Uncountably many groups of type FP

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Abstract

We construct uncountably many discrete groups of type FP; in particular we construct groups of type FP that do not embed in any finitely presented group. We compute the ordinary, ℓ^2, and compactly-supported cohomology of these groups. For each $n \geq 4$ we construct a closed aspherical n-manifold that admits an uncountable family of acyclic regular coverings with non-isomorphic covering groups.

1 Introduction

An Eilenberg-Mac Lane space or $K(G,1)$ for a group G is a connected CW-complex whose fundamental group is G and whose universal cover is contractible. Equivalently a $K(G,1)$ is an aspherical CW-complex whose fundamental group is G. The group G is of type F if there is a finite $K(G,1)$, and is of type F_n if there is a $K(G,1)$ with finite n-skeleton. These conditions are generalizations of two standard group-theoretic conditions: G can be finitely generated if and only if G is type F_1, and G can be finitely presented if and only if G is type F_2.

There are a number of other finiteness conditions on groups that are implied by type F or type F_n. The universal covering space of a $K(G,1)$ is a contractible free G-CW-complex, with one G-orbit of cells for each cell in the given $K(G,1)$. For a non-trivial ring R, a group G is type $FH(R)$ if there is an R-acyclic free G-CW-complex with finitely many orbits of cells. Similarly, G is type $FL(R)$ (resp. $FP(R)$) if there is a finite resolution of R by finitely generated free (resp. finitely generated projective) RG-modules. There are corresponding analogues of the properties F_n. If G is type F then G satisfies each of these other finiteness conditions for any R. It has long been known that G is type F if and only if G is both F_2 and $FL(\mathbb{Z})$.

The question of whether $FP(\mathbb{Z})$ implies F_2 was solved by Bestvina and Brady [2]. To each finite flag complex L, they associated a group BB_L in such a way that the finiteness properties of BB_L were controlled by properties of L. This gave a systematic way to construct groups enjoying different finiteness properties. In particular, in the case when L is acyclic but is not contractible, they showed that BB_L is $FP(\mathbb{Z})$, and even $FH(\mathbb{Z})$, but is not F_2.

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Since there are only countably many finite presentations of groups, there are only countably many finitely presented groups up to isomorphism. On the other hand it is well-known that there are uncountably many finitely generated groups. Since a countable group contains at most countably many finitely generated subgroups, it follows that not every finitely generated group is a subgroup of a finitely presented group [23]. Higman showed that a group G embeds in a finitely presented group if and only if G admits a recursive presentation, i.e., a presentation in which the sets of generators and relators are recursively enumerable [15].

Once one knows that there are $FP_2(\mathbb{Z})$ groups that are not finitely presented, it becomes natural to ask how many there are. There are only countably many finite flag complexes L, and hence only countably many Bestvina-Brady groups BB_L. Moreover each BB_L is defined as a normal subgroup of a right-angled Artin group A_L and this group is of type F for every finite L. Thus the examples constructed in [2] leave open the question of whether every group of type $FP_2(\mathbb{Z})$ embeds in a finitely presented group, and whether there are uncountably many groups of type $FP_2(\mathbb{Z})$.

We answer these questions; we construct an uncountable family of groups of type $FP(\mathbb{Z})$, and determine which of these groups are isomorphic to subgroups of finitely presented groups. Our construction uses the techniques introduced in [2]. For any connected finite flag complex L and any subset S of $\mathbb{Z}$ we construct a group $G_L(S)$. Moreover, the construction is functorial in S, in the sense that for $S \subseteq T \subseteq \mathbb{Z}$ there is a natural surjective group homomorphism $G_L(S) \to G_L(T)$. The finiteness properties of $G_L(S)$ are controlled by properties of L and its universal cover $\widetilde{L}$. Our main example, an uncountable family $G_L(S)$ of groups of type $FP(\mathbb{Z})$, arises from the case when L and $\widetilde{L}$ are both acyclic but L is not contractible. The groups $G_L(S)$ can be viewed as interpolating between the groups $G_L(\emptyset)$ and $G_L(\mathbb{Z})$, which were known previously: $G_L(\mathbb{Z})$ is BB_L and $G_L(\emptyset)$ is isomorphic to the natural semidirect product $BB_{\widetilde{L}} \rtimes \pi_1(L)$.

Just as in [2], the proofs of our main results are geometric, involving Morse functions on CAT(0) cubical complexes. For each S and L, the group $G_L(S)$ acts as a group of automorphisms of a CAT(0) cubical complex $X_L^{(S)}$. This cubical complex is a branched cover of the complex X_L that plays a crucial role in [2], and the Morse function on X_L induces a Morse function on $X_L^{(S)}$. Just as in [2], the group $G_L(S)$ acts freely cocompactly on a level set for this Morse function, and Morse theory is used to establish the connectivity properties of this level set. In contrast to the groups BB_L in [2], the groups $G_L(S)$ for general S cannot be described as subgroups of other previously known groups. We give a number of ways of describing $G_L(S)$, each of which will play a role in our proofs. Some of these ways simplify if either $0 \in S$ or if L has nlcp, which stands for 'no local cut points', a property defined below in Definition 8. In particular $G_L(S)$ is isomorphic to:

- the group of deck transformations of the regular branched covering $X_L^{(S)} \to X_L/BB_L$;
- the group of deck transformations of the regular covering $X_t^{(S)} \to X_t/BB_L$, where $X_t^{(S)}$ denotes a non-integer level set in $X_L^{(S)}$ and similarly X_t in X_L;
- the fundamental group of a space obtained from X_L/BB_L by removing some vertices, in the case when L has nlcp;

- the group given by a presentation $P_L(\Gamma, S)$, in the case when $0 \in S$.

In Theorem 22 we will *define* $G_L(S)$ as the group of deck transformations of the regular branched covering $X_L^{(S)} \to X_L/BB_L$.

However, defining the CAT(0) cubical complex $X_L^{(S)}$ will occupy a significant proportion of our paper. For this reason, we give the presentations $P_L(\Gamma, S)$ before stating our main theorems. These presentations realize the groups $G_L(S)$ and the homomorphisms $G_L(S) \twoheadrightarrow G_L(T)$ for $0 \in S \subseteq T$, in the sense that the generating set depends only on L while the relators in $P_L(\Gamma, S)$ are a subset of the relators in $P_L(\Gamma, T)$. Since there are isomorphisms $G_L(S) \cong G_L(T)$ whenever $T = S + n$ is a translate of S, the presentations $P_L(\Gamma, S)$ describe the isomorphism type of each $G_L(S)$ except for $G_L(\emptyset)$, which can be described as a semidirect product $BB_{\widetilde{L}} \rtimes \pi_1(L)$.

Recall that a directed edge in a simplicial complex is an ordered pair (v, v') of vertices such that the corresponding unordered pair is an edge. A directed loop $\gamma = (a_1, \ldots, a_l)$ of length l is an ordered l-tuple of directed edges whose endpoints match up, in the sense that there are vertices $v_0, \ldots, v_l$ with $v_0 = v_l$ and $a_i = (v_{i-1}, v_i)$ for each i. In the following definition, we assume that L is a finite connected flag complex, and that Γ is a finite collection of directed edge loops in L that normally generates $\pi_1(L)$; equivalently if one attaches discs to L along the loops in Γ one obtains a simply-connected complex.

Definition 1. *For L and Γ as above and for any set S with $0 \in S \subseteq \mathbb{Z}$, the presentation $P_L(\Gamma, S)$ has as generators the directed edges of L, subject to the following relations.*

- *(Edge relations.) For each directed edge $a = (x, y)$ with opposite edge $\overline{a} = (y, x)$, the relation $a\overline{a} = 1$;*
- *(Triangle relations.) For each directed triangle (a, b, c) in L, the relations $abc = 1$ and $a^{-1}b^{-1}c^{-1} = 1$;*
- *(Long cycle relations.) For each $n \in S - \{0\}$, and each $(a_1, a_2, \ldots, a_l) \in \Gamma$, the relation $a_1^n a_2^n \cdots a_l^n = 1$.*

In the case when L is simply connected, we may take Γ to be empty, in which case $P_L(\Gamma, S)$ is independent of S. This reflects the fact that when L is simply-connected, the natural map $G_L(S) \to G_L(\mathbb{Z}) \cong BB_L$ is an isomorphism for each S. Our first main result addresses all other cases.

Theorem A. *Let L be a fixed finite connected flag complex that is not simply-connected.*

1. *There are uncountably many (in fact $2^{\aleph_0}$) isomorphism types of group $G_L(S)$.*
2. *For $0 \in S$, and any Γ that normally generates $\pi_1(L)$, the presentation $P_L(\Gamma, S)$ is a presentation of the group $G_L(S)$.*
3. *$G_L(S)$ is finitely presentable if and only if S is finite.*
4. *$G_L(S)$ is isomorphic to a subgroup of a finitely presented group if and only if S is recursively enumerable.*

Our second main result summarizes the finiteness properties of the groups $G_L(S)$. Some of our results concerning the cases when either S or $\mathbb{Z} - S$ is finite are omitted to simplify the statement. All finiteness properties mentioned in the statement will be defined in a subsequent section.

Theorem B. *Let L be a connected finite flag complex, let R be any ring in which $1 \neq 0$, and let S_0 be any subset of $\mathbb{Z}$ such that both S_0 and $\mathbb{Z} - S_0$ are infinite. The following are equivalent:*

- *L and $\widetilde{L}$ are R-acyclic;*
- *For each $S \subseteq \mathbb{Z}$, $G_L(S)$ is type $FH(R)$;*
- *For each $S \subseteq \mathbb{Z}$, $G_L(S)$ is type $FP(R)$;*
- *$\pi_1(L)$ and $G_L(S_0)$ are type $FP(R)$.*

For each $n \geq 2$, the following are equivalent:

- *L and $\widetilde{L}$ are $(n-1)$-R-acyclic;*
- *For each $S \subseteq \mathbb{Z}$, $G_L(S)$ is type $FH_n(R)$;*
- *For each $S \subseteq \mathbb{Z}$, $G_L(S)$ is type $FP_n(R)$;*
- *$\pi_1(L)$ and $G_L(S_0)$ are type $FP_n(R)$.*

Bestvina-Brady groups have been studied extensively, and many of the results that have been obtained for BB_L have analogues for $G_L(S)$. In particular, our proof of Theorem B uses some techniques from [2, 8]. For $G_L(S)$ we prove some of the results that have been proved for BB_L in [10, 12, 13, 20]. With some hypotheses on L, we compute the ordinary cohomology of $G_L(S)$ in terms of that of BB_L, which was calculated in [20], and we compute the compactly supported and ℓ^2-cohomology of $G_L(S)$ following [12]. We use Davis's trick [10, 11] to construct, for each $n \geq 4$, a closed aspherical n-manifold that admits an uncountable family of acyclic regular covers with non-isomorphic groups of deck transformations. We also give a second proof that $G_L(S)$ is $FP_2(\mathbb{Z})$ whenever $\pi_1(L)$ is perfect that does not use CAT(0) cubical complexes, along the lines of [13].

The paper is structured as follows. Sections 2–5 describe background material concerning flag complexes, finiteness conditions, Artin groups and Morse functions respectively. Sections 6–8 concern links in CAT(0) cubical complexes, especially the complex X_L, and Sections 9–10 contain the construction of $X_L^{(S)}$, its Morse function, and the connectivity of its level sets. Sections 11–13 complete the proof of Theorem B; they discuss realizing $G_L(S)$ in terms of fundamental groups, the definition of 'sheets' in $X_L^{(S)}$ and the application of Brown's criterion respectively. Section 14 gives presentations for $G_L(S)$, and Section 15 introduces a set-valued invariant for a group and a finite sequence of elements, which may be of independent interest. These two sections complete the proof of Theorem A. Sections 16–18 give analogues for $G_L(S)$ of various results that have been obtained for BB_L. In more detail, Section 16 considers classifying spaces and ordinary cohomology; Section 17 considers compactly supported and ℓ^2-cohomology; Section 18 establishes the existence of uncountably many (non-finitely presented) Poincaré duality groups. Sections 19 and 20 describe alternative proofs of some of the results: Section 19 gives a proof that $G_L(S)$ is $FP_2(\mathbb{Z})$ whenever $\pi_1(L)$ is perfect that does not use $X_L^{(S)}$, and Section 20 describes the homotopy type of level sets in $X_L^{(S)}$ in the case when $\pi_1(L)$ is finite. Finally, Section 21 discusses a small number of open problems.

The author first heard the question of whether there can be uncountably many groups of type FP from a conversation with Martin Bridson, Peter Kropholler, Robert Kropholler and Charles Miller III. Martin Bridson and the author subsequently discussed the problem, and this work benefitted from these discussions and a conversation with Ashot Minasyan. The author thanks these five for their helpful comments on this work.

2 Examples of flag complexes

To apply Theorems A and B, we require certain finite flag complexes L. The purpose of this section is to establish that such complexes exist.

A *flag complex* or *clique complex* is a simplicial complex with the property that any finite set of mutually adjacent vertices spans a simplex. The realization of any poset is a flag complex and so in particular the barycentric subdivision of any simplicial complex is a flag complex.

If L is connected, then $\widetilde{L}$ is acyclic if and only if L is aspherical, i.e., if and only if L is an Eilenberg-Mac Lane space. A theorem of Baumslag-Dyer-Heller tells us that for any finite complex K, there is a finite Eilenberg-Mac Lane space L and a homology isomorphism $L \to K$ [1, 18]. Thus the assumption that $\widetilde{L}$ is acyclic places no restrictions on the homology of L. This provides a source of examples for applying our main theorems. In particular if L is the Eilenberg-Mac Lane space for a non-trivial acyclic group, then Theorems A and B imply that the groups $G_L(S)$ comprise uncountably many isomorphism types of groups of type $FP(\mathbb{Z})$.

For a more explicit example, consider the presentation 2-complex for Higman's group

$$\langle a, b, c, d \ : \ a^b = a^2,\ b^c = b^2,\ c^d = c^2,\ d^a = d^2 \rangle.$$

This is a polygonal cell complex consisting of one 0-cell, four 1-cells and four pentagonal 2-cells. This 2-complex is both acyclic and aspherical, and has non-trivial fundamental group [1]. Any flag triangulation of this 2-complex has nlcp (as in Definition 8) and can be used in our main theorems. For example, the second barycentric subdivision of the given polygonal structure is a suitable flag complex L consisting of 240 triangles, 336 edges and 97 vertices. For other examples of 2-complexes that are acyclic and aspherical, see [18, section 3] or [4, section 4].

3 Finiteness properties for groups

We briefly recall the definitions of some of the homological finiteness properties for groups and give some results that we will use to establish these properties. The algebraic properties were introduced by Bieri [3, 7], F and F_n by Wall [24], and the FH and FH_n properties by Bestvina-Brady [2]. Recall that a CW-complex X is said to be R-acyclic if its reduced homology groups satisfy $\overline{H}_i(X; R) = 0$ for all $i \geq -1$. Similarly, a CW-complex X is said to be n-R-acyclic if $\overline{H}_i(X; R) = 0$ for $i \leq n$.

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Definition 2. *Let G be a discrete group, and let R be a nontrivial ring.*

- *G is type F if there is a finite $K(G,1)$, or equivalently there is a contractible free G-CW-complex with finitely many orbits of cells.*
- *G is type $FH(R)$ if there is an R-acyclic free G-CW-complex with finitely many orbits of cells.*
- *G is type $FL(R)$ if there is some $n \geq 0$ and a long exact sequence of RG-modules*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

in which each P_i is a finitely generated free module.

- *G is type $FP(R)$ if there is a long exact sequence as above in which each P_i is a finitely generated projective module.*

Definition 3. *Let G be a discrete group, R a non-trivial ring, and $n \geq 0$ an integer.*

- *G is type F_n if there is a $K(G,1)$ with finite n-skeleton.*
- *G is type $FH_n(R)$ if there is an $(n-1)$-R-acyclic free G-CW-complex with finitely many orbits of cells.*
- *G is type $FL_n(R)$ if there is a long exact sequence of RG-modules*

$$\to P_m \to P_{m-1} \to \cdots \to P_0 \to R \to 0$$

in which P_i is a finitely generated free module for $i \leq n$.

- *G is type $FP_n(R)$ if there is a long exact sequence as above in which P_i is a finitely generated projective module for $i \leq n$.*

A contractible space is R-acyclic for any R, the cellular chain complex for a free G-CW-complex is a chain complex of free modules, and free modules are projective. From these observations it follows that for any R, $F \implies FH(R) \implies FL(R) \implies FP(R)$, and similarly $F_n \implies FH_n(R) \implies FL_n(R) \implies FP_n(R)$. It can be shown that $FL_n(R)$ is equivalent to $FP_n(R)$, for each n and R. It follows from the universal coefficient theorem that whenever there is a ring homomorphism $R \to S$, $FX(R) \implies FX(S)$, where $FX(-)$ denotes any of the above finiteness conditions that involves a ring. In particular, $FX(\mathbb{Z}) \implies FX(R)$ for any R. For any non-trivial ring R, $FP_1(R)$ is equivalent to F_1, and for $n \geq 2$, $FP_n(\mathbb{Z})$ together with F_2 is equivalent to F_n [7, VIII.7.1].

G is said to have cohomological dimension at most n over R if there is a projective resolution of R as an RG-module of length n:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0.$$

If G is type $FP(R)$, then G has finite cohomological dimension, and conversely if G has cohomological dimension n over R and G is $FP_n(R)$ then G is $FP(R)$. If $H \leq G$, then the cohomological dimension of H is a lower bound for that of G. Since non-trivial finite

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groups have infinite cohomological dimension, it follows that every group of type $FP(\mathbb{Z})$ is torsion-free.

For a general ring R, the difference between $FP(R)$ and $FL(R)$ is well understood in terms of K-theory; in particular if every finitely generated projective RG-module is stably free then G is $FP(R)$ if and only if G is $FL(R)$. No examples are known that distinguish between the properties $FP(\mathbb{Z})$, $FL(\mathbb{Z})$ and $FH(\mathbb{Z})$. For larger rings however, there is a difference. If G is any finite group, then G is $FP(\mathbb{Q})$ by Maschke's theorem, but G is not $FL(\mathbb{Q})$ unless G is the trivial group. No examples are known to distinguish $FL(\mathbb{Q})$ and $FH(\mathbb{Q})$, but again there is a difference for larger fields. If $K = \mathbb{Q}[e^{2\pi i/3}]$ is the field obtained by adjoining a cube root of 1 to $\mathbb{Q}$, there is a (3-dimensional crystallographic group) G which is $FL(K)$, $FP(\mathbb{Q})$ and not $FL(\mathbb{Q})$. Since $FH(K) \Leftrightarrow FH(\mathbb{Q})$ it follows that this group cannot be $FH(K)$ [17].

A sequence $\to A_m \to A_{m+1} \to$ of abelian groups and morphisms indexed by $\mathbb{N}$ is *essentially trivial* if for all m there exists $m' > m$ so that the composite map $A_m \to A_{m'}$ is trivial. The following statement is a simplified version of K. S. Brown's criterion [6], which will suffice for our purposes.

Theorem 4. *Suppose that X is a finite-dimensional R-acyclic G-CW-complex, and that G acts freely except possibly that some vertices have isotropy subgroups that are of type $FP(R)$ (resp. $FP_n(R)$). Suppose also that $X = \bigcup_{m\in\mathbb{N}} X(m)$, where $X(m) \subseteq X(m+1) \subseteq \cdots \subseteq X$ is an ascending sequence of G-subcomplexes, each of which contains only finitely many orbits of cells. In this case G is $FP(R)$ (resp. $FP_n(R)$) if and only if for all i (resp. for all $i < n$) the sequence $\overline{H}_i(X(m); R)$ of reduced homology groups is essentially trivial.*

Proposition 5. *If a group Q is a retract of a group G, and G is of type $FP(R)$ (resp. type $FP_n(R)$) then so is Q.*

Proof. A group Γ is $FP(R)$ if and only if Γ has finite cohomological dimension, n, over R and Γ is type $FP_n(R)$. Since Q is isomorphic to a subgroup of G, the cohomological dimension of G is an upper bound for the cohomological dimension for Q. Thus the claim for type $FP(R)$ will follow from the claim for type $FP_n(R)$ for all $n \in \mathbb{N}$.

According to [3, Theorem 1.3], a group Γ is $FP_n(R)$ if and only if the following condition holds: for any directed system M_* of $R\Gamma$-modules with $\lim M_* = 0$, one has that $\lim H^k(\Gamma; M_*) = 0$ for all $k \leq n$. Now suppose that M_* is a directed system of RQ-modules such that $\lim M_* = 0$, and that we have maps $Q \rightarrowtail G \twoheadrightarrow Q$ whose composite is the identity. The surjection $G \twoheadrightarrow Q$ enables us to view M_* as a directed system of RG-modules, and the direct limit of this system is still zero, since its underlying abelian group is unchanged. To see that for each $k \leq n$, $\lim H^k(Q; M_*) = 0$, note that the identity map of this group factors through $\lim H^k(G; M_*)$, which is zero because G is type $FP_n(R)$. □

4 Classifying spaces for right-angled Artin groups

Recall that a flag complex is a simplicial complex such that every finite mutually adjacent set of vertices spans a simplex. The right-angled Artin group A_L associated to a flag complex L is the group given by a presentation with generators the vertex set L^0 of L, subject only to the relations that the ends of each edge commute.

$$A_L = \langle v \in L^0 \quad : \quad [v,w] = 1 \text{ for all } \{v,w\} \in L^1 \rangle$$

For example, if L is an n-simplex then A_L is free abelian of rank $n+1$, while if L is 0-dimensional, A_L is a free group of rank $|L^0|$.

For a vertex $v \in L^0$, let $\mathbb{T}_v$ be a copy of the circle $\mathbb{R}/\mathbb{Z}$, and give $\mathbb{T}_v$ a cellular structure with one 0-cell corresponding to the image of $\mathbb{Z}$ and one 1-cell. For a simplex $\sigma = (v_0, \ldots, v_n)$, let $\mathbb{T}_\sigma$ be the direct product $\mathbb{T}_\sigma = \prod_{i=0}^n \mathbb{T}_{v_i}$. If τ is a face of σ then $\mathbb{T}_\tau$ can be identified with a subcomplex of $\mathbb{T}_\sigma$:

$$\mathbb{T}_\tau \cong \prod_{v\in\tau} \mathbb{T}_v \times \prod_{v\in\sigma-\tau} \{v\}.$$

Now let Δ be a simplex with the same vertex set as L, and view L as a subcomplex of Δ. For each simplex σ of L, we can view $\mathbb{T}_\sigma$ as a subtorus of $\mathbb{T}_\Delta$. Define the Salvetti complex $\mathbb{T}_L$ to be the union of these tori:

$$\mathbb{T}_L = \bigcup_{\sigma\in L} \mathbb{T}_\sigma \subseteq \mathbb{T}_\Delta.$$

The given cell structure on each $\mathbb{T}_v$ induces a cell structure on $\mathbb{T}_L$, with one 0-cell and for each $n > 0$, one n-cell for each $(n-1)$-simplex of L. Furthermore, the n-cell corresponding to the $(n-1)$-simplex σ is naturally a copy of the n-cube I^n, in such a way that the n-torus $\mathbb{T}_\sigma$ is obtained by identifying opposite faces of I^n. The 2-skeleton of $\mathbb{T}_L$ is the presentation 2-complex for the defining presentation of A_L, and $\mathbb{T}_L$ is an Eilenberg-Mac Lane space $K(A_L, 1)$. If we identify the circle $\mathbb{T}_v$ with the group $\mathbb{R}/\mathbb{Z}$, so that the zero cell is the group identity, then we obtain a map $l_\Delta : \mathbb{T}_\Delta \to \mathbb{R}/\mathbb{Z}$ defined by

$$(t_1, \ldots, t_n) \mapsto t_1 + \cdots + t_n \quad \text{for all } (t_1, \ldots, t_n) \in \mathbb{T}_\Delta.$$

This map restricts to $\mathbb{T}_L \subseteq \mathbb{T}_\Delta$ to define a map $l_L : \mathbb{T}_L \to \mathbb{R}/\mathbb{Z}$. The induced map on fundamental groups is the homomorphism $l_* : A_L \to \mathbb{Z}$ that sends each of the standard generators for A_L to $1 \in \mathbb{Z}$, and the kernel of this homomorphism is by definition the Bestvina-Brady group BB_L. For each $v \in L^0$, the map l_L restricts to a homeomorphism $l_v : \mathbb{T}_v \to \mathbb{R}/\mathbb{Z}$.

The link of the point 0 in $\mathbb{R}/\mathbb{Z}$ is a copy of the 0-sphere, corresponding to travelling in the positive and negative directions along $\mathbb{R}$. Similarly, the link of the 0-cell in $\mathbb{T}_v$ is a copy of the 0-sphere, which we shall denote by $\mathbb{S}(v) = \{v^+, v^-\}$. Under the isomorphism of links induced by the homeomorphism $l_v : \mathbb{T}_v \to \mathbb{R}/\mathbb{Z}$, the point v^+ (resp. v^-) corresponds to the positive direction (resp. negative direction). The link of the vertex in $\mathbb{T}_\Delta$ is the $(|L^0| - 1)$-sphere $\mathbb{S}(\Delta)$, equal to the join $\mathbb{S}(\Delta) = \ast_{v\in\Delta} \mathbb{S}(v)$. For each simplex σ of L, there is an inclusion $\mathbb{S}(\sigma) \subseteq \mathbb{S}(\Delta)$, and the link $\mathbb{S}(L)$ of the unique vertex of $\mathbb{T}_L$ is equal to the union of these images:

$$\mathbb{S}(L) = \bigcup_{\sigma\in L} \mathbb{S}(\sigma) \subseteq \mathbb{S}(\Delta).$$

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5 A Morse function on X_L

Let $X = X_L$ denote the universal cover of $\mathbb{T}_L$, and let $f = f_L : X_L \to \mathbb{R}$ be the map of universal covers induced by $l : \mathbb{T}_L \to \mathbb{R}/\mathbb{Z}$. Each lift of each cell of $\mathbb{T}_L$ is an embedded cube in X, and so X is a cubical complex, in which the link of each vertex is isomorphic to $\mathbb{S}(L)$. Since $\mathbb{S}(L)$ is a flag complex (Proposition 11), it follows that the path metric on X defined by using the standard Euclidean metric on each cube to measure lengths of PL paths is CAT(0). This metric gives one way to prove that $\mathbb{T}_L$ is a $K(A_L, 1)$ [5].

The map $f : X_L \to \mathbb{R}$ has the following properties: its restriction to each n-cube (identified with $[0,1]^n \subseteq \mathbb{R}^n$) is an affine map; the image of each vertex is in $\mathbb{Z}$; the image of each n-cube is an interval of length n. Any map from a cubical complex to $\mathbb{R}$ having these properties will be called a *Morse function*; the definition can be made more general, but this will suffice for our purposes. For a Morse function $f : X \to \mathbb{R}$ and v a vertex of X, the ascending or $\uparrow$-link $\mathrm{Lk}^{\uparrow}_X(v)$ is the subcomplex of $\mathrm{Lk}_X(v)$ coming from the cubes C containing v for which $f : C \to \mathbb{R}$ attains its minimum at v; the descending or $\downarrow$-link $\mathrm{Lk}^{\downarrow}_X(v)$ is defined similarly with 'minimum' replaced by 'maximum'. The Morse function $f : X_L \to \mathbb{R}$ has the property that for each vertex v, both $\mathrm{Lk}^{\uparrow}_X(v)$ and $\mathrm{Lk}^{\downarrow}_X(v)$ are naturally identified with L, being the full subcomplexes of $\mathbb{S}(L)$ spanned by $\{v^+; v \in L^0\}$ and $\{v^-; v \in L^0\}$ respectively. We will use the following theorem concerning Morse functions on cubical complexes [2].

Theorem 6. *Let $g : Y \to \mathbb{R}$ be a Morse function on a cubical complex, let $a < b < c < d \in \mathbb{R}$, and define $Y_{[b,c]} = g^{-1}([b,c])$ and similarly $Y_{[a,d]} = g^{-1}([a,d])$. Up to homotopy $Y_{[a,d]}$ is obtained from $Y_{[b,c]}$ by coning off a copy of $\mathrm{Lk}{\downarrow}_Y(v)$ for each vertex v with $g(v) \in (c,d]$ and coning off a copy of $\mathrm{Lk}^{\uparrow}_Y(v)$ for each vertex v with $g(v) \in [a,b)$.*

Let $\tilde{l} : X_L/BB_L \to \mathbb{R}$ be the map induced by $f : X \to \mathbb{R}$, or equivalently the pullback of the maps $l : \mathbb{T}_L \to \mathbb{R}/\mathbb{Z}$ and $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Note that the map $\tilde{l}$ gives a bijection between the vertex set of X_L/BB_L and $\mathbb{Z}$.

6 L and $\mathbb{S}(L)$

Proposition 7. *For a simplicial complex L without isolated vertices, the following properties are equivalent:*

- *Every 1-simplex is contained in a 2-simplex and every vertex link is connected;*
- *The topological realization $|L|$ has no local cut point.*

Proof. For $n \geq 2$ the n-simplex has no local cut points; it follows that the only possible local cut points are either vertices of L or points in an edge of L that is not contained in any triangle. A non-isolated vertex is a local cut point if and only if its link is not connected, and the midpoint of an isolated edge is always a local cut point. □

Definition 8. *Say that L has nlcp if it satisfies the equivalent conditions listed above.*

Lemma 9. *For any flag complex L without isolated vertices, there is a flag complex $M = M(L)$ with $L \subseteq M$, and having the following properties.*

1. *M has nlcp;*

2. *L is a full subcomplex of M;*

3. *The inclusion $L \subseteq M$ is a homotopy equivalence;*

4. *Provided that L is at least 2-dimensional, M has the same dimension as L.*

The construction $L \mapsto M(L)$ can be chosen to be functorial for maps $L_1 \to L_2$ that do not collapse any simplex.

Proof. Any triangulation of $|L| \times [0,1]$ for which $|L| \times \{0\}$ is realized as a full subcomplex isomorphic to L will have properties (1)–(3). For property (4) to hold as well, we instead take $M(L)$ to be a suitable triangulation of the mapping cylinder of the inclusion of L^1 in L. It is convenient to define $M(L)$ as a full subcomplex of a triangulation $N(L)$ of $|L| \times [0,1]$, so that

$$|M(L)| = |L| \times \{0\} \cup |L^1| \times [0,1] \subseteq |N(L)| = |L| \times [0,1].$$

A construction for $N(L)$ appears implicitly in a well-known proof of excision for singular homology [14, theorem 2.20]. The vertices of $N(L)$ are the vertices of L, together with the simplices of L, distinguishing between the vertex v and the 0-simplex $\{v\}$. For $-1 \leq k \leq n$, a set $\{v_0, \ldots, v_k, \sigma_{k+1} \ldots, \sigma_n\}$ of $k+1$ vertices and $n-k$ simplices of L spans a simplex of $N(L)$ if and only if the following conditions hold: $v_0, \ldots, v_k$ span a simplex of L; the simplices $\sigma_{k+1}, \ldots, \sigma_n$ are totally ordered by inclusion; each v_i is a vertex of each σ_j. There is a homeomorphism from $|N(L)|$ to $|L| \times [0,1]$ that sends v to $(v,0)$ and sends σ to $(\hat{\sigma}, 1)$, where $\hat{\sigma}$ denotes the barycentre in $|L|$ of the the simplex σ. It is easily checked that this construction has properties (1)–(3), and is functorial for simplicial maps that do not collapse any simplex. Furthermore if $M(L)$ is defined to be the full subcomplex of $N(L)$ with vertex set the vertices of L together with the 0- and 1-simplices of L, then $M(L)$ is a triangulation of $|L| \times \{0\} \cup |L^1| \times [0,1] \subseteq |L| \times [0,1]$, and $M(L)$ has properties (1)–(4). □

Remark 10. *Note that the set of embeddings $L \subseteq M$ having properties (1)–(4) is* directed*, in the sense that if $L \subseteq M_1$ and $L \subseteq M_2$ have these properties then so does $L \subseteq M_1 \cup_L M_2$.*

Proposition 11. *L is a flag complex if and only if $\mathbb{S}(L)$ is a flag complex. $L \subseteq M$ is a full subcomplex if and only if $\mathbb{S}(L) \subseteq \mathbb{S}(M)$ is a full subcomplex.*

Proof. See [2, lemma 5.8] for the first statement. The second statement is proved similarly. □

Proposition 12. *For any L, L is a retract of $\mathbb{S}(L)$.*

Proof. For any function $g : L^0 \to \{+,-\}$, the map $v \mapsto v^{g(v)}$ induces a simplicial map $L \to \mathbb{S}(L)$ which is a pre-inverse to the projection map $\pi(v^\epsilon) = v$. □

Proposition 13. *$\mathbb{S}(L)$ is connected if and only if L is connected and not equal to a single point.*

Proof. Since L is a retract of $\mathbb{S}(L)$, $\mathbb{S}(L)$ cannot be connected in the case when L is not. In the case when L is a single point, $\mathbb{S}(L)$ is two points. Conversely, suppose that v^ϵ and $w^{\epsilon'}$ are vertices of $\mathbb{S}(L)$ with $\epsilon, \epsilon' = \pm$. Given any path $v = v_0 \dots, v_l = w$ in L with $l > 0$, the path $v^\epsilon, v_1^+, v_2^+, \dots, v_{l-1}^+, w^{\epsilon'}$ is a path from v^ϵ to $w^{\epsilon'}$. □

Proposition 14. *If L has nlcp, then for any vertex v of L, $\mathbb{S}(\mathrm{St}_L(v))$ is simply-connected.*

Proof. The fact that L has nlcp implies that $\mathrm{Lk}_L(v)$ is connected and not a single point. By Proposition 13, it follows that $\mathbb{S}(\mathrm{Lk}_L(v))$ is connected. Since $\mathbb{S}(\mathrm{St}_L(v))$ can be identified with the join $\{v^+, v^-\} * \mathbb{S}(\mathrm{Lk}_L(v))$, the claim follows, since the join of a non-empty space and a connected space is simply-connected. □

Proposition 15. *The natural map $\pi_1(\mathbb{S}(L)) \to \pi_1(L)$ is always surjective. If L has nlcp, then this map is an isomorphism.*

Proof. The first statement is immediate from Proposition 12. For the second statement, it suffices to show that every edge loop in $\mathbb{S}(L)$ based at a positive vertex v_0^+ is based homotopic to an edge loop contained in the positive copy of L. Let $v_0^+ = v_0^{\epsilon_0}, v_1^{\epsilon_1}, \dots, v_{l-1}^{\epsilon_{l-1}}, v_l^{\epsilon_l} = v_0^+$ be such an edge loop in $\mathbb{S}(L)$. If $\epsilon_i = +$ for each i with $0 < i < l$ then this loop already lies in the positive copy of L. Otherwise, pick some i so that $\epsilon_i = -$. The subpath $v_{i-1}^{\epsilon_{i-1}}, v_i^{\epsilon_i}, v_{i+1}^{\epsilon_{i+1}}$ is contained in $\mathbb{S}(\mathrm{St}_L(v_i))$, which is simply-connected by Proposition 14. Hence this path is homotopic relative to its end points to the path $v_{i-1}^{\epsilon_{i-1}}, v_i^+, v_{i+1}^{\epsilon_{i+1}}$, and the original loop is based homotopic to a based loop containing fewer negative vertices. □

7 Coverings of L and $\mathbb{S}(L)$

Proposition 16. *Let L be a connected flag complex with universal covering $\pi : \widetilde{L} \to L$. The following diagram is a pullback square.*

$$\begin{array}{ccc} \mathbb{S}(\widetilde{L}) & \xrightarrow{\mathbb{S}(\pi)} & \mathbb{S}(L) \\ \downarrow & & \downarrow \\ \widetilde{L} & \xrightarrow{\pi} & L \end{array}$$

Proof. The vertex set of $\mathbb{S}(\widetilde{L})$ is easily seen to be the pullback of the other three vertex sets. Furthermore, a set of $n+1$ vertices of $\mathbb{S}(\widetilde{L})$ spans an n-simplex if and only if its images in both $\widetilde{L}$ and $\mathbb{S}(L)$ span n-simplices. □

Corollary 17. *For L connected and not a single point, $\mathbb{S}(\pi) : \mathbb{S}(\widetilde{L}) \to \mathbb{S}(L)$ is a covering of $\mathbb{S}(L)$ with fundamental group equal to the kernel of $\pi : \pi_1(\mathbb{S}(L)) \to \pi_1(L)$.*

Proof. $\mathbb{S}(\pi)$ is a covering map, since it is the pullback of the covering map $\pi : \widetilde{L} \to L$, and since the map $\mathbb{S}(\widetilde{L}) \to L$ factors through $\widetilde{L}$, the fundamental group of $\mathbb{S}(\widetilde{L})$ has trivial image in $\pi_1(L)$. Conversely, if γ is an edge loop in $\mathbb{S}(L)$ whose image in L is null-homotopic, then the image of γ lifts to an edge loop γ' in $\widetilde{L}$. The pair (γ, γ') defines an edge loop γ'' in $\mathbb{S}(\widetilde{L})$ with $\mathbb{S}(\pi)(\gamma'') = \gamma$. □

When the complex $\mathbb{S}(L)$ arises as the vertex link in $\mathbb{T}_L$, its vertices are naturally partitioned into 'opposite pairs' v^+, v^-. This partition is not, in general, determined by the isomorphism type of the simplicial complex $K = \mathbb{S}(L)$. For example, if $L = \mathbb{S}(\Delta)$ for Δ an n-simplex and $K = \mathbb{S}(L)$, then K is isomorphic to the join of $(n+1)$ sets of four vertices, and K itself gives no clue as to how these four element sets pair up. However, we show that the opposite pairs in $\mathbb{S}(\widetilde{L})$ are determined completely by the covering map $\mathbb{S}(\pi) : \mathbb{S}(\widetilde{L}) \to \mathbb{S}(L)$ and the opposite pairs in $\mathbb{S}(L)$.

Proposition 18. *For L connected and not a single point the vertices v, w of $\mathbb{S}(\widetilde{L})$ form an opposite pair if and only if the following conditions hold:*

- *there is an edge path in $\mathbb{S}(\widetilde{L})$ of length two from v to w;*
- *$\mathbb{S}(\pi)(v)$ and $\mathbb{S}(\pi)(w)$ form an opposite pair in $\mathbb{S}(L)$.*

Proof. If v and w form an opposite pair in $\mathbb{S}(\widetilde{L})$ then so do their images in $\mathbb{S}(L)$. Also for any adjacent vertices x, y of $\widetilde{L}$, x^+, y^+, x^- is an edge path of length two in $\mathbb{S}(\widetilde{L})$ joining the opposite pair x^+, x^-.

Now suppose that there are edge paths of length two in $\mathbb{S}(\widetilde{L})$ joining v to both w and w', so that $\mathbb{S}(\pi)(w) = \mathbb{S}(\pi)(w')$ forms an opposite pair with $\mathbb{S}(\pi)(v)$. By concatenating these we obtain a length four path w, u, v, u', w' from w to w', which maps to a loop in $\mathbb{S}(L)$. Since w, v and w' all map to the same vertex of L, the image of this loop in L is null-homotopic. By Corollary 17, the original path in $\mathbb{S}(\widetilde{L})$ must be a closed loop and hence $w = w'$. $\square$

8 Local properties of CAT(0)-cube complexes

Note that in a cubical complex, the link of a vertex is naturally isomorphic to the sphere of radius 1/4 around that vertex. In a CAT(0) cubical complex, we can say more.

Theorem 19. *For any CAT(0) cubical complex X and any vertex v of X, $\mathrm{Lk}_X(v)$ is a strong deformation retract of $X - \{v\}$.*

Proof. For any $x \in X - \{v\}$, there is a unique $y \in \mathrm{Lk}_X(v)$ such that the geodesic arcs $[v, y]$ and $[v, x]$ are nested: if $d(x, v) = 1/4$ then $y = x$; if $d(x, v) > 1/4$ then y is the unique point of $\mathrm{Lk}_X(v) \cap [x, v]$, and if $d(x, v) < 1/4$ this follows from [5, I.7.16]. Define $r : X - \{v\} \to \mathrm{Lk}_X(v)$ by $r(x) = y$, for y defined as above. If i denotes the inclusion of the link as the sphere of radius 1/4, $i : \mathrm{Lk}_X(v) \to X - \{v\}$, then $r \circ i$ is the identity map $\mathrm{Id}_{\mathrm{Lk}_X(v)}$, and one can define a homotopy from Id_X to $i \circ r$ by moving from x to $r(x)$ at constant speed along the geodesic arc $[x, r(x)]$. $\square$

Corollary 20. *Let Z be a locally CAT(0) cube complex with all vertex links connected, and let W be a set of vertices of Z. Then $Z - W$ is connected and for any $w \in W$, the inclusion map $i : \mathrm{Lk}_Z(w) \to Z - W$ is π_1-injective.*

Proof. Z is path-connected by definition, and connectivity of vertex links implies that any PL path between points of $Z - W$ may be homotoped to avoid W. Now let $p : X \to Z$ be the universal covering map, so that X is a CAT(0) cubical complex, and let $V = p^{-1}(W)$, a set of vertices of X, and fix some v with $p(v) = w$. Since $i : \mathrm{Lk}_Z(w) \to Z$ is homotopic to the constant map sending every point of $\mathrm{Lk}_Z(w)$ to w, it follows that $p : \mathrm{Lk}_X(v) \to \mathrm{Lk}_Z(w)$ is a homeomorphism. Now $p : X - V \to Z - W$ is a covering of connected spaces, and so the induced map $p_* : \pi_1(X - V) \to \pi_1(Z - W)$ is injective. Hence it suffices to show that the map $i' : \mathrm{Lk}_X(v) \to X - V$ is injective on fundamental groups. This follows from Theorem 19, since the composite $\mathrm{Lk}_X(v) \to X - V \to X - \{v\}$ is split injective on fundamental groups. $\square$

Theorem 21. *Let X be a locally CAT(0) cube complex with universal covering $p : \widetilde{X} \to X$, suppose that all vertex links in X are connected, and let V be a set of vertices of X. There is a CAT(0) cubical complex $\overline{X}$ and a regular branched covering map $c : \overline{X} \to X$, with branching locus contained in V and such that*

$$\mathrm{Lk}_{\overline{X}}(w) = \begin{cases} \mathrm{Lk}_X(c(w)) & \textit{if } c(w) \notin V \\ \widetilde{\mathrm{Lk}_X(c(w))} & \textit{if } c(w) \in V. \end{cases}$$

Moreover, the covering map c factors through $\widetilde{X}$, in the sense that $c = p \circ b$ for some regular branched covering map $b : \overline{X} \to \widetilde{X}$.

Proof. We shall define $\overline{X}$ by adding vertices to the universal cover Y of $X - V$. Let $c : Y \to X - V$ denote the covering map. Since the map $\pi_1(X - V) \to \pi_1(X)$ is surjective, it follows that $c : Y \to X - V$ factors through $\widetilde{X} - p^{-1}(V)$, so we define $b : Y \to \widetilde{X} - p^{-1}(V)$ so that $c = p \circ b$. The connected components of the inverse images of cubes of $\widetilde{X}$ split Y as a disjoint union of open cubes. This gives Y the structure of a cubical complex with some missing vertices. Let W' be the set of those ends of edges of Y that do not have a vertex incident on them. Generate an equivalence relation on W' by the relation $e \sim e'$ if there is a corner of a square of Y that contains the edge-ends e and e'. Now define W to be $W'/\sim$, the set of equivalence classes, and let $\overline{X} = Y \coprod W$. For $n > 0$, each n-cube of $\overline{X}$ is an n-cube of Y. The vertex set of $\overline{X}$ consists of W together with the vertices of Y. A vertex $w \in W$ is incident on an edge-end e if and only w is the equivalence class defined by e. For $w \in W$, $b(w)$ is defined as follows: pick some edge-end e incident on w, and define $b(w)$ to be the vertex of $\widetilde{X}$ incident on the edge-end $b(e)$, and define $c(w) = p(b(w))$. The claimed structure for the vertex links in $\overline{X}$ is now easily verified. Since Y is simply-connected and the links of vertices in W are connected, it follows that $\overline{X}$ is simply-connected, and hence that $\overline{X}$ is CAT(0) as claimed. $\square$

There is an alternative description of $\overline{X}$ as the metric completion of Y in the path metric induced by $c : Y \to X - V$, but we prefer to emphasize the combinatorial nature of $\overline{X}$.

9 The construction of $X_L^{(S)}$

Theorem 22. *For any connected finite L and any $S \subseteq \mathbb{Z}$, the locally CAT(0) cubical complex X_L/BB_L admits a regular branched cover $c : X_L^{(S)} \to X_L/BB_L$ with the following properties:*

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- *c factors through X_L: there is a regular branched cover $b : X_L^{(S)} \to X_L$ whose composite with the natural map $X_L \to X_L/BB_L$ is equal to c;*
- *The function $f^{(S)} := f \circ b = \tilde{l} \circ c$ is a Morse function on $X^{(S)}$;*
- *The links of vertices of $X^{(S)}$ are either $\mathbb{S}(L)$ or $\mathbb{S}(\widetilde{L})$, depending on their height:*

$$\mathrm{Lk}_{X^{(S)}}(v) = \begin{cases} \mathbb{S}(L) \text{ if } f^{(S)}(v) \in S \\ \mathbb{S}(\widetilde{L}) \text{ if } f^{(S)}(v) \notin S \end{cases}$$

- *The ascending and descending links of vertices of $X^{(S)}$ are either L or $\widetilde{L}$ depending on their height:*

$$\mathrm{Lk}{\downarrow}_{X^{(S)}}(v) = \mathrm{Lk}{\uparrow}_{X^{(S)}}(v) = \begin{cases} L \text{ if } f^{(S)}(v) \in S \\ \widetilde{L} \text{ if } f^{(S)}(v) \notin S \end{cases}$$

The group $G_L(S)$ is by definition the group of deck transformations of the branched covering $c : X_L^{(S)} \to X_L/BB_L$.

Proof. In the case when L has nlcp, Proposition 15 shows that $\mathbb{S}(\widetilde{L})$ is the universal cover of $\mathbb{S}(L)$; in this case $X_L^{(S)}$ and b can be constructed by applying Theorem 21 to the locally CAT(0) cube complex X_L/BB_L, with V equal to the vertices of X_L/BB_L whose height lies in $\mathbb{Z} - S$. Since X_L/BB_L has exactly one vertex of each height we note that $G_L(S)$ acts transitively on the vertices of $X_L^{(S)}$ of each height.

In the general case, we realize $X_L^{(S)}$ as a subcomplex of $X_M^{(S)}$, where $M = M(L)$ is as described in Lemma 9. We have X_L/A_L embedded as a subcomplex of X_M/A_M. Define Y to be one of the connected components of the inverse image of $X_L/A_L \subseteq X_M/A_M$ under the composite map $X_M^{(S)} \to X_M \to X_M/A_M$. Since the group $G_M(S)$ acts transitively on the vertices of $X_M^{(S)}$ of a given height and Y contains vertices of all heights, any other connected component will be isomorphic to Y. Since L is a full subcomplex of M, $\mathbb{S}(L)$ is a full subcomplex of $\mathbb{S}(M)$, and the inverse image of $\mathbb{S}(L) \subseteq \mathbb{S}(M)$ under the universal covering map $\mathbb{S}(\widetilde{M}) \to \mathbb{S}(M)$ is $\mathbb{S}(\widetilde{L})$. It follows that vertices of Y of height in S have link $\mathbb{S}(L)$ and vertices of Y of height not in S have link $\mathbb{S}(\widetilde{L})$ as required.

Since each vertex link in Y is a full subcomplex of the vertex link in $X_M^{(S)}$, it follows that Y is a convex subspace of $X_M^{(S)}$, and so Y is simply-connected and hence is itself CAT(0). By construction, we have branched covering maps $b : Y \to X_L$ and $c : Y \to X_L/BB_L \subseteq X_M/BB_M$. It remains to show that the second of these coverings is regular. For this, it suffices to show that whenever $g \in G_M(S)$ is such that $gY \cap Y \neq \emptyset$, then $gY = Y$. This is because the action of g preserves the 'type' of cubes. Suppose that y, z are vertices of Y and that $y \in gY \cap Y$, and let $e_1, \ldots, e_l$ be an edge path in Y from y to z. Equivalently, $e_1, \ldots, e_l$ is an edge path in $X_M^{(S)}$ such that each e_i has image contained in the subspace $X_L/A_L \subseteq X_M/A_M$. The edge path $g^{-1}e_1, \ldots, g^{-1}e_n$ starts at $y' \in Y$ and has image contained in the subspace X_L/A_L of X_M/A_M, and hence its end point z' must lie in Y. From this we see that the edge path $e_1, \ldots, e_l$ is contained in $Y \cap gY$, so that $z \in gY$ and hence that $Y = gY$. This establishes that the group

$$G_L(S) = \{g \in G_M(S) : gY \cap Y \neq \emptyset\} = \{g \in G_M(S) : gY = Y\}$$

acts regularly on the branched covering $c : Y \to X_L/BB_L$.

It remains to show that this construction does not depend on the choice of M. Suppose that $L \subseteq M_1$ and $L \subseteq M_2$ are two choices of embedding as in Lemma 9, yielding Y_1 and Y_2 with groups G_1 and G_2 of deck transformations. If we define $M_3 := M_1 \cup_L M_2$, then $L \subseteq M_3$ is also an embedding with the properties of Lemma 9, and we may construct $Y_3 \subseteq X_{M_3}^{(S)}$ with an action of $G_3 \leq G_{M_3}(S)$. The naturality properties lead to maps $Y_1 \to Y_3$ and $Y_2 \to Y_3$ which are local isomorphisms and hence (since all three are simply-connected) isomorphisms. This in turn yields group isomorphisms $G_1 \to G_3$ and $G_2 \to G_3$ with respect to which the given isomorphisms $Y_i \to Y_3$ are equivariant. $\square$

Corollary 23. *The cellular action of the group $G_L(S)$ on $X_L^{(S)}$ is free, except that each vertex whose height is not in S is stabilized by a subgroup isomorphic to $\pi_1(L)$.*

Proof. Immediate from the construction of $X_L^{(S)}$. $\square$

Fix some connected L, and for $t \in \mathbb{R}$, let $X_t = X_{L,t}$ denote the level set $f_L^{-1}(t) \subseteq X_L$.

Corollary 24. *If $t \notin \mathbb{Z}$, or $t \in S$, then then $c : X_t^{(S)} \to X_t/BB_L$ is a regular cover without branching, and induces an isomorphism $X_t^{(S)}/G_L(S) \cong X_t/BB_L$.*

Proof. No branch point of $c : X^{(S)} \to X/BB_L$ has height equal to t. $\square$

10 Connectivity of level sets in $X_L^{(S)}$

As at the end of the previous section, we fix a choice of connected L, and for $t \in \mathbb{R}$ and $S \subseteq \mathbb{Z}$ we write $X_t^{(S)}$ for the level set ${f^{(S)}}^{-1}(t) \subseteq X_L^{(S)}$. Furthermore, we write $X_{[a,b]}^{(S)}$ for ${f^{(S)}}^{-1}([a,b]) \subseteq X_L^{(S)}$.

Proposition 25. *1. Each $X_t^{(S)}$ is connected.*

2. If both L and $\widetilde{L}$ are R-acyclic, then $X_t^{(S)}$ is R-acyclic.

3. If L and $\widetilde{L}$ are both m-R-acyclic then so is $X_t^{(S)}$.

Proof. (1) Let $[a,b]$ be any interval in $\mathbb{R}$ containing t. The ascending and descending links for each vertex of $X^{(S)}$ are connected, from which it follows that each inclusion $X_t^{(S)} \to X_{[a,b]}^{(S)}$ induces a bijection on the set of path components. Since $X^{(S)}$ is contractible, it follows that each level set must be connected, as claimed.

(2) and (3) Under the hypotheses the ascending and descending links for each vertex in $X^{(S)}$ are either R-acyclic, or have reduced R-homology that vanishes in degree at most m. Theorem 6 implies that for each interval $[a,b]$ with $a \leq t \leq b$, the relative homology groups $H_i(X_{[a,b]}^{(S)}, X_t^{(S)}; R)$ are zero either for all i or for $i \leq m+1$. It follows that the map $H_i(X_t^{(S)}; R) \to H_i(X^{(S)}; R)$ is an isomorphism, either for all i or for $i \leq m$. Since $X^{(S)}$ is contractible the result follows. $\square$

This already allows us to establish part of the main theorem.

Corollary 26. *If L and $\widetilde{L}$ are R-acyclic (resp. $(n-1)$-R-acyclic) then for each $S \subseteq \mathbb{Z}$, $G_L(S)$ is $FH(R)$ (resp. $FH_n(R)$).*

Proof. $G_L(S)$ acts freely cocompactly on the level set $X_t^{(S)}$, which is R-acyclic (resp. $(n-1)$-R-acyclic) by the proposition. $\square$

Proposition 27. *For any t, $X_t^{(\emptyset)}$ is simply-connected. For $t = 0$, $X_0^{(\{0\})}$ is simply-connected.*

Proof. In each of these cases, every vertex not of height t has ascending and descending links isomorphic to $\widetilde{L}$, which is simply-connected. From this it follows that the inclusion of X_t into $X_{[a,b]}$ induces an isomorphism of fundamental groups for each interval $[a, b]$ containing t. Since X is simply-connected, the claim follows. $\square$

Corollary 28. *For any $t \notin \mathbb{Z}$, $G_L(\emptyset)$ is isomorphic to $\pi_1(X_t/BB_L)$. For $t = 0$, $G_L(\{0\})$ is isomorphic to $\pi_1(X_0/BB_L)$.*

Proof. Define S by $S = \emptyset$ for $t \notin \mathbb{Z}$ or $S = \{0\}$ for $t = 0$. In each case, $X_t^{(S)}$ is the universal covering of X_t/BB_L, with $G_L(S)$ as its group of deck transformations. $\square$

Corollary 29. *$G_L(\emptyset) \cong BB_{\widetilde{L}} \rtimes \pi_1(L)$.*

Proof. It suffices to identify $BB_{\widetilde{L}} \rtimes \pi_1(L)$ with the fundamental group of X_t/BB_L for $t \notin \mathbb{Z}$. We start by constructing a branched covering of X_L by $X_{\widetilde{L}}$. The action of $\pi_1(L)$ on $\widetilde{L}$ by deck transformations induces an action on $\mathbb{T}_{\widetilde{L}}$ which fixes the unique vertex and is free at other points. Furthermore, there is a natural identification $\mathbb{T}_{\widetilde{L}}/\pi_1(L) = \mathbb{T}_L$. This exhibits a branched regular covering map $\mathbb{T}_{\widetilde{L}} \to \mathbb{T}_L$, with branching only at the vertex and with $\pi_1(L)$ as its group of deck transformations. The induced map of universal covering spaces is a branched covering $X_{\widetilde{L}} \to X_L$, which is regular and equivariant for the actions of the groups $A_{\widetilde{L}} \rtimes \pi_1(L) = \pi_1(\mathbb{T}_{\widetilde{L}}) \rtimes \pi_1(L)$ and A_L respectively. The composite of the branched covering map and the Morse function $f : X_L \to \mathbb{R}$ induces a Morse function on $\widetilde{f} : X_{\widetilde{L}} \to \mathbb{R}$, in which each ascending and descending link is isomorphic to $\widetilde{L}$. Furthermore, the subgroup of $A_{\widetilde{L}} \rtimes \pi_1(L)$ that preserves level sets of this Morse function is $BB_{\widetilde{L}} \rtimes \pi_1(L)$.

Since all ascending and descending links in $X_{\widetilde{L}}$ are 1-connected, we see that for any t, the level set $X_{\widetilde{L},t} = \widetilde{f}^{-1}(t)$ is simply-connected. For $t \notin \mathbb{Z}$, we see that $X_{\widetilde{L},t} \to X_t/BB_L$ is a universal covering map, with $BB_{\widetilde{L}} \rtimes \pi_1(L)$ as its group of deck transformations. Hence we get isomorphisms $BB_{\widetilde{L}} \rtimes \pi_1(L) \cong \pi_1(X_t/BB_L) \cong G_L(\emptyset)$ by comparing the two descriptions of $\pi_1(X_t/BB_L)$. $\square$

Corollary 30. *For each $S \subseteq \mathbb{Z}$, $G_L(S)$ is finitely generated.*

Proof. $G_L(\emptyset)$ is the fundamental group of the compact polyhedral complex X_t/BB_L for any $t \notin \mathbb{Z}$, and so is finitely generated. Each $G_L(S)$ can be viewed as the group of deck transformations of a regular covering of X_t/BB_L, and hence as a quotient of $G_L(\emptyset)$. $\square$

11 $G_L(S)$ as a fundamental group

In this section we realize $G_L(S)$ as either a fundamental group (if L has nlcp) or as the image of a map of fundamental groups (in general). This clarifies G_L as a functor of S and allows us to exhibit isomorphisms between $G_L(S)$ and $G_L(T)$ in some cases.

Recall that X_L/BB_L is equipped with a height function $l : X_L/BB_L \to \mathbb{R}$. The height function l establishes a bijection between the vertex set of X_L/BB_L and the integers. For $S \subseteq \mathbb{Z}$ we define $V(S) = V_L(S)$ to be the set of vertices whose height is *not* in S:

$$V(S) = \{v \text{ a vertex of } X_L/BB_L\colon l(v) \notin S\}.$$

Proposition 31. *If L has nlcp, then $G_L(S) \cong \pi_1(X_L/BB_L - V(S))$.*

Proof. Let $W(S)$ denote the set of vertices of $X_L^{(S)}$ that have height not in S, so that $W(S)$ consists of the vertices of $X_L^{(S)}$ with link $\mathbb{S}(\widetilde{L})$. Note also that $W(S)$ is equal to the set of points of $X_L^{(S)}$ with non-trivial stabilizer in $G_L(S)$. Since L has nlcp, $\mathbb{S}(\widetilde{L})$ is 1-connected, from which it follows that $X_L^{(S)} - W(S)$ is simply-connected, and so $G_L(S)$ can be identified with the group of deck transformations of the universal covering of $X_L^{(S)} - W(S)$. □

To describe $G_L(S)$ in terms of fundamental groups when L does not have nlcp, we embed L as a full subcomplex of $M = M(L)$ as in the statement of Lemma 9.

Proposition 32. *In general, $G_L(S)$ is isomorphic to the image of the map*

$$\pi_1(X_L/BB_L - V_L(S)) \to \pi_1(X_M/BB_M - V_M(S)).$$

Proof. Recall that in the general case, we defined $X_L^{(S)}$ as a 1-connected subspace of $X_M^{(S)}$ for $M = M(L)$ as in Lemma 9, and we defined $G_L(S)$ as the subgroup of $G_M(S)$ that preserves this subspace. As before, $G_L(S)$ acts freely on $X_L^{(S)} - W_L(S)$, and although this space need not be simply-connected, it admits a $G_L(S)$-equivariant map to the simply-connected space $X_M^{(S)} - W_M(S)$. Since $G_L(S) \leq G_M(S)$, this gives an identification of $G_L(S)$ with the claimed image. □

Remark 33. *For each $S \subseteq T \subseteq \mathbb{Z}$, there are inclusions of spaces $X_L/BB_L - V(S) \subseteq X_L/BB_L - V(T)$ and $X_M/BB_M - V(S) \subseteq X_M/BB_M - V(T)$. The induced maps of fundamental groups give a way to view $S \mapsto G_L(S)$ as a functor from subsets of $\mathbb{Z}$ with inclusion to groups and surjective homomorphisms.*

An affine isometry of $\mathbb{Z}$ is by definition a map of the form $\phi(n) = m+n$ or $\phi(n) = m-n$ for some $m \in \mathbb{Z}$.

Corollary 34. *If $\phi(S) = T$ for some affine isometry ϕ, then $G_L(S) \cong G_L(T)$.*

Proof. Extend ϕ to an affine isometry of $\mathbb{R}$ given by the same formula, and as usual let $l : X_L/BB_L \to \mathbb{R}$ be the height function on X_L/BB_L. Define a new height function on X_L/BB_L as $l' = \phi \circ l$, and note that $l(v) \in S$ if and only if $l'(v) \in T$. □

Provided that L is not simply-connected, we know of no cases when $G_L(S)$ is isomorphic to $G_L(T)$ except those that arise as from this corollary.

Corollary 35. *For any $S \subseteq T \subseteq \mathbb{Z}$, $X_L^{(S)}$ is a regular branched cover of $X_L^{(T)}$, branched only at vertices of height in $T - S$.*

Proof. The viewpoint of Remark 33 gives a map $X_L^{(S)} - W(S) \to X_L^{(T)} - W(T)$ which is equivariant for the map $G_L(S) \twoheadrightarrow G_L(T)$. (Here as before, we let $W(S)$ denote the vertices of $X_L^{(S)}$ that have height not in S.) If K denotes the kernel of the map $G_L(S) \to G_L(T)$, then K acts freely on $X_L^{(S)} - W(S)$ and one obtains an isomorphism $(X_L^{(S)} - W(S))/K \to X_L^{(T)} - W(T)$. Completing this map gives an isomorphism $X_L^{(S)}/K \cong X_L^{(T)}$. The vertices of $X_L^{(S)}$ that are fixed by some non-trivial element of K are precisely the vertices of height in $T - S$. □

12 Sheets in $X_L^{(S)}$

An n-flat in a CAT(0) complex is a subcomplex isometric to $\mathbb{R}^n$. In [2] an n-sheet in X_L is defined to be an n-flat whose image in $\mathbb{T}_L$ is a single subtorus $\mathbb{T}_\sigma$ for some $(n-1)$-simplex σ of L. Since we do not have a cocompact group action on every $X_L^{(S)}$ this definition does not apply directly.

An n-sheet in $X_L^{(\emptyset)} = X_{\widetilde{L}}$ can be defined to be an n-flat whose image in $\mathbb{T}_{\widetilde{L}}$ is a subtorus A_σ for some $(n-1)$-simplex of $\widetilde{L}$. Now an n-sheet in $X_L^{(S)}$ can be defined to be the image of any n-sheet of $X_L^{(\emptyset)}$ under the branched covering map $X_L^{(\emptyset)} \to X_L^{(S)}$. We prefer the following equivalent definition: an n-sheet in $X^{(S)}$ is an n-flat F with the property that for every vertex v of $X^{(S)}$, the subcomplex $\mathrm{Lk}_F(v) \subseteq \mathrm{Lk}_{X^{(S)}}(v)$ is closed under taking opposite pairs. Here we use Proposition 18 to define opposite pairs in $\mathrm{Lk}_{X^{(S)}}(v)$.

Under the branched covering map $X_L^{(S)} \to \mathbb{T}_L$, each n-sheet will map to an n-subtorus $\mathbb{T}_\sigma$, however for $S \neq \mathbb{Z}$ there are many n-flats with this property that are not n-sheets.

For v a vertex of $X_L^{(S)}$, the link $\mathrm{Lk}_X(v)$ is a copy of $\mathbb{S}(L)$ if $f(v) \in S$ or $\mathbb{S}(\widetilde{L})$ if $f(v) \notin S$. For each vertex v with $f(v) \in S$ (resp. with $f(v) \notin S$) and each simplex σ of L (resp. of $\widetilde{L}$) there is a unique sheet $C(v,\sigma)$ containing v and such that $\mathrm{Lk}_C(v) = \mathbb{S}(\sigma) \subseteq \mathrm{Lk}_X(v)$. Let Λ denote $\mathbb{R}^n$ with its standard tesselation by unit cubes, and let $\phi : \Lambda \to C(v,\sigma)$ be an isomorphism with the following properties: $\phi((0,0,\ldots,0)) = v$, and $f(\phi(t_1,\ldots,t_n)) = f(v) + \sum_{i=1}^n t_i$. Now define the upward and downward parts of the sheet by

$$C_\uparrow(v,\sigma) = \phi((\mathbb{R}_{\geq 0})^n), \qquad C_\downarrow(v,\sigma) = \phi((\mathbb{R}_{\leq 0})^n).$$

Thus $C_\uparrow(v,\sigma)$ is the points x of $C(v,\sigma)$ for which the geodesic from v to x leaves v in a point of $\mathrm{Lk}_\uparrow(v) \cap \mathbb{S}(\sigma)$ and similarly for the downward part.

If $f(v) \in S$ (resp. $f(v) \notin S$), and K is any subcomplex of L (resp. $\widetilde{L}$), define

$$C(v,K) = \bigcup_{\sigma \in K} C(v,\sigma). \qquad C_\uparrow(v,K) = \bigcup_{\sigma \in K} C_\uparrow(v,\sigma), \qquad C_\downarrow(v,K) = \bigcup_{\sigma \in K} C_\downarrow(v,\sigma).$$

For $f(v) > t$, the shadow $S_{v,K}$ is $X_t \cap C_\downarrow(v,K)$, and for $f(v) < t$ the shadow $S_{v,K}$ is $X_t \cap C_\uparrow(v,K)$.

Proposition 36. *For any $t < f(v)$, and any K, the retraction map $X - \{v\} \to \mathrm{Lk}_X(v)$ induces a homeomorphism from $S_{v,K}$ to $K \subseteq \mathrm{Lk}_X^\downarrow(v) \subseteq \mathrm{Lk}_X(v)$. Similarly, if $t > f(v)$*

then the same retraction map induces a homeomorphism from $S_{v,K}$ to $K \subseteq \mathrm{Lk}_X^{\uparrow}(v) \subseteq \mathrm{Lk}_X(v)$.

Proposition 37. *Suppose that $t \notin \mathbb{Z}$ and that either $f(v) \in S$ and $K = L$ or $f(v) \notin S$ and $K = \widetilde{L}$. Let $t \in [a, b]$, and let $X_{[a,b]}$ denote $f^{-1}([a, b])$.*

- *If $f(v) \in [a, b]$ then $S_{v,K}$ is null-homotopic in $X_{[a,b]}$;*
- *If $f(v) \notin [a, b]$ then $S_{v,K}$ is a retract of $X_{[a,b]}$.*

Proof. We consider only the case $f(v) > t$, the contrary case being similar. In this case the retraction map $r_v : X - \{v\} \to \mathrm{Lk}_X(v)$ restricts to $S_{v,K}$ as a homeomorphism $h : S_{v,K} \to \mathrm{Lk}_X^{\downarrow}(v)$. Let $q : \mathrm{Lk}_X(v) \to \mathrm{Lk}_X^{\downarrow}(v)$ be the simplicial map sending each vertex to the downward member of its opposite pair. If $f(v) \notin [a, b]$ then the composite

$$h^{-1} \circ q \circ r_v : X_{[a,b]} \to S_{v,K} \subseteq X_{[a,b]}$$

is the required retraction. If on the other hand $f(v) \in [a, b]$, then by moving points of $S_{v,K}$ at constant speed along the geodesic joining them to v we get a homotopy from the identity map of $S_{v,K}$ to the constant map with image $\{v\}$. □

13 Brown's criterion applied to $X_L^{(S)}$

To be able to apply Brown's criterion to decide whether $G_L(S)$ is type $FP(R)$ or type $FP_n(R)$, we will need to know something about the finiteness properties of the cell stabilizers in $X_L^{(S)}$. By Corollary 23, the only non-trivial cell stabilizers are the stabilizers of the vertices whose height is not in S. The following proposition shows that each of the conditions stated to be equivalent in Theorem B implies the required conditions on $\pi_1(L)$.

Proposition 38. *If $\widetilde{L}$ is R-acyclic, or if $G_L(\emptyset)$ is type $FP(R)$, then $\pi_1(L)$ is type $FP(R)$.*
If $\widetilde{L}$ is $(n-1)$-R-acyclic or if $G_L(\emptyset)$ is type $FP_n(R)$, then $\pi_1(L)$ is type $FP_n(R)$.

Proof. $\pi_1(L)$ acts freely on $\widetilde{L}$ by deck transformations. Hence if $\widetilde{L}$ is R-acyclic (resp. $(n-1)$-R-acyclic) then $\pi_1(L)$ is by definition $FH(R)$ (resp. $FH_n(R)$), and hence $\pi_1(L)$ is $FP(R)$ (resp. $FP_n(R)$).

Since $G_L(\emptyset) \cong BB_{\widetilde{L}} \rtimes \pi_1(L)$, we see that $\pi_1(L)$ is a retract of $G_L(\emptyset)$. By Proposition 5 it follows that $\pi_1(L)$ is $FP(R)$ (resp. $FP_n(R)$) whenever $G_L(\emptyset)$ is. □

Theorem 39. *Suppose that $\pi_1(L)$ is type $FP(R)$ (resp. type $FP_n(R)$). Then $G_L(S)$ is type $FP(R)$ (resp. type $FP_n(R)$) if and only if the set of heights of vertices whose ascending links are not R-acyclic (resp. are not $(n-1)$-R-acyclic) is finite.*

Proof. By Proposition 38, the action of $G = G_L(S)$ on $X = X_L^{(S)}$ satisfies the hypotheses of Theorem 4, our statement of Brown's criterion. Now consider the filtration of X by the G-subcomplexes $X(m) = X_{[-m-1/2,m+1/2]} = f^{-1}([-m + 1/2, m + 1/2])$ for $m \in \mathbb{N}$. G acts cocompactly on each $X(m)$, and so G is $FP(R)$ (resp. $FP_n(R)$) if and only if for each i (resp. for each $i < n$) the system $\overline{H}_i(X(m); R)$ is essentially trivial.

If the set of vertices whose ascending links are not R-acyclic (resp. are not $(n-1)$-R-acyclic) is finite, pick $m_0 \in \mathbb{N}$ so that $|f(v)| > m_0$ implies that $\mathrm{Lk}^{\uparrow}(v) \cong \mathrm{Lk}^{\downarrow}(v)$ is R-acyclic (resp. $(n-1)$-R-acyclic). Then for all $m' > m > m_0$ and for all i (resp. for all $i < n$) $H_i(X(m'), X(m); R) = 0$. Hence in this case G is $FP(R)$ (resp. $FP_n(R)$) by Brown's criterion.

Conversely, suppose that for each $m \in \mathbb{N}$ there exists $m' > m+1$ and v with $|f(v)| = m'$ such that $K = \mathrm{Lk}_X^{\uparrow}(v)$ is not $(n-1)$-R-acyclic. By Proposition 36, $S_{v,K}$ is a retract of $X(m)$ but is null-homotopic in $X(m')$. Hence there exists i (resp. $i < n$) so that $\overline{H}_i(X(m); R) \to \overline{H}_i(X(m'); R)$ has non-trivial kernel, and so by Brown's criterion G cannot be $FP(R)$ (resp. $FP_n(R)$). □

Corollary 40. *Suppose that $\pi_1(L)$ is type $FP(R)$ (resp. type $FP_n(R)$).*

- *If S is finite then $G_L(S)$ is $FP(R)$ (resp. $FP_n(R)$) if and only if $\widetilde{L}$ is R-acyclic (resp. $(n-1)$-R-acyclic).*
- *If $\mathbb{Z} - S$ is finite then $G_L(S)$ is $FP(R)$ (resp. $FP_n(R)$) if and only if L is R-acyclic (resp. $(n-1)$-R-acyclic).*
- *If S and $\mathbb{Z} - S$ are both infinite, then $G_L(S)$ is $FP(R)$ (resp. $FP_n(R)$) if and only if both L and $\widetilde{L}$ are R-acyclic (resp. $(n-1)$-R-acyclic).*

This corollary together with Corollary 26 completes the proof of Theorem B. There are some cases with either S or $\mathbb{Z} - S$ finite in which the above corollary shows that $G_L(S)$ is $FP(R)$ and yet Corollary 26 does not apply. For the cases when S is finite we will show that $G_L(S)$ is in fact $FH(R)$ in Proposition 51.

14 Presentations for $G_L(S)$

In this section we study presentations for the groups $G_L(S)$. Since we want to realize the surjective group homomorphism $G_L(S) \to G_L(T)$ by a map of presentations, it is convenient to fix a finite generating set, and for this reason we focus on the case when $0 \in S$.

Theorem 41. *For any S with $0 \in S$ and any collection Γ of directed loops in L whose normal closure generates $\pi_1(L)$, the presentation $P_L(S, \Gamma)$ is a presentation of the group $G_L(S)$. If $S \subseteq T$ then the pair $P_L(S, \Gamma) \subseteq P_L(T, \Gamma)$ realizes the surjection $G_L(S) \to G_L(T)$.*

Remark 42. *The case $S = \{0\}$ of this statement first appeared in [16] and was stated in [13]. A proof of the case $S = \mathbb{Z}$ without using the cube complex $\mathbb{T}_L$ was given in [13].*

Proof. We start with the case $S = \{0\}$. By Corollary 28, $G_L(\{0\})$ may be viewed as the fundamental group of X_0/BB_L. The presentation $P_L(\{0\}, \Gamma)$, which does not depend on Γ, arises from the 2-skeleton of the natural polyhedral cell complex structure on X_0/BB_L. Cells of the polyhedral structure on X_0 are intersections $C \cap X_0$, where C is a cube of X. The vertices of X_0 form a single G-orbit. Edges meet X_0 only at their end points, and for each $n \geq 2$, each A_L-orbit of n-cubes of X contributes $(n-1)$ distinct G-orbits of

$(n-1)$-cells of X_0, corresponding to the distinct integer heights at which members of the orbit can appear. Each A-orbit of squares contributes one G-orbit of edges, each A-orbit of 3-cubes contributes two G-orbits of triangles, and each A-orbit of 4-cubes contributes two G-orbits of tetrahedra plus one G-orbit of octahedra. The 1-skeleton of X_0/BB_L is a rose consisting of one vertex and edges in bijective correspondence with the edges of L, or equivalently the squares of $\mathbb{T}_L$. If x and y are commuting Artin generators, then the two directions along the corresponding edge represent xy^{-1} and yx^{-1}. The 2-skeleton is formed by attaching two triangles for each triangle of L; if (a, b, c) is the edge loop around a directed triangle of L then the attaching maps are abc and $a^{-1}b^{-1}c^{-1}$, giving the claimed presentation. □

Before proving the general case, we record a useful lemma. For each S with $0 \in S$, the edges of $X_0^{(S)}/G_L(S)$ are naturally bijective with the edges of L. Hence given any initial vertex w in $X_0^{(S)}$, any word in the directed edges of L describes a unique directed edge path starting at w.

Lemma 43. *Let $(a_1, \ldots, a_l)$ be any directed edge path in L. For any S with $0 \in S$, for $n \in \mathbb{Z} - \{0\}$ and any initial vertex in $X_0^{(S)}$, the directed edge path defined by the word $a_1^n a_2^n \cdots a_l^n$ is contained in the shadow of a vertex v of height n, and is equal to the image of the path $(a_1, \ldots, a_l)$ under the isomorphism $L \to S_{v,L}$.*

Proof. First consider the case when $l = 1$, and suppose that $n > 0$. If a is the directed edge in L from vertex x to vertex y, then for any vertex $\textsc{w} \in X_0^{(S)} \subseteq X = X_L^{(S)}$ there is an $n \times n$ square with vertices $\textsc{w}, \textsc{n}, \textsc{e}, \textsc{s}$ of heights $0, n, 0, -n$ respectively such that the height 0 diagonal from $\textsc{w}$ to $\textsc{e}$ represents the word a^n, while the boundary paths from $\textsc{w}$ to $\textsc{n}$ and from $\textsc{s}$ to $\textsc{e}$ represent the word x^n and the boundary paths from $\textsc{e}$ to $\textsc{n}$ and from $\textsc{s}$ to $\textsc{w}$ represent the word y^n. The square is of course contained in the 2-sheet through $\textsc{w}$ defined by the edge a, and is uniquely determined by a, n and any one of its vertices. The edge paths a^n and a^{-n} are each the image of a single edge under the shadow map from either $\textsc{n}$ or $\textsc{s}$. This proves the claim whenever $l = 1$.

For the general case, fix $n > 0$ and fix i with $1 \leq i < l$, let $a = a_i$ and $b = a_{i+1}$, where a is the directed edge from vertex x to vertex y and b is the directed edge from vertex y to vertex z. As before there are $n \times n$ squares with vertices $\textsc{w}, \textsc{n}, \textsc{e}, \textsc{s}$ and $\textsc{w}', \textsc{n}', \textsc{e}', \textsc{s}'$ such that the respective height 0 diagonals represent the words a^n and b^n. If $\textsc{e} = \textsc{w}'$, so that the concatenated path from $\textsc{w}$ to $\textsc{e}'$ represents $a^n b^n$, then we see that $\textsc{n} = \textsc{n}'$, since both are vertices of X that can be reached from $\textsc{e} = \textsc{w}'$ by moving along the path y^n. Similarly, if $\textsc{e}' = \textsc{w}$ so that the concatenated path from $\textsc{e}$ to $\textsc{w}'$ represents $a^{-n}b^{-n}$, then $\textsc{s} = \textsc{s}'$. Thus the word $a_1^n \cdots a_l^n$ is the image of $(a_1, \ldots, a_l)$ under the shadow map from $\textsc{n}$ and $a_1^{-n} \cdots a_l^{-n}$ is the image of the same path under the shadow map from $\textsc{s}$. Since $\textsc{n}$ has height n and $\textsc{s}$ has height $-n$, the claim follows. □

Proof. (Theorem 41, general case) For general S with $0 \in S$, we use Morse theory to compare $G_L(\{0\}) = \pi_1(X_0^{(S)}/G_L(S))$ with $G_L(S) = \pi_1(X^{(S)}/G_L(S))$. Since $X_0^{(S)}$ is a regular covering space of X_0/BB_L (Corollary 24), we may view words in the directed edges of L as defining edge paths in $X_0^{(S)}$, and a word represents the identity in $G_L(S)$ if and only if the corresponding path in $X_0^{(S)}$ is a closed loop. We build $X^{(S)}$ as the direct limit of the $G_L(S)$-subspaces $X(m) = X^{(S)}_{[-m-1/2, m+1/2]}$ for $m \in \mathbb{N}$. Up to homotopy,

$X(m+1)$ is obtained from $X(m)$ by attaching cones to the shadows of vertices of height $m+1$ in $X_m \subseteq X(m)$ and attaching cones to the shadows of vertices of height $-(m+1)$ in $X_{-m} \subseteq X(m)$. Up to homotopy, this is the same as attaching cones to the shadows of these vertices in X_0. If vertices of height m (resp. height $-m$) have link $\mathbb{S}(\widetilde{L})$, then since $\widetilde{L}$ is simply connected, the fundamental group is left unchanged. If the vertices of a given height have link $\mathbb{S}(L)$, then attaching a cone to the shadow in X_0 of the vertices of that height has the same effect on the fundamental group as attaching discs to the images under the shadow map of all edge loops in Γ. Hence we see that the fundamental group of $X(m)/G_L(S)$ has presentation $P_L(S \cap [-m, m], \Gamma)$, and hence that $P_L(S, \Gamma)$ is a presentation for $G_L(S) = \pi_1(X/G_L(S))$. □

Lemma 44. *Suppose that $0 \in S$ and that $\gamma = a_1, \ldots, a_l$ is a homotopically non-trivial edge loop in L. The relation $a_1^n a_2^n \cdots a_l^n = 1$ holds in $G_L(S)$ if and only if $n \in S$.*

Proof. Since $0 \in S$ the case when $n = 0$ is trivial. If $n \neq 0$, the relation holds if and only if any edge path in $X_0^{(S)}$ described by the word $a_1^n a_2^n \cdots a_l^n$ is a closed loop. By Lemma 43, any such edge path is the image, under the shadow map from a vertex v of height n, of the edge path $(a_1, \ldots, a_l)$ in either the ascending or descending link of v (depending whether n is less than or greater than 0). If $f(v) \in S$ then this edge path is a closed loop in a complex isomorphic to L. If on the other hand $f(v) \notin S$ then this edge path lies in a complex isomorphic to $\widetilde{L}$ and is not closed. The claim follows since the shadow maps are homeomorphisms. □

Corollary 45. *$G_L(S)$ is finitely presentable if and only if S is finite.*

Proof. If S is finite, then for any finite set of edge loops Γ that normally generates $\pi_1(L)$, $P_L(S, \Gamma)$ is a finite presentation of $G_L(S)$. For the converse, fix a choice of Γ, and consider the presentation $P_L(S, \Gamma)$. If $G_L(S)$ has a finite presentation then these finitely many relators are consequences of finitely many of the relators in the presentation $P_L(S, \Gamma)$. It follows that when $G_L(S)$ is finitely presentable, then there must be a finite subset $S' \subseteq S$ so that $P_L(S', \Gamma)$ also presents $G_L(S)$. By Lemma 44 this cannot happen unless $S' = S$. □

If T is non-empty but does not contain 0, then $G_L(T)$ is isomorphic to $G_L(S)$ where S is a translate of T that contains zero. Thus we obtain a presentation for $G_L(T)$ whenever T is non-empty, albeit with a non-canonical choice of generating set. We finish this section by giving a finite presentation for $G_L(\emptyset)$.

Since $\widetilde{L}$ is simply-connected, the group $G_{\widetilde{L}}(S)$ is isomorphic to $BB_{\widetilde{L}}$ for each $S \subseteq \mathbb{Z}$. Hence Theorem 41 gives a presentation for $BB_{\widetilde{L}}$: the generators are the directed edges of $\widetilde{L}$, subject only to the edge relations and the pairs of triangle relations coming from the triangles of $\widetilde{L}$. Of course, this presentation $P_{\widetilde{L}}(\{0\}, \emptyset)$ is infinite whenever $\pi_1(L)$ is, but the generators and relators are both permuted freely by $\pi_1(L)$ and lie in finitely many orbits. From this we obtain a finite presentation of the group $G_L(\emptyset) = BB_{\widetilde{L}} \rtimes \pi_1(L)$. Choose a finite presentation for $\pi_1(L)$, and choose a finite fundamental domain K in $\widetilde{L}$ for the action of $\pi_1(L)$, i.e., a finite subcomplex K that contains at least one simplex from each $\pi_1(L)$-orbit. The generators in the presentation are the generators of the given presentation for $\pi_1(L)$, together with the directed edges of K, subject to four types of relation: the finite set of relators coming from the given presentation for $\pi_1(L)$; the edge relators for each of the finitely many edges in K; the pair of triangle relators for each of the finitely many triangles in K; conjugacy relations: whenever a is a directed edge of K and $g \in \pi_1(L) - \{1\}$ is an element such that the image of a under g is another edge b of K, the relation $gag^{-1} = b$.

The map $G_L(\emptyset) \to G_L(\{0\})$ is easily described in terms of this presentation: the elements of $\pi_1(L)$ are sent to the identity and the directed edges of $K \subseteq \widetilde{L}$ are sent to the corresponding directed edges of L.

15 A set-valued invariant

Definition 46. *For a group G and a finite sequence $\mathbf{g} = (g_1, \ldots, g_l)$ of elements of G, define a set $\mathcal{R}(G, \mathbf{g})$ with $\{0\} \subseteq \mathcal{R}(G, \mathbf{g}) \subseteq \mathbb{Z}$ by*

$$\mathcal{R}(G, \mathbf{g}) = \{n \in \mathbb{Z} \colon g_1^n g_2^n \cdots g_l^n = 1\}.$$

Proposition 47. *Let $\mathbf{g} = (g_1, \ldots, g_l)$ be a sequence of elements of a group G.*

1. *If $G \leq H$, then $\mathcal{R}(G, \mathbf{g}) = \mathcal{R}(H, \mathbf{g})$;*
2. *If G is finitely presented then the set $\mathcal{R}(G, \mathbf{g})$ is recursively enumerable;*
3. *For any fixed isomorphism type of countable group G, the invariant $\mathcal{R}(G, \mathbf{g})$ can take at most countably many distinct values.*

Proof. If $G \leq H$ then any equation between elements of G holds in G if and only if it holds in H.

Suppose we are given a finite presentation of the group G, consisting of a set $x_1, \ldots, x_d$ of generators, together with a set $r_1, \ldots, r_m$ of relators, given as words in $x_i^{\pm 1}$. Suppose also that we are given words $w_1, \ldots, w_l$ in $x_i^{\pm 1}$ such that $g_i = w_i$. We work in the free group freely generated by $x_1, \ldots, x_d$, where there is an easy algorithm to replace a word by a reduced word. Now for each fixed $N > 0$, construct the list $\mathcal{L}(N)$ consisting of the reductions of words obtained as a product of at most N words of the form $wr_i^{\pm 1}w^{-1}$, where w denotes a word in $x_1^{\pm 1}, \ldots, x_d^{\pm 1}$ of length at most N. For each n with $|n| \leq N$, check whether the reduction of the word $w_1^n \cdots w_l^n$ is in $\mathcal{L}$; whenever this happens, output n. Keep repeating this programme for larger and larger values of N.

If G is countable then G contains only countably many finite sequences $\mathbf{g}$ of elements, so the function $\mathbf{g} \mapsto \mathcal{R}(G, \mathbf{g})$ can take only countably many values. If $\phi : G \to H$ is an isomorphism and $\phi(\mathbf{g})$ denotes the sequence $(\phi(g_1), \ldots, \phi(g_l))$ of elements of H, then clearly $\mathcal{R}(H, \phi(\mathbf{g})) = \mathcal{R}(G, \mathbf{g})$. □

Lemma 48. *Suppose that $\gamma = (a_1, \ldots, a_l)$ is a homotopically non-trivial edge loop in L, and let $\mathbf{a}$ be the sequence $(a_1, \ldots, a_l)$ of elements of $G_L(S)$ for any S containing zero. In this case, $\mathcal{R}(G_L(S), \mathbf{a}) = S$.*

Proof. This is immediate from Lemma 44. □

Corollary 49. *If $\pi_1(L)$ is non-trivial then there are uncountably many (in fact $2^{\aleph_0}$) isomorphism types of group $G_L(S)$.*

Proof. Follows from part 3 of Proposition 47 and Lemma 48. $\square$

Corollary 50. *If $\pi_1(L)$ is non-trivial then $G_L(S)$ is a subgroup of a finitely presented group if and only if S is recursively enumerable.*

Proof. First note that $G_L(\emptyset)$ is finitely presented. Next note that $S \subseteq \mathbb{Z}$ is recursively enumerable if and only if $S+m$ is, and $G_L(S) \cong G_L(S+m)$. By these observations it suffices to consider the case when $0 \in S$. If $G_L(S)$ is a subgroup of a finitely presented group, then by Proposition 47, $S = \mathcal{R}(G_L(S), \mathbf{a})$ must be recursively enumerable. For the converse, if S is recursively enumerable and Γ is any finite set of edge loops that normally generate $\pi_1(L)$ then $P_L(S,\Gamma)$ is a recursive presentation for $G_L(S)$. Higman's embedding theorem implies that $G_L(S)$ embeds in a finitely presented group [15]. $\square$

This completes the proof of Theorem A.

16 A classifying space for $G_L(S)$

The group $G_L(S)$ does not act freely on $X_L^{(S)}$ except when $S = \mathbb{Z}$ or $\pi_1(L)$ is trivial. However, since only vertices have non-trivial stabilizers it is comparatively easy to construct a classifying space for $G_L(S)$. A suitable small neighbourhood in $X_L^{(S)}$ of a singular vertex is $\pi_1(L)$-equivariantly isomorphic to the mapping cylinder of the map $\mathbb{S}(\widetilde{L}) \to *$, where $*$ denotes a point with trivial $\pi_1(L)$-action. If at each such vertex we replace this mapping cylinder by the mapping cylinder of the map $\mathbb{S}(\widetilde{L}) \to E\pi_1(L)$ then we obtain a contractible complex with a free $G_L(S)$ action. Note that the map $\mathbb{S}(\widetilde{L}) \to E\pi_1(L)$ factors through $\widetilde{L}$. There are other variations on this construction that have useful corollaries.

Proposition 51. *If S is finite and $\widetilde{L}$ is R-acyclic (resp. $(n-1)$-R-acyclic) then $G_L(S)$ is $FH(R)$ (resp. $FH_n(R)$).*

Proof. To simplify the notation we concentrate on the case when $\widetilde{L}$ is R-acyclic. As in the proof of Theorem 39, let $X(m) = f^{-1}([-m-1/2, m+1/2]) \subseteq X_L^{(S)}$. Provided that m is sufficiently large that $S \subseteq [-m, m]$, the inclusion of $X(m)$ into $X_L^{(S)}$ is an R-homology isomorphism, and so $X(m)$ is R-acyclic, and $G = G_L(S)$ acts cocompactly but not freely. We modify $X(m)$ to obtain a new R-acyclic space Y with a free G-action as follows. Remove from $X(m)$ an open ball of radius $1/4$ around each vertex v with non-trivial stabilizer. The closure of each such ball is G_v-equivariantly isomorphic to the mapping cylinder $M(\mathbb{S}(\widetilde{L}) \to *)$, where G_v denotes the stabilizer of v. Replace the removed ball by a copy of the mapping cylinder $M(\mathbb{S}(\widetilde{L}) \to \widetilde{L})$ to obtain Y. Since $\widetilde{L}$ is R-acyclic this replacement does not change the R-homology, and hence Y is R-acyclic. By construction, G acts freely on Y, and the quotient Y/G is obtained from the finite complex $X(m)/G$ by replacing finitely many copies of $M(\mathbb{S}(L) \to *)$ by copies of $M(\mathbb{S}(L) \to L)$; in particular Y/G is finite and hence G is $FH(R)$.

In the case when $\widetilde{L}$ is only $(n-1)$-R-acyclic, the argument is similar, except that now we only know that the mapping cylinders $M(\mathbb{S}(\widetilde{L}) \to *)$ and $M(\mathbb{S}(\widetilde{L}) \to \widetilde{L})$ have the same R-homology in degrees less than or equal to $(n-1)$. Hence in this case the free G-complex Y is only $(n-1)$-R-acyclic, and we deduce only that G is $FH_n(R)$. $\square$

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If L is R-acyclic and $\mathbb{Z} - S$ is finite but $\widetilde{L}$ is not R-acyclic, then Corollary 40 implies that $G_L(S)$ is $FP(R)$. In this case we do not know whether $G_L(S)$ is $FH(R)$, even if we assume that $\pi_1(L)$ is $FH(R)$. Let E be an R-acyclic complex with a free cocompact action of $\pi_1(L)$. Given a $\pi_1(L)$-equivariant map $\mathbb{S}(\widetilde{L}) \to E$, the technique used above would show that G is $FH(R)$.

Theorem 52. *If L is R-acyclic then for any S, there is an isomorphism of ordinary group homology*

$$H_*(G_L(S); R) \cong H_*(G_L(\mathbb{Z}); R) \oplus \bigoplus_{\mathbb{Z}-S} \overline{H}_*(\pi_1(L); R).$$

This isomorphism is natural for inclusions $S \subseteq T$.

Remark 53. *The ordinary homology and cohomology of $BB_L \cong G_L(\mathbb{Z})$ was computed for all L in [20]. In the case when L is R-acyclic, $H_i(G_L(\mathbb{Z}); R)$ is a free R-module isomorphic to the $(i-1)$-cycles in the augmented chain complex for L.*

Proof. Let Y be the free $G = G_L(S)$-complex in which $X_L^{(S)}$ is desingularized by replacing a ball around each singular vertex with the mapping cylinder of the $\pi_1(L)$-equivariant map $\mathbb{S}(\widetilde{L}) \to \widetilde{L}$. There is a G-equivariant map $Y \to X$ given by collapsing the copies of $\widetilde{L}$ to the vertices that they replaced. Let E be the universal covering space of any model for the classifying space for $\pi_1(L)$, fix a classifying map $\widetilde{L} \to E$, and let Z be built from $X_L^{(S)}$ by replacing a ball around each singular vertex with the concatenation of the two mapping cylinders for the following $\pi_1(L)$-equivariant maps:

$$\mathbb{S}(\widetilde{L}) \to \widetilde{L} \to E.$$

By construction, Z is contractible, and G acts freely on Z so that Z/G is a classifying space for G.

Now $X_L^{(S)}/G \cong X_L/BB_L = X_L/G_L(\mathbb{Z})$, and Y/G is obtained from $X_L^{(S)}/G$ by replacing a ball around each vertex with height in $\mathbb{Z} - S$ by the mapping cylinder of $\mathbb{S}(L) \to L$. Since L is R-acyclic, this implies that the map $Y/G \to X/G$ (given by collapsing each copy of L down to the vertex it replaces) is an R-homology isomorphism. Now Z/G is obtained from Y/G by attaching a copy of the mapping cylinder of $L \to E/\pi_1(L)$ to the copy of L that replaced the vertex at each height in $\mathbb{Z} - S$. Since L is R-acyclic, the Mayer-Vietoris sequence expressing Z/G as the union of Y/G and $\coprod_{\mathbb{Z}-S} M(L \to E/\pi_1(L))$ gives the claimed isomorphism. □

Corollary 54. *In the case when L and $\widetilde{L}$ are both R-acyclic, for each $S \subseteq T$ the natural map $H_*(G_L(S); R) \to H_*(G_L(T); R)$ is an isomorphism.*

Proof. If $\widetilde{L}$ is R-acyclic, then $H_*(L; R)$ is isomorphic to the R-homology of the group $\pi_1(L)$. Hence when L is also R-acyclic, the group $\pi_1(L)$ is R-acyclic. □

Theorem 55. *If L and $\widetilde{L}$ are both R-acyclic, then for each $S \subseteq T$, the kernel of the map $G_L(S) \to G_L(T)$ is R-acyclic.*

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Proof. Let K be the kernel of the map $G_L(S) \to G_L(T)$. By Corollary 35, K acts on $X_L^{(S)}$ freely except that each vertex of height in $T - S$ has stabilizer isomorphic to $\pi_1(L)$. As above, a suitable small neighbourhood of each such vertex can be identified with the mapping cylinder of the map $\mathbb{S}(\widetilde{L}) \to *$. By replacing each such neighbourhood by a copy of the mapping cylinder for the map $\mathbb{S}(\widetilde{L}) \to \widetilde{L}$ we obtain a space X on which K acts freely. Since $\widetilde{L}$ is R-acyclic, X has the same R-homology as $X_L^{(S)}$ and so X is R-acyclic. Hence X can be used to compute group cohomology with coefficients in any RK-module. By Corollary 35, X/K is homeomorphic to the space Y constructed as follows. In $X_L^{(T)}$, each vertex of height in T has a neighbourhood that can be identified with the mapping cylinder of $\mathbb{S}(L) \to *$. The space Y is obtained by removing such a neighbourhood for each vertex of height in $T - S$, and replacing it with a copy of the mapping cylinder for $\mathbb{S}(L) \to L$. Since L is R-acyclic, the R-homology and R-cohomology of Y is isomorphic to that of the contractible space $X_L^{(T)}$, which proves the claim. $\square$

Corollary 56. *Let R be a principal ideal domain and suppose that L and $\widetilde{L}$ are both R-acyclic. For any $S \subseteq T$ and any $RG_L(T)$-module M, the natural map*

$$H^*(G_L(T); M) \to H^*(G_L(S); M)$$

is an isomorphism.

Proof. Let K be the kernel of the homomorphism $G_L(S) \twoheadrightarrow G_L(T)$. Since the $G_L(S)$-action on M is defined in terms of the quotient map, every element of K acts as the identity on M. Hence $H^0(K; M) = M$ and by the universal coefficient theorem, $H^i(K; M) = 0$ for $i > 0$. Now consider the Lyndon-Hochschild-Serre spectral sequence for the group extension $K \rightarrowtail G_L(S) \twoheadrightarrow G_L(T)$ with coefficients in M. In this spectral sequence $E_2^{i,j} = H^i(G_L(T); H^j(K; M))$ and the terms $E_\infty^{i,j}$ form a filtration of $H^{i+j}(G_L(S); M)$. Since $E_2^{i,j} = 0$ for $j \neq 0$, the spectral sequence collapses at the E_2-page. The claimed isomorphism follows. $\square$

17 Compactly-supported and ℓ^2-cohomology

In [12], Davis and Okun computed the cohomology with compact supports and the ℓ^2-cohomology of X_t for each finite flag complex L and any non-integer t. (Throughout this section we will write X instead of X_L to simplify the notation.) Their methods extend almost without change to a similar computation for $X_t^{(S)}$ for any $S \subseteq \mathbb{Z}$, L and $t \notin \mathbb{Z}$. In particular, this computes the ℓ^2-cohomology of the group $G_L(S)$ whenever both L and $\widetilde{L}$ are $\mathbb{Q}$-acyclic, and computes (a filtration of) $H^*(G_L(S); RG_L(S))$ whenever L and $\widetilde{L}$ are R-acyclic. We indicate briefly how to extend their methods to our situation and state the results obtained in this way.

Davis and Okun decompose X as a union of BB_L-orbits of sheets, and rely on the fact that the stabilizer of each sheet is a free abelian group. Following their approach, we fix a non-integer t, and define a *level sheet* or *t-level sheet* to be an intersection $C_t = C \cap X_t^{(S)}$, for some sheet C of $X^{(S)}$. If C is an n-dimensional sheet in $X^{(S)}$, then the image of C in $X^{(S)}/G_L(S) \cong X/BB_L$ is isomorphic to $\mathbb{R}^n/\mathbb{Z}^{n-1}$, and the image of C_t in X_t/BB_L is isomorphic to the $(n-1)$-torus $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. It follows that the stabilizer of the level

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sheet C_t is isomorphic to $\mathbb{Z}^{n-1}$. Although sheets in the same $G_L(S)$-orbit can intersect, they intersect in at most a single vertex, so the t-level sheets in a $G_L(S)$-orbit do not intersect. Since each n-simplex σ of L gives rise to a $G_L(S)$-orbit of $(n+1)$-sheets, we see that there is a natural bijection between the n-simplices of L and the $G_L(S)$-orbits of n-dimensional level sheets in $X_t^{(S)}$.

Put a polyhedral cell structure on the level set $X_t^{(S)}$ in which the cells are the intersections of cubes of $X^{(S)}$ with the level set; this cell structure is $G_L(S)$-equivariant and respects the decomposition into level sheets. For σ a simplex of L, let Y_σ be the subcomplex of $X_t^{(S)}$ comprising the union of all level sheets that correspond to σ. Each Y_σ is a $G_L(S)$-subcomplex of $X_t^{(S)}$ and the stabilizer of each component of Y_σ is a copy of $\mathbb{Z}^n$, where n is the dimension of the simplex σ.

Since $X_t = X_t^{(S)}$ is equal to the union of the subcomplexes Y_σ, there is a surjective map of cellular chain complexes

$$\bigoplus_{\sigma\in L} C_*(Y_\sigma) \to C_*(X_t^{(S)}).$$

Of course, some cells of X_t arise in more than one Y_σ. If c is an n-dimensional cell of X_t, then there is a unique n-dimensional simplex σ for which $c \in Y_\sigma$, and for any $\tau \in L$, we have that $c \in Y_\tau$ if and only if σ is a face of τ. To compensate for the multiple counting of cells of X_t, Davis and Okun introduce a double complex C_{**}, in which C_{i*} consists of the direct sum of one copy of $C_*(Y_{\sigma_0})$ for each chain $\sigma_0 < \sigma_1 < \cdots < \sigma_i$ of simplices of L. The indexing set for the summands of C_{i*} is equivalently the set of i-simplices of the barycentric subdivision L' of L. The bigraded homology of the double complex C_{**}, taken in the horizontal direction, is zero for $i > 0$ and is $C_j(X_t)$ for $i = 0$; essentially this follows from the fact that the Euler characteristic of the star of any simplex of L' is equal to 1.

The case $S = \mathbb{Z}$ of the following theorem is [12, theorem 4.4].

Theorem 57. *The ℓ^2-Betti numbers of the space $X_t^{(S)}$ for $t \notin \mathbb{Z}$ are given in terms of the ordinary reduced Betti numbers of vertex links in L as follows:*

$$\beta^n_{(2)}(X_t^{(S)}; G_L(S)) = \sum_{v\in L^0} \beta^n(\mathrm{St}_L(v), \mathrm{Lk}_L(v)) = \sum_{v\in L^0} \overline{\beta}^{n-1}(\mathrm{Lk}_L(v)).$$

In the case when both L and $\widetilde{L}$ are $\mathbb{Q}$-acyclic, these are the ℓ^2-Betti numbers of the group $G_L(S)$, and so $\beta^n_{(2)}(G_L(S))$ does not depend on S.

Proof. Consider the spectral sequences coming from the double cochain complex

$$E_0^{ij} = \mathrm{Hom}_G(C_{ij}, \mathcal{N}(G)),$$

where $\mathcal{N}(G)$ is the group von Neumann algebra of $G = G_L(S)$. The spectral sequence in which the horizontal differential is taken first collapses to give $H^*(\mathrm{Hom}_G(C_*(X_t), \mathcal{N}(G)))$, which has the same von Neumann dimension (in Lück's extended definition [21]) as the ℓ^2-cohomology of $X_t^{(S)}$. The spectral sequence in which the vertical differential is taken first is the one appearing in the statement of [12, lemma 2.1]. The E_1-page of this spectral sequence is identified with simplicial cochains on L' with certain non-constant coefficients; the graded coefficient module for the simplex $\sigma_0 < \sigma_1 < \cdots < \sigma_i$ in degree j is $H^j(Y_{\sigma_0};\mathcal{N}(G))$. Since the ℓ^2-cohomology of $\mathbb{Z}^n$ vanishes for $n > 0$ and is one copy of $\mathbb{C}$ in dimension zero for $n = 0$, it follows that $H^j(Y_\sigma;\mathcal{N}(G))$ is equal to $\mathcal{N}(G)$ for $j = \dim(\sigma) = 0$ and otherwise has von Neumann dimension zero. Up to modules of von Neumann dimension zero, the E_1-page is identified with the direct sum $\bigoplus_{v\in L^0} C^*(\mathrm{St}_L(v), \mathrm{Lk}_L(v);\mathcal{N}(G))$ concentrated in the line $j = 0$. It follows that E_2^{i0} is isomorphic to $\bigoplus_{v\in L^0} H^i(\mathrm{St}_L(v), \mathrm{Lk}_L(v);\mathcal{N}(G))$, while other terms in the E_2-page have von Neumann dimension zero. Since modules of von Neumann dimension zero form a Serre class, no subsequent differential can affect the von Neumann dimensions, which gives the claimed result. □

Similarly, the case $S = \mathbb{Z}$ of the following theorem is [12, theorem 4.3]. In the case of cohomology with compact supports, the relevant spectral sequence still collapses at the E_2-page, but the E_2-page contains many non-zero terms in the same total degree, which makes the statement more complicated. In the statement, we write G_σ for the stabilizer in $G_L(S)$ of a level sheet of type σ. The group G_σ is free abelian of rank equal to the dimension of σ and it is well-defined up to conjugacy.

Theorem 58. *For $t \notin \mathbb{Z}$, for any ring R and for $n \geq 0$, there is a finite sequence of $RG_L(S)$-submodules of the compactly supported cohomology $H^n_c(X^{(S)}_t; R)$ so that the associated graded module is*

$$\mathrm{Gr}H^n_c(X^{(S)}_t; R) \cong \bigoplus_{\sigma\in L} H^{n-\dim\sigma}(\mathrm{St}_L(\sigma), \mathrm{Lk}_L(\sigma); R) \otimes_R RG_L(S)/G_\sigma.$$

In the case when L and $\widetilde{L}$ are both R-acyclic, note that $H^n_c(X^{(S)}_t; R) \cong H^n(G; RG)$.

In the statement the filtration grading is by $n - \dim(\sigma)$, so that the contribution coming from vertices of L is a quotient of the compactly supported cohomology H^n_c and simplices of largest possible dimension contribute a submodule of H^n_c.

Proof. In this case, one considers the spectral sequences arising from the double cochain complex

$$E_0^{ij} = \mathrm{Hom}_G(C_{ij}, RG).$$

As before, the spectral sequence in which the horizontal differential is taken first has $E_1^{ij} = 0$ for $i > 0$, and E_2^{0j} is isomorphic to $H^j_c(X^{(S)}; R)$. In the spectral sequence in which the vertical differential is taken first E_1^{ij} is the ith simplicial cochains on L' with graded non-constant coefficients in which the simplex $\sigma_0 < \cdots < \sigma_i$ is assigned $H^j_c(Y_{\sigma_0}; R)$ in degree j. As in [12, theorem 4.3], since Y_{σ_0} is a disjoint union of level sheets, and G_{σ_0} is free abelian of rank $\dim(\sigma_0)$, it follows that $H^j_c(Y_{\sigma_0}; R)$ is equal to RG/G_{σ_0} for $j = \dim(\sigma)$ and is zero otherwise. The computation of the E_2-page and the fact that the higher differentials are trivial is more delicate than in the previous theorem, but it works in exactly the same way for $G_L(S)$ as for BB_L; essentially it follows from the fact that the map $H^j_c(\mathbb{R}^n; R) \to H^j_c(\mathbb{R}^m; R)$ is the zero map for each j whenever $\mathbb{R}^m$ is a proper subspace of $\mathbb{R}^n$. Given this, the E_1-page splits as a direct sum of cochain complexes and

$$E_2^{ij} \cong \bigoplus_{\dim(\sigma)=j} H^i(\mathrm{St}_L(\sigma), \mathrm{Lk}_L(\sigma); R) \otimes_R RG_L(S)/G_\sigma.$$

The claim follows. $\square$

Corollary 59. *The following are equivalent.*

- *For each $S \subseteq \mathbb{Z}$, $G_L(S)$ is an n-dimensional duality group.*
- *L and $\widetilde{L}$ are acyclic and for every simplex σ of L, the reduced homology of $\mathrm{Lk}_L(\sigma)$ is free and is non-zero only in degree $n-1-\dim(\sigma)$.*

Proof. A duality group is a group Γ of type FP such that $H^*(\Gamma; \mathbb{Z}\Gamma)$ is torsion-free and concentrated in a single degree [7, theorem 10.1]. The claim follows from Theorem B and Theorem 58. $\square$

18 Poincaré duality groups

As in [3, 7], define a Poincaré duality group to be a group Γ of type FP such that $H^*(\Gamma; \mathbb{Z}\Gamma)$ is isomorphic (as abelian group) to $\mathbb{Z}$. If $H^n(\Gamma; \mathbb{Z}\Gamma) \cong \mathbb{Z}$, then Γ is an n-dimensional Poincaré duality group. The fundamental group of a closed aspherical n-manifold is a finitely presented Poincaré duality group.

In [10], Davis constructed the first non-finitely presentable Poincaré duality groups. More precisely, he applied his 'reflection group trick' to show that every group of type FH is a retract of a Poincaré duality group; see also [9, 11, 22]. This theorem combined with the Bestvina-Brady examples gave the new Poincaré duality groups. Similarly Davis's theorem applied to the groups $G_L(S)$ gives rise to an uncountable family of Poincaré duality groups. The following stronger result can be proved in a similar way.

Theorem 60. *For each $n \geq 4$ there is a closed aspherical n-manifold M that admits a family of acyclic regular coverings $M(S)$ indexed by the subsets S of $\mathbb{Z}$, in such a way that for $S \subseteq T \subseteq \mathbb{Z}$, $M(S)$ is a covering of $M(T)$. Members of this family give rise to uncountably many non-isomorphic groups of deck transformations. Each such group of deck transformations is an n-dimensional Poincaré duality group.*

Before beginning the proof, we recall some of the material concerning Coxeter groups that we will use; for more details see [11]. Given a flag complex K, the right-angled Coxeter group W_K is the group generated by elements in bijective correspondence with the vertices of K, subject only to the relations that each generator has order two and that generators corresponding to vertices that span an edge commute with each other. If L is a full subcomplex of K, the natural map on generating sets embeds W_L as a subgroup of W_K; in particular each $(i-1)$-simplex σ of K gives rise to a subgroup $W_\sigma \leq W_K$ which is isomorphic to the direct product $(C_2)^i$. A *special subgroup* of W_K is either the trivial subgroup or a subgroup W_σ for σ a simplex of K; it is convenient to view the empty set as the unique -1-simplex in K and define $W_\emptyset$ to be the trivial subgroup of W_K.

The Davis complex Σ_K is a CAT(0) cubical complex with a properly discontinuous cellular action of W_K. It is slightly easier to define first the barycentric subdivision of Σ_K. This is the simplicial complex obtained as the realization of the poset whose elements are the cosets of the special subgroups, ordered by inclusion. For any $(i-1)$-simplex σ of K and any $w \in W$, the subposet of cosets contained in wW_σ is isomorphic to the poset of faces of an i-cube, and these are the i-cubes of Σ_K. The complex Σ_K is simply-connected,

W_K acts freely transitively on its vertex set, and the link of each vertex is K. It follows that Σ_K is CAT(0). All cube stabilizers are finite, and so any torsion-free subgroup of W_K acts freely on Σ_K.

Lemma 61. *Let X be a finite 2-complex. For any $n \geq 4$ there is a compact triangulable manifold V homotopy equivalent to X.*

Proof. See [11, theorem 11.6.4]. □

Lemma 62. *There is a finite aspherical 2-complex X that admits a family of acyclic regular covers $X(S)$ indexed by the subsets S of $\mathbb{Z}$, with functoriality as in the statement of Theorem 60.*

Proof. Let L be a flag 2-complex that is acyclic and aspherical but not simply-connected, and satisfies nlcp; for example the 2-complex described in Section 2. Let X_t be a non-integer level set in the cubical complex X_L, and define X to be the finite 2-complex X_t/BB_L. By Corollary 28, the fundamental group of X is $G_L(\emptyset)$. By Corollary 24, for each $S \subseteq \mathbb{Z}$, X has a regular covering space $X_t^{(S)}$, with $G_L(S)$ as its group of deck transformations. Define $X(S)$ to be this covering space of X. By Corollary 35 this construction is functorial in S in the following sense. For $S \subseteq T \subseteq \mathbb{Z}$, the natural surjective homomorphism $G_L(S) \to G_L(T)$ can be used to define an action of $G_L(S)$ on $X(T)$, and there is a regular covering map $X(S) \to X(T)$ which is $G_L(S)$-equivariant. Moreover, for each $S \subseteq \mathbb{Z}$, $X(S)/G_L(S) = X$. □

Proof. (of Theorem 60) Let L be a flag complex as used in the proof of Lemma 62, and let X be the aspherical 2-complex constructed in Lemma 62 with fundamental group $G_L(\emptyset)$. Now fix $n \geq 4$, and let V be a compact n-manifold homotopy equivalent to X as in Lemma 61. Let K be the barycentric subdivision of a triangulation of the boundary ∂V of V. The fundamental group of V is isomorphic to $G_L(\emptyset)$. For $S \subseteq \mathbb{Z}$, let $V(S)$ denote the cover of V that is homotopy equivalent to $X(S)$, and similarly let $K(S)$ be the induced triangulation of $\partial V(S)$. Thus $G_L(S)$ acts freely on $V(S)$ and on $K(S)$, with quotient spaces V and K respectively, and these spaces enjoy the same functoriality as the spaces $X(S)$. Since K is a barycentric subdivision, it admits a simplicial map to an $(n-1)$-simplex Δ that is injective on each simplex; the vertices of Δ are labelled $0, 1, \ldots, n-1$, and the map sends the barycentre of a simplex τ to $\dim(\tau)$. There is also an induced map $K(S) \to \Delta$ for each $S \subseteq \mathbb{Z}$.

Now let W denote the finitely-generated Coxeter group W_K, and for $S \subseteq \mathbb{Z}$, let $W(S)$ be the Coxeter group $W_{K(S)}$. For $S \subseteq T$, the maps $K(S) \to K(T)$ and $K(S) \to K$ induce surjective group homomorphisms $W(S) \to W(T)$ and $W(S) \to W$, defining a functor. The maps to Δ define surjective homomorphisms $W(S) \to W_\Delta$ and $W \to W_\Delta$, and we define $W'(S)$ and W' to be the kernels of these maps. Thus $W'(S)$ and W' are normal subgroups of index 2^n in $W(S)$ and W respectively.

Let Σ denote the Davis complex Σ_K, and for $S \subseteq \mathbb{Z}$, let $\Sigma(S)$ denote the Davis complex $\Sigma_{K(S)}$ associated to the Coxeter group $W(S)$. For $S \subseteq T$ the surjective homomorphisms $W(S) \to W(T)$ and $W(S) \to W$ induce equivariant maps $\Sigma(S) \to \Sigma(T)$ and $\Sigma(S) \to \Sigma$. Since each special subgroup of W or $W(S)$ maps isomorphically to a subgroup of $W_\Delta \cong C_2^n$, the group $W'(S)$ acts freely on $\Sigma(S)$ and W' acts freely on Σ.

Now $G_L(S)$ acts freely on $K(S)$, which induces an action by automorphisms of $G_L(S)$ on $W(S)$ (one might describe this as an action by 'diagram automorphisms'). Since the $G_L(S)$-action preserves the set of special subgroups there is an induced action of the semi-direct product $W(S) \rtimes G_L(S)$ on $\Sigma(S)$. To simplify notation, let $J(S) = W(S) \rtimes G_L(S)$. For $S \subseteq T$ there is a surjective group homomorphism $J(S) \to J(T)$, and the natural map $\Sigma(S) \to \Sigma(T)$ is equivariant for this map. Since $K = K(S)/G_L(S)$, there is also a surjective group homomorphism $J(S) \to W$, and the natural map $\Sigma(S) \to \Sigma$ is equivariant for this homomorphism.

Now let $H(S) = W'(S) \rtimes G_L(S)$, so that $H(S)$ is an index 2^n normal subgroup of $J(S)$, and let $H = W'$. This H acts freely cocompactly on Σ. The action of $H(S) \leq J(S)$ on $\Sigma(S)$ is free except that each vertex of $\Sigma(S)$ has stabilizer a $J(S)$-conjugate of the standard copy of $G_L(S) \leq H(S)$. Nevertheless, the action is cocompact, and the natural maps induce isomorphisms of finite cube complexes $\Sigma(S)/H(S) \cong \Sigma/H$. Since the actions of $H(S)$ and $H(T)$ are free except at the vertices, the equivariant maps $\Sigma(S) \to \Sigma(T)$ and $\Sigma(S) \to \Sigma$ are branched coverings, with branching only at the vertices.

The link of each vertex of $\Sigma(S)$ is a copy of the $(n-1)$-manifold $K(S)$ and the link of each vertex of Σ is a copy of K. From this it follows that $\Sigma(S)$ and Σ are locally homeomorphic to $\mathbb{R}^n$, except at their vertices. Remove from $\Sigma(S)$ the open balls of radius $1/4$ with centres the vertices of $\Sigma(S)$. These balls have disjoint closures, and the resulting space is a $J(S)$-manifold whose boundary is equivariantly homeomorphic to $J(S) \times_{G_L(S)} K(S)$. Define $M(S)$ by attaching $J(S) \times_{G_L(S)} V$, which has the same boundary. Let $\overline{M}$ be defined similarly by removing open balls around the vertices from Σ and identifying the boundary of the resulting manifold with the boundary of $W \times V$. Now define M to be $\overline{M}/H$. Since H acts freely cocompactly on $\overline{M}$, M is a closed manifold, built by identifying the boundaries of 2^n copies of V. Since each $V(S)$ is acyclic, the homology of $M(S)$ is isomorphic to the homology of $\Sigma(S)$, and so each $M(S)$ is an acyclic manifold. Since $V(\emptyset)$ is simply connected, it follows that $M(\emptyset)$ is simply connected too, and hence $M(\emptyset)$ is contractible. For each S, $H(S)$ acts freely cocompactly on the acyclic manifold $M(S)$ with $M(S)/H(S) \cong M$, and for $S \subseteq T$ the map $M(S) \to M(T)$ is an equivariant covering map. The universal covering space of M is the contractible manifold $M(\emptyset)$, which implies that M is aspherical as claimed.

It remains to show that there are uncountably many isomorphism types of group $H(S)$. This follows from the fact that there are uncountably many isomorphism types of group $G_L(S)$ since $G_L(S)$ is a subgroup of $H(S)$; alternatively one can apply the invariant $\mathcal{R}$ directly to a suitable sequence of elements of $H(S)$. □

19 FP_2 via presentations

In this section we briefly outline an alternative approach to the groups $G_L(S)$, along the lines of the treatment of BB_L given in [13]. This uses the presentation $P_L(S, \Gamma)$ and the geometry of L, but not $X_L^{(S)}$. These results represent our first approach to the groups $G_L(S)$ and they motivated our discovery of $X_L^{(S)}$.

As usual, L is a finite connected flag complex, and Γ a finite collection of directed edge loops in L such that $\pi_1(L)$ can be killed by attaching discs to these loops. It will be useful to reduce the size of the sets of generators and relations, so we fix a choice

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of orientation for each edge a of L, and take the presentation $P'_L(S,\Gamma)$ in which the generator $\overline{a}$ corresponding to the opposite orientation is eliminated, together with the length two relator $a\overline{a}$. To avoid having to write $(a_1^{\epsilon_1},\ldots,a_l^{\epsilon_l})$ for a directed edge loop, we make the convention that a single letter such as a_i may stand for either one of our preferred generators or its inverse.

Lemma 63. *For any directed edge loop $\beta=(b_1,\ldots,b_m)$ in L, and any $n\in S$, the relation $b_1^n b_2^n\cdots b_m^n=1$ is a consequence of the relations in $P'_L(S,\Gamma)$.*

Proof. Identical to the proof in [13], which considered only the case $S=\mathbb{Z}$; we give a brief account mainly to establish notation. If β is null-homotopic then there is a simplicial map from a disc D^2 to L so that the boundary of the disc is the loop β. This gives rise to a van Kampen diagram D in the group presented by $P'_L(S,\Gamma)$, with $b_1^n\cdots b_m^n$ as its boundary word. The interior regions of D are triangles, each edge is labelled a^n for some directed edge a of L, or possibly $1=1^n$, and the labels around each triangle are either $a^nb^nc^n$ for a directed triangle (a,b,c) of L, or $1^n1^n1^n$ or $1^na^na^{-n}$. These relations are all consequences of the short relations in P'. Similarly, if β is homotopic to some $\gamma=(a_1,\ldots,a_l)\in\Gamma$, a simplicial homotopy can be used to build a van Kampen diagram with $b_1^n\cdots b_m^n$ as its outer boundary, in which all regions are triangles with boundaries as above except for one larger disc with boundary word $a_1^na_2^n\cdots a_l^n$. An arbitrary edge loop β can be expressed as a concatenation of loops of the above types. □

By the above lemma, the group $G_L(S)$ defined by the presentation $P'_L(S,\Gamma)$ does not depend on choice of Γ. Now let $\mathcal{C}$ be the Cayley 2-complex of the presentation P', and let $\mathcal{C}_t$ be the subcomplex consisting of the 1-skeleton and the 2-cells coming from triangle relators. Let $C_2\to C_1\to C_0$ be the cellular chain complex for $\mathcal{C}$, with $d:C_2\to C_1$ being the boundary map. Let C_2^t denote the $G_L(S)$-submodule of C_2 coming from the triangle relators and let C_2^l denote the $G_L(S)$-submodule coming from the 'long relators' $a_1^n\cdots a_l^n$ for $n\in S$ and $\gamma\in\Gamma$. Note that C_2^t is finitely generated, and that $C_2=C_2^l\oplus C_2^t$.

Theorem 64. *In the case when $\pi_1(L)$ is perfect, $d(C_2^l)\le d(C_2^t)$. Hence in this case $G_L(S)$ is type $FP_2(\mathbb{Z})$.*

Proof. Fix $n\in S$ and let $\gamma=(a_1,a_2,\ldots,a_l)$ be a directed edge loop in Γ. By hypothesis, γ represents the trivial element of $H_1(L)$, and so we can find a compact orientable surface F, with boundary ∂F a single circle, and a simplicial map from F to L so that γ is the image of ∂F. This gives rise to a 'generalized van Kampen diagram' in the presentation P', which we shall also call F, by a slight abuse of notation. Thus we have a labelling of the edges of F by elements a^n or $1=1^n$, so that the word around ∂F spells out $a_1^n\cdots a_l^n$ and so that each triangle is labelled by a relator that is a consequence of the triangle relators. Fix as basepoint $p\in F$ the vertex of ∂F that corresponds to the start of the word $a_1^n\cdots a_l^n$. Note also that any edge path in F is mapped to an edge path in L. For each vertex v of F and each $n\in S$, define $g(v,n)\in G_L(S)$ by $g(v,n)=b_1^n\cdots b_m^n$, where $(b_1,\ldots,b_m)$ is any directed edge path in L that is the image of a directed edge path in F from p to v. By Lemma 63, $g(v,n)$ depends only on v and $n\in S$ but not on the choice of edge path. Orient the triangles of F consistently and pick a base point $v(\sigma)$ for each triangle σ of F.

Now fix a base vertex x in the Cayley 2-complex $\mathcal{C}$. For each triangle σ of F, the word $a^n b^n c^n$ around the boundary of σ is a consequence of the triangle relators. Hence there is a disc $\phi(\sigma)$ in $\mathcal{C}^t$ that bounds the path that starts at the vertex $g(v(\sigma), n).x$ and follows the edge path $a^n b^n c^n$. The union of all of these discs can be used to define a map from F to $\mathcal{C}^t$ such that ∂F maps to edge path starting at x and given by the word $a_1^n \cdots a_l^n$. It follows that the attaching map for each 2-cell is contained in $d(C_2^t)$ and so $d(C_2^l) \subseteq d(C_2^t)$ as claimed. □

For any group presentation, a van Kampen diagram can be viewed as describing a map from a disc into the Cayley 2-complex. However, a 'generalized van Kampen diagram' on a surface F does not in general define a map from F to the Cayley 2-complex; the fact that this works for the presentation $P'_L(S, \Gamma)$ relies crucially on Lemma 63.

20 The homotopy type of level sets

In the original Bestvina-Brady paper [2], some parts of the main theorem depend upon a study of the homotopy types of level sets $X_{[a,b]} = f^{-1}([a,b]) \subseteq X_L$, which comprises section 8 of [2]. This is not essential to our argument, which is closer to that of [8]. Nevertheless, we state the analogue of theorem 8.6 of [2].

Theorem 65. *Suppose that $\pi_1(L)$ is finite. For any $a \leq b \in \mathbb{R}$, the level set $X^{(S)}_{[a,b]}$ is homotopy equivalent to a wedge of copies of L, indexed by the vertices of $X^{(S)}$ of height in $S - [a,b]$ and copies of $\widetilde{L}$, indexed by the vertices of height in $(\mathbb{Z} - S) - [a,b]$.*

Proof. Identical to the proof of [2, thm 8.6]. For each vertex v of $X^{(S)}$, let K_v be the union of all sheets containing v. Analogues of lemmas 8.1–8.5 of [2] hold; in particular each K_v is a convex subset of $X^{(S)}$. The hypothesis that $\pi_1(L)$ is finite implies that $X^{(S)}$ is locally finite, which allows us to enumerate its vertex set as $v = v_1, v_2, v_3, \ldots$ in such a way that $d(v, v_i) \leq d(v, v_j)$ whenever $i < j$. Such an enumeration plays an important role in the proof of [2, lemma 8.3] and its analogue for $X^{(S)}$. □

We believe that the above theorem should hold without the hypothesis that $\pi_1(L)$ be finite. This would give an alternative treatment of some of our results.

21 Closing remarks

It would be easy to generalize some of our results to the case in which $\widetilde{L}$ is replaced by some other regular covering $\widehat{L}$ of L. To do this, embed L as a full subcomplex of some M with nlcp in such a way that the induced map $\pi_1(L) \to \pi_1(M)$ is a surjection with kernel $\pi_1(\widehat{L})$. Given such an M, the statement and ('general case') proof of Theorem 22 go through unchanged.

Ignoring our results concerning higher finiteness properties, our main theorem gives a new proof that there are uncountably many finitely generated groups. The simplest family of such groups arising from our theorem is given by the presentations

$$\langle a, b, c, d \,:\, a^n b^n c^n d^n = 1 \ \forall n \in S \rangle.$$

These are the presentations $P_L(\Gamma, S)$ for $0 \in S \subseteq \mathbb{Z}$ in the case when L is a circle, triangulated as the boundary of a square.

We reiterate the remark made after Proposition 51: In the case when $\mathbb{Z} - S$ is finite and L is R-acyclic, we know that $G_L(S)$ is $FP(R)$, but we do not know whether $G_L(S)$ is necessarily $FH(R)$.

In an earlier version of this article, we drew attention to two questions related to this work. Does every group of type F_n embed in a group of type F_{n+1}? Does every group of type $FP_n(R)$ embed in a group of type $FP_{n+1}(R)$? These questions are open for each $n \geq 2$. We have recently shown that every countable group embeds in a group of type $FP_2(\mathbb{Z})$ [19], resolving the case $n = 1$ of the second question.

References

[1] G. Baumslag, E. Dyer and A. Heller, *The topology of discrete groups*, J. Pure Appl. Algebra **16** (1980), 1–47.

[2] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445–470.

[3] R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes, Mathematics Department, Queen Mary College London (1976).

[4] M. R. Bridson and F. J. Grunewald, *Grothendieck's problems concerning profinite completions and representations of groups*, Ann. of Math. **160** (2004) 359–373.

[5] M. R. Bridson and A. Haefliger, *Metric spaces of nonpositive curvature*, Grundlehren der Math. Wiss. **319**, Springer-Verlag (1999).

[6] K. S. Brown, *Finiteness properties of groups*, J. Pure Appl. Algebra **44** (1987) 45–75.

[7] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math. **87**, Springer-Verlag (1982).

[8] K.-U. Bux and C. Gonzalez, *The Bestvina-Brady construction revisited: geometric computations of Σ-invariants for right-angled Artin groups* J. London Math. Soc. **60** (1999) 793–801.

[9] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. **117** (1983) 293–324.

[10] M. W. Davis, *The cohomology of a Coxeter group with group ring coefficients*, Duke Math. J. **91** (1998) 297–314.

[11] M. W. Davis, *The geometry and topology of Coxeter groups*, London Math. Society Monographs Series **32** Princeton University Press (2008).

[12] M. W. Davis and B. Okun, *Cohomology computations for Artin groups, Bestvina-Brady groups, and graph products*, Groups Geom. Dyn. **6** (2012) 485–531.

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Version 1 - Submitted to PLMS

[13] W. Dicks and I. J. Leary, *Presentations for subgroups of Artin groups*, Proc. Amer. Math. Soc. **127** (1999), 343–348.

[14] A. Hatcher, *Algebraic Topology*, Cambridge University Press 2002.

[15] G. Higman, *Subgroups of finitely presented groups*, Proc. Roy. Soc. Ser. A **262** (1961) 455–475.

[16] J. Howie, *Bestvina-Brady groups and the plus construction*, Math. Proc. Camb. Phil. Soc. **127** (1999), 487–493.

[17] I. J. Leary, *The Euler class of a Poincaré duality group*, Proc. Edinburgh Math. Soc. **45** (2002) 421–448.

[18] I. J. Leary, *A metric Kan-Thurston theorem*, J. Topology **6** (2013) 251–284.

[19] I. J. Leary, *Subgroups of almost finitely presented groups*, preprint, arXiv:1610.05813 8 pages (2016).

[20] I. J. Leary and M. Saadetoğlu, *The cohomology of Bestvina-Brady groups*, Groups Geom. Dyn. **5** (2011) 121-138.

[21] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory* Ergebnisse der Mathematik und ihrer Grenzgebiete **44**, Springer (2002).

[22] G. Mess, *Examples of Poincaré duality groups*, Proc. Amer. Math. Soc. **110** (1990), 1145-1146.

[23] B. H. Neumann, *Some remarks on infinite groups*, Proc. London Math. Soc. **12** (1937) 120–127.

[24] C. T. C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math. **81** (1965) 56–69.

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