Jan 17

# Right Angularity

Polyhedral Products:
Generalization of $P_L$.

**Def** Data

$L$ = simplicial cx
$I$ = Vert$(L)$

$$\underline{A} = \{(A_i, B_i)\}_{i \in I}$$

$$*_i \in B_i \subset A_i$$

$$\underline{A}^L \subset \prod_{i \in I} A_i$$

$\underline{A}^L \ni (x_i)$

(a) $x_i = *_i$ except for finitely many $i$

$$\boxed{\text{(b) } \sigma = \{ i \in I \mid x_i \in A_i - B_i \} \quad \sigma = \text{simplex of } L.}$$

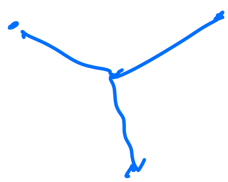
(0)

(1) $(A_i, B_i) = ([-1,1], \{\pm 1\})$

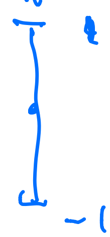
$$A^L = \prod_{i \in I} [-1,1]$$
$$\quad \| $$
$$P_L$$

(2) $E_i$ = discrete set (usually finite set)

$$A_i = \text{Cone } E_i = \frac{E_i \times [0,1]}{\sim}$$
$$B_i = \cancel{tx}\ E_i \times$$



$E_i = 2$ pts



$$\underline{A} = \{(\text{Cone } E_i, E_i)\}_{i \in I}$$

$$Z_L = \underline{A}^L$$

"Right angled building"

(3) "Salvetti complex for Right angled ARTIN Gp"

$$A_i = S^1 \qquad B_i = *_i \in S^1$$

$$A^L \subset \prod_{i \in I} S^1 = \underbrace{T^I}_{\text{torus}}$$

$L = \sigma =$ simplex $\Delta$

$$A^\sigma = \left(\prod_{i \in \sigma} A_i\right) \times \prod_{i \in I - \sigma} B_i$$

$$A^L = \bigcup A^\sigma$$

$$Z_L = A^L \subset \prod S^1$$

$A^\sigma =$ subtorus $= T^\sigma$

$\cancel{B}$ $Z_L = \bigcup T^\sigma$

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(0) $A_i = [0, 1]$

$B_i = +1$

$$\underset{\overset{\|}{K}}{K(L)} = \underline{A}^L \subset \prod_{i \in I} [0,1]$$

$K$ = a cone (cone pt $= (1,1,\dots 1)$ ) $\swarrow$ ao

$i \in I$ $\quad K_i = K \cap \{x_i = 1\}$ $\nwarrow$ $\{x_i = 0\}$

$\sigma \in L$ $\quad K_\sigma = \bigcap_{i \in \mathrm{Vert}(\sigma)} K_i$

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~~S~~ Def'n Strict fund. domain

$G$ = discrete gp

$G \curvearrowright X$

$Y \subset X$ closed subset is strict fund domain

- $Y$ ~~A~~ intersects each orbit in exactly 1 ~~pt~~.

$\Rightarrow G(x) = \{gx\}_{g \in G}$

$Y \cap G(x) = $ singleton

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr, "\bar{\pi}", "1-1"'] & X \arrow[d, "\pi"] \\ & X/G \end{tikzcd}$$

<u>Fact</u> $\bar{\pi}$ is a homeomorphism

Recover $X$ by gluing together copies of $Y$

$$G \times Y \longrightarrow X$$
$$(g, y) \longmapsto gy$$

Define equiv. on $G \times Y$

and $(G \times Y)/\sim \; = X$

<u>Isotropy subgp</u> at $x \in X$

$$:= \{ g \in G \mid gx = x \}$$

$$(g_1, y_1) \sim (g_2, y_2)$$

$$\iff y_1 = y_2$$

$$g_1 G_{y_1} = g_2 G_{y_1}$$

$$\cancel{g_1^{-1} g_2} \quad g_2^{-1} g_1 \in G_y$$

$$\boxed{(g, y) \sim (gh, y)}$$

$y \in G_y$

$[g, y]$ = equivalence class

Define $D(G, Y) = (G \times Y)/\sim$

basic construction in Brid Haefliger

Fact $G \times Y \to X$

induces $D(G, Y) \to X$

a $G$-equivariant homeomorphism

$K = K(L)$ This is a fund domain for

$(C_2)^I \curvearrowright P_L$. $\quad G = (C_2)^I$

$$P_L = D(G, K)$$

$$\parallel$$

$$(G \times K)/\sim$$

$$(g, x)$$

$$\sigma(x) = \{ i \mid x \in K_i \}$$

$$G_{\sigma(x)} = \langle \Delta_i \rangle_{i \in \sigma(x)}$$

~~g(x)~~ $(g, x) \sim (gh, x)$

$h \in G_{\sigma(x)}$.

Jan 22

- Graph Product of groups
- Action on $\mathcal{Z}_L$
- $\pi_1$ (polyhedral product) = graph product.

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$i \in I = \mathrm{Vert}(L)$

$(A_i, B_i) \ni x_i \qquad \underline{A} = (A_i, B_i)_{i \in I}$

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$$\boxed{\underline{A}^L \subset \prod_{i \in I} A_i}$$

$\{ i \in I \mid x_i \in A_i - B_i \}$

= simplex $\in L$

(0) $K \qquad A_i = [0,1], \ B_i = 0$

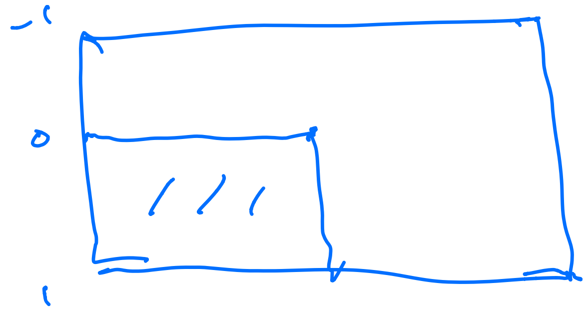
(1) $P_L \qquad A_i = [-1, 1] \quad B_i = \{\pm 1\}$

(2) $\mathcal{Z}_L = (A_i, B_i) = (\mathrm{Cone}\, E_i, E_i)$

$B_i$ is discrete set

(3) $(A_i, B_i) = (S^1, *_i)$

$K$ = fund domain for $(\mathbb{Z}_2)^I$ action $P_L$



$[-1,1]^2$

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Assume $E_i = G_i$ a discrete gp

$\mathbb{Z}_L \le (\text{Cone } G_i)^I$

$O_i$ = cone pt in $[0,1]\times G_i / \sim$

$G = \sum_{i \in I} G_i = \bigoplus \subseteq \prod G_i$

$G_i \curvearrowright \text{Cone } G_i$

$G \curvearrowright \prod \operatorname{Cone} G_i$

$Z_L$ is $G$-stable

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<u>Def</u> $(G_i)_{i \in I}$ is collection of groups

$L' =$ simple graph

$\prod_{L'} G_i = \Gamma$ is graph product

Take free product of $G_i$. Divide by Normal subgp generated by commutators $[g_i, g_j]$

$g_i \in G_i$, $g_j \in G_j$, $\{i, j\} \in \operatorname{Edge} L'$

$G_i$ and $G_j$ commute if $i$ and $j$ connected by edge

$$\varphi : \Gamma \longrightarrow \sum_{i \in I} G_i = \prod G_i = G$$

$\widetilde{\Sigma}_L$ = univ cover

$$\begin{array}{c} \widetilde{\Sigma}_L \\ \downarrow \\ \Sigma_L \end{array} \circlearrowleft G$$

Let $H$ = group of all lifts of $G$-action to $\widetilde{\Sigma}_L$

<u>Prop</u> $H = \Gamma$ (the graph product $\sum G_i$)

i.e.

$$1 \to \pi_1(\Sigma_L) \to \Gamma \longrightarrow G \longrightarrow 1$$

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$K$ = fund. domain for $G$-action on $\Sigma_L \subset \prod \mathrm{Cone}\, G_i$

$x_i \in E_i \qquad x = \prod x_i$

$$\begin{array}{c} Z_L \\ \downarrow p \\ K = Z_L/G \end{array}$$

$\tilde K$ = component of $p^{-1}(K)$ containing ~~lift~~ of $\tilde x$ (= lift of base pt)

$O_i \in \operatorname{Cone} G_i$

$\tilde O_i$ = corresponding pt in $\tilde K$

<u>Pf</u> <u>prop</u>: Let $H_i$

= gp of lifts of $G_i$-action

which fix $\tilde O_i$

$\{i, j\} \in \operatorname{Edge} L^1$

$G_i \times G_j$ fixes $O_i \times O_j \in Z_L$

The gp of $G_i \times G_j$ ~~to~~

fixing $\tilde O_i \times \tilde O_j$ is

$\cancel{H_i \times H_j}$ projects to

$G_i \times G_j \quad \ni \quad i \; = H_i \times H_j$

$$G_i \cong H_i \hookrightarrow H$$

$\{i,j\} = \mathrm{Ed}_{\Gamma}$ – $\Gamma \supset G_i \times G_j \cong H_i \times H_j \hookrightarrow H$

By universal prop. of direct limit this extends to $\Gamma \longrightarrow H$

Isomorphism? ~~[illegible]~~

$\Sigma G_i \curvearrowright$ simply transitively on vertices of $Z_L$

$\wr\wr$

$\Sigma H_i$ lifts acts $H \curvearrowright$ ~~[illegible]~~ simply ~~t~~ransit. on vertices of $\tilde{Z}_L$

$\therefore \Gamma \to H$ injective

$\square$

~~Pro~~

Suppose $A_i$ = connected CW cx

$B_i^{\cdot\cdot}$ $*_i$ = $\tilde{A}$ basept.

$\underline{A} = (A_i, *_i)$

**Prop** $\pi_1(\underline{A}^L) = \Gamma$ = graph product

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$\tilde{A}_i$ $p_i^{-1}(*_i) \cong G_i = \pi_1(A_i)$

$\downarrow p_i$

$A_i \ni *_i$

Not a proof

**Pf** $\pi_1(A^L) \xrightarrow{\varphi} \sum G_i \to 1$

Consider the covering space induced by $\varphi$ call it

$$A_\varphi^{L} = \prod_{L'} (\tilde{A}_i, p^{-1}(*_i))$$

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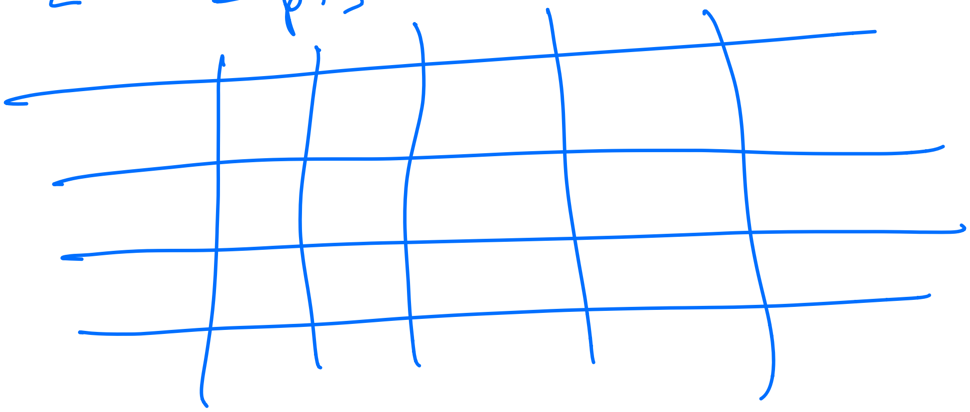
Ex: $A_i = S^1 \ni *$

$\tilde{A}_i = \mathbb{R}^1$ $p^{-1}(*) = \mathbb{Z}$

$2^1 =$ 2 pts



Covering space corresponding to

$$\pi_1(A^L) \to \sum G_i$$

From this you get

$\tilde{A}^L_k$ is covering space of $(A^1)^L$

$\therefore \ldots \quad \Gamma = \pi_1(A^L_k) \circlearrowright$

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Above is not a proof

What I wanted to prove is following

Jan 24

$L$ = flag cx, $I$ = {vertices}

$\{G_i\}_{i \in I}$.

$BG_i$ = classifying space

$EG_i$ = univ cover = contractible

$\Gamma = \prod_{L^1} G_i$ the graph product

<u>Thm 1.</u>

$$B\Gamma = (BG_i, *_i)^L$$

<u>Example</u>: Each $G_i = \mathbb{Z}$

$\Gamma = A_L$ = RAAG

$B\mathbb{Z} = S^1$, show $BA_L = (S^1)^L$

Last time

$\Gamma = \prod_L G_i$

$Z_L = (\text{Cone } G_i, G_i)^L$

$\Gamma \curvearrowright \tilde{Z}_L$.

Should have proved

Thm 2. L is a flag cx

then $Z_L$ is NPC

so $\tilde{Z}_L$ is contractible.

Pf Links are flag complexes?

Pf of Thm 1 $G = \sum_i G_i$

Embedding $Z_L = (\text{Cone}\, G_i, G_i)^L \hookrightarrow \prod \text{Cone}\, G_i$

Gives $G$-action on $(\text{Cone}\, G_i, G_i)^L$

$EG_i \xrightarrow{p_i} BG_i$ is universal cover.

Covering space

$$\begin{array}{c} EG_i,\ p_i^{-1}(*_i) \\ \downarrow \\ (BG_i,\ *_i) \end{array}$$

hence covering space

$$\begin{array}{ccc} (EG_i,\ p_i^{-1}(*_i))^L & \subset & \prod EG_i \\ \downarrow & & \downarrow \\ (BG_i,\ *_i)^L & \subset & \prod BG_i \end{array}$$

$$(EG_i,\ p_i^{-1}(*_i)) \overset{h.e}{\sim} (\mathrm{Cone}\, G_i, G_i)^{B}$$

$$\mathbb{Z}_L$$

$$\therefore\ (EG_i,\ p^{-1}(*_i))^L \sim \mathbb{Z}_L$$

Pulling Back universal cover

$\sim$

of $\mathbb{Z}_L$ we get

universal cover of

$(EG_i, p_i^{-1}(*_\lambda))^L$ is

contractible.

$\therefore$ $B\Gamma^L = (BG_i, *_\lambda)^L$