb{Z}$ for which $\mathrm{vcd}G = \mathcal{F}\text{-}\mathrm{cd}G = 2m$ and $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G = 3m$, and that there are groups G of type VF for which $\mathrm{vcd}G = \mathcal{F}\text{-}\mathrm{cd}G = 3m$ and $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G = 4m$. These groups answer the question known as Brown's conjecture ([15], p. 32 or [43], 7.10).

If the pair L and Q satisfy condtions (i)–(iii) as above, L cannot be a model for $\underline{E}Q$. Hence our methods do not answer the following.

Question 2. Is there a group G admitting a finite $\underline{E}G$ for which the minimal dimension of any model for $\underline{E}G$ is strictly greater than $\mathrm{vcd}G$?

8. Presentations and linearity

For a flag complex L, the associated right-angled Coxeter group W_L is the group with generators the vertices of L, subject to the relations that each generator has order two and that the ends of every edge commute with each other. Just as in the case of the right-angled Artin group G_L, this construction is functorial in L. There is of course a quotient map $G_L \twoheadrightarrow W_L$, but another map (to be described below) is of greater interest to us.

Recall that a flag complex is determined by its 1-skeleton, since its higher simplices are just the complete subgraphs of its 1-skelelton. If L is a flag complex, define a flag complex $S(L)$ as follows. For each vertex v of L, there are two vertices v', v'' (or $v^{(1)}$, $v^{(2)}$) of $S(L)$. There is never an edge between v' and v''. If $v \neq w$ and $1 \leq i, j \leq 2$, then there is an edge in $S(L)$ joining $v^{(i)}$ and $w^{(j)}$ if and only if there is an edge in L joining v and w.

For example, if L is the 2-simplex, $S(L)$ is the octahedron. In general, each n-simplex of L gives rise to 2^{n+1} distinct n-simplices in $S(L)$. The construction of $S(L)$ is functorial in L, and in particular an action of a group Q on L gives rise to an action of Q on $S(L)$ defined by $q \cdot v^{(i)} = (q \cdot v)^{(i)}$.

Theorem 7. *Let L be a finite flag complex with an effective action of Q. The map defined on generators of G_L by $v \mapsto v'v''$ gives an embedding of $G_L \rtimes Q$ into $W_{S(L)} \rtimes Q$.*

Proof. The fact that this defines an embedding of G_L in $W_{S(L)}$ is proved by Hsu and Wise in [33]. This embedding is equivariant for the action of Q, and so extends to an embedding of $G_L \rtimes Q$ in $W_{S(L)} \rtimes Q$ as claimed. □

Remark. An alternative embedding of G_L in a right-angled Coxeter group $W_{T(L)}$ has been given by M. Davis and T. Januszkiewicz [22]. This embedding can also be used to prove Theorem 7, and has the advantage that G_L embeds as a finite-index subgroup of $W_{T(L)}$. The vertex set of $T(L)$ is equal to that of $S(L)$. T contains an edge between v' and w' if and only if there is an edge in L between v and w. For any $v \neq w$, there are edges in T between v'' and w'' and between v' and w''.

Corollary 8. *Suppose L and Q are as in Theorem 7, and that L has n vertices. Then $G_L \rtimes Q$ embeds in the integral special linear group $SL_{2n}(\mathbb{Z})$.*

Proof. For any flag complex M with m vertices, an embedding of W_M in $SL_m(\mathbb{Z})$ is described in [18], p. 55. Under this embedding, there is a natural correspondence between the standard $\mathbb{Z}$-basis for $\mathbb{Z}^m$ and the generators for W_M. If P is a group of automorphisms of M, then this embedding extends to a homomorphism $\rho : W_M \rtimes P \to GL_m(\mathbb{Z})$ by sending elements of the 'standard copy of P' to permutation matrices. In the case when $M = S(L)$ and $P = Q$, the image $\rho(W_{S(L)} \rtimes Q)$ is contained in $SL_{2n}(\mathbb{Z})$ because the Q-set $S(L)^0$ is isomorphic to a disjoint union of two copies of L^0, and so Q acts on $S(L)^0$ via even permutations. Finally, the composite of ρ and the inclusion of $G_L \rtimes Q$ in $W_{S(L)} \rtimes Q$ is an embedding, because $\rho(G_L) \cong G_L$, $\rho(Q) \cong Q$, and the intersection of $\rho(G_L)$ and $\rho(Q)$ is trivial because G_L is torsion-free and Q is finite. □

Next we recall from [23] a theorem giving a presentation for H_L whenever L is connected, and a finite presentation whenever L is 1-connected. The 1-connected case also appears in [32], and the general case is implicit in the work of Bestvina and Brady [6]. (Note that the proof of Theorem 9 in [23] is independent of [6].)

Theorem 9. *Let L be a finite connected flag complex. There is a presentation for H_L with one generator for each directed edge e of L subject to the following relations.*

1. *If $\bar{e}$ is the same edge as e with the opposite orientation, then $\bar{e} = e^{-1}$;*
2. *For every directed triangle (e, f, g), the relations $efg = gfe = 1$;*
3. *For every $l \geq 4$, every directed cycle $(e_1, \ldots, e_l)$ in L and every $n \neq 0$, the relation $e_1^n e_2^n \cdots e_l^n = 1$.*

If L is 1-connected the relations of type 3 are not required. In general, if $c_1, \ldots, c_r$ are cycles such that L can be made 1-connected by coning of these cycles, then one only need take the type 3 relations for these cycles (and for all $n \neq 0$).

In terms of the generators original generators for G_L, the directed edge e corresponds to the element vw^{-1}, where v is the initial vertex of e and w is the terminal vertex of e. □

As an example, if L is the boundary of an n-gon for $n \geq 4$, then H_L has the following infinite presentation:

$$H_L = \langle a_1, \ldots, a_n \;:\; a_1^m \cdots a_n^m = 1, \quad \text{for all } m \neq 0 \rangle.$$

A very similar presentation for this group in the case when $n = 4$ is given in [60].

If Q acts as a group of automorphisms of L, then Q permutes the generators and relators in the presentations for H_L given in Theorem 9. This gives an easy way to produce presentations for the groups $H_L \rtimes Q$: take a 'fundamental domain' M for the action of Q (i.e., a subcomplex whose images under Q form the whole of L), and eliminate all the edges and type 2 relations except those coming from M. One example of this is the following.

Corollary 10. *Let L be a finite 1-connected complex, with an effective action of Q, and let M be any 1-connected subcomplex of L whose images under Q cover L. A presentation for the group $H_L \rtimes Q$ is given by: take the presentation of H_M from Theorem 9, use this to make a presentation for the free product $H_M * Q$, and for each edge $e \in M$ and each $q \in Q$ such that the edge $e' = q \cdot e$ also lies in M, add the relation $qeq^{-1} = e'$.* □

In the examples in the next section, M can be chosen to be an n-simplex for some n, and so $H_L \rtimes Q$ will be presented as a quotient of the free product $\mathbb{Z}^n * Q$.

9. More examples

In this section, we adopt the same notation for various specific groups as in Sect. 7. The examples discussed here are special cases of the ones mentioned already in Sect. 7, although now we are interested in providing explicit triangulations with either very few vertices or very few Q-orbits of simplices.

Example 1. $Q = C_n$ for some $n > 3$, acting freely on L, the boundary of an n-gon. In this case, $\widetilde{H} = H_L \rtimes Q$ is a finitely generated (but not finitely presented) group containing, for each $m > 1$ that divides n, infinitely many conjugacy classes of elements of order m. Moreover, $\widetilde{H}$ embeds in $SL_{2n}(\mathbb{Z})$. Let x be a generator for C_n, and suppose that (using the presentation for H_L given at the end of the previous section) $x^{-1}a_ix = a_{i+1}$ for each i, where indices should be taken modulo n. Eliminating $a_2, \ldots, a_n$ gives the following presentation for $\widetilde{H}$.

$$\widetilde{H} = \langle a, x \;:\; x^n = 1, \; (a^m x^{-1})^{n-1} a^m x^{n-1} = 1, \quad \text{for all } m \neq 0 \rangle.$$

This can be simplified to give

$$\widetilde{H} = \langle a, x \;:\; (a^m x^{-1})^n = 1, \quad \text{for all } m \in \mathbb{Z} \rangle.$$

One can also find explicit words for elements of given order that lie in distinct conjugacy classes. Write $v_1, \dots, v_n$ for the generators of G_L, so that $a_i = v_{i-1}v_i^{-1}$ and $x^{-1}v_ix = v_{i+1}$ for all i, where as before indices are taken mod n. Since L^Q is empty, each element of $G_L \rtimes Q$ fixes a single point of X_L. A study of the proof of Theorem 3 shows that the elements $v_1^{-l}xv_1^l$ for $l \in \mathbb{Z}$ fix points that are at different heights, and so these elements generate distinct conjugacy classes of cyclic subgroups of $\widetilde{H}$. It remains to express these elements in terms of the generators for $\widetilde{H}$, but we have that $xv_1x^{-1} = v_n$, and so $xv_1^l = v_n^l x$. Hence

$$v_1^{-l}xv_1^l = v_1^{-l}v_n^l x = a^{-l}x.$$

Thus we see that the elements $\{\, a^l x \ : \ l \in \mathbb{Z}\}$ generate distinct conjugacy classes of subgroups of $\widetilde{H}$ of order n, and that any subgroup of $\widetilde{H}$ of order n is conjugate to one of these.

Example 2. Actions of $Q = C_2$ on the $(n-1)$-sphere L, triangulated as a join of n pairs of points. Here H_L is the subgroup of $(\mathrm{Fr}_2)^n$ that is F_{n-1} but not F_n that was introduced by Stallings for $n \leq 3$ and Bieri for general n [60,7]. For any $m \leq n$, there is an admissible action of C_2 on L which swaps the members of m of the pairs of points, and fixes the members of the other pairs. These actions give rise to index two extensions of H_L containing one of:

(a) infinitely many conjugacy classes of elements of order two (in the case $m = n$);
(b) elements of order two whose centralizers are F_{n-m-1} but not F_{n-m} (in the case $m < n$).

Corollary 8 implies that each of these groups embeds in $GL_{4n}(\mathbb{Z})$, but this can be improved. In the case $n = 1$, the group $G_L \rtimes C_2$ is $\mathrm{Fr}_2 \rtimes C_2 \cong \mathbb{Z} * C_2$, and for general n, these groups are subgroups of $(\mathbb{Z} * C_2)^n$. The group $\mathbb{Z} * C_2$ embeds in $GL_2(\mathbb{Z})$ via the map that sends the generators of the factors to the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence it follows that each of these groups embeds in $GL_{2n}(\mathbb{Z})$. In particular, the antipodal action of C_2 on the square gives rise to a finitely generated subgroup of $GL_4(\mathbb{Z})$ containing infinitely many conjugacy classes of elements of order two, and the antipodal action of C_2 on the octahedron gives rise to a finitely presented subgroup of $GL_6(\mathbb{Z})$ that contains infinitely many conjugacy classes of elements of order two. In [29], F. Grunewald and V. Platonov constructed a finitely generated subgroup of $SL_4(\mathbb{Z})$ containing infinitely many conjugacy classes of elements of order 4, and asked if there could be a finitely presented subgroup of $GL_n(\mathbb{Z})$ for any n having this property (see also Problem 4 of [54]. Using the example of Grunewald and Platonov,

M. Bridson has constructed a finitely presented subgroup of $SL_6(\mathbb{Z})$ containing infinitely many conjugacy classes of elements of order 4 [11]. The examples of Grunewald-Platonov and Bridson contain only finitely many conjugacy classes of elements of order two.

We consider presentations only in the case $n = 3$. As remarked earlier, the presentations coming from Theorem 9 are simplest when there are very few Q-orbits of simplices in L. Hence we consider the case of $(C_2)^3$ acting on the octahedron L by reflections in three orthogonal planes that each contain four of its vertices. For this action, the 2-simplices of the octahedron form a single orbit, and the edges form three distinct orbits. Let a and b be edges of a 2-simplex of the octahedron, with the terminal vertex of a and the initial vertex of b equal (so that in H_L the third edge of this 2-simplex is equal to ab). Let x, y, and z be reflections in the planes containing a, b and ab respectively. The following describes a presentation for $\widetilde{H} = H_L \rtimes (C_2)^3$. Take the free product of $\mathbb{Z}^2$ generated by a and b with $(C_2)^3$ generated by x, y and z, and add the relations:

$$[a, x] = [b, y] = [ab, z] = 1.$$

This group $\widetilde{H}$ is an index 8 extension of Stallings' group that is finitely presented but not F_3. The centralizer of the element x is finitely generated but not finitely presented, and the centralizer of the element xy is not finitely generated. There are infinitely many conjugacy classes of elements of elements of order two in $\widetilde{H}$ that map to xyz in the quotient of $\widetilde{H}$ by the normal subgroup generated by a and b (which is isomorphic to $(C_2)^3$).

There is an explicit embedding of this group $\widetilde{H}$ into $(GL_2(\mathbb{Z}))^3 \leq GL_6(\mathbb{Z})$ given by

$$a \mapsto \begin{pmatrix} 1 & 2 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -2 & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & -2 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

and sending x, y and z to the permutation matrices corresponding to the transpositions $(5, 6)$, $(1, 2)$ and $(3, 4)$ respectively.

Finally, for $m \in \mathbb{Z}$, the elements $a^m ya^{-m}xz$ have order two and lie in distinct conjugacy classes in $\widetilde{H}$, although they are conjugate in $\widetilde{G} = G_L \rtimes (C_2)^3$. This follows from an explicit calculation, as in Example 1, but it is simpler to apply the general statement given below as Lemma 11. Even checking directly that this element has order two is not trivial, but one can check that

$$(a^m ya^{-m}xz)^2 = a^m ya^{-m}za^m ya^{-m}z = a^m(ya^{-m}y)(yza^m zy)(za^{-m}z) = 1,$$

where the final equality follows because the word as bracketed is equal to a 'type 3 relation' in the presentation for H_L.

Lemma 11. *Let L and Q be as in Theorem 7, suppose that L is connected, and take the presentation for H_L as described in Theorem 9. If $q \in Q$ and v is one of the vertices of L viewed as an element of G_L, then the element $v^m q v^{-m}$ can be expressed in terms of the generators for $H_L \rtimes Q$ as $e_1^m \cdots e_l^m q$, where $(e_1, \ldots, e_l)$ is a directed path from v to the vertex $w = q \cdot v = qvq^{-1}$.*

Proof. $v^m q v^{-m} = v^m q v^{-m} q^{-1} q = v^m w^{-m} q = (e_1^m \cdots e_l^m) q$. □

Remark. After writing an account of these groups, we learned that M. Feighn and G. Mess had also constructed, for each $n \geq 1$, a group G_n of type F_{n-1} containing infinitely many conjugacy classes of elements of order two [27]. In [27], the group G_n is given as a subgroup of $PGL_2(\mathbb{R})^n$. Define an action of $C_2 = \langle \tau \rangle$ on $\mathrm{Fr}_2 = \langle x, y \rangle$ by $x^\tau = x^{-1}$, $y^\tau = y$. For this action, the semi-direct product $\mathrm{Fr}_2 \rtimes C_2$ requires three generators, so it is not isomorphic to $\mathbb{Z} * C_2$. The group G_n may be described as a subgroup of $(\mathrm{Fr}_2 \rtimes C_2)^n$. Since Fr_2 embeds in $SL_2(\mathbb{Z})$, any index-two extension of Fr_2 may be embedded in $SL_4(\mathbb{Z})$. Hence the Feighn-Mess group G_n embeds in $SL_{4n}(\mathbb{Z})$, and so affords an answer to the questions of Platonov and of Grunewald and Platonov [54,29].

Example 3. The group C_n for $n \geq 4$ acting freely on S^{2m-1}. Here it is convenient to triangulate S^{2m-1} as a join of m copies of the n-gon (as in Example 1). In this way one obtains a subgroup of $SL_{2mn}(\mathbb{Z})$ that is type F_{2m-1} (but not type F_{2m}) and contains infinitely many conjugacy classes of elements of order l for each $l > 1$ dividing n. We give a presentation only in the case $m = 2$. To minimize the number of relators, it is more convenient to let $C_n \times C_n$ act on $L = S^3$. Let a (resp. c) be an edge of an n-gon, and let x (resp. y) be a generator for a copy of C_n that rotates this n-gon and sends the initial vertex of a (resp. c) to the terminal vertex. Suppose that x fixes c and that y fixes a. Finally, let b be the edge in the join of the two n-gons that goes from the terminal vertex of a to the initial vertex of c. The 3-simplex containing the edges a, b and c is a fundamental domain for the action of $C_n \times C_n$ on L. The group $\widetilde{H} = H_L \rtimes (C_n)^2$ has the following presentation. Take the free product of a copy of $\mathbb{Z}^3$, generated by a, b and c, and $(C_n)^2$ generated by x and y. Add the relations

$$[c, x] = [a, y] = 1, \qquad x^{-1} b x = ab, \qquad y b y^{-1} = bc.$$

The reason why these relations are slightly more complicated than in Example 2 is that four of the edges of the fundamental domain lie in the same orbit. The edges are a, b, c, ab, bc and abc. The action of y sends b to bc and the action of x^{-1} sends b to ab. (Also $x^{-1}y$ sends b to abc, but the corresponding relation can be eliminated from the presentation.) The subgroup of $\widetilde{H}$ generated by a and x is the group described in Example 1, and

$$C_{\widetilde{H}}(y) = \langle a, x \rangle \times \langle y \rangle.$$

The centralizer of $x^{-1}y$, on the other hand is the copy of $(C_n)^2$ generated by x and y. By Lemma 11, the elements $c^l x^{-1} y$ for $l \in \mathbb{Z}$ all lie in distinct conjugacy classes.

Example 4. A fixed-point free action on a contractible complex. There is an example (due to E.E. Floyd and R.W. Richardson [28], but see also [10], p. 55–57, [61], p. 34 and [9], Sect. 4) of a 2-dimensional acyclic simplicial complex M with an action of A_5 without a global fixed point, and this will be our starting point. M is described as the barycentric subdivision of a polyhedral complex. Recall that there are two conjugacy classes of 12 elements of order 5 in A_5, and that each element x of order 5 is conjugate to x^{-1} (but not to x^2 or x^3), so that each conjugacy class consists of 6 pairs of inverse elements. Take the 1-skeleton of a simplex with vertex set $\{1, \ldots, 5\}$, and attach 6 pentagonal 2-cells along the unoriented cycles corresponding to one of the two conjugacy classes of elements of order five. By construction, this complex admits an action of A_5 without a global fixed point. It may be checked that the complex is acyclic and that its fundamental group is isomorphic to $SL_2(\mathbb{F}_5)$, the perfect group of order 120.

Its barycentric subdivision, M, has 21 0-simplices in 3 A_5-orbits, 80 1-simplices in 3 A_5-orbits and 60 2-simplices in a single, free A_5-orbit. As in Example 12 of Sect. 7, it follows that the group $\widetilde{H} = H_M \rtimes A_5$ is a group of type VFP with $\mathrm{vcd}\widetilde{H} = 2$ but such that any model for $\underline{E}\widetilde{H}$ has dimension at least three.

The join of M with any non-empty simplicial complex is contractible. For example, if we take L to be the join of M with a set of 5 points permuted transitively by A_5, then L is contractible, has 26 vertices, and $L^{A_5} = \emptyset$. Hence $\widetilde{H} = H_L \rtimes A_5$ is a subgroup of $SL_{52}(\mathbb{Z})$ of type VF containing infinitely many conjugacy classes of subgroup isomorphic to A_5. On the other hand, if we take L to be the join of M and two points (with trivial A_5 action), then $\widetilde{H} = H_L \rtimes A_5$ is a subgroup of $SL_{46}(\mathbb{Z})$ of type VF in which the centralizer of an A_5 is not finitely generated (and is in fact free of infinite rank).

From the point of view of finding presentations, the simplest example seems to be to take $Q = A_5 \times C_2$, and to take L to be the join of M (with Q acting via the A_5 action) and two points (permuted freely by C_2). To be completely explicit, choose the conjugacy class of 5-cycles in A_5 containing $(1,2,3,4,5)$ to define M, and take the 2-simplex with vertices (1), $(1,2)$ and $(1,2,3,4,5)$ as the fundamental domain for the A_5-action on M. Write N and S for the two extra vertices permuted freely by C_2, let τ denote the generator of C_2, and write elements of A_5 in cycle notation. In L, let a denote the edge from N to $(1,2,3,4,5)$, let b denote the edge from $(1,2,3,4,5)$ to $(1,2)$, and let c denote the edge from $(1,2)$ to (1). Then the group $\widetilde{H} = H_L \rtimes (A_5 \times C_2)$ has the following presentation. Take the free product of $\mathbb{Z}^3$ generated by a, b and c with $A_5 \times C_2$ and add the following twelve commutators as relators.

$[a, (1, 2, 3, 4, 5)]$	$[a, (1, 2)(3, 5)]$	$[b, \tau]$	$[b, (1, 2)(3, 5)]$
$[c, \tau]$	$[c, (3, 4, 5)]$	$[ab, (1, 2)(3, 4)]$	$[ab, (3, 4, 5)]$
$[bc, \tau]$	$[bc, (2, 5)(3, 4)]$	$[abc, (2, 3)(4, 5)]$	$[abc, (2, 3, 4)]$

This gives an explicit presentation of a group $\widetilde{H} = H_L \rtimes Q$ of type *VF* that contains infinitely many conjugacy classes of subgroup isomorphic to $A_5 \times C_2$, and which can be embedded in $SL_{46}(\mathbb{Z})$. The group $\widetilde{H}$ is of virtual cohomological dimension three, and any model for $\underline{\mathrm{E}}\widetilde{H}$ has dimension at least four. The centralizer of the element τ is *VFP* over $\mathbb{Z}$, but is not finitely presented. Writing A for the standard copy of A_5 in $\widetilde{H} = H_L \rtimes Q$, the conjugates (in $G_L \rtimes Q$) of $A \times \langle\tau\rangle$ by powers of N are the groups $A \times \langle a^m \tau a^{-m}\rangle$. Hence these groups, for $m \in \mathbb{Z}$, are representatives of the distinct conjugacy classes of $A_5 \times C_2$ in $H_L \rtimes Q$. The centralizer $C_{\widetilde{H}}(A)$ can also be described in terms of these elements. It is the group $H_{\{N,S\}} \rtimes \langle\tau\rangle$, which is isomorphic to an infinite free product of copies of C_2 generated (freely) by the elements $a^m \tau a^{-m}$ for $m \in \mathbb{Z}$.

10. K-theory

In this section, we construct groups G of type *VF* for which the rational homology groups of centralizers of certain elements of finite order are not finite-dimensional. We deduce that these groups have the properties that their higher algebraic K-groups $K_i(RG)$ for various rings R and the (topological) K-groups of their reduced group C^*-algebras $K_*^{\mathrm{Top}}(C_r^*(G))$ are not of finite rank.

Proposition 12. *Let L be a finite flag complex, and Q a group of automorphisms of L. The Baum-Connes conjecture (with coefficients) holds for the groups $H_L \rtimes Q$ and $G_L \rtimes Q$.*

Proof. N. Higson and G. Kasparov have shown that the Baum-Connes conjecture holds for any discrete group admitting a properly discontinuous action on a Hilbert space by affine isometries [31,35]. (Groups admitting such an action are said to be ‘a-T-menable’ or to ‘have the Haagerup property’.) An argument due to G. Niblo and L. Reeves shows that any group admitting a proper action on a locally finite CAT(0) cube complex has the Haagerup property (see the ‘Technical Lemma’ in [50], where the Haagerup property is not explicitly mentioned, or Sect. 1.4 of [35]). Since $G_L \rtimes Q$ acts properly on the cube complex X_L, it follows that the Baum-Connes conjecture holds for $G_L \rtimes Q$ and for each of its subgroups. □

Proposition 13. *Let L be a flag complex homeomorphic to the* $(n-1)$*-sphere for some* $n > 0$*. Then the rational homology group* $H_n(H_L; \mathbb{Q})$ *is infinite dimensional.*

Proof. The argument is based on the one used by Stallings in the case when L is the octahedron [60], see also Corollary 6 of [23]. Throughout this proof, homology is to be understood to be taken with rational coefficients, which will be omitted. The complex X_L is an n-dimensional model for EH, and by the Bestvina-Brady theorem (quoted as Theorem 1), H_L is of type F_{n-1}. Hence the homology groups $H_i(H_L)$ are finite-dimensional for $i < n$ and are trivial for $i > n$.

Now suppose that $H_n(H_L)$ is also finite-dimensional. In this case, H_L has a well-defined Euler characteristic $\chi(H_L)$. The expression for G_L as an extension with kernel H_L and quotient $\mathbb{Z}$ gives rise to a long exact sequence

$$\cdots \to H_i(H_L) \to H_i(H_L) \to H_i(G_L) \to H_{i-1}(H_L) \to H_{i-1}(H_L) \to \cdots ,$$

and so the Euler characteristic $\chi(G_L)$ must be zero. However, a count of the cells in X_L/G_L shows that $\chi(G_L) = 1 - \chi(L) = (-1)^n$. This contradiction shows that $H_n(H_L)$ cannot be finite-dimensional. □

The formula $\chi(G_L) = 1 - \chi(L)$, which holds for any L, predates the work of Bestvina and Brady. It was proved by C. Droms in [24].

Proposition 14. *Suppose that the flag complex L is the disjoint union of non-empty subcomplexes M and N. Then H_L is isomorphic to the free product of H_M, H_N and a free group of infinite rank. In symbols, $H_L \cong H_M * H_N * \mathrm{Fr}_\infty$.*

Proof. The group G_L is isomorphic to the free product $G_M * G_N$. Let Y and Z be models for $K(G_M, 1)$ and $K(G_N, 1)$ respectively, and as a model for $K(G_L, 1)$ take X consisting of Y and Z joined by an interval I. Consider the infinite cyclic cover $\widetilde{X}$ of X corresponding to the subgroup H_L. Since ϕ_L restricted to G_M (resp. to G_N) is equal to ϕ_M (resp. ϕ_N), the part of $\widetilde{X}$ lying above Y (resp. Z) is a model for $K(H_M, 1)$ (resp. $K(H_N, 1)$). Thus $\widetilde{X}$ consists of a copy of $K(H_M, 1)$ and a copy of $K(H_N, 1)$ joined together by infinitely many intervals. The fundamental group of this space is as claimed. □

Corollary 15. *For each $n \geq 1$, there is a flag complex M with the properties that $\chi(M) = 1$ and $H_n(H_M; \mathbb{Q})$ is not finite-dimensional.*

Proof. When n is even, take M to be a triangulation of the disjoint union of S^{n-1} and a point. When n is odd, take M to be a triangulation of the disjoint union of S^{n-1} and a bouquet of two circles. The claim concerning the homology of H_M follows from Propositions 13 and 14. □

Example. For any finite flag complex M with $\chi(M) = 1$, there is an action of the cyclic group C_6 on a finite contractible flag complex L such that $L^{C_6} = M$ (see [52]). In $\widetilde{H} = H_L \rtimes C_6$, the centralizer of any element of order 6 is isomorphic to $H_M \times C_6$. For any $n > 0$, if we choose M as in Corollary 15, we obtain a group $\widetilde{H} = \widetilde{H}(n)$ of type VF and an element $g \in \widetilde{H}$ of order six such that $H_n(C_{\widetilde{H}}(g); \mathbb{Q})$ is infinite-dimensional.

In Theorem 0.1 of [44], W. Lück constructs, for any discrete group G, a homomorphism

$$\bigoplus_{g\in\mathcal{C}} \bigoplus_{i+j=n} H_i(C_G(g);\mathbb{C}) \otimes K_j^{\mathrm{Top}}(\mathbb{C}) \longrightarrow \mathbb{C} \otimes K_n^{\mathrm{Top}}(C_r^*(G)). \qquad (3)$$

Here $K_*^{\mathrm{Top}}(C_r^*(G))$ denotes the topological K-groups of the reduced C^*-algebra for G, the indexing set $\mathcal{C}$ is a set of representatives for the conjugacy classes of elements of finite order in G, and $C_G(g)$ denotes the centralizer of g. This homomorphism is an isomorphism whenever the Baum-Connes conjecture is valid for G. From Proposition 12, we see that the group $\widetilde{H} = \widetilde{H}(n)$ constructed above has the property that $K_n^{\mathrm{Top}}(C_r^*(\widetilde{H}))$ has infinite rank.

One may also apply these techniques to deduce that there are groups of type *VF* for which the algebraic K-groups $K_*(RG)$ have infinite rank. Of course, this is only of interest in the case when R is such that each $K_i(R)$ has finite rank: if R is a field of characteristic zero, and G is free abelian of rank $n-1$, then $K_n(RG)$ has infinite rank, since it contains $K_1(R)$ as a direct summand.

Two theorems of Quillen (see 5.3.12 and 5.3.27 in [55]) together imply that if R is obtained from the ring of integers in any algebraic number field by inverting finitely many primes, then each $K_i(R)$ is finitely generated.

M. Matthey has recently proved the following [47,46]. Let m be a positive integer, and let G be a group in which the order of every torsion element divides m. For any subring R of $\mathbb{C}$ that contains both $1/m$ and $\exp(2\pi i/m)$, there is an injective map

$$\bigoplus_{g\in\mathcal{C}} H_n(C_G(g);\mathbb{C}) \to \mathbb{C} \otimes K_n(RG), \qquad (4)$$

where the indexing set $\mathcal{C}$ is as in (3). In particular, for $R = \mathbb{Z}[1/6, \exp(2\pi i/3)]$ and $\widetilde{H} = \widetilde{H}(n)$ as above, the group $K_n(R\widetilde{H})$ is not of finite rank, although each $K_i(R)$ is finitely generated. M. Matthey has shown the authors a modified version of (4) that applies in the case $R = \mathbb{Z}[1/m]$, which implies that $K_n(\mathbb{Z}[1/6]\widetilde{H})$ is not of finite rank.

References

1. Adem, A.: Characters and K-theory of discrete groups. Invent. Math. **114**, 489–514 (1993)
2. Aschbacher, M., Segev, Y.: A fixed point theorem for groups acting on finite 2-dimensional acyclic simplicial complexes. Proc. Lond. Math. Soc., III. Ser. **67**, 329–354 (1993)
3. Bass, H.: Euler characteristics and characters of discrete groups. Invent. Math. **35**, 155–196 (1976)
4. Bass, H.: Traces and Euler characteristics. In: Homological Group Theory (Proc. Sympos., Durham 1977), London Math. Soc. Lecture Notes **36**. Cambridge Univ. Press 1979, 1–26

5. Baum, P., Connes, A., Higson, N.: Classifying spaces for proper actions and K-theory of group C^*-algebras. In: Contemp. Math. **167**, R.S. Doran, ed., Am. Math. Soc. 1994
6. Bestvina, M., Brady, N.: Morse theory and finiteness properties of groups. Invent. Math. **129**, 445–470 (1997)
7. Bieri, R.: Homological dimension of discrete groups. Queen Mary College Mathematics Notes, London 1976
8. Bouc, S.: Le complexe de chaînes d'un G-complexe simplicial acyclique. J. Algebra **220**, 415–436 (1999)
9. Brady, N., Leary, I.J., Nucinkis, B.E.A.: Geometric and algebraic dimensions for groups with torsion. J. Lond. Math. Soc., II. Ser. **64**, 489–500 (2001)
10. Bredon, G.E.: Introduction to compact transformation groups. Academic Press, New York, London 1972
11. Bridson, M.: Finiteness properties for subgroups of $GL_n(\mathbb{Z})$. Math. Ann. **317**, 629–633 (2000)
12. Bridson, M., Haefliger, A.: Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften **319**, Springer Verlag 1999
13. Brown, K.A., Lorenz, M.: Colimits of functors and Grothendieck groups of infinite group algebras. In: Abelian groups and noncommutative rings, Contemp. Math. **130**, Am. Math. Soc. 1992, 89–109
14. Brown, K.S.: Euler characteristics of discrete groups and G-spaces. Invent. Math. **24**, 229–264 (1974)
15. Brown, K.S.: Groups of virtually finite dimension. In: Homological Group Theory, London Math. Soc. Lecture Notes **36**, Cambridge Univ. Press 1979, 27–70
16. Brown, K.S.: Cohomology of groups. Graduate Texts in Mathematics **87**, Springer 1982
17. Brown, K.S., Geoghegan, R.: An infinite-dimensional torsion-free FP_∞ group. Invent. Math. **77**, 367–381 (1984)
18. Brown, K.S.: Buildings. Springer 1989
19. Connolly, F.X., Koźniewski, T.: Finiteness properties of classifying spaces of proper Γ-actions. J. Pure Appl. Algebra **41**, 17–36 (1986)
20. Crisp, J.S.: Symmetrical subgroups of Artin groups. Adv. Math. **152**, 159–177 (2000)
21. Davis, J.F., Lück, W.: Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. K-theory **15**, 201–252 (1998)
22. Davis, M.W., Januszkiewicz, T.: Right-angled Artin groups are commensurable with right-angled Coxeter groups. J. Pure Appl. Algebra **153**, 229–235 (2000)
23. Dicks, W., Leary, I.J.: Presentations for subgroups of Artin groups. Proc. Am. Math. Soc. **127**, 343–348 (1999)
24. Droms, C.: Subgroups of graph groups. J. Algebra **110**, 519–522 (1987)
25. Dunwoody, M.J.: Accessibility and groups of cohomological dimension one. Proc. Lond. Math. Soc., III. Ser. **38**, 193–215 (1979)
26. Farrell, F.T., Jones, L.E.: Isomorphism conjectures in algebraic K-theory. J. Am. Math. Soc. **6**, 249–298 (1993)
27. Feighn, M., Mess, G.: Conjugacy classes of finite subgroups of Kleinian groups. Am. J. Math. **113**, 179–188 (1991)
28. Floyd, E.E., Richardson, R.W.: An action of a finite group on an n-cell without stationary points. Bull. Am. Math. Soc. **65**, 73–76 (1959)
29. Grunewald, F., Platonov, V.: Rigidity results for groups with radical cohomology of finite groups and arithmeticity problems. Duke Math. J. **100**, 321–358 (1999)
30. Henn, H.-W.: Unstable modules over the Steenrod algebra and cohomology of groups. In: Proc. Symp. in Pure Math. **63**, 'Group representations: cohomology, group actions and topology', Am. Math. Soc. 1998, 277–300
31. Higson, N., Kasparov, G.: Operator K-theory for groups which act properly and isometrically on Hilbert space. Electron. Res. Announc. Am. Math. Soc. **3**, 131–142 (1997)
32. Howie, J.: Bestvina-Brady groups and the plus construction. Math. Proc. Camb. Philos. Soc. **127**, 487–493 (1999)
33. Hsu, T., Wise, D.T.: On linear and residual properties of graph products. Mich. Math. J. **46**, 251–259 (1999)

34. Jones, L.E.: The converse to the fixed point theorem of P.A. Smith: I. Ann. Math. **94**, 52–68 (1971)
35. Julg, P.: Travaux de N. Higson et G. Kasparov sur la conjecture de Baum-Connes. Séminaire Bourbaki Vol. 1997/98, Exp. No. 841. Astérisque **252**, 151–183 (1998)
36. Kropholler, P.H.: On groups of type FP_∞. J. Pure Appl. Algebra **90**, 55–67 (1993)
37. Kropholler, P.H., Wall, C.T.C.: private communication from P. H. Kropholler
38. Leary, I.J., Nucinkis, B.E.A.: Every CW-complex is a classifying space for proper bundles. Topology **40**, 539–550 (2001)
39. Leary, I.J., Nucinkis, B.E.A.: Bounding the orders of finite subgroups. Publ. Math. **45**, 259–264 (2001)
40. Leary, I.J.: Some more groups of type *VF*. In preparation
41. Lück, W.: Transformation groups and algebraic K-theory. Lecture Notes in Mathematics **1408**, Springer 1989
42. Lück, W.: Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants. Math. Ann. **309**, 247–285 (1997)
43. Lück, W.: The type of the classifying space for a family of subgroups. J. Pure Appl. Algebra **149**, 177–203 (2000)
44. Lück, W.: Chern characters for proper equivariant homology theories and applications to K- and L-theory. J. Reine Angew. Math. **543**, 193–234 (2002)
45. MacLane, S.: Homology. Springer 1963
46. Matthey, M.: K-theories, C^*-algebras and assembly maps. PhD Thesis, Neuchâtel 2000
47. Matthey, M.: A delocalization property for assembly maps and an application in algebraic K-theory of group rings. K-theory **24**, 87–107 (2001)
48. Meier, J., Meinert, H., VanWyk, L.: Higher generation subgroup sets and the Σ-invariants of graph groups. Comment. Math. Helv. **73**, 22–44 (1998)
49. Mislin, G.: On the classifying space for proper actions. Cohomological methods in homotopy theory (Bellaterra 1998). Progr. Math. **196**, Birkhäuser, Basel 2001, 263–269
50. Niblo, G.A., Reeves, L.: Groups acting on CAT(0) cube complexes. Geom. Topol. **1**, 1–7 (1997)
51. Nucinkis, B.E.A.: Is there an easy algebraic characterization of universal proper G-spaces? Manuscr. Math. **102**, 335–345 (2000)
52. Oliver, R.: Fixed-point sets of group actions on finite acyclic complexes. Comment. Math. Helv. **50**, 155–177 (1975)
53. Oliver, R.: Group actions on disks, integral permutation representations and the Burnside ring. Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ.) Part I, Am. Math. Soc. 1978, 339–346
54. Platonov, V.: Splittable linear groups. Sov. Math. Dokl. **4**, 1002–1005 (1963)
55. Rosenberg, J.: Algebraic K-theory and its applications. Graduate Texts in Mathematics **147**, Springer-Verlag, New York, Heidelberg, Berlin 1994
56. Schneebeli, H.R.: On virtual properties and group extensions. Math. Z. **159**, 159–167 (1978)
57. Serre, J.-P.: Cohomology des groupes discrets. Ann. Math. Stud. **70**, Princeton Univ. Press 1971, 77–169
58. Serre, J.-P.: Linear representations of finite groups. Graduate Texts in Mathematics **42**, Springer-Verlag, New York, Heidelberg, Berlin 1977
59. Smith, P.A.: Fixed-points of periodic transformations. Colloq. Publ., Am. Math. Soc. **27**, 350–373 (1942)
60. Stallings, J.R.: A finitely presented group whose 3-dimensional integral homology is not finitely generated. Am. J. Math. **85**, 541–543 (1963)
61. Thurston, W.P.: Three-dimensional geometry and topology, Vol. 1. S. Levy, ed., Princeton Math. Series **35**, Princeton University Press 1997
62. tom Dieck, T.: Orbittypen und äquivariante Homologie I. Arch. Math. **23**, 307–317 (1972)
63. tom Dieck, T.: Transformation groups. Stud. Math. **8**, de Gruyter 1987
64. Wall, C.T.C.: Problem list in Homological Group Theory. London Math. Soc. Lecture Notes **36**, Cambridge Univ. Press 1979, 27–70