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# Actions of infinite groups on polyhedra

– Monograph –

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# Contents

# Chapter 1
# Introduction

The road between topology and group theory goes through the fundamental group. Suppose $X$ is a space that admits a universal covering space (i.e., $X$ is semi-locally 1-connected). Its fundamental group acts as the group of deck transformations on the universal cover $\widetilde{X}$. Given a presentation for a group $\pi$, there is a standard construction of a connected, 2-dimensional CW complex $X$ with $\pi_1(X) = \pi$. (Start with a wedge of circles, one for each generator; then attach a 2-cell for each relation.) One can continue to attach cells to $X$ of dimension greater than 2 to kill all the higher homotopy groups $\pi_i(X)$, $i \geq 2$. The result is a CW complex with fundamental group $\pi$ and all higher homotopy groups $= 0$. We denote such a space $B\pi$ and call it the classifying space of $\pi$. Standard arguments in homotopy theory show that $B\pi$ is unique up to homotopy equivalence, e.g., see [80]. Thus, algebraic topological properties of $B\pi$ such as its cohomology are properties of the group $\pi$.The universal cover of $B\pi$, denoted $E\pi$, is contractible. The space $B\pi$ is also called an Eilenberg-MacLane space $K(\pi, 1)$. Such a space with vanishing higher homotopy groups is said to be *aspherical*.

Usually, the CW complex $B\pi$ cannot be taken to have a finite number of cells. For example, if $\pi$ has nontrivial torsion, then $B\pi$ must be infinite dimensional. later

Suppose a group $G$ acts on a cell complex $X$ by an action that permutes the cells. We write $G \curvearrowright X$ to mean that $G$ acts on $X$. There are several different situations to consider. First, the complex $X$ could be locally infinite. In this case, cell stabilizers might well be infinite. For example this is often the case in Serre's theory of groups acting on trees. If $X$ is locally finite, then $\mathrm{Aut}(G)$ is a locally compact, totally disconnected group. Although the cell stabilizers might still be infinite, we will primarily be interested in the case when $G$ is a discrete subgroup of $\mathrm{Aut}(X)$. The group $G$, is then said to act *properly* on $X$. Finally, the cell stabilizers in $G$ might be trivial. In this case, the $G$-action is *free*. The $G$-action is *cocompact* if the orbit space $X/G$ is compact.

If a group $G$ has nontrivial torsion, then $BG$ cannot be finite dimensional. (The proof is due to P.A. Smith: if $H$ is a finite cyclic subgroup of $G$, then $BH \to BG$ is a covering space. If $H$ is nontrivial, $BH$ has cohomology in arbitrarily high degrees. So, $BH$ and a fortiori $BG$ must be an infinite dimensional.)

A group $G$ is *virtually torsion-free* if it has a torsion-free subgroup $\Gamma$ of finite index. The *virtual cohomological dimension of* $G$, denoted by $\mathrm{vcd}\, G$, is the cohomolgical dimension of such a $\Gamma$. The group $G$ is type $VF$ if there is a finite index subgroup $\Gamma$ of type $F$. (These notions are independent of the choice of $\Gamma$.)

When a group $G$ has nontrivial torsion, there is a notion of a "universal space for proper actions" which is sometimes more useful than the universal space $EG$.

**Definition 1.1.** For a discrete group $G$, its *universal proper $G$-space* is a CW complex $\underline{E}G$ together with a proper cellular action, $G \curvearrowright \underline{E}G$, so that for any finite subgroup $H < G$, the subcomplex, $(\underline{E}G)^H$, which is fixed by $H$, is contractible. (In particular, if every finite subgroup of $G$ is trivial, then $\underline{E}G = EG$.) The quotient complex $\underline{E}G/G$ is denoted by $\underline{B}G$. (The complex $\underline{B}G$ is an orbihedron.)

Definition and next 2 paragraphs might go in intro

For example, if $Y$ a locally finite $\mathrm{CAT}(0)$ polyhedron and $G \curvearrowright Y$ by isometries, then $Y = \underline{E}G$.

The universal property of $\underline{E}G$ is that for any CW complex $Y$ with a cellular, proper $G$-action (i.e., with finite isotropy groups), there is an equivariant map $Y \to \underline{E}G$, unique up to equivariant homotopy.

# Chapter 2
# Polyhedral preliminaries

## 2.1 Cell complexes, links

By a *cell complex* we will mean a space $X$ formed by gluing together convex polytopes (in some euclidean space) via isometries of their faces. The decomposition of $X$ into polytopes ("cells") is part of the structure. The word "polyhedron" will be used synonymously with "cell complex".

In classical definitions of a convex cell complex, one often adds the conditions that each cell is embedded Delta and that the intersection of two cells is either empty or a common face of each. We call these two requirements the *classical conditions*. The first of these conditions rules out the possibility of a 1-gon while the second rules out a 2-gon. (Even when $X$ does not satisfy the classical conditions, the induced cell structure on its universal cover often will satisfy them.) Two types of cell complexes are of particular interest: first, if each cell is required to be a simplex we have "$\Delta$-complex" in the sense of Hatcher [80], second if each cell is a cube, we have a "cube complex". A $\Delta$-complex or a cube complex can have "multiple faces," e.g., in a $\Delta$-complex two different simplices can have the same vertex set. If a $\Delta$-complex satisfies the classical conditions, then it is a *simplicial complex*. Similarly, if a cube complex satisfies the classical conditions, then it is sometimes called a *simple cube complex* (for example this terminology is used in [79]).

Given a face $F$ of a convex polytope $P$ and a point $x$ in its relative interior, the *normal cone* $N(F,P)$ is the set of all inward-pointing tangent vectors at $x$ that are orthogonal to $F$. It is a convex polyhedral cone in the the tangent space $T_xP$. The dimension of $N(F,P)$ is the codimension of $F$ in $P$. The *link of $F$ in $P$*, denoted $\mathrm{Lk}(F,P)$, is set of unit tangent vectors in $N(F,P)$. In other words, $\mathrm{Lk}(F,P)$ is the space of inward-pointing normal directions to $F$ in $P$. It is a geodesically convex polytope in the unit sphere of $T_xP$. If $F < F'$, where $F'$ is another face of $P$, then $\mathrm{Lk}(F,F')$ is identified with a face of $\mathrm{Lk}(F,P)$. Given a cell $\tau$ of a cell complex $X$, we can then glue together the corresponding links to to form a cell complex:

$$\mathrm{Lk}(\tau,X) := \bigcup_{\substack{\sigma \\ \tau<\sigma}} \mathrm{Lk}(\tau,\sigma), \tag{2.1}$$

called the *link of $\tau$ in $X$*. (The definition given in (2.1) needs to be slightly modified when there are self-gluings. For example, $P$ is the polytope corresponding to the cell $\sigma$ and faces $F_1$, $F_2$ of $P$ are identified to get a single cell $\tau$ of $X$, then $\mathrm{Lk}(\tau,\sigma)$ be interpreted to be $\mathrm{Lk}(F_1,P)\cup\mathrm{Lk}(F_2,P)$')

For example, if $P$ an $n$-simplex and $F < P$ is a $k$-dimensional face, then $\mathrm{Lk}(F,P)$ is a spherical $(n-k-1)$-simplex. So, if $X$ is a $\Delta$-complex and $\sigma$ is a cell in $X$, then $\mathrm{Lk}(\sigma,X)$ is a $\Delta$-complex. Moreover, if $X$ is a simplicial complex, then so is $\mathrm{Lk}(\sigma,X)$. Similarly, if $F$ is a $k$-dimensional face of an $n$-cube $P$, then $\mathrm{Lk}(F,P)$

is a $(n-k-1)$-simplex. Hence, if $X$ is a cube complex, then $\mathrm{Lk}(\sigma, X)$ is a $\Delta$-complex and if $X$ is a simple cube complex, then $\mathrm{Lk}(\sigma, X)$ is a simplicial complex.

One can define the notion a geodesically convex polytope in any simply connected space of constant curvature $\kappa$. In dimension $n \geq 2$ the possibilities are euclidean $n$-space $\mathbb{E}^n$ for $\kappa = 0$, the $n$-sphere of curvature $\kappa$, $\mathbb{S}^n_\kappa$, for $\kappa > 0$, and hyperbolic $n$-space of curvature $\kappa$, $\mathbb{H}^n_\kappa$, , , A cell complex $X$ made by gluing together polytopes in a space of constant curvature $\kappa$ is called a constant curvature $\kappa$ is called a *constant curvature polyhedron*. (See [23] or [48] for more details.) We shall be interested only in the cases $\kappa = +1$, 0, or $-1$. We shall say that $X$ is *piecewise spherical*, *piecewise euclidean* or *piecewise hyperbolic*, respectively. If a constant curvature polyhedron $X$ is connected, then it inherits a natural length metric as follows. The length of a geodesic segment in a polytope of constant curvature $\kappa$ is defined as usual. If $x, y \in X$, then $d(x,y)$ is defined to be the infimum of the lengths of all piecewise linear paths from $x$ to $y$. If $X$ is locally finite, then this gives $X$ the structure of a *geodesic space* meaning that the infimum $d(x,y)$ is always realized by a geodesic
Is this correct? path (= isometric embedding), $\gamma : [0,d] \to X$ with $\gamma(0) = x$, $\gamma(d) = y$. (For more details see [23, Ch I.3].)

Since the link of a face of any polytope in a constant curvature space is always a spherical polytope in $\mathbb{S}^n_1$, we see that the link of any cell of a constant curvature polyhedron is always a piecewise spherical polyhedron ($\kappa = 1$). Similarly, for any point $x \in X$, let $\mathrm{Lk}(x,X)$ be the union of all unit tangent vectors which point into a cell of $X$. (If $x$ belongs to the relative interior of a $k$-cell $\sigma$, then $\mathrm{Lk}(x,\sigma) = S^{k-1}$. When this is the case it follows that $\mathrm{Lk}(x,X)$ can be identified with $\mathbb{S}^{k-1} * \mathrm{Lk}(\sigma, X)$, the $k$-fold suspension of $\mathrm{Lk}(\sigma, X)$.) Let $N_\varepsilon(x,X)$ (resp. $\partial N_\varepsilon(x,X)$) denote the ball (resp. sphere) of radius $\varepsilon$ about $x$, i.e., $N_\varepsilon(x,X) = \{y \in X \mid d(x,y) \leq \varepsilon\}$ and $\partial N_\varepsilon(x,X) = \{y \in X \mid d(x,y) = \varepsilon\}$. One sees $\partial N_\varepsilon(x,X)$ can be identified with $\mathrm{Lk}(x,X)$ and that

$$N_\varepsilon(x,X) \cong \mathrm{Cone}_\varepsilon(\mathrm{Lk}(x,X)), \tag{2.2}$$

where $\mathrm{Cone}_\varepsilon(A)$ stands for the *cone of radius* $\varepsilon$, that is, $A \times [0,\varepsilon]$ with $A \times 0$ collapsed to a point. Formula (2.2) means that $\mathrm{Lk}(x,X)$ determines the local topology of $X$. For example, $X$ is a PL $n$-manifold if and only if $\mathrm{Lk}(x,X)$ is piecewise linearly homeomorphic to $S^{n-1}$ for each $x \in X$.

## 2.2 The CAT(0)-inequality

Using ideas of Alexandrov and Busemann, Gromov [76] defined what it means for a geodesic metric space $X$ to have curvature bounded above by a real number $\kappa$ (where usually, $\kappa \in \{-1,0,+1\}$). First, a geodesic triangle in $X$ satisfies the $\mathrm{CAT}(\kappa)$*-inequality*, if the distance between any two points of the triangle is $\leq$ the distance between the corresponding points of a comparison triangle in the simply connected 2-manifold of constant curvature $\kappa$, i.e., in the hyperbolic plane, the euclidean plane, or the 2-sphere, as $\kappa = -1, 0$ or $+1$. A geodesic space $X$ is $\mathrm{CAT}(\kappa)$ if each triangle satisfies the $\mathrm{CAT}(\kappa)$-inequality. The $\mathrm{CAT}(\kappa)$-inequality implies the $\mathrm{CAT}(\kappa')$-inequality for $\kappa' > \kappa$. In particular, if a space is $\mathrm{CAT}(\kappa)$ with $\kappa \leq 0$, then it is $\mathrm{CAT}(0)$.

*Remark 2.1.* he terminology "$\mathrm{CAT}(\kappa)$-inequality" is due to Gromov [76, p.106]. The acronym CAT stands for "comparison inequality of Alexandov and Toponogov," although sometimes the "C" is said to stand for "Cartan."

The uniqueness of the geodesic connecting two points in a $\mathrm{CAT}(0)$ space $X$ follows immediately from the definitions. With a little work one can see that the constant map $X \to X$ which each point to a basepoint is homotopic to the identity via geodesic contraction. This gives the following.

**Theorem 2.1.** *If a complete geodesic space is* $\mathrm{CAT}(0)$*, then it is contractible.*

The space $X$ has *curvature* $\leq \kappa$ if it satisfies the $\mathrm{CAT}(\kappa)$-inequality locally. We say that $X$ is *nonpositively curved* (abbreviated NPC) if this is true for $\kappa = 0$. (See [23].)

**Theorem 2.2.** (Globalization Theorem of Cartan-Hadamard, cf. [76, p. 119], [23, p. 193], or [4]). *Suppose $X$ is a complete geodesic space of curvature $\leq \kappa$*

*(i) Suppose $\kappa \leq 0$. Then $X$ is $\mathrm{CAT}(\kappa)$ if and only if it is simply connected.*

*(ii)* ([76, p. 122]. *Suppose $\kappa > 0$. Then $X$ is $\mathrm{CAT}(\kappa)$ if and only if it has no closed geodesic of length $\leq 2\pi/\sqrt{\kappa}$.*

An immediate corollary to the first part of this theorem is the following.

**Corollary 2.1.** *Suppose a complete geodesic space $X$ has curvature $\leq \kappa$ with $\kappa \leq 0$. Then its universal cover $\tilde{X}$ is $\mathrm{CAT}(\kappa)$.* omit

The relevance of nonpositive curvature to geometric group theory is explained by the next theorem, which combines Theorem 2.1 and Corollary 2.1.

**Theorem 2.3.** *If a complete geodesic space is NPC, then it is aspherical.*

Just as in the case where a polyhedron is obtained by gluing together euclidean polytopes, one can consider a polyhedron $X$ where each cell is identified with some convex polytope in a space of constant curvature $\kappa$. In general we call such an $X$ a *piecewise constant curvature polyhedron* and we say that it is *piecewise hyperbolic*, *piecewise euclidean*, or *piecewise spherical* as $\kappa = -1, 0, or +1$, respectively. As before any such $X$ is a length space and in fact, a geodesic space whenever it is locally finite.

*Example 2.1.* (*Dimension one*). If $X$ is a connected graph with finitely many edges and we assign a length to each edge, then $X$ is a constant curvature polyhedron for any curvature $\kappa$ for any $\kappa$. It is locally $\mathrm{CAT}(\kappa)$. So, if each edge of $X$ is assigned length 1, then $X$ is an NPC cube complex. If $X$ is a tree and each edge has length 1, then, by Theorem 2.2 (i), $X$ is a $\mathrm{CAT}(0)$ cube complex. We can also regard the metric on a graph $X$ as giving it a piecewise spherical structure. In this case, by part (ii) of Theorem 2.2, $X$ is $\mathrm{CAT}(1)$ if and only if it has no circuit (= closed geodesic) of length $\leq 2\pi$. Should be Definitionstyle

We turn now to the question of when a piecewise constant curvature polyhedron (with the curvature constant $= \kappa$) has curvature $\leq \kappa$. The answer is provided by the "Link Condition," stated in Theorem 2.4 below. The proof of the Link Condition is a combination of three facts: first, the link of any cell in a piecewise constant curvature polyhedron $X$ is naturally a piecewise spherical polyhedron; second, an open ball centered at a point $x \in X$ is isometric to an open neighborhood of the cone point in the cone (or "$\kappa$-cone") on a piecewise spherical polyhedron (essentially the link of $x$), and third, the $\kappa$-cone on a piecewise spherical polyhedron $L$ is $\mathrm{CAT}(\kappa)$ if and only if $L$ is $\mathrm{CAT}(1)$. Since such graphs occur as links in 2-dimensional constant curvature polyhedra, we get, via the Link Condition in Theorem 2.4, a condition for a 2-dimensional polyhedron to be locally $\mathrm{CAT}(\kappa)$.

As was pointed out in Section 2.1, if $F$ is a face of a geodesically convex polytope $P$ in constant curvature space, then $\mathrm{Lk}(F,P)$ is a polytope in the unit sphere of the tangent space. Hence, if $\tau$ is a cell of a constant curvature polyhedron $X$, then $\mathrm{Lk}(\tau,X)$ has a natural piecewise spherical structure. If $x$ is any point of a face $F$ of a polytope $P$, then let $\mathrm{Lk}(x,P)$ be the space of inward-pointing tangent vectors at $x$. This space is a join $S_x(F) * \mathrm{Lk}(F,P)$ where $S_x$ is the unit sphere in $T_xF$. This space has a length metric as a subspace of the unit sphere in $T_xP$. (This is the metric of the "spherical join" defined in [23, p. 63]. The union of these spherical joins at a point $x$ of the polyhedron $X$ is denoted $\mathrm{Lk}(x,X)$ and is called the *space of directions at $x$*.

**Lemma 2.1.** *Let $X$ be a piecewise constant curvature polyhedron, $\tau$ a cell of $X$ and $x$ a point in $X$. Then* $\mathrm{Lk}(\tau,X)$ *and* $\mathrm{Lk}(x,X)$ *are piecewise spherical polyhedra.*

Suppose $L$ is a piecewise spherical polyhedron. Define the *euclidean cone* on $L$ by

$$\mathrm{Cone}\, L := (L \times [0,\infty))/\sim, \tag{2.3}$$

where $L \times 0$ is identified to a point. If $(\theta, r) \in \mathbb{S}^{n-1} \times [0,\infty)$ are polar coordinates in $\mathbb{R}^n$, then by using the Law of Cosines one can write a formula for the euclidean distance on $\mathbb{R}^n$. By using $(\theta, r) \in L \times [0,\infty)$ as polar coordinatest he same formula defines a metric on the euclidean cone, $\mathrm{Cone}(L)$. is defined using $(\theta, r) \in L \times [0,\infty)$ as polar coordinates and the Law of Cosines in euclidean space. Similarly, for any real number $\kappa$ and nonnegative real number $r$, define the *$\kappa$-cone of radius $r$ on $L$*, by

$$\mathrm{Cone}^{\kappa}_{r}(L) := (L \times [0,r])/\sim, \tag{2.4}$$

where the metric is defined using polar coordinates and the Law of Cosines in the space of constant curvature $\kappa$. (See [23, p. 59] for more details.) If $\kappa > 0$, we require the radius of the cone to be $\leq \pi/\sqrt{\kappa}$.

**Lemma 2.2.** ([23, pp. 206-207]). $\mathrm{Cone}^{\kappa}_{r}(L)$ *is* $\mathrm{CAT}(\kappa)$ *if and only if $L$ is* $\mathrm{CAT}(1)$.

Lemma 2.2 immediately implies the second sentence of the next lemma.

**Lemma 2.3.** ([23, pp. 206-207]). *If $x$ is a point of a piecewise constant curvature polyhedron $X$, then for small enough $\varepsilon$ the ball of radius $\varepsilon$ centered at $x$ is isometric to* $\mathrm{Cone}^{\kappa}_{\varepsilon}(\mathrm{Lk}(x,X))$. *Hence, $X$ has curvature $\leq \kappa$ if and only if for all $x \in X$,* $\mathrm{Lk}(x,X)$ *is* $\mathrm{CAT}(1)$.

**Definition 2.1.** A piecewise constant curvature polyhedron $X$ satisfies the *Link Condition* if for each vertex $v$, each connected component of $\mathrm{Lk}(v,X)$ is $\mathrm{CAT}(1)$.

**Theorem 2.4.** (The Link Condition, cf. [23, Theorem 5.4, p. 206]). *A piecewise constant curvature polyhedron $X$ has curvature $\leq \kappa$ if and only if it satisfies the Link Condition.*

Theorem 2.4 is almost the same as Lemma 2.3. The only difference is that in Lemma 2.3 links are $\mathrm{CAT}(1)$ at all points $x \in X$, while in Theorem 2.4 this need only be true at vertices.

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*Example 2.2.* (*The Link Condition for* 2*-dimensional piecewise euclidean polyhedra*).

a) Let $X$ be a piecewise euclidean polyhedron of dimension 2. If $X$ is a surface, then the link of any vertex $v$ is then a circle. Each edge of the link corresponds to a 2-cell containing $v$. With its natural spherical structure, the length of the edge is equal to the interior angle of the 2-cell at $v$. So, we get the following well-known statement statement from folklore: the surface $X$ is NPC if and only if the sum of the angles at each vertex is $\geq 2\pi$. If $X$ is a general 2-dimensional piecewise euclidean polyhedron, then the link of each vertex is a graph and the Link Condition says that $X$ is NPC if and only if for each vertex $v$, $\mathrm{Lk}(v,X)$ contains no circuit of length $< 2\pi$ (cf. Example 2.1).

b) If $X$ is a square complex, then since each interior angle in a square is $\pi/2$, the Link Condition says each link must be simple graph without 3-circuits. More generally, if each cell of $X$ is a regular euclidean $m$-gon, then each interior angle is $(m-2)\pi/2m$ and hence, this is the length of each edge in $\mathrm{Lk}(v,X)$. So, when $m \geq 6$, we see that $X$ is NPC whenever each link is a simple graph (i.e., no circuits of length 1 or 2.

*Example 2.3.* (*Regular* $2k$*-gonal complexes*, cf. Examples 4.4 and 4.6 of Section 4.2). Suppose $L^1$ is a simple graph and that $k$ is an integer $\geq 2$. In Section 4.2 by using the theory of Coxeter groups we shall see that there is a simply connected 2-complex $X_k$ so that each 2-cell is a regular euclidean $2k$-gon and the link of each is $L^1$. Since the vertex angle of a regular euclidean $2k$-gon is $\pi - \pi/k$, $L^1$ should be given a piecewise spherical structure with each edge having length $\pi - \pi/k$. When $k = 2$, $L^1$ is $\mathrm{CAT}(1)$ if and only if it has no circuits of length 3, and when $k > 2$, it is always $\mathrm{CAT}(1)$. Thus, for $k > 2$ the 2-complex $X_k$ is $\mathrm{CAT}(0)$ for any $L^1$, while for $k = 2$ it is $\mathrm{CAT}(0)$ provided $L^1$ has no circuits of length 3.

To prove Theorem 2.4 we need Lemmas 2.4 and 2.5 below. Before stating Lemma 2.4 we need the notion of a "spherical join". For $i = 1, 2$, suppose $\sigma_i$ is a spherical polytope in $\mathbb{S}^{m_i}$. The join these two polytopes, denoted $\sigma_1 * \sigma_2$, is naturally a spherical polytope in $\mathbb{S}^{m_1+m_2+1}$. The euclidean cones $\mathrm{Cone}^0(\sigma_i)$ are polyhedral cones in $\mathbb{R}^{m_i+1}$ and the product $\mathrm{Cone}^0(\sigma_1) \times \mathrm{Cone}^0(\sigma_2)$ is the polyhedral cone in $\mathbb{R}^{m_1+m_2+1}$ corresponding to the spherical polytope $\sigma = \sigma_1 * \sigma_2$. In other words, the link of cone point of $\mathrm{Cone}^0(\sigma_1) \times \mathrm{Cone}^0(\sigma_2)$ is the spherical join $\sigma = \sigma_1 * \sigma_2$, where the spherical polytope $\sigma_1 * \sigma_2$ has its natural spherical metric. If, for $i = 1, 2$, $L_i$ is a piecewise spherical polyhedron, then the *spherical join*, $L_1 * L_2$, is defined by

$$L_1 * L_2 := \bigcup_{(\sigma_1,\sigma_2)} \sigma_1 * \sigma_2, \tag{2.5}$$

(Here $(\sigma_1, \sigma_2) \in \mathcal{P}(L_1) \times \mathcal{P}(L_2)$, where $\mathcal{P}(L_i)$ means the poset of cells in $L_i$, including the empty cell, and either $\sigma_1$ or $\sigma_2$ is nonempty.)

The proof of the next lemma is easy.

**Lemma 2.4.** *Suppose $L_1$ and $L_2$ are piecewise spherical polyhedra. Then $L_1 * L_2$ is* $\mathrm{CAT}(1)$ *if and only if both $L_1$ and $L_2$ are* $\mathrm{CAT}(1)$.

**Lemma 2.5.** *If $X$ is a constant curvature polyhedron, then either of the following conditions is equivalent to the Link Condition for $X$.*

(1) *For each $x \in X$,* $\mathrm{Lk}(x,X)$ *is* $\mathrm{CAT}(1)$.
(2) *For each cell $\tau$ of $X$,* $\mathrm{Lk}(\tau,X)$ *is* $\mathrm{CAT}(1)$.
(3) *For each cell $\tau$ of $X$,* $\mathrm{Lk}(\tau,X)$ *has no closed geodesic of length $\leq 2\pi$.*

*Proof.* If $x$ lies in the relative interior of a $k$-dimensional cell $\tau$ of $X$, then $\mathrm{Lk}(x,X)$ is isometric to the spherical join $\mathbb{S}^{k-1} * \mathrm{Lk}(\tau,X)$. By Lemma 2.5, either $\mathrm{Lk}(x,X)$ and $\mathrm{Lk}(\tau,X)$ are both $\mathrm{CAT}(1)$ or neither is $\mathrm{CAT}(1)$.

*Proof (of Theorem 2.4).* Given a vertex $v \in X$, put $L = \mathrm{Lk}(v,X)$. If $\tau$ is a cell of $X$ containing $v$, then there is fix
a corresponding cell $\tau'$ in $L$ (with $\dim \tau' = \dim \tau - 1$) and $\mathrm{Lk}(\tau,X)$ can be identified with $\mathrm{Lk}(\tau',L)$. If $L$ is locally $\mathrm{CAT}(1)$, then $\mathrm{Lk}(\tau',L)$ is $\mathrm{CAT}(1)$. Hence, if the link of each vertex of $X$ is $\mathrm{CAT}(1)$, then $\mathrm{Lk}(\tau,X)$ is $\mathrm{CAT}(1)$ for each cell $\tau$ of $X$ and therefore, by Lemma 2.5, $\mathrm{Lk}(x.X)$ is $\mathrm{CAT}(1)$ for all points $x \in X$. By Lemma 2.3, this is equivalent to $X$ having curvature $\leq \kappa$.

### *2.2.1 Piecewise hyperbolic polyhedra*

One also can consider polyhedra of constant curvature $-1$ (i.e., *piecewise hyperberolic polyhedra*). Each cell of such a piecewise hyperbolic cell complex is a convex polytope in some hyperbolic space $\mathbb{H}^n$. As before, the link of each cell in this cell complex is piecewise spherical complex and by Theorem 2.4, $X$ has curvature $\leq -1$ if and only if each link is $\mathrm{CAT}(1)$.

Next we consider the possibility of deforming the metric on a piecewise euclidean cell complex $X_e$ to a piecewise hyperbolic cell complex $X_h$. If $X_e$ is NPC, then when will $X_h$ be $\mathrm{CAT}(-1)$? The problem is that since the angles change when we deform a euclidean polytope to one which is hyperbolic, the links in $X_e$ might not remain $\mathrm{CAT}(1)$ in $X_h$. If $P_e$ is a small euclidean polytope, say with all edge lengths $< \varepsilon$, then a small deformation will be small hyperbolic polytope $P_h$. The link of a face in $P_h$ will be a small deformation of the link of the corresponding face of $P_h$. A precise formulation of this result can be found in [102]. However, after even a small deformation, the link of the cell in $X_h$ may fail to be $\mathrm{CAT}(1)$. The reason is that if a closed geodesic in the link of a cell in a piecewise euclidean cell complex has length $= 2\pi$, then under a small deformation its length can become $< 2\pi$. For example, each angle of a regular convex 4-gon in $\mathbb{H}^2$ is $< \pi/2$. So, if a euclidean square complex has a vertex link with a circuit with 4-edges, then after deformation this circuit will become a closed geodesic in the link of length $< 2\pi$; hence, the hyperbolic square complex will fail to be locally $\mathrm{CAT}(-1)$. The condition that the link have no such 4-circuits is the "no $\square$ condition" discussed below. More generally, given a compact 2-dimensional piecewise euclidean cell complex $X$, it can deformed to a piecewise hyperbolic complex of curvature $\leq -1$ provided that for each vertex $v$ of $X$, each circuit of $\mathrm{Lk}(v,X)$ has length $> 2\pi$, cf. Example 2.2. (It may be necessary be necessary to first rescale the metric on $X$ so that all its edge lengths are small.)

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**Definition 2.2.** A piecewise spherical complex $L$ is *large* if it is $\mathrm{CAT}(1)$. The cell complex $L$ is *extra large* if the length of the shortest closed geodesic in $L$ is strictly greater than $2\pi$ and if the same holds true for $\mathrm{Lk}(\sigma,L)$ for any cell $\sigma$ in $L$.

**Lemma 2.6.** (Moussong [102, Lemma 5.11]). *Suppose $L$ is a piecewise spherical cell complex with finitely many cells. If $L$ is extra large, then any sufficiently small deformation of it will be* $\mathrm{CAT}(1)$

A corollary of this lemma is the following.

**Proposition 2.1.** (Moussong [102]). *Suppose $X$ is a NPC piecewise euclidean cell complex with finitely many shapes of cells so that for each cell $\sigma$ of $X$,* $\mathrm{Lk}(\sigma,L)$ *is extra large. Then, after rescaling the metric, $X$ can be deformed to a piecewise hyperbolic cell complex with curvature $\leq -1$.*

### 2.2.2 A few remarks concerning word hyperbolic groups

The most important chapter in geometric group theory begins with Gromov's work on hyperbolic groups in [76]. He first defines what it means for metric space to be hyperbolic - his definition encapsulates the notion of having negative curvature in the large. The definition is easiest to state for a geodesic metric spaces. A geodesic in triangle in a geodesic metric space $X$ is $\delta$*-thin* if the distance from any point $x$ on one side of the triangle to the union of the other two sides is $\leq \delta$. The geodesic space $X$ is *hyperbolic* if there is a constant $\delta > 0$ so that every triangle in $X$ is $\delta$-thin. (See [23, pp. 408–409], [76].) A finitely generated group $G$ with a finite set of generators $S$ is *word hyperbolic* if its Cayley graph $\mathrm{Cay}(G,S)$ is a hyperbolic metric space. The property of being hyperbolic is invariant under quasi-isometry. It follows that the property of a group being word hyperbolic is independent of the choice of generating set $S$. The action of a group on a metric space $X$ by isometries is *geometric* if the action is proper and cocompact. The Fundamental Lemma of Geometric Group Theory states that if a group $G$ acts geometrically on two metric spaces $X$ and $X'$ then they are quasi-isometric; in particular, any such $X$ is quasi-isometric to $\mathrm{Cay}(G,S)$. Since the hyperbolic plane $\mathbb{H}^2$

is a hyperbolic, any $\mathrm{CAT}(-1)$-space is a hyperbolic metric space; hence, any group which acts properly on a $\mathrm{CAT}(-1)$-space is word hyperbolic. The above discussion is summarized in the next lemma.

**Lemma 2.7.** *If a group $G$ acts geometrically on a piecewise hyperbolic* $\mathrm{CAT}(-1)$ *cell complex $X$, then $G$ is word hyperbolic.*

### 2.2.3 Reshetnyak's Gluing Lemma

## 2.3 Flag complexes and Gromov's Lemma

A $\Delta$-complex $L$ is a *flag complex* if it is a simplicial complex and if it satisfies the following: any finite set of vertices that are pairwise connected by edges of $L$, spans a simplex of $L$. In other words, any clique in the 1-skeleton $L^1$ spans a simplex of $L$. (A *clique* in a simplicial graph $\Gamma$ is a finite set of vertices of $\Gamma$ such that the induced subgraph is a complete graph. Note that a complete graph is the 1-skeleton of a simplex.)

*Remark 2.2.* (*Comments about terminology*). Combinatorialists use the term "clique complex" instead of "flag complex". The *clique complex* of a simplicial graph $\Gamma$ is the simplicial complex whose simplices are the cliques in $\Gamma$. So, a simplicial complex $L$ is a flag complex if and only if $L$ is equal to the clique complex determined by $L^1$.

Here is a slick version of this definition. A simplicial complex $L$ is a flag complex if and only if every non-simplex contains a non-edge (a set of vertices is a "non-simplex" if it is not the vertex set of a simplex).

A simplicial graph is a *circuit of length $m$* if it is isomorphic to the boundary complex of an $m$-gon). A circuit $\Gamma$ of length $m$ is a flag complex if and only if $m > 3$. More generally, a simplicial graph $\Gamma$ is a flag complex if and only if it contains no circuits of length 3.

Instead of "flag complex" Gromov [76] used the terminology that $L$ satisfies "the no $\Delta$ condition" (pronounced "the no triangle condition").

A natural way in which flag complexes arise is as order complexes of partially ordered sets.

*Example 2.4.* (Barycentric subdivisions). If $X$ is a classical convex cell complex, then the order complex of its poset of cells is a simplicial complex. The geometric realization of the order complex is the barycentric subdivision of $X$. It follows that the requirement of being a flag complex puts no restriction on the topological type of $X$.

The next lemma is left as an exercise for the reader.

**Lemma 2.8.** (Some properties of flag complexes).

(i) *A full subcomplex of a flag complex is a flag complex.*
(ii) *The join of two flag complexes is a flag complex.*
(iii) *The link of a simplex in a flag complex is a flag complex.*

**Lemma 2.9.** (Gromov's Lemma, cf. [76, p. 122], [23, p. 211] or [48, Lemma I.6.1]). *Suppose $L$ is a finite dimensional all right, piecewise spherical simplicial complex (or $\Delta$-complex). Then $L$ is* $\mathrm{CAT}(1)$ *if and only if it is a flag complex.*

*Example 2.5.* (An all right simplicial complex that is not $\mathrm{CAT}(1)$). Start with the triangulation of $\mathbb{S}^2$ as the boundary complex of an octahedron in which each 2-simplex is all right. Let $L$ be the complement of the interior of one of the 2-simplices $\Delta$. Of course, $L$ is not a flag complex since it has an empty triangle, namely $\partial\Delta$. Note that $\partial\Delta$ is a short closed geodesic in $L$ (its length is $3\pi/2$). (To see that $\partial\Delta$ is a closed geodesic, it suffices to show that it is a local geodesic. This follows from the fact that at each vertex of $\partial\Delta$ the link is a circular arc of length $3\pi/2$.)

Combining Gromov's Lemma with the observation in Example 2.4 about barycentric subdivisions, we get the following well-known result of Berestovskii.

**Corollary 2.2.** (Berestovskii [11]). *Any finite dimensional polyhedron can be given a piecewise spherical* $\mathrm{CAT}(1)$ *metric.*

The form in which Lemma 2.9 will most often be used is that in the following corollary. This corollary will also be referred to as "Gromov's Lemma."

**Corollary 2.3.** (Gromov's Lemma for cube complexes). *A piecewise euclidean cube complex is NPC if and only if the link of each vertex is a flag complex.*

*Proof (of Gromov's Lemma).* If $L$ has curvature $\leq 1$ and is not a flag complex, then an argument similar to that in Example 2.5 shows that $L$ has a short geodesic. It folllows by induction on dimension that if $L$ is not a flag complex, then at least one of its links has a short closed geodesic of length $3\pi/2$.

In all right simplicial complex the closed star of a vertex $v$ is the closed ball $B_{\pi/2}(v)$. An abbreviated version of Gromov's argument goes as follows. Suppose $L$ is a flag complex and that $\gamma$ is a closed geodesic which intersects the interior of $B_{\pi/2}(v)$ in a geodesic arc $\gamma'$. The claim is that $l(\gamma') = \pi$. (This is proved by developing the surface determined by $v$ and $\gamma'$ onto the northern hemisphere of $\mathbb{S}^2$.) Hence, if $l(\gamma) < 2\pi$, then it cannot intersect two disjoint open stars about vertices. So, if $V$ denotes the set of vertices $v$ such that $\gamma$ intersects the interior of $B_{\pi/2}(v)$, then any two element of $V$ are connected by an edge. Since $L$ is a flag complex $V$ must span a simplex of $L$. But an all right spherical simplex does not contain a closed geodesic. So, $l(\gamma) \geq 2\pi$.

**The no $\square$ condition**. When is an all right flag complex $L$ extra large as in Definition 2.2? Since all edge lengths are $= \pi/2$, all closed geodesic of length $2\pi$ must correspond to an "empty 4-circuit" in $L$, i.e., a 4-circuit in $L$ so that there is no other edge of $L$ connecting a pair of diagonally opposite edges in the 4-circuit. So, the answer is simple: $L$ cannot contain any empty 4-circuit. (In [76] this is called the *no $\square$ condition*.) Proposition 2.1 then yields the following.

**Lemma 2.10.** *A cube complex can be given a piecewise hyperbolic structure with curvature* $\leq -1$ *if and only if the link of each vertex is a flag complex satisfying the no $\square$ condition.*

## 2.4 Some first examples of NPC cube complexes

### *2.4.1 Products*

repeat from earlier

$G = \Omega$?

Any graph is a 1-dimensional NPC cube complex. Hence, any tree is a $\mathrm{CAT}(0)$ cube complex (by Theorem 2.2 (i)).

A product of graphs is naturally an NPC cube complex. To see this, suppose $X = G_1 \times \cdots \times G_n$ is a product of graphs. The cubes in $X$ correspond to collections $\{c_1, \dots, c_n\}$, where $c_i$ is either an edge or a vertex of $G_i$. The corresponding cube $\Box(c_1, \dots, c_n)$ is isometric to $c_1 \times \cdots \times c_n$. Its dimension is the number of $c_i$ which are edges. If $v = (v_1, \dots, v_n)$ is a vertex of $X$, then its link is the $n$-fold join:

$$\mathrm{Lk}(v, X) = \mathop{\ast}_{i=1}^{i=n} \mathrm{Lk}(v_i, G_i) \tag{2.6}$$

Noting that the link of a vertex in a graph is a discrete space, we see that $\mathrm{Lk}(v, X)$ is a simplicial complex in which each $(k-1)$-simplex is a join of $k$ points. With the metric of a spherical join, each such simplex is all right. By Lemma 2.8 (ii), $\mathrm{Lk}(v, X)$ is a flag complex. By Gromov's Lemma 2.3, $X$ is an NPC cube complex. A similar argument gives the following result.

**Proposition 2.2.** *The product of cube complexes is a cube complex. If $X = X_1 \times \cdots \times X_n$ is such a product and each $X_i$ is NPC, then $X$ is NPC.*

*Remark 2.3.* One can prove directly, without mentioning the link condition, that a product of NPC spaces is NPC. (See [23, Exercise 1.9 (c), p. 163 and Examples 1.15(3), p. 167].)

*Example 2.6.* (The cubical complex $P_L$). Given a finite simplicial complex $L$ with vertex set $I$, in section 3.1 of Chapter 3 we define a subcomplex of the cube $[-1, 1]^I$ such that the link at each vertex is $L$. It follows from Gromov's Lemma that if $L$ is a flag complex, then $P_L$ is NPC.

### *2.4.2 Alternating link complements*

### *2.4.3 Square complexes*

**Higman groups**. In Serre's book. Paper by Martin Burger-Mozes. Wise put in bibliography

### *2.4.4 Configuration spaces of a graph*

Given a space $Y$ and a natural number $n \in \mathbb{N}$, the *configuration space* of $n$ ordered points in $Y$ is the space of $n$-tuples $(y_1, \dots, y_n) \in Y^n$ such that $y_i \neq y_j$ for $1 \leq i < j \leq n$, in other words, it is the complement in $Y^n$ of the fat diagonal. When $Y$ is a cell complex we shall often remove a neighborhood of the fat diagonal instead. For example, suppose $Y$ is a connected graph $G$. Define $\mathcal{C}_n(G)$ to be the subcomplex of $G^n$ consisting of all cubes of the form $\Box(c_1, \dots, c_n)$, where $\{c_1, \dots, c_n\}$ is a collection of pairwise disjoint cells of $G$. In other words, $\mathcal{C}_n(G)$ is the union of all cubes in $G^n$ which do not meet the fat diagonal. After replacing $G$ by a suitable subdivision, it is easy to see that $\mathcal{C}_n$ is a deformation retract of the complement of the fat diagonal in $G^n$. So, $\mathcal{C}_n(G)$ is a model for the configuration of $n$ ordered points in $G$.

$\Gamma$ instead of $G$?
reference [132] Świ

The symmetric group $\Sigma_n$ acts freely and cellularly on $\mathcal{C}_n(G)$. The orbit space $\mathcal{C}_n(G)/\Sigma_n$ is denoted by $\overline{\mathcal{C}}_n(G)$ and is called the *unordered configuration space of* $G$. Since $\Sigma_n$ acts freely on $\mathcal{C}_n(G)$, the unordered configuration space is also a cube complex. A $k$-cube in $\overline{\mathcal{C}}_n(G)$ corresponds to a set $\{x_1, \dots, x_k, u_1, \dots, u_{n-k}\}$, where the $x_i$ are edges of $G$ and the $u_i$ are vertices. The cube is denoted $\Box\{x_1, \dots, x_k, u_1, \dots, u_{n-k}\}$.

**Proposition 2.3.** *The cube complexes $\mathcal{C}_n(G)$ and $\overline{\mathcal{C}}_n(G)$ are NPC.*

*Proof.* After subdividing we may assume that $G$ is a simplicial graph. Since $\mathcal{C}_n(G)$ is a covering space of $\overline{\mathcal{C}}_n(G)$, it suffices to consider $\mathcal{C}_n(G)$ . Put $\mathcal{C} = \mathcal{C}_n(G)$. According to Gromov's Lemma 2.3, we only need show that the link of each vertex of $\mathcal{C}$ is a flag complex. A vertex $x$ of $\mathcal{C}$ has the form, $x = (x_1, \dots, x_n)$, where $\{x_1, \dots, x_n\}$ is a collection of $n$ disjoint vertices in $G$. By (2.6), $\mathrm{Lk}(x, G^n)$ is the join $L_1 * \cdots * L_n$, where $L_i = \mathrm{Lk}(x_i, G)$. since $\mathcal{C}$ obtained from $G^n$ by deleting certan open cells, $\mathrm{Lk}(x, \mathcal{C})$ is similarly, obtained from $\mathrm{Lk}(x, G^n)$. First, notice that certain vertices must be deleted from $\mathrm{Lk}(x, G^n)$. This happens whenever $x_i$ and $x_j$ are adjacent vertices in $G$. Indeed, if $c = [x_i, x_j]$ denotes the edge from $x_i$ to $x_j$, then all faces of the square $c \times c$, except for the vertices $(x_i, x_j)$ and $(x_j, x_i)$, are deleted. Let $u_{ij}$ be the vertex of $L_i$ corresponding tot the direction from $x_i$ to $x_j$. (Similarly, $u_{jI} \in L_j$ corresponds to the direction from $x_j$ to $x_i$.) Let $L'_i$ denote the complement of $\bigcup u_{ij}$ in $L_i$ (a discrete set of vertices). Then $\mathrm{Lk}(x, \mathcal{C})$ is a subcomplex of the subjoin,

$$L'_1 * \cdots * L'_n < L_1 * \cdots * L_n = \mathrm{Lk}(x, G^n).$$

To obtain $\mathrm{Lk}(x, \mathcal{C})$ we must delete more simplices from $L'_1 * \cdots * L'_n$. For example, suppose the distance in $G$ from $x_i$ to $x_j$ is 2. Let $v$ be a vertex on the geodesic between them. The square $[x_i, v] \times [x_j, v]$ has vertices $(x_i, x_j)$, $(x_i, v)$, $(v, v)$, $(v, x_j)$. Since interior of this square is deleted; however, the interiors of the edges $[x_i, v]$ and$[x_j, v]$ are not deleted. If $u_{iv} \in L'_i$ and $u_{jv} \in L'_j$ are thedirections correspondin to these edges , then the interior of the 1-simplex $u_{iv} * u_{jv}$ is deleted from $L'_i * L'_j$. Similarly, if $y_0, \dots, y_k$ are distinct vertices in $\{x_1, \dots, x_n\}$ and each is another vertex $v$ adjacent to all of them (as in Figure **??**), and if $u_{iv}$ are corresponding vertices of $\mathrm{Lk}(x, \mathcal{C})$ then the interior of the $k$-simplex $u_{0v} * \cdots * u_{kv}$ is deleted. However, the point is that the interior of each positive-dimensional subface of $u_{0v} * \cdots * u_{kv}$ is also deleted. Since $L'_1 * \cdots * L'_n$ is a join, it is a flag complex (by Lemma 2.8 (ii)). The simplicial complex $\mathrm{Lk}(x, \mathcal{C})$ is obtained from this by deleting open simplices and each edge of such a simplex is also deleted, so there are no "empty simplices". It follows that $\mathrm{Lk}(x, \mathcal{C})$ is a flag complex.

The fundamental groups of $\mathcal{C}_n(G)$ and $\overline{\mathcal{C}}_n(G)$ are called, respectively, the *pure braid group* and *braid group* of $G$ and denoted by $PB_n(G)$ and $B_n(G)$, [39]. (See also [2], [72], [132].)

**The symmetric product of a graph**. For any space $Y$, the symmetric group $\Sigma_n$ acts on the $n$-fold product $Y^n$. The orbit space $Y^n/\Sigma_n$ is called the $n$-fold *symmetric product* of $Y$.

### *2.4.5 Graph braid groups embed in* RAAG*s*

later, after RAAGs

The discussion follows [39].

Given a graph $G$, define another graph $\Delta(G)$. The vertex set of $\Delta(G)$ is the edge set of $G$. Given $e \in \mathrm{Edge}(G)$, denote by $\overline{e}$ the corresponding vertex of $\Delta(G)$. Vertices $\overline{e}$ and $\overline{f}$ span an edge of $\Delta(G)$ if and only if $e$ and $f$ are disjoint.

Let $A_{\Delta(G)}$ be the RAAG associated to $\Delta(G)$ and let $BA_{\Delta(G)}$ be the standard model for its classifying space. The is a 1-cell of $BA_{\Delta(G)}$ for each vertex of $\Delta(G)$. Choose an orientation for edge of $G$. Define a cubical map $\Phi : \overline{\mathcal{C}}_n(G) \to BA_{\Delta(G)}$ by sending the $k$-cube $\square\{x_1, \dots, x_k, u_1, \dots, u_{n-k}\}$ to the cube in $BA_{\Delta(G)}$ spanned by the edges corresponding to $\overline{x}_1, \dots, \overline{x}_k$ while respecting the orientations of all edges. (The image of the $k$-cube in $BA_{\Delta(G)}$ is a $k$-torus.) It is proved in [39, Theorem 2] that $\Phi$ is a local isometry and hence, is $\pi_1$-injective.) This gives the following.

**Proposition 2.4.** (Crisp-Weist [39]). *The map $\Phi$, defined above, induces an injection, $B_n(G) \hookrightarrow A_{\Delta(G)}$.*

In other words, any graph braid group is a subgroup of a RAAG.

# Chapter 3
# Right-angled complexes and groups

Many examples of cubical complexes can be defined using the notion of a "polyhedral product" with respect to a simplicial complex $L$. Under suitable conditions the universal cover of such a polyhedral product often will be $\mathrm{CAT}(0)$.

The fundamental group of a polyhedral product is the "graph product" of the fundamental groups of its factors.

Dual cones, dual of a simplicial complex, strict fundamental domain

## 3.1 Polyhedral products

Let $L$ be a simplicial complex with vertex set $I$. The poset of simplices (including the empty simplex) is denoted by $\mathcal{S}(L)$. If $\sigma$ is a simplex of $L$, then $I(\sigma)$ denotes its vertex set. Change $I$ to $V$? Vert $\sigma$?

Suppose that $\mathbf{A} = \{(A_i, B_i)\}_{i\in I}$ is a collection of pairs of spaces with base points $*_i$ chosen in $B_i$.

**Definition 3.1.** The *polyhedral product* of $\mathbf{A}$ with respect to $L$ is the subset $\mathbf{A}^L$ of the product $\prod_{i\in I} A_i$ consisting of the points $(x_i)_{i\in I}$ satisfying the following two conditions:

(a) $x_i = *_i$ for all but finitely many $i$,
(b) $\{i \in I \mid x_i \notin B_i\}$ is a simplex of $L$.

For pairs $M < M'$ of subcomplexes of $L$, the base points determine natural inclusions $\mathbf{A}^M \hookrightarrow \mathbf{A}^{M'}$. When $L$ is finite, the polyhedral product inherits its topology from the product $\prod_{i\in I} A_i$. In general $\mathbf{A}^L$ acquires the topology of the direct limit, $\lim \mathbf{A}^M$, as $M$ runs through the finite subcomplexes.

One can also view $\mathbf{A}^L$ as a union of products as follows. For each simplex $\sigma$ in $\mathcal{S}(L)$ put

$$\mathbf{A}^\sigma = \prod_{i\in \operatorname{Vert}\sigma} A_i \times \prod_{i\notin \operatorname{Vert}\sigma} B_i$$

So, if $\sigma$ is the empty simplex, then $A^\sigma = \prod B_i$. Also, note that if each $B_i = *_i$, then $\mathbf{A}^\sigma$ is just the product of the $A_i$, $i \in \operatorname{Vert}\sigma$. Then

$$\mathbf{A}^L = \bigcup_{\sigma\in\mathcal{S}(L)} A^\sigma \,. \tag{3.1}$$

The topology on $\mathbf{A}^L$ also can be explained using (3.1). If $\Delta$ denotes the full simplex on $I$, then $\mathbf{A}^\Delta$ is the subset of the product of the $A_i$ satisfying (a). So, $\mathbf{A}^\Delta$ is the union of all $\mathbf{A}^\sigma$ over all finite subsets of $\sigma \leq I$. Give $\mathbf{A}^\Delta$ the direct limit topology and $A^L \leq A^\Delta$ is the induced topology.

$Z_L$? The main examples of this book are when:

(1) each $A_i$ is the interval $[0,1]$, $B_i = *_i = 1$,
(2) each $A_i$ is the interval $[-1,1]$, $B_i = \partial[-1,1] = \{\pm 1\}$ and $*_i = 1$,
(3) each $B_i$ is a discrete space $E_i$ and $A_i = \operatorname{Cone} E_i$ and $*_i \in E_i$
(4) each $A_i = S^1$ and $B_i = *_i = 1$.

In (3) $\operatorname{Cone} E_i$ means the cone on $E_i$, i.e., it is the quotient space $(E_i \times [0,1])/\sim$. For example, if $E_i = \{\pm 1\}$, then $\operatorname{Cone} E_i$ can be identified with $[-1,1]$. The *cone point* of $E_i$ is the point $0_i$ corresponding to $0 \in [0,1]$ Choose an element $*_i \in E_I$ to be the base point. Let $\mathbf{Cone}\,\mathbf{E}$ denote the $I$-tuple $(\operatorname{Cone} E_i, E_i, *_i)_{i \in I}$.

Denote the polyhedral products in (1),(2), (3), (4) by $K_L$, $P_L$, $Z_L$, and $\mathbb{T}^L$, respectively, i.e.,

$$K_L = (0,1],1)^L, \tag{3.2}$$
$$P_L = ([-1,1],\{\pm 1\})^L, \tag{3.3}$$
$$Z_L = (\mathbf{Cone}\,\mathbf{E},\mathbf{E})^L, \tag{3.4}$$
$$\mathbb{T}_L = (S^1,1)^L. \tag{3.5}$$

When the pairs $(A_i,B_i)$ are independent of $i$ so that $(A_i,B_i) = (A,B)$, we have writen simply $(A,B)^L$ instead of using boldface (for example, this is the case in (3.2), (3.3) and (3.5)).

### *3.1.1 The cube complexes $K_L$, $P_L$, $Z_L$, and $\mathbb{T}_L$*

Since polyhedral products are subspaces of products, $P_L$ is a cubical subcomplex of the cube $[-1,1]^I$ while $\mathbb{T}_L$ is a subcomplex of the torus, $(S^1)^I$. Regarding $S^1$ as a 1-cube complex with one edge and one vertex, the product $(S^1)^I$ becomes a cube complex and $\mathbb{T}_L$ is a cubical subcomplex. The cone, $\operatorname{Cone} E_i$, is the union of intervals, one for each point of $E_i$, glued together at the cone point $0_i$. So, $\operatorname{Cone} E_i$ is a 1-cube complex. Hence, the product $\prod_{i\in I} \operatorname{Cone} E_i$ is a cube complex. It follows that the polyhedral product $Z_L$, defined by (3.4), is also a cube complex. Each of these cube complexes, $K_L$, $P_L$, $Z_L$ and $\mathbb{T}_L$ has the same dimension, namely, $1+\dim(L)$.

**The complex** $P_L$. Let $\mathbf{C}_2 = \{\pm 1\}$ be the cyclic group of order 2 and denote by $(\mathbf{C}_2)^I$ (or by $\sum_{i\in I} \mathbf{C}_2$ the direct sum of $I$ copies of $\mathbf{C}_2$. Similarly, $[-1,1]^I$ is the set of all $(t_i)$ in the product satisfying condition (a) in Definition 3.1. The group $(\mathbf{C}_2)^I \curvearrowright [-1,1]^I$ as a reflection group. Each element of $I$ can be regarded as a standard basis vector for the euclidean space $\mathbb{R}^I$. So, each $(k-1)$-face $\sigma$ of the full simplex on $I$ determines a $k$-dimensional subspace $\mathbb{R}^\sigma$ spanned by the vertex set $I(\sigma)$ of $\sigma$. The cube complex $P_L$ is stable under the $(\mathbf{C}_2)^I$-action on $[-1,1]^I$. Any $k$-dimensional face of $[-1,1]^I$ is parallel to some $\mathbb{R}^\sigma$. We denote such a face by $\square^\sigma$. Moreover, given $\sigma$ the corresponding set of $k$-cubes forms a single $(\mathbf{C}_2)^I$-orbit which we denote by $\operatorname{orb}(\square^\sigma)$. In other words, $\operatorname{orb}\square^\sigma \cong (\mathbf{C}_2)^{I-I(\sigma)} \times [-1,1]^{I(\sigma)}$. Thus,

$$P_L = \bigcup_{\sigma\in\mathcal{S}(L)} \operatorname{orb}\square^\sigma. \tag{3.6}$$

So, orb$\square^{\emptyset}$ means the $(\mathbf{C}_2)^I$-orbit of a vertex, i.e., orb$\square^{\emptyset} = \{\pm 1\}^I$.

*Example 3.1.* some kind of examples

(i) If $L$ consists of $n$ vertices and no higher dimensional cells, then $P_L$ is the 1-skeleton of an $n$-cube.
(ii) If $L = \partial \Delta^n$ is the boundary complex of an $n$-simplex, then $P_L$ is the boundary complex of an $(n+1)$-cube, i.e., $P_L \cong S^n$
(iii) Suppose $L$ is an $m$-circuit (that is, a circle triangulated as the boundary of an $m$-gon). The square complex $P_L$ is a closed 2-manifold. (since the link of each vertex is $S^1$ and the link of each edge is $S^0$). It is easily seen to be connected and orientable. Its Euler characteristic is $2^m - m2^{m-1} + m2^{m-2} = 2^m(1-m/4)$. This example appeared already in Coxeter's 1938 paper [37, p. 57]. (Thanks to Alex Suciu for pointing out this reference.) If $m = 4$, $P_L$ is a flat 2-torus.

If the 1-skeleton $L^1$ of $L$ is not a complete graph, then $P_L$ is not simply connected. In general, $\pi_1(P_L)$ depends only on $L^1$ (cf. [51]).

**The complex** $Z_L$. The polyhedral product, $Z_L := (\mathbf{Cone\,E})^L$, defined by (3.4), is a generalization of the previous example $P_L$ of a polyhedral product of intervals.

*Example 3.2.* Suppose, for $i = 1, 2$, that $E_i$ is a finite set of $m_i$ points. If $L$ is a 1-simplex $\sigma^1$, then the polyhedral product $Z_{\sigma^1} = \operatorname{Cone} E_1 \times \operatorname{Cone} E_2$ is a square complex. Each square has the form $\operatorname{Cone} e_1 \times \operatorname{Cone} e_2$, where $e_i \in E_i$. This square complex also has the structure of cone, namely the cone on $Z_{\partial\sigma^1} = (\operatorname{Cone} E_1 \times E_2) \cup (E_1 \times \operatorname{Cone} E_2)$. The space $Z_{\partial\sigma^1}$ can be identified with the join $E_1 * E_2$ (a complete bipartite graph). (See Figure **??**.)

More generally, suppose $\sigma$ is an $(n-1)$-simplex with vertex set $I = \{1, \dots, n\}$ and that $E_i$ is a finite set of cardinality $m_i$. Then $Z_\sigma$ is the product, $\operatorname{Cone} E_1 \times \cdots \times \operatorname{Cone} E_n$, which is a cube complex. Moreover, $Z_{\partial\sigma}$ is homeomorphic to the join, $E_1 * \cdots * E_n$. If $I(\sigma)$ is the vertex set of $\sigma$ regarded as a simplicial complex with no edges, then $Z_{I(\sigma)}$ is a graph – the star of a vertex lying above $i \in I$ is isomorphic to $\operatorname{Cone} E_i$. Why say the last sentence?

**The fundamental chamber** $K_L$. Suppose each $E_i$ is a singleton, necessarily the base point. The polyhedral product $Z_L$ becomes

$$K_L := (\operatorname{Cone} *_i, *_i)^L. \tag{3.7}$$

Since $(\operatorname{Cone} *_i, *_i) \cong ([0,1], 1)$, $K_L$ is naturally a cube complex. Since each cube contains the base point $* = (*_i)_{i \in I}$, $K_L$ is a cone – in fact it can be identified with the cone on the barycentric subdivision of $L$. Note that $K_L$ is a subcomplex of $Z_L$. In particular, it is a subcomplex of $P_L$. We have that $P_L$ is a subcomplex of $[-1,1]^I$ and $K_L$ can be identified with the intersection $P_L \cap [0,1]^I$. So, it is a strict fundamental domain, in the sense defined below, for the $(\mathbf{C}_2)^I$-action on $P_L$. For this reason, $K_L$ is called the *fundamental chamber*.

To simplify notation, put $K = K_L$ and for each vertex $i \in I$, define the *mirror* $K_i$ to be the intersection of $K$ with the hyperplane $x_i = 0$. Similarly, for each simplex $\sigma \in \mathcal{S}(L)$, put

$$K_\sigma = \bigcap_{i \in \operatorname{Vert}\sigma} K_i \ = \{x \in K \mid x_i = 0 \text{ for all } i \in I(\sigma)\}.$$

later in links

$\mathbb{T}_L$**, the standard classifying space for a** RAAG. The cube complex $\mathbb{T}_L$, defined by (3.5), is a polyhedral product of circles. If $L = \Delta$ is an $(n-1)$-simplex, then $\mathbb{T}_\Delta$ is the $n$-torus, while $\mathbb{T}_{\partial\Delta}$ is the $(n-1)$-skeleton of the $n$-torus. If $L$ is a flag complex, then $\mathbb{T}_L$ is the *Salvetti complex*. Thus, $\mathbb{T}_L$ is a subcomplex of $\mathbb{T}^\Delta$ where

$\Delta$ denotess the full simplex on $I$. In other words, $\mathbb{T}_L$ is the union of subtori $\mathbb{T}_\sigma$, where $\sigma \in \mathcal{S}(L)$, cf. (3.1). (In some contexts, we might define the Salvetti complex to be the covering space of $\mathbb{T}_L$ with group of deck transformations the RACG associated to $L$.)

**Definition of a strict fundamental domain**.

**Definition 3.2.** Suppose $G$ is a discrete group acting on a space $X$. A closed subspace $Y < X$ is a *strict fundamental domain* if it intersects each $G$-orbit in exactly one point.

do we need to assume proper?

We leave the proof of the next lemma as an exercise for the reader.

**Lemma 3.1.** *Suppose $G \curvearrowright X$ with strict fundamental domain $Y$. Let $\pi : X \to X/G$ be projection to the orbit space and let $\overline{\pi} = \pi|_Y$. Then $\overline{\pi} : Y \to X/G$ is a homeomorphism.*

One can reconstruct the $G$-action on $X$ from the group $G$, the strict fundamental domain $Y$, and the knowledge of the isotropy subgroups $G_y$ at each point $y \in Y$ (where $G_y = \{g \in G \mid gy = y\}$). Define an equivalence relation $\sim$ on $G \times Y$ by

$$(g,y) \sim (g',y') \iff y = y' \text{ and } g^{-1}g' \in G_y \,. \tag{3.8}$$

The *basic construction* is the $G$-space $D(G,Y)$ defined by $D(G,Y) = (G \times Y)/\sim$. Let $[g,y]$ denote the equivalence class of $(g,y)$.

**Lemma 3.2.** (Properties of the basic construction).

(i) *The natural map $y \mapsto [1,y]$ from $Y$ into $D(G,Y)$ is an embedding (which we regard as an inclusion).*
(ii) *$G \curvearrowright D(G,Y)$ and $Y$ is a strict fundamental domain.*

**Links of vertices**. One of the main features of $P_L$ is that the link of each of its vertices can be identified with $L$. We state this as follows.

**Lemma 3.3.** *For each vertex $v$ of $P_L$,* $\mathrm{Lk}(v,P_L) = L$.

Consider the 1-cube complex $\mathrm{Cone}\,E_i$. Let $0_i$ denote the cone point. There are two types of vertices in $\mathrm{Cone}\,E_i$: the cone point $0_i$ is a vertex as is each element $e \in E_i$. As for links, $\mathrm{Lk}(0_i, \mathrm{Cone}\,E_i) = E_i$, while for each $e \in E_i$, $\mathrm{Lk}(e, \mathrm{Cone}\,E_i)$ is a point. Next consider the general case, $Z_L := (\mathbf{Cone}\,\mathbf{E})^L$. Any vertex $v$ of $Z_L$ has the form $v = (v_i)_{i\in I}$, where $v_i$ is a vertex of $\mathrm{Cone}\,E_i$, i.e., either $v_i \in E_i$ or $v_i = 0_i$. Put

$$\tau(v) = \{i \in \mid v_i = 0_i\}. \tag{3.9}$$

So, $\tau(v)$ is a simplex of $L$ (possibly the empty simplex). This proves the following lemma which describes links in $Z_L$.

**Lemma 3.4.** *For a vertex $v$ of $Z_L$ let $\tau(v)$ be defined by* (3.9)*. The link of $v$ in $Z_L$ is the join:*

$$\mathrm{Lk}(v,Z_L) = \mathrm{Lk}(\tau(v),L) * \mathop{\ast}_{i\in I(\tau(v))} E_i. \tag{3.10}$$

*In particular, if each $v_i$ lies in some $E_i$, then* $\mathrm{Lk}(v,Z_L) = L$.

earlier

*Remark 3.1.* If each $E_i$ consists of two points, then $Z_L = P_L$. However, the cubical structure on $P_L$ is coarser then the one we get by identifying it with $Z_L$. The reason is that if we identify $[-1,1]$ with $\mathrm{Cone}\{\pm 1\}$, then we should subdivide $[-1,1]$ into two intervals, $[-1,0]$ and $[0,1]$. Thus, when thought of as $Z_L$, each $n$-cube in $P_L$ is subdivided into $2^n$ cubes.

earlier **Definition 3.3.** (Compare Definition 3.1 and formula (3.1)). Suppose $\{X_i\}_{i\in I}$ is a collection of spaces indexed by the set $I$ of verices of a simplicial complex $L$. For any simplex $\sigma$ of $L$, let $X(\sigma)$ denote the join $\ast_{i\in I(\sigma)} X_i$. The *polyhedral join over $L$* of $\{X_i\}_{i\in I}$ is defined by

$$\ast_L X_i := \bigcup_{\sigma\in\mathcal{S}(L)} X(\sigma). \tag{3.11}$$

**Lemma 3.5.** *The link of each vertex $v$ in $\mathbb{T}_L$ is a polyhedral join of* 0*-spheres,* $\mathrm{Lk}(v,\mathbb{T}_L) = \ast_L S^0$.

**Lemma 3.6.** *The polyhedral join of flag complexes is a flag complex.*

The next result follows from Lemma 2.3 (Gromov's Lemma).

**Theorem 3.1.** *The cube complexes $K_L$, $P_L$, $Z_L$ and $\mathbb{T}_L$ are NPC if and only if $L$ is a flag complex.*

*Proof.* The issue is to show that links of vertices in $P_L$, $Z_L$ and $\mathbb{T}_L$ are flag complexes if and only if $L$ is a flag complex. So, suppose $L$ is a flag complex. Since the link of each vertex in $P_L$ is $L$ (by Lemma 3.3), this is immediate for $P_L$. By Lemma 3.4, the link of a vertex $v$ of $Z_L$ has the form $\mathrm{Lk}(\tau(v),L) \ast \ast_{i\in I(\tau(v))} E_i$. By part (iii) of Lemma 2.8, $\mathrm{Lk}(\tau(v),L)$ is a flag complex and by part (ii), so is $\ast_{i\in I(\tau(v))} E_i$. Another application of part (ii) gives that the join, $\mathrm{Lk}(\tau(v),L) \ast \ast_{i\in I(\tau(v))} E_i$, is a flag. complex. The complex $K_L$ is the special case of $Z_L$ where each $E_i$ is a singleton. As for $\mathbb{T}_L$, if $L$ is a flag complex, then, by Lemma 3.6, so is $\ast_L S^0$; hence, by Lemma 3.5, the link of each vertex in $\mathbb{T}_L$ is a flag complex.

**Universal covers**. More important for us than the cube complexes $P_L$, $Z_L$ and $\mathbb{T}_L$ will be their universal covers, denoted $\tilde{P}_L$, $\tilde{Z}_L$ and $\tilde{\mathbb{T}}_L$, respectively. If $L$ is a flag complex, then by the Cartan-Hadamard Theorem (Theorem 2.2 (i)), these universal covers are $\mathrm{CAT}(0)$ cube complexes. The complex $\tilde{P}_L$ is the *Davis complex* for the right-angled Coxeter group associated to $L$, when each $E_i$ has a least two points the complex $\tilde{Z}_L$ is a *right-angled building*, and $\tilde{\mathbb{T}}_L$ is the universal cover of the Salvetti complex for the right-angled Artin group associated to $L$. In order to describe the groups which act on these complexes we need the notion of a "graph product" of groups, developed in the next subsection.

Here is another straightforward application of Gromov's Lemma.

**Lemma 3.7.** *For each $i \in I$ suppose $A_i$ is an NPC cube complex and its basepoint $\ast_i \in A_i$ is a vertex of $A_i$. Put $\mathbf{A} = (A_i, \ast_i)$. Then $\mathbf{A}^L$ is naturally a cube complex. Moreover, it is NPC if and only if $L$ is a flag complex.*

### *3.1.2 Polyhedral products of NPC cube complexes*

**Theorem 3.2.** *A polyhedral product over a flag complex $L$ of NPC cube complexes is an NPC cube complex.*

Examples? RACGs. Start with a RACG and replace each generator with another RACG, eg., the infinite dihedral group $D_\infty$ (cf..

### *3.1.3 Graph products of groups*

Suppose $\{G_i\}_{i\in I}$ is a collection of groups indexed by the vertex set $I$ of a simplicial graph $L^1$. Their *graph product*, denoted $\prod_{L^1} G_i$, is the quotient of the free product of the $G_i$ by the normal subgroup generated by all commutators of the form $[g_i, g_j]$, where $\{i, j\} \in \operatorname{Edge} L^1$, $g_i \in G_i$ and $g_j \in G_j$. (These were first studied by Droms [65].)

As examples, if $\operatorname{Edge} L^1 = \emptyset$, then the graph product is the free product, while if $L^1$ is a complete graph, then the graph product is the direct sum: $\prod_{L^1} G_i = \sum_{i\in I} G_i$. (The *direct sum* $\sum_{i\in I} G_i$ is the subset of the direct product where only finitely many coordinates are not equal to the identity element of $G_i$.) There is a canonical epimorphism $p : \prod_{L^1} G_i \to \sum_{i\in I} G_i$.

Given a system of groups, the notion of their "direct limit" is defined, for example, on page 1 of [128]. The graph product alternatively can be described as the direct limit of certain system of groups indexed by the poset of cells in $L^1$:

$$G_i,\ i \in I, \quad G_i \times G_j,\ \{i, j\} \in L^{(1)},$$

together with the natural inclusions $G_i \hookrightarrow G_i \times G_j$. The graph product $\prod_{L^1} G_i$ is then the direct limit of this system of groups. (In Chapter 5 this system of groups will be presented as a nice example of a "simple complex of groups".)

examples *Example 3.3.* (RACG*s and* RAAG*s*). If each $G_i$ is the cyclic group of order two, then the graph product is called the *right-angled Coxeter group* associated to $L^1$. (The term "right-angled Coxeter group" will often be abbreviated to RACG.) If each $G_i$ is the infinite cyclic group, then $\prod_{L^1} \mathbb{Z}$ is the *right-angled Artin group* (abbreviated to RAAG) associated to $L^1$.

Let $G = \sum_{i\in I} G_i$ denote the direct sum. Suppose $G_i \curvearrowright E_i$. Then $G \curvearrowright (\mathbf{Cone}\,\mathbf{E})^{\Delta}$, where $\Delta$ means the full simplex on $I$. The polyhedral product $Z_L = (\mathbf{Cone}\,\mathbf{E})^L$ is stable under the $G$-action. Choose base points $*_i \in E_i$ so that the $I$-tuple $* = (*_i)_{i\in I}$ is a base point for the polyhedral product $Z_L$. Let $K_L$ be the subcomplex of $Z_L$ corresponding to this choice of base points. The cone point $0_i$ of $\operatorname{Cone} E_i$ lies in $K_L$ and its isotropy subgroup in $G$ is $G_i$. Let $p : \tilde{Z}_L \to Z_L$ be the universal cover and let $\tilde{*}$ be a lift of $*$ in $\tilde{Z}_L$. Since $K_L$ is simply connected, each component of $p^{-1}(K_L)$ is mapped homeomorphically onto $K_L$. Let $\tilde{K}$ denote the component containing $\tilde{*}$ and let $\tilde{0}_i$ denote the inverse image of $0_i$ in $\tilde{K}$.

**Lemma 3.8.** *With notation as above, let $G = \sum G_i$ be the direct sum and let $\Gamma = \prod_{L^1} G_i$ be the graph product. Let $Z_L = (\mathbf{Cone}\,\mathbf{E})^L$ be the polyhedral product and let $\tilde{Z}_L$ be its universal cover. For each $i \in I$, suppose the $G_i$-action on $E_i$ is simply transitive. Then $\Gamma$ is the group of all lifts of the $G$-action to $\tilde{Z}_L$. Hence, there is a short exact sequence,*

$$1 \to \pi_1(Z_L) \to \Gamma \to G \to 1\ .$$

fix *Proof.* Let $H$ be the group of all lifts of the $G$-action to $\tilde{Z}_L$ and let $\varphi : H \to$ be the canonical projection. The subgroup $H_i$ consisting of all lifts which fix the point $\tilde{c}_i$ projects isomorphically onto $G_i$. If $\{i, j\} \in L^{(1)}$, then $\operatorname{Cone} E_i \times \operatorname{Cone} E_j$ is a subcomplex of $Z_L$ and there is an isomorphic copy of this subcomplex in $\tilde{K}$ containing $\tilde{*}$. In particular, this subcomplex contains the square $[\tilde{*}_i, \tilde{c}_i] \times [\tilde{*}_j, \tilde{c}_j]$. The subgroup $H_i \times H_j$ fixes the point $\tilde{c}_i \times \tilde{c}_j$ and projects isomorphically to $G_i \times G_j$. So, $H_i \times H_j$ is the group of all lifts of $G_i \times G_j$ which fix $\tilde{c}_i \times \tilde{c}_j$. In particular, this shows that whenever $\{i, j\}$ is an edge of $L^1$, the subgroups $H_i$ and $H_j$ commute. It follows from this and the universal property of the direct limit (cf. **??**), that the natural inclusions, $G_i \to H_i \hookrightarrow H$ and $G_i \times G_j \to H_i \times H_j \hookrightarrow H$, with $|[i, j\} \in \operatorname{Edge} L^1$, extend to a homomorphism $\phi : G \to H$ from the graph product to the group of lifts. Since ${G_i}^I$ acts simply transitively on $(E_i)^I$, $H$ acts simply transitively on $p^{-1}((E_i)^I)$. A straighforward argument implies that $\phi : G \to H$ is an isomorphism.

*Example 3.4.* Let $L$ be a finite dimensional flag complex on vertex set $I$. For each $i \in I$, suppose $G_i$ is a nontrivial finite group. Let $Z_L = (\operatorname{Cone} G_i, *_i)^L$ be the polyhedral product and $\tilde{Z}_L$ the associated RAB. Let $\Gamma$ be the graph product. Then $\Gamma$ acts properly, each finite subgroup is conjugate to a finite subgroup of $G = \sum_{i\in I} G_i$. It follows that $\tilde{Z}_L$ is the universal proper $\Gamma$-space $\underline{E}G$.

Next consider a polyhedral product, where each $A_i = BG_i$, the classifying space of a discrete group $G_i$.

**Proposition 3.1.** *Suppose $L$ is a flag complex. Then the polyhedral product $(BG_i, *_i)^L$ is the classifying space for the graph product $\Gamma = \prod_{L^1} G_i$.*

*Proof.* Let $p_i : EG_i \to BG_i$ be the universal cover. The induced map of polyhedral products, $(EG_i, p^{-1}(*_i))^L \to (BG_i, {}_i)^L$, is then a covering projection. Since $(EG_i, p_i^{-1}(*_i))$ is contractible, $(EG_i, p_i^{-1}(*_i))$ and $(\mathrm{Cone}\, G_i, G_i)$ are homotopy equivalent as pairs; so, the polyhedral products $Z_L = (\mathrm{Cone}\, G_i, G_i)^L$ and $Y_L = (EG_i, p_i^{-1}(*_i))^L$ also are homotopy equivalent. Hence, their universal covers $\tilde{Z}_L$ and $\tilde{Y}_L$ are homotopy equivalent. Moreover, the group of lifts of the $(\sum_{i \in I} G_i)$-action on $Y_L$ to $\tilde{Y}_L$ is also identified with the graph product $\Gamma$. So, the composition $\tilde{Y}_L \to Y_L \to (BG_i, *_i)^L$ is the universal cover and $\Gamma$ is the group of deck transformations. By Theorem 3.1, since $L$ is a flag complex, $Z_L$ is NPC and hence, by the Cartan-Hadamard Theorem, $\tilde{Z}_L$ is contractible. (Alternatively, it is proved in [49] that $\tilde{Z}_L$ is the standard realization of a right-angled building (a "RAB") and hence, is contractible (since the standard realization of any RAB is contractible, see **??**). Since $\tilde{Y}_L$ is homotopy equivalent to $\tilde{Z}_L$, it also is contractible. Hence, $B\Gamma \sim (BG_i, *_i)^L$.

Examples?

Someplace else

*Example 3.5.* (*Standard classifying spaces for* RAAG*s and* RACG*s*, cf. Examples 3.3). Suppose each $G_i$ is infinite cyclic so that $BG_i = S^1$. For each simplex $\sigma \in \mathcal{S}(L)$, the polyhedral product $(S^1, *_i)^\sigma$ is a torus with basis $\mathrm{Vert}\,\sigma$. By Proposition 3.1, when $L$ is a flag complex, $\mathbb{T}_L = (S^1, *_i)^L$ is the classifying space $BA_L$ for the RAAG associated to $L^1$. (The NPC cube complex $\mathbb{T}_L$ from (3.5) in Subsection'3.1.1 is the Salvetti complex of $A_L$.)

If each $G_i$ is $\mathbf{C}_2$, then $B\mathbf{C}_2 = \mathbb{R}P^\infty$. When $L$ is a flag complex, $(\mathbb{R}P^\infty, *_i)^L$ is the classifying space $BW_L$ for the RACG associated to $L^1$.

Proposition 3.1 suggests that the fundamental group of an arbitrary polyhedral product with each $B_i = *_i$ is a graph product of the fundamental groups of the factors. In fact, this is proved in [51, Theorem 2.18, p. 248]. We state this as the following proposition without giving the proof.

**Proposition 3.2.** ([51]). *Suppose $\{A_i\}_{i \in I}$ is a collection of path connected CW complexes indexed by the vertex set of a simplicial complex $L$. Let $*_i$ be a base point and put $\mathbf{A} = (A_i, *_i)$ and $G_i = \pi_1(A_i, *_i)$. Then $\pi_1(\mathbf{A}^L)$ is the graph product $\Gamma = \prod_{L^1} G_i$. (In particular, $\pi_1(\mathbf{A}^L)$ depends only on the 1-skeleton of $L$.)*

Actually a more general fact is proved in [51]: if the $B_i$ is only required to be a subspace of $A_i$, then $\pi_1(\mathbf{A}^L)$ is an appropriately defined "generalized graph product".

### 3.1.4 Graph wreath products

Suppose $L^1$ is a simple graph on vertex set $I$ (which need not be finite) and $\Gamma = \prod_{L^1} G$ denotes the graph product. Let $J$ be a group of automorphisms $L^1$. The $J$-action on $L^1$ naturally induces a $J$-action on $\Gamma$ via group automorphisms. The semidirect product $\Gamma \rtimes J$ is called the (restricted) *graph wreath product* of the G. (The terminology was introduced in [91].) When $L^1$ is the complete graph, $\Gamma$ is the direct sum $\sum_{i \in I} G$ and $\Gamma \rtimes J$ is the ordinary restricted wreath product. ("Restricted" refers to the fact that we are using the direct sum rather than the direct product.)

The graph wreath product $\Gamma \rtimes J$ acts naturally on 1) the universal cover $\tilde{Z}_L$ of the polyhedral product $Z_L = (\mathrm{Cone}\, G, G)$; it also acts on 2) the classifying space $E\Gamma$ for $\Gamma$. To see the action in case 1), first consider the action of the ordinary wreath product $\sum G \rtimes J$ on the weak product, $(\mathrm{Cone}\, G, G)^{\Delta(I)}$, where $\Delta(I)$ is the full simplex on $I$. The subspace $(\mathrm{Cone}\, G, G)^L$ is stable under the action of $\sum G \rtimes J$. The group of lifts of this action to $\tilde{Z}^L$ is the graph wreath product $\Gamma \rtimes J$. (To see this it is sufficient to note that since $J$ fixes the base point of $(\mathrm{Cone}\, G, G)^L$, its group of lifts of $J$ at a base point of $\tilde{Z}_L$ projects isomorphically to $J$). Suppose $M = L^J$ denotes the subcomplex of $L$ fixed by $J$. It follows that $(\tilde{Z}_L)^J = (\tilde{Z}_M)$: in particular, since this fixed point set contains a given lift of the base point, it is nonempty. It follows that the action of $\Gamma \rtimes J$ on $\tilde{Z}_L$ is proper if and only $J$ is a finite group.

One can check case 2) by following the construction in the proof of Proposition **??**. The covering space $(EG, p^{-1}(*))^L \to (BG, *)^L$ corresponds to the homomorphism $\pi_1((BG,*)^L) = \Gamma \to \sum G$. The wreath product $\sum G \rtimes J$ acts on $(EG, p^{-1}(*))^L$ and $\Gamma \rtimes J$ is the group of lifts to the universal cover $E\Gamma$. If $J$ is finite, then action of $\Gamma \rtimes J$ on this model of $E\Gamma$ is proper. Since $E\Gamma^J$ is homotopy equivalent to $\tilde{Z}_{L^J}$ which is contractible, $E\Gamma^J$ is also contractible. Hence, when $J$ is finite, $E\Gamma$ is a model for $\underline{E}(\Gamma \rtimes J)$, the universal space for proper actions of the graph wreath product. We record this as the following.

**Proposition 3.3.** *Suppose $J$ is a finite group of automorphisms of $L$ and that $\Gamma \rtimes J$ is the graph wreath product. Then the universal cover of $(BG, *)^L$ is a model for $\underline{E}(\Gamma \rtimes J)$.*

### *3.1.5 Application: cohomological dimension versus geometric dimension*

The geometric dimension of a group $G$, denoted $\mathrm{gd}\, G$, is the smallest dimension of a model for $BG$ by a CW complex. If $EG$ denotes the universal cover of $BG$, then its cellular chin complex $C_* := \{C_i(EG)\}$ is a chain complex of free $\mathbb{Z}G$-modules. In fact, $C_i(EG)$ is isomorphic to a direct sum of copies of $\mathbb{Z}G$ where there is one summand for each $i$-cell in $BG$. Since $EG$ is contractible, the chain complex $C_*$ is acyclic. In fact, if we equip it with the augmentation homorphism $C_0 \to \mathbb{Z}$ then $C_*$ gives a free resolution of $\mathbb{Z}$.

The cohomological dimension of $G$, denoted $\mathrm{cd}\, G$ is the shortest length of a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}G$-modules. (Here one can replace the word “projective” by “free”.) It follows that $\mathrm{gd}\, G \geq \mathrm{cd}\, G$. In the 1950 s Eilenberg and Ganea [66] raised the question if these two numbers are equal and in almost all cases they proved that they are equal. More precisely, we have the following.

**Theorem 3.3.** (The Eilenberg-Ganea Theorem [66]).

(1) *If* $\mathrm{cd}\, G = 1$*, then* $\mathrm{gd}\, G = 1$.
(2) *If* $\mathrm{cd}\, G \geq 3$*, then* $\mathrm{gd}\, G = \mathrm{cd}\, G$*.*
(3) *If* $\mathrm{cd}\, G = 2$*, then either* $\mathrm{gd}\, G = 2$ *or* $\mathrm{gd}\, G = 3$

Actually statement (1) follows from results of Stallings and Swan that show that groups of cohomological ! are free. The Eilenberg-Ganea Conjecture is the conjecture that we always have equality between $\mathrm{gd}\, G$ and $\mathrm{cd}\, G$.

**Acyclic versus contractible**.

**Probable counterexamples to the Eilenberg-Ganea Conjecture.**

## 3.2 Reflection groups

There is a well-known, classical theory of groups generated by reflections on the constant curvature spaces, $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$. Since much of the intuition for the constructions of polyhedral products in section 3.1, we briefly will give some highlights of this theory here. A fuller account can be found in **??**.

Where should this be in relation to ???
Maybe it stays here

A *dihedral group* is a group generated by two involutions There are two basic examples of geometric actions of dihedral actions to have in mind. First, the group generated by the reflections across an interval in $\mathbb{E}^1$ gives an action of the infinite dihedral group $D_\infty$ with fundamental domain the interval. The resulting tessellation of $\mathbb{E}^1$ by intervals gives it the structure of a 1-cube complex. Second consider the group generated by reflections across two lines through the origin in $\mathbb{R}^2$. If the angle between the two lines is $\pi/m$ the group is the finite dihedral group $D_m$ of order $2m$. A strict fundamental domain for the action is the sector bounded by two rays making an angle of $\pi/m$. If we restrict the $D_m$-action to the unit circle $\mathbb{S}^1$, a strict fundamental domain is a circular arc of length $\pi/m$.

Before considering more of the classical geometric pictures, let us discuss a fairly general situation of a reflection group acting on connected smooth manifold $X$. A *reflection* on $X$ is a smooth involution $r : X \to X$ such the its fixed set $X^r$ disconnects $X$ into two connected components, the closure of one of these components is called a *half-space* and the fixed set $X^r$ is a *wall*. One sees that each wall is a submanifold of codimension one and that each half-space is a manifold with boundary and a strict fundamental domain for the action of $\mathbf{C}_2$ $(= \langle\rangle)$ on $X$. Suppose that $W$ is a discrete group acting properly on $X$ and that $W$ is generated by the set $R$ of all reflections in $W$. Choose a $K^\circ$ denote a connected component of $X - \bigcup_{r\in R} X^r$ and let $K$ denote the closure of $K^\circ$. Then $K$ is a *chamber* for the reflection group $W$. One says say that $X^r$ is a *wall of* $K$ if $K \cap X^r$ is codimension zero in $X^r$. It follows that $K$ is the intersection of half-spaces bounded by the walls of $K$. Let $S$ be the set of reflections across the walls of $K$. Here are the main facts.

(1) $R$ is the set of all conjugates of elements of $S$ by elements of $W$. In particular, since $R$ generates $W$, $S$ is a set of generators for $W$.
(2) The group $W$ is a *Coxeter group*. This means that $W$ has a presentation of the form $\langle S \mid \mathcal{R}\rangle$ where $S$ is the above set of generators and the relations $\mathcal{R}$ consists of all words of the form: $s^2$, with $s \in S$ and $(st)^{m(s,t)}$, where $\{s,t\}$ ranges over all unordered pairs of distinct elements of $S$ and where $m(s,t)$ is the order of $st$ in $W$ (and if $m(s,t) = \infty$ we omit the relation.) The pair $(W,S)$ is called a *Coxeter system*If all the $m(s,t)$ are equal to 2 or $\infty$, then $W$ is a right-angled Coxeter group.
(3) $K$ is a strict fundamental domain for the $W$-action (see Definition 3.2). It follows from considering the basic geometric examples on $\mathbb{E}^1$ and $\mathbb{R}^2$ that $K$ has codimension one faces where the isotropy group is $\mathbf{C}_2$. In fact it will follow from **??** below that $K$ is a manifold with corners. Hence, $X$ together with the $W$-action can be recovered via the basic construction $D(W,K)$ (cf. (3.8) and the following definition).

Next we specialize to the geometric examples where $X = \mathbb{S}^n$, $\mathbb{E}^n$ of $\mathbb{H}^n$, reflections are isometries and half-spaces are actual geodesically convex half-spaces. Since the fundamental chamber $K$ is an intersection of half-spaces, it is a convex polytope. We require $K$ to be compact.

(a) $(X = \mathbb{S}^n)$. Assume $W$ acts without global fixed points on $\mathbb{S}^n$. Then $K$ is a spherical simplex. (Actually this is a result in linear algebra, cf. [48, Lemma 6.3.3, p. 78].) This result implies that when $X = \mathbb{E}^n$ or $\mathbb{H}^n$, the polytope $K$ is *simple*, which means that the link of each nonempty face of $K$ is a simplex. Similarly, in the case where $X$ is a general manifold, it follows that $K$ is a manifold with corners.
(b) $(X = \mathbb{E}^n)$.
(c) $(X = \mathbb{H}^n)$.

### 3.2.1 Davis complex

- Geometric examples with simple polytopes

### 3.2.2 Semidirect products

subsumed in graph wreath products

### 3.2.3 The reflection group trick

This is method for showing any group of type $F$ is a retract of the fundamental group of a closed aspherical manifold. In fact, any group of type $FH$ is a retract of a Poincaré duality group.

**Corollary 3.1.** *(Mess, Weinberger, Sapir)*

(i) *The fundamental group of a closed aspherical manifold need not have solvable word problem.*
(ii) *The fundamental group of a closed aspherical manifold need not be residually finite.*
(iii) [100] *The fundamental group of a closed aspherical manifold need not have solvable word problem.*
(iv) *It can contain a "Tarski monster" (*[123]*)*

## 3.3 Right-angled buildings

A product of trees is a RAB (provided each vertex has degree $\geq 2$). So, the universal cover of a product of graphs without degree 1 vertices is a RAB.

The RABs of form $\tilde{Z}_\Delta$ are very symmetric RABs. The $\tilde{Z}_L$ are highly symmetric in that automorphism group is chamber-transitive.

The space $Z_L$ is usually not simply connected. (In fact, if each $E_i$ has at least two elements, then $Z_L$ is simply connected if and only if $L^1$ is a complete graph.)

**Definition 3.4.** Suppose $\mathrm{Card}(E_i) \geq 2$ for each $i \in I$ and suppose $L$ is a flag complex. Consider the polyhedral product $Z_L = (\mathbf{Cone}\,\mathbf{E}, \mathbf{E})^L$ from (3.4). Then $\tilde{Z}_L$, the universal cover of $Z_L$, is called (the standard realization of) a *right-angled building*, and abbreviated as RAB.

In fact, each RAB $\tilde{Z}_L$ is the standard realization of a Tits building (see [?]); the associated Coxeter group is the RACG, $W_L$, associated to the the 1-skeleton of $L$; each apartment is isomorphic to its Davis complex $\tilde{P}_L$.

## 3.4 Right-angled Artin groups

Salvetti complex

### 3.4.1 The standard cube complex

### 3.4.2 Residually torsion-free nilpotent, left orderable

# Chapter 4
# Coxeter groups, Artin groups, buildings

Suppose $L^1$ is a simple graph with its edges labeled by integers $\geq 2$. Let $S = \operatorname{Vert} L^1$. The edge labeling is a function $m : \operatorname{Edge}(L^1) \to \{2, 3 \dots\}$ written as $\{s,t\} \mapsto m(s,t)$. For $s = t$, put $m(s,s) = 1$. Then $m(s,t)$ is a symmetric $(S \times S)$-matrix called the *Coxeter matrix* associated to $(L^1, m)$. This data is enough to define two groups, a Coxeter group $W$ and an Artin group $A$. Define a presentation of $W$ so that its set of generators $S$ coincides with the vertex set of $L^1$. The relations are given by

$$s^2, \text{ for all } s \in S, \quad \text{ and } \quad (st)^{m(s,t)}, \text{ for all } \{s,t\} \in \operatorname{Edge} L^1. \tag{4.1}$$

That is to say, each generator of $S$ is an involution and every other relation is a word involving only the two letters that correspond to the end points of an edge of $L^1$. Then $W$ is called the *Coxeter group* and the pair $(W, S)$ is the *Coxeter system* associated to the labeled graph $(L^1, m)$. For any subset $T$ of $S$ the subgroup $W_T$ generated by $T$ is a *special subgroup*. It turns out that $(W_T, T)$ is itself a Coxeter system. If $W_T$ is finite, then the subset $T$ and the subgroup $W_T$ that it generates are said to be *spherical*. Denote by $\mathcal{S}$ (or $\mathcal{S}(W,S)$) the set of spherical subsets of $S$. There is an associated abstract simplicial complex $L(W,S)$ called the *nerve* of $(W,S)$; its vertex set is $S$ and its poset of simplices (including the empty simplex) is $\mathcal{S}$.

To define the presentation for the associated Artin group $A$ we introduce symbols $\{\sigma_s\}_{s \in S}$ for its generators. The relations are given by omitting the relations corresponding to the $s^2$ in (4.1) and by rewriting the relations corresponding to $(st)^{m(s,t)}$ as

$$\underbrace{\sigma_s \sigma_t \cdots}_{m(s,t) \text{ terms}} = \underbrace{\sigma_t \sigma_s \cdots}_{m(s,t) \text{ terms}} \tag{4.2}$$

In other words, the alternating word of length $m(s,t)$ in $\sigma_s$ and $\sigma_t$ is equal to the alternating word of the same length written in the other order. The relation in (4.2) is called the *Artin relation* (or the *braid relation* when $m(s,t) = 3$). Since each $s$ has order two, it follows that the Artin relation is simply a rewriting of $(st)^{m(s,t)}$. Hence, the map $\sigma_s \mapsto s$ gives a canonical epimorphism $A \to W$.

A *spherical coset* in $W$ is a left coset in $W/W_T$, where $W_T$ is a spherical special subgroup. Let

$$W\mathcal{S} := \bigsqcup_{T \in \mathcal{S}} W/W_T \tag{4.3}$$

denote the set of spherical cosets. The set $W\mathcal{S}$ can be given the structure of a poset. The order relation is inclusion of one coset in another. Similarly, a spherical coset in $A$ is a left coset in $A/A_T$, where $A_T$ denotes the spherical special subgroup of $A$ generated by $\{\sigma_s\}_{s \in T}$. Let

$$A\mathcal{S} := \bigsqcup_{T\in\mathcal{S}} A/A_T \tag{4.4}$$

denote the poset of spherical cosets in $A$. The geometric realization of the order complex of $W\mathcal{S}$ is the barycentric subdivision of the Davis-Moussong complex $\Sigma(W,S)$ described in Subsection 4.2.3. The geometric realization of the order complex of $A\mathcal{S}$ is the barycentric subdivision of the Deligne complex $\Lambda(W,S)$ described in Subsection 4.3.3.

The poset of spherical cosets $W\mathcal{S}$ also plays an important role in the theory of buildings. As will be explained in Section 4.4, a building is a set $\mathcal{C}$ of "chambers" together with some extra structure. In particular, every building has an associated Coxeter system $(W,S)$. The standard geometric realization of the building $\mathcal{C}$ is the order complex of its "poset of spherical residues." An "apartment" in $\mathcal{C}$ is a subposet isomorphic to $W\mathcal{S}$.

Roughly speaking, the term "simple complex of groups" means a poset of groups, i.e., an assignment of a group $G_\sigma$ to each element of a poset $\mathcal{Q}$. The relevant definitions and the theory of simple complexes of groups are the subject of Chapter 5. Our goal in this chapter is to explain three basic examples of simple complexes of groups associated to a Coxeter system $(W,S)$. For all three examples the underlying poset is the same: the poset $\mathcal{S}$ of spherical subsets of $S$. (Actually to be in compliance with the definitions in [23] we will eventually want to replace $\mathcal{S}$ by its opposite poset $\mathcal{S}^{\mathrm{op}}$ where the order relations are reversed.) Here are the three types of examples.

(1) The complex $\{W_T\}_{T\in\mathcal{S}}$ of special spherical subgroups of $W$.
(2) The complex $\{A_T\}_{T\in\mathcal{S}}$ of special spherical subgroups of an Artin group $A$.
(3) The complex $\{G_T\}_{T\in\mathcal{S}}$ of residue stabilizers in building $\mathcal{C}$ of type $(W,S)$ with a chamber transitive automorphism group $G$ (cf. Example **??**).

## 4.1 Some simple complexes of groups

Maybe not needed at this point

The theory of simple complexes of groups is developed in [23]. Basic examples of simple complexes of groups are provided by Coxeter groups, Artin groups and buildings. So, we will make a few comments concerning the general theory here and postpone a fuller discussion to Appendix **??**

Let $\mathcal{P}$ be a poset. A *chain* in $\mathcal{P}$ is a totally ordered subset. In other words, a finite subset $\{p_0,\dots p_k\}$ is a chain if, possibly after reordering, $p_0 < \cdots < p_k$. The *order complex* of $\mathcal{P}$ is the set of all chains in $\mathcal{P}$. It is an abstract simplicial complex; a $k$ simplex is a chain of length $k+1$. Denote the geometric realization of the order complex by $|\mathcal{P}|$. Reversing the order relation on $\mathcal{P}$ one obtains the *opposite poset* $\mathcal{P}^{\mathrm{op}}$. Note that the order complexes of $\mathcal{P}$ and $\mathcal{P}^{\mathrm{op}}$ are identical

Here is the definition from the book of Bridson-Haefliger.

**Definition 4.1.** (cf. Bridson-Haefliger [23, II.12.11, p. 375]). A *simple complex of groups* $G\mathcal{Q}$ over a poset $\mathcal{Q}$ is a collection of groups $\{G_\sigma\}_{\sigma\in\mathcal{Q}}$ and monomorphisms $\phi_{\sigma\tau}: G_\sigma \to G_\tau$ defined whenever $\tau<\sigma$. The $G_\sigma$ are the *local groups*. Furthermore, $G\mathcal{Q}$ must be a cofunctor from $\mathcal{Q}$ to the category of groups and monomorphisms in the sense that $\phi_{\sigma\tau}\phi_{\tau\mu} = \phi_{\sigma\mu}$ whenever $\mu<\tau<\sigma$. (A "cofunctor" is a contravariant functor, i.e., a functor on $\mathcal{Q}^{\mathrm{op}}$.)

Simple complexes of groups arise from group actions with strict fundamental domains. Suppose $G \curvearrowright Y$ with strict fundamental domain $|\mathcal{Q}|$. The space $|\mathcal{Q}|$ has a stratification with strata indexed by $\mathcal{Q}$, where $|\mathcal{Q}|_\sigma = |\mathcal{Q}_{\le\sigma}|$. (For example if $\mathcal{Q}$ is the poset of cells in a cell complex, then each stratum is a cell.) The local groups are the stabilizers of the strata (one assumes the $G$-action is without inversions meaning that the setwise stabilizer of a stratum fixes each point of the stratum. So, $G_\sigma$ is the isotropy subgroup of $|\mathcal{Q}|_\sigma$. (The reason a simple complex of groups is defined as a cofunctor rather than a functor is that for group actions, a smaller stratum should correspond to a larger isotropy subgroup.) The simple complex of groups is *developable* if it is induced from a $G$-action.

The *direct limit*, $\lim G\mathcal{Q}$, is defined in the usual fashion by taking the free product of the $G_\sigma$ and then quotienting by relations to insure that $G_\sigma$ is identified with its image under $\phi_{\sigma\tau}$ in $G_\tau$ and that $\phi_{\sigma\tau}\phi_{\tau\mu}(g) = \phi_{\sigma\mu}(g)$ whenever $\mu < \tau < \sigma$, for all $g \in G_\mu$. A *simple morphism* $\psi = (\psi_\sigma)$ from $G\mathcal{Q}$ to a group $G$ is a function which assigns to each $\sigma \in \mathcal{Q}$ a homomorphism $\psi_\sigma : G_\sigma \to G$ such that $\psi_\tau = \psi_\sigma\phi_{\sigma\tau}$ whenever $\tau < \sigma$. It turns out that $G\mathcal{Q}$ is developable if and only if the canonical simple morphism $G_\sigma \to \lim G\mathcal{Q}$ is injective for each $\sigma \in \mathcal{Q}$.

The next result uses the basic construction of Subsection 3.1.1 to show that every developable $G\mathcal{Q}$ arises from an action with strict fundamental on the basic construction.

**Theorem 4.1.** (The basic construction, cf. [23]). *Suppose that $G\mathcal{Q}$ is developable and $G = \lim G\mathcal{Q}$. Then there is a poset $D(G,\mathcal{Q})$ such that $G \curvearrowright D(G,\mathcal{Q})$ with strict fundamental poset $\mathcal{Q}$. Taking geometric realizations, we get a polyhedron $D(G,|\mathcal{Q}|)$ such that $G \curvearrowright D(G,|\mathcal{Q}|)$ with strict fundamental domain $|\mathcal{Q}|$.*

The basic construction $D(G,|\mathcal{Q}|)$,

$$D(G,|\mathcal{Q}|) = (G \times |\mathcal{Q}|)/\sim \tag{4.5}$$

is defined as in Subsection 3.1.1. Given $x \in |\mathcal{Q}|$, let $\sigma(x)$ be the index the smallest stratum $|\mathcal{Q}|_\sigma$ such that $x \in |\mathcal{Q}|_\sigma$. the equivalence relation $\sim$ is then defined the same way as in (3.8):

$$(g,x) \sim (g',x') \iff x = x' \text{ and } g^{-1}g' \in G_{\sigma(x)}\ , \tag{4.6}$$

in other words, $gG_{\sigma(x)}$ and $g'G_{\sigma(x)}$ are the same coset in $G/G_{\sigma(x)}$.

It also turns out that when $|\mathcal{Q}|$ is simply connected, then so is $D(G,|\mathcal{Q}|)$ (see **??**). So, when $|\mathcal{Q}|$ is simply connected, $G = \lim G\mathcal{Q}$ should be regarded as the "fundamental group of $G\mathcal{Q}$".

Given a Coxeter system $(W,S)$, we define two simple complexes of groups over $\mathcal{S}^{\mathrm{op}}$, where $\mathcal{S} = \mathcal{S}(W,S)$, the poset of spherical subsets of $S$. Denote the order relation on $\mathcal{S}^{\mathrm{op}}$ by $\prec$, so that $T' \prec T$ if $T$ is a proper subset of $T'$. The *fundamental chamber* $K$ is defined as $|\mathcal{S}^{\mathrm{op}}|$. The stratum $K_T$ is the union of all simplices in $|\mathcal{S}^{\mathrm{op}}|$ with maximum vertex $= T$.

*Example 4.1.* (*The complex of spherical Coxeter subgroups of $W$.*) There is a simple complex of groups $\mathcal{W}\mathcal{S}^{\mathrm{op}}$ over $\mathcal{S}^{\mathrm{op}}$ called the *complex of spherical subgroups of $W$*. It is defined as the cofunctor $T \mapsto W_T$, so that if $T' \prec T$, then $W_T$ is a subgroup of $W_{T'}$. It is obvious that the direct limit, $\lim \mathcal{W}\mathcal{S}^{\mathrm{op}}$, is equal to$W$. On the level of posets the basic construction $D(W,\mathcal{S}^{\mathrm{op}}$ is equal to the poset of spherical cosets $W\mathcal{S}$ defined in (4.3). On the level of spaces it is the cell complex $\Sigma(W,S)$ of Subsection 4.2.3 (actually, the strata of $D(W,K)$ are dual cones to the cells in $\Sigma(W,S)$).

*Example 4.2.* (*The complex of spherical Artin subgroups of $A$.*) The *complex of spherical subgroups of $A$* is the simple complex of groups $\mathcal{A}\mathcal{S}^{\mathrm{op}}$ over $\mathcal{S}^{\mathrm{op}}$ defined by $T \mapsto A_T$. If $T' \prec T$, then $A_T$ is a subgroup of $A_{T'}$. We also have $\lim \mathcal{A}\mathcal{S}^{\mathrm{op}} = A$. On the level of posets, $D(A,\mathcal{A}\mathcal{S}^{\mathrm{op}}) = A\mathcal{S}$ where $A\mathcal{S}$ is poset of spherical cosets from (4.4) and on the level of spaces, $D(A,K)$ is the Deligne complex $\Lambda(W,S)$ of Subsection 4.3.3 below.

## 4.2 Coxeter groups

The material in this section can be found in [48].

### *4.2.1 Spherical Coxeter groups*

Suppose $(W,S)$ is a Coxeter system with $\mathrm{Card}(W) < \infty$. The group $W$ has a representation as an orthogonal linear reflection group on $\mathbb{R}^n$, with $n = \mathrm{Card}(S)$, so that each $s \in S$ corresponds to reflection across a supporting hyperplane of a simplicial cone. When $(W,S)$ is irreducible this representation is unique up to homothety (e.g., see [15, p. 70]). It is called the *canonical representation* . The set of reflections $R$ in $W$ is precisely the set of conjugates of elements of $S$. A hyperplane $H_r$ fixed by a reflection $r \in R$ is a *wall* and the set of all such hyperplanes, $\mathcal{A} := \{H_r\}_{r\in R}$ is the associated *reflection arrangement*. The arrangement $\mathcal{A}$ cuts $\mathbb{R}^n$ into simplicial cones; this collection of cones together and their faces is called the *fan*, $\mathrm{Fan}(\mathcal{A})$, associated to $\mathcal{A}$. Each top-dimensional simplicial cone in $\mathrm{Fan}(\mathcal{A})$ is called a *chamber*. There are two chambers (antipodal to each other) bounded by the walls indexed by $S$. Let $C$ be one of them. Then $C$ is a strict fundamental domain for the $W$-action on $\mathbb{R}^n$. A *codimension-one face* $C_s$ of $C$ is the intersection of $C$ with the wall fixed by $s$. The intersection of $\mathrm{Fan}(\mathcal{A})$ with $\mathbb{S}^{n-1}$ is called the *spherical fan* and denoted $\mathbb{S}\mathrm{Fan}(\mathcal{A})$. It is a tessellation of $\mathbb{S}^{n-1}$ by spherical simplices. The underlying simplicial complex is the *Coxeter complex* of $(W,S)$.

The intersection $C\cap\mathbb{S}^{n-1}$ is denoted by $\sigma(W,S)$ (or sometimes simply by $\sigma$) and is called the *fundamental spherical simplex* (or the *fundamental chamber*). The codimension-one faces of $\sigma$ $(= \sigma(W,S))$ are the faces $\sigma_s := \sigma \cap H_s$ and for each proper subset $T < S$ we have a face $\sigma_T$ of $\sigma$ defined by $\sigma_T := \bigcap_{s\in T}$. The isotropy subgroup at a point in the relative interior of $\sigma_T$ is a special subgroup $W_T$. Since $W \curvearrowright \mathbb{S}^{n-1}$ with $\sigma(W,S)$ as a strict fundamental domain, the sphere $\mathbb{S}^{n-1}$ can be identified with the basic construction $D(W,\sigma)$ defined Subsection 3.1.2, where, as in (3.8), the equivalence relation $\sim$ on $W\times\sigma$ is defined by

$$(w,x) \sim (w',x') \iff x = x' \text{ and } wW_{S(x)} = w'W_{S(x)}. \tag{4.7}$$

Here $S(x) = \{s \in S \mid x \in \sigma_s\}$.

For each $s \in S$, let $u_s$ be the outward-pointing unit normal vector to the the face $C_s$ of $C$ (or equivalently, the outward-pointing normal vector to $\sigma$). Since the angle between $u_s$ and $u_t$ is the exterior dihedral angle along $C_s \cap C_t$, this angle is $\pi - \pi/m(s,t)$. Hence, the matrix of inner products $\langle u_s, u_t\rangle$ is equal to the *cosine matrix* $c(s,t)$ of $(W,S)$. This is the $(S\times S)$-symmetric matrix defined by

$$c(s,t) = -\cos(\pi/m(s,t)). \tag{4.8}$$

When $s = t$, $m(s,s) = 1$ so that $c(s,s) = 1$. (The formula in (5.6) for the cosine matrix $c(s,t)$ makes sense for any Coxeter system $(W,S)$, provided that when $m(s,t) = \infty$, we interpret $-\cos(\pi/\infty)$ to be $-\cos(0)$ $(= -1)$.) Since $\{u_s\}_{s\in S}$ is a basis for $\mathbb{R}^n$ when $(W,S)$ is spherical we see that the cosine matrix is positive definite whenever $W$ is finite. The converse is also true and this gives the following well-known characterization for a Coxeter group to be finite.

**Lemma 4.1.** (e.g. see [48, Theorem 6.12.9]). *The Coxeter group $W$ is finite if and only if the cosine matrix from* (5.6) *is positive definite.*

The *fundamental dual simplex* $\sigma^*(W,S)$ is the spherical simplex in $\mathbb{S}^{n-1}$ spanned by the unit vectors $\{u_s\}_{s\in S}$. The simplex $\sigma^*(W,S)$ actually is dual to $\sigma(W,S)$ in the sense that it consists of all points $y \in \mathbb{S}^{n-1}$ of distance $\geq \pi/2$ from $\sigma(W,S)$, i.e., $\sigma^*(W,S) = \{y \in \mathbb{S}^{n-1} \mid \langle y,x\rangle \leq 0 \text{ for all } x \in \sigma(W,S)\}$. The length of the circular arc connecting the vertices $u_s$ and $u_t$ of $\sigma^*(W,S)$ is $\pi - \frac{\pi}{m(s.t)}$.

### 4.2.2 Coxeter zonotopes

Associated a spherical Coxeter system $(W,S)$ there is a Coxeter zonotope $Z$ $(= Z(W,S))$ that is dual to the reflection arrangement $\mathcal{A}$. It is a simple convex polytope such that $\partial Z$ is dual to the simplicial complex $\mathbb{S}\mathrm{Fan}(\mathcal{A})$. Here is an explicit description. Let $C$ be the fundamental chamber bounded by the walls indexed by $S$. Choose a base point $x_0$ in the interior of $C$ and consider its $W$-orbit, $Wx_0$. Then $Z(W,S)$ can be defined to be the convex hull of $Wx_0$. Coxeter zonotope

**Lemma 4.2.** (e.g. see [33, Lemma 2.1.3]). *Suppose that $(W,S)$ is a spherical Coxeter system and that $Z(W,S)$ is the corresponding Coxeter zonotope. Then the face poset of $Z(W,S)$ is isomorphic to the poset of spherical cosets $W\mathcal{S}$ defined in* (4.3). *The face corresponding to the coset $wW_T$ has vertex set $(wW_T)x_0$ and this face is isomorphic to the zonotope $Z(W_T,T)$.*

As examples, if $(W,S)$ is the symmetric group on $(n+1)$ letters (i.e., if its Coxeter diagram is $\mathbf{A}_n$), then $Z$ is a permutohedron. When $n=2$ a permutohedron is a hexagon. If $W=(\mathbf{C}_2)^n$, then $Z$ is an $n$-cube. N.B. The action of $W$ on $Z$, regarded as a cell complex $Z$ "has inversions," e.g., the maximum cell $Z$ is stabilized by the entire group $W$ but is not pointwise fixed.

The link, $\mathrm{Lk}(x_0,Z)$ of the vertex $x_0$ in $Z$ is naturally identified with the dual spherical simplex $\sigma^*(W,S)$. The edges meeting at the vertex $x_0$ are parallel to the basis of unit normal vectors, $\{u_s\}_{s\in S}$. The metric on $Z(W,S)$ depends on the position of $x_0$ in the interior of $C$. We can normalize this position by choosing $x_0$ to be the unique point of distance $=1/2$ from each wall $H_s$, $s\in S$. This will have the effect of giving each edge of $Z(W,S)$ a length of 1. The face corresponding to $wW_T$ is said to be of *type* $T$; it is a cell of dimension $=\mathrm{Card}(T)$.

**Figure of hexagon and pemutohedron**

Zonotopes play the same role in constructing the standard CAT(0) complexes for general Coxeter groups as do cubes for RACGs. They are also used in the construction of the Salvetti complex for a general Artin groups.

### 4.2.3 The Davis-Moussong complex

We return to the situation where $(W,S)$ is an arbitrary Coxeter system. Let $\mathcal{S}=\mathcal{S}(W,S)$ be its poset of spherical subsets and let $W\mathcal{S}$ be the set of spherical cosets (cf. (4.3)). There is a poset structure on $W\mathcal{S}$ defined by inclusion of cosets, so that

$$uW_T < vW_{T'} \iff T<T' \text{ and } v^{-1}u\in W_{T'}. \tag{4.9}$$

The purpose of this subsection is to describe a cell complex $\Sigma=\Sigma(W,S)$ together with a proper action $W \curvearrowright \Sigma$. The poset of cells of $\Sigma(W,S)$ is $W\mathcal{S}$; the cell corresponding to the spherical coset $wW_T$ is isomorphic to the zonotope $Z(W_T,T)$ (cf. Lemma 4.2). It follows that the barycentric subdivision of $\Sigma(W,S)$ is the geometric realization $|W\mathcal{S}|$ of the order complex of $W\mathcal{S}$. (The *order complex* of a poset $\mathcal{P}$ is the simplicial complex whose $k$-simplices are the chains, $p_0<\cdots<p_k$ of length $k+1$.)

Since the complex $\Sigma(W,S)$ plays a central role in [48] and is described in detail there, we will content ourselves with giving a briefer description here. In the case where $W$ is right-angled, $\Sigma(W,S)$ is the cubical complex $\tilde{P}_L$ explained at the end of Subsection 3.1.1. Some properties of $\Sigma$ are listed in the next proposition.

**Proposition 4.1.** *Let $\Sigma=\Sigma(W,S)$.*

(1) *The poset of cells of in the cell complex* $\Sigma$ *is equal to* $W\mathcal{S}$*.*
(2) *Each cell of* $\Sigma(W,S)$ *is a Coxeter zonotope. The cell corresponding to the spherical coset* $wW_T$ *is isomorphic to the Coxeter zonotope* $Z(W_T,T)$*; it is a convex polytope of dimesion* $\operatorname{Card} T$.
(3) *The group* $W$ *acts properly on* $\Sigma$*. The stabilizer of the cell* $wZ(W_T,T)$ *is the subgroup* $wW_Tw^{-1}$.
(4) *The* $W$*-action on* $\Sigma$ *has a strict fundamental domain homeomorphic to the geometric realization* $|\mathcal{S}|$ *of the poset* $\mathcal{S}$*. Hence, when* $S$ *is finite,* $\Sigma/W$ *is compact.*
(5) *The natural piecewise euclidean metric on* $\Sigma$ *is* $\mathrm{CAT}(0)$. *In particular, by Theorem 2.1,* $\Sigma$ *is contractible.*

In order to appreciate the naturalness and beauty of $\Sigma$, we spell out its construction in some detail. Its 0-skeleton, $\Sigma^0$, is identified with $W$ (i.e., with the spherical coset $W/W_\emptyset$). Its 1-skeleton is the Cayley graph $\mathrm{Cay}(W,S)$ – there is an edge connecting $w$ to $ws$ whenever $\{w,ws\}$ is a coset of $W_{\{s\}}$. So, there is an orbit of edges for each generator $s\in S$. Next we fill in the 2-cells. There is an orbit of 2-cells of the form $Z(W_{\{s,t\}},\{s,t\})$, whenever $\operatorname{Card}\{s,t\}=2$ and $m(s,t)<\infty$. Each such 2-cell a 2-dimensional Coxeter zonotope, in this case, it is a polygon with $2m(s,t)$ sides. The orbits of these 2-cells correspond precisely to the relations in (4.1). Essentially, $\Sigma^2$ is the Cayley 2-complex of $(W,S)$. In particular, $\Sigma^2$ is simply connected. We continue by filling in an orbit of 3-dimensional zonotopes of the form $Z(W_T,T)$ for each spherical subset $T\in\mathcal{S}$ with 3 elements. Miraculously, after filling in all such 3-cells, the resulting 3-skeleton, $\Sigma^3$ is 2-connected. The complex $\Sigma$ is formed by filling in an orbit of Coxeter zonotopes for each spherical coset in $W\mathcal{S}$. It turns out that $\Sigma$ is contractible. One way to to show this is to prove statement 5 in the above proposition: $\Sigma$ is $\mathrm{CAT}(0)$. This is a theorem of Moussong which will be discussed in the next subsection. We note that when $W$ is finite, $\Sigma(W,S)$ is equal to the single zonotope $Z(W,S)$ (cf. Lemma‘4.2).

The *nerve* of $(W,S)$ is the simplicial complex $L(W,S)$ with vertex set $S$ such that a nonempty subset $T\le S$ spans a simplex if and only if $T\in\mathcal{S}(W,S)$. There is natural piecewise spherical metric on $L(W,S)$ where the simplex corresponding to $T$ is identified with the fundamental dual spherical simplex $\sigma^*(W_T,T)$ defined above. In particular, the edge $\{s,t\}$ that corresponding to the spherical subset $T=\{s,t\}$ has length $\pi-\pi/m(s,t)$.

A spherical simplex $\sigma$ *has size* $\ge\pi/2$ if the distance from any vertex to the opposite face is $\ge\pi/2$.

**Definition 4.2.** (*Metric flag complexes*). Suppose $L$ is a simplicial complex with a piecewise spherical metric so that whenever $e$ is an edge of $L$, its length, $l(e)$, lies in the interval $[\pi/2,\pi]$ (this follows from the assumption that each simplex of $L$ has size $\ge\pi/2$). Then $L$ is a *metric flag complex* if whenever $\Gamma$ is a subcomplex of $L$ isometric to the 1-skeleton of a spherical $k$-simplex $\sigma$, then there must actually exist a $k$-simplex $\sigma$ in $L$ with edge lengths specified by the $l(e)$.

The condition in this definition also can be expressed as follows. Suppose $T$ is the set of vertices of a subcomplex isomorphic to a complete graph $\Gamma$ and that for distinct vertices $t$, $t'$ of $\Gamma$, the length $l(t,t')$ of the edge $\{t,t'\}$ lies $[\pi/2,\pi)$. Let $c(t,t')$ be the "cosine matrix" defined by

$$c(t,t')=\begin{cases}1, & \text{if } t=t';\\ \cos(t,t'), & \text{if } t\neq t'.\end{cases} \tag{4.10}$$

The condition that eac Then $L$ is a metric flag complex if and only if the $(T\times T)$-symmetric matrix $c(t,t')$ is positive definite whenever $T$ is vertex set of a simplex of $L$. In other words, the simplices of $L$ are determined by the metric on $L^1$. By Lemma 4.1, for each $T\in\mathcal{S}$, the corresponding cosine matrix is positive definite; so, $L(W,S)$, with its natural piecewise spherical structure $L(W,S)$ is a metric flag complex.

*Example 4.3.* (*Metric flag complexes of dimension* 1). When is the graph $L^1$ a $\mathrm{CAT}(1)$ cell complex? The answer was alluded to in Examples 2.2: the length of each circuit in $L^1$ must be $\ge 2\pi$. If all edge lengths lie in $[\pi/2, \pi]$ and if $L^1$ is a simplicial graph, then this condition is vacuous except for circuits with 3 edges, say $e_1$,$e_2$, $e_3$, in which case the condition reads:

$$l(e_1) + l(e_2) + l(e_3) \geq 2\pi. \tag{4.11}$$

This is precisely the condition that the cosine matrix for the 3-circuit fails to be positive definite. Thus, $L^1$ is a metric flag complex if and only if (4.11) holds for each 3-circuit in $L^1$ and this is equivalent to the condition that $L^1$ be $\mathrm{CAT}(1)$. Moussong's Lemma, which is stated in the next subsection, is a generalization of this to higher dimensions.

*Example 4.4.* (*$\Sigma$ is a 2-dimensional* $\mathrm{CAT}(0)$ *cell complex*, cf. Example 2.3). Suppose $(L^1, m)$ is a labeled simplicial graph with edge labeling $m : \mathrm{Edge}\, L^1 \to \{2, 3, \dots\}$ with associated Coxeter system $(W, S)$. As in the previous example suppose each spherical subset $T \in \mathcal{S}$ has cardinality $\leq 2$ so that $L(W, S) = L^1$. Then $\dim \Sigma = 2$. The 2-dimensional zonotope corresponding to an edge $e$ is a regular euclidean $2m(e)$-gon. The interior angle at a vertex of such a $2m(e)$-gon is $\pi - \pi/m(e)$. If $\{e_1, e_2, e_3\}$ is a 3-circuit of $L^1$, then the condition in (4.11) is equivalent to

$$\sum_{i=1}^{3} \frac{1}{m(e_i)} \leq 1. \tag{4.12}$$

So, $\dim \Sigma = 2$ exactly when (4.12) holds for all 3-circuits in $L^1$. Thus, $\Sigma$ is a 2-dimensional $\mathrm{CAT}(0)$ cell with an orbit of $2m(e)$-gons for each $e \in L^1$.

Next, consider the question of when does $\Sigma$ admit a piecewise hyperbolic structure that is $\mathrm{CAT}(-1)$. First, there is the question of how to make the zonotopes hyperbolic. If $W_T$ is a spherical Coxeter group, then we can represent it as a finite reflection group on $\mathbb{H}^m$, where $m = \mathrm{Card}\, T$. Let $C^h$ be a fundamental cone for the $W_T$-action on $\mathbb{H}^m$ and choose a point $x_0$ in the interior of $C$ of distance $\frac{1}{2\varepsilon}$ from each wall. Let $Z^h_\varepsilon(W_T, T)$ $(= Z^h_\varepsilon)$ denote the convex hull of the orbit of $x_0$. Then $Z^h_\varepsilon(W_T, T)$ is the hyperbolic zonotope of with all edge lengths $= \varepsilon$. As before, we can glue together the $Z^h_\varepsilon(W_T, T)$ to obtain a piecewise hyperbolic model $\Sigma^h(W, S)$ of the Davis-Moussong complex. When $\varepsilon$ is small, the effect on $\mathrm{Lk}(x_0, Z^h_\varepsilon(W_T, T)$ will be to make a small change in the dual spherical simplex $\sigma^*(W_T, T)$ Hence, there also will only be a small change in the metric on $\mathrm{Lk}(v, \Sigma^h_\varepsilon)$ (cf. [102, Lemma 5.11] or [48, Lemma I.6.7]).

*Example 4.5.* (*2-dimensional* $\mathrm{CAT}(-1)$ *cell complexes*). Suppose $\dim \Sigma = 2$. Then a small deformation of the metric on $L^1$ is guaranteed to be $\mathrm{CAT}(1)$ if and only if the sum of the edge lengths in each circuit is strictly greater than $2\pi$. As in Example 4.4 we only consider circuits with $\leq 4$ edges. If the number of edges is 3 we must modify (4.12) by requiring the inequality to be strict. If the number of edges is 4, the only case which leads to a closed geodesic of length $2\pi$ is when each edge has length exactly $\pi/2$. We exclude that possibility by requiring that for 4-circuits at least one edge has $m(e) \neq 2$. (This is a more general version of the no $\square$ condition of Section 2.3.) Following the discussion in Subsection 2.2.1 we see that a small deformation of the piecewise euclidean structure on $\Sigma$ to a piecewise hyperbolic $\Sigma^h_\varepsilon$ is $\mathrm{CAT}(-1)$ if and only if the above two conditions hold for for $L^1$, i.e., if $L^1$ is extra large in the sense of Definition 2.2, in which case each new vertex links will be $\mathrm{CAT}(1)$. Hence, for small enough $\varepsilon$, $\Sigma^h_\varepsilon$ will be $\mathrm{CAT}(-1)$ if and only if $L^1$ is extra large.

*Example 4.6.* (*Gromov polyhedra*). Here is a corollary to the previous example. Suppose $m$ is an integer $\geq 2$ and $n \geq 3$. Then there is a simply connected, 2-dimensional polyhedron $X_{n,2m}$ so that each 2-cell is a $2m$-gon and so that the link of each vertex is a complete graph on $n$-vertices. (Apply the construction in the previous example to the case where $L^1$ the complete graph and each edge is labelled by the same integer $m$, and $X_{n,m} = \Sigma(W, S)$, or its 2-skeleton when $m = 2$.) If $m \geq 3$, then $X_{n,m}$ is $\mathrm{CAT}(0)$. If $m \geq 4$, it can be given a $\mathrm{CAT}(-1)$ metric where each polygon is hyperbolic.

### *4.2.4 Moussong's Lemma*

**Lemma 4.3.** (Moussong's Lemma, cf. [102] or [48]). *A piecewise spherical simplicial complex $L$ with all simplices of size $\geq \pi/2$ is* CAT(1) *if and only if it is a metric flag complex.*

Moussong's Theorem is a corollary of this lemma.

**Theorem 4.2.** (Moussong [102], also cf. [62] or [48, Lemma I.7.4]). *The piecewise euclidean cell complex $\Sigma(W,S)$ is* CAT(0).

*Proof (Comments on).* As explained in Subsection 4.2.2, the link of a vertex in the zonotope $Z(W_T,T)$ is isometric to the dual spherical simplex $\sigma^*(W_T,T)$. It follows that the link of each vertex in $\Sigma(W,S)$ is isometric to $L(W,S)$. By Moussong's Lemma, this link is CAT(1); hence, $\Sigma(W,S)$ is NPC. Since the 2-skeleton of $\Sigma$ is the Cayley 2-complex for $(W,S)$, we see that $\Sigma$ is simply connected; hence, by Theorem 2.2, $\Sigma$ is CAT(0).

When $(W,S)$ is right-angled, its nerve $L(W,S)$ is a flag complex and it has an all-right, piecewise spherical structure. By Lemma 2.10, $L(W,S)$ is extra large if and only if it satisfies the no $\square$ condition. So, the cubical metric on $\Sigma(W,S)$ can be deformed to a piecewise hyperbolic metric that is CAT$(-1)$ if and only if $L(W,S)$ satisfies the no $\square$ condition. Basically, this establishes the following proposition. We will give details of its proof later in Theorem 4.3 below.

**Proposition 4.2.** *Suppose $(W,S)$ is a right-angled Coxeter system (a "*RACS*"). Then $(W,S)$ is word hyperbolic if and only if $L(W,S)$ satisfies the no $\square$ condition.*

Moussong continued along the line begun in Lemma 4.3 by determining when a piecewise spherical, metric flag complex $L$ of size $\geq \pi/2$ is extra large (cf. Definition 2.2). As usual $S = \mathrm{Vert}L$. For $T < S$ let $L_T$ denote the full subcomplex spanned by $T$. Then $L_T$ is a totally geodesic subspace of $L$. There are two obvious types of subcomplexes $L_T$ that can lead to a closed geodesic of length $2\pi$ in $L$. They are the following.

(i) The complex $L_T$ is isometric to the polar dual of a euclidean simplex so that $L_T$ is isometric to the round sphere of dimension $\mathrm{Card}(T)-2$ and $LT$ combinatorially isomorphic to the boundary complex of a simplex of dimension $\mathrm{Card}(T)-1$.
(ii) The complex $L_T$ decomposes as a spherical join $L_{T_1} * L_{T_2}$, where neither $L_{T_1}$ nor $L_{T_2}$ is a spherical simplex.

To see that (ii) yields a closed geodesic of length $2\pi$, for $i = 1,2$, choose points $x_i, y_i$ in $L_{T_1}$ of distance at least $\pi$. Then the join $\{x_1,y_1\} * \{x_2,y_2\}$ is a 4-circuit where each edge has length $\pi/2$. Moussong proved in [102, Lemma 10.3] that the metric flag complex $L$ has no closed geodesics of length $2\pi$ if and only if it contains no subcomplexes $L_T$ of type (i) or (ii). Applying this in the special case $L = L(W,S)$ we get the following.

**Lemma 4.4.** *The piecewise spherical complex $L(W,S)$ is extra large if for each subset $T$ of $S$ neither of the following conditions holds for $W_T$.*

(i) *The special subgroup $W_T$ is an irreducible euclidean reflection group (generated by the reflections across the faces of a euclidean simplex of dimension $\geq 2$.*
(ii) *$(W_T,T)$ decomposes as $(W_{T_1} \times W_{T_2}, T_1 \sqcup T_2)$, where both $W_{T_1}$ and $W_{T_2}$ are infinite.*

If both conditions (i) and (ii) of this lemma fail to hold, we say that $(W,S)$ satisfies *Moussong's Condition*.

**Theorem 4.3.** (Moussong's characterization of word hyperbolic Coxeter groups). *The following conditions on a (finitely generated) Coxeter system $(W,S)$ are equivalent.*

(a) *The group $W$ is word hyperbolic.*
(b) *The group $W$ does not contain $\mathbb{Z}\times\mathbb{Z}$ (the product of two infinite cyclic groups.*
(c) *The Coxeter system $(W,S)$ satisfies Moussong's Condition.*
(d) *The* $\mathrm{CAT}(0)$ *metric on $\Sigma(W,S)$ can be deformed to a piecewise hyperbolic,* $\mathrm{CAT}(-1)$ *metric.*

*Proof.* Since a word hyperbolic group cannot contain $\mathbb{Z}\times\mathbb{Z}$, (a) $\implies$ (b). If a special subgroup $W_T$ is a euclidean reflection group, then its translation subgroup is free abelian of rank $\mathrm{Card}(T)-1$ and this rank is $\geq 2$ when (i) holds. Similarly, if $W_T = W_{T_1}\times W_{T_2}$, where both factors are infinite, then $\mathbb{Z}\times\mathbb{Z} < W_T$. So, (b) $\implies$ (c). By Lemma 4.4, Moussong's Condition implies that $L(W,S)$ is extra large. As in Example 4.5, Proposition 2.1 implies that $\Sigma(W,S)$ can be given a piecewise hyperbolic $\mathrm{CAT}(-1)$ metric. So, (c) $\implies (d)$. By Lemma 2.7, (d) $\implies (a)$.

**Examples of word hyperbolic Coxeter groups.** Moussong, Benoist.

### *4.2.5 Piecewise euclidean metrics on $K$*

Let $K = |\mathcal{S}^{\mathrm{op}}|$ $(= |\mathcal{S}|)$ be the fundamental chamber defined in Section 4.1 and put $K_T = \mathcal{S}^{\mathrm{op}}_{\preceq T}$. In other words, $K_T$ is the union of simplices in the order complex of $\mathcal{S}^{\mathrm{op}}$ with maxmum vertex $T$. Since $W \curvearrowright \Sigma$ with strict fundamental domain $K$ (cf. Definition 3.2), we get a different description of $\Sigma$ in terms of the basic construction of (5.2): $\Sigma = D(W,K)$. Here $D(W,K) = (W\times K)/\sim$, where $\sim$ is defined by: $(u,x)\sim(v,x')$ if and only if $x = x'$ and $uW_{S(x)} = vW_{S(x)}$, where $S(x)$ is the index of the smallest stratum containing $x$. This description is dual to the cellulation of $\Sigma$ by Coxeter zonotopes described in Proposition 4.1.

The fundamental domain $K$ is a cube complex. For any $T\in\mathcal{S}$ , the poset $\mathcal{S}_{\leq T}$ $(= \mathcal{S}^{\mathrm{op}}_{\succeq T})$ is the power set of $T$. This means that the order complex of $\mathcal{S}^{\mathrm{op}}_{\succeq T}$ can be identified with a standard subdivision of a cube $\square^T$ of dimension $\mathrm{Card}(T)$. More generally, for any subset $T'$ of $T$ consider the interval $[T,T']$ in $\mathcal{S}^{\mathrm{op}}$ defined by $[T,T'] = \{J\in\mathcal{S} \mid T'\leq J\leq T\}$. So, $[T,T']$ is a standard subdivision of a cube $\square^{T-T'}$ $(\square^{[T,T']})$ of dimension $\mathrm{Card}(T-T')$. Explicitly, $\square^{[T,T']}$ is the subcomplex of $|\mathcal{S}^{\mathrm{op}}$ consisting of all simplices in $|\mathcal{S}^{\mathrm{op}}|$ with minimum vertex $T$ and maximum vertex $T'$. In other words, the simplices in $|\mathcal{S}^{\mathrm{op}}|$ $(= K)$ can be amalgamated into cubes, giving $K$ the structure of a cube complex. By taking the union of all translates of such cubes in $|W\mathcal{S}|$ $(= \Sigma)$ we see that $\Sigma$ is also a cube complex. We state this as follows.

**Lemma 4.5.** *The fundamental chamber $K$ has the structure of a cube complex as does the Davis-Moussong complex $\Sigma$.*

The subcomplexes $\square^{[T,T']}$ are only combinatorially equivalent to cubes. What is the relationship between these combinatorial cubes and the Coxeter zonotopes in $\Sigma$. Suppose $Z(W_T,T)$ is the Coxeter zonotope in $\mathbb{R}^T$ corresponding to $W_T$ and let $\mathrm{Fan}(W_T)$ be the fan cut out by the reflecting hyperplanes. A top dimensional simplicial cone in $\mathrm{Fan}(W_T)$ is called a *sector*. The intersection $B_T := Z(W_T,T)\cap\mathrm{Fan}(W_T)$ is called a *Coxeter block*; it is combinatorially isomorphic to the cube $\square^{[T,\emptyset]}$. However, $B_T$ need not be isometric to to a unit cube. (The link of the vertex the Coxeter block corresponding to $T\in\mathcal{S}_{\leq T}$ (the cone point in the fan) is isometric to the spherical simplex $\sigma(W_T,T)$ while the link at the opposite vertex corresponding to $\emptyset\in\mathcal{S}_{\leq T}$ is the dual spherical simplex $\sigma^*(W_T,T)$ defined near the end of Subsection 4.2.1.) The *natural piecewise euclidean metric on* $\Sigma$ is the path metric defined by giving each zonotopal cell in $\Sigma$ its natural metric as a

Coxeter zonotope (say, with each edge of length $= 2$). Similarly, the *natural piecewise euclidean metric on K* is the path metric defined by giving each Coxeter block its natural metric as a convex subset of the Coxeter zonotope. We just showed in Theorem 4.2, that with its natural piecewise euclidean metric $\Sigma$ is CAT(0). On the other hand, we could give $K$ the piecewise euclidean metric where each subcomplex $\square^{[T,T']}$ is isometric to the unit cube. Denote $K$ with this cubical metric by $K^{\square}$. Similarly, the cubical metric on $\Sigma$ is denoted $\Sigma^{\square}$. The cube complexes $K^{\square}$ and $\Sigma^{\square}$ may fail to be CAT$(0)$ since links need not be flag complexes. the condition that we need to insure that all klinks are flag complexes is given in the following definition.

**Figure of pentagon with square, Figure of hexagon with square**

**Definition 4.3.** A Coxeter system $(W,S)$ is *type FC* if its nerve $L(W,S)$ is a flag complex. Similarly, an Artin group is type FC if its associated Coxeter system is FC.

The cubical structures on these polyhedra induce the piecewise euclidean cubical metrics, denoted by $K^{\square}$ and $\Sigma^{\square}$, respectively. N.B. In general, $K^{\square}$ and $\Sigma^{\square}$ need not be CAT$(0)$. On the other hand, we just showed in Theorem 4.2, that $\Sigma$ is CAT$(0)$ when given the metric induced from its zonotopal cells. We shall see in Proposition **??** below, the condition that we need to insure that the cube complex $\Sigma^{\square}$ is CAT$(0)$ is that the nerve $L(W,S)$ of the Coxeter system is a flag complex.

**Proposition 4.3.** *The cube complex $\Sigma^{\square}$ is* CAT$(0)$ *if and only if the Coxeter system $(W,S)$ is type FC.*

*Proof.* Of course, this should follow from the version of Gromov's Lemma stated as Corollary 2.3; however, it does not quite immediately follow from the fact the fact that $L(W,S)$ $(= L)$ is a flag complex. Although it is not true that the link of every cube in $\Sigma$ is isomorphic to $L$ or a sublink of the form $\mathrm{Lk}(\sigma, L)$, it is true that the link of every 0-cube in $\Sigma$ is isomorphic to a join $\mathbb{S}\mathrm{Fan}(W_T,T) * \mathrm{Lk}(\sigma_T, L)$, where $\mathbb{S}\mathrm{Fan}(W_T,T)$ is the spherical Coxeter complex and $\sigma_T$ is the simplex corresponding to $T$. Since any spherical Coxeter complex is a triangulation of a sphere as a flag complex (cf. [33]), both factors of the join are flag complexes and hence, so is the join.

This shows that when $(W,S)$ is type FC one can prove, without using Moussong's Lemma, that $\Sigma(W,S)$ has a CAT$(0)$ structure. In Subsection 4.3.2 essentially the same observation will be used to show that if an Artin group is FC, then its Salvetti complex has a CAT$(0)$ cubical structure.

### *4.2.6 The Tits representation*

When $W$ is infinite, it also has a representation on $\mathbb{R}^n$, with $n = \mathrm{Card}\, S$, as a group generated by linear reflections (which need not be orthogonal linear transformations). Let $c(s,t)$ be the cosine matrix defined as in (5.6) with the proviso that $c(s,t) = -1$ when $m(s,t) = \infty$. Then there is a symmetric bilinear form $B : \mathbb{R}^S \times \mathbb{R}^S \to \mathbb{R}$ defined by $B(e_s, e_t) = c(s,t)$. For each $s \in S$, let $\rho_s$ be the linear reflection on $\mathbb{R}^S$ defined by

$$\rho_s(x) := x - 2B(e_s, x)e_s. \tag{4.13}$$

(The eigenvalue $-1$ of $\rho_s$ has eigenvector $e_s$.) The map $s \mapsto \rho_s$ extends to a homomorphism $\rho : W \to GL(\mathbb{R}^S)$, called the *canonical representation*. More interesting to us is the dual of the canonical representation, $\rho^* : W \to GL((\mathbb{R}^S)^*)$, which we call the *geometric representation* or the *Tits representation* of $W$. The main fact about the geometric representation is that it defines a $W$-action on a certain open convex set $\Omega$ with a strict fundamental domain. This allows us to conclude that the representation $\rho^*$ is discrete and faithful.

To see this, first define a simplicial cone $C$ in $(\mathbb{R}^S)^*$ by the inequalities: $e_s(v) \leq 0$, for all $s \in S$. (Each basis vector $e_s$ gives linear form, $v \mapsto e_s(v)$ on $(\mathbb{R}^S)^*$.) For each subset $T \leq S$, let $C_T$ be the face of $C$ defined by $e_s(v) = 0$, for all $s \in T$ and let $C^f$ denote the complement in $C$ of the union of all faces $C_T$ with $T \notin \mathcal{S}$. That is to say, $C^f = \{x \in C \mid S(x) \in \mathcal{S}\}$, where $S(x) = \{s \in S \mid x \in C_s\}$. The *Tits cone* $\overline{\Omega}$ is defined to be $\bigcup_{w \in W} wC$. Let $\Omega$ denote the interior of $\overline{\Omega}$. Tits established the following facts (e.g., see [15, Prop. 6, p. 102]).

(a) For any nontrivial $w \in W$, $w(\mathrm{int}\, C) \cap (\mathrm{int}\, C) = \emptyset$.
(b) $\overline{\Omega}$ is a convex cone. So, its interior $\Omega$ is an open $W$-stable subset in $\mathbb{R}^n$ $(= (\mathbb{R}^S)^*)$.
(c) $\Omega = \bigcup_{w \in W} wC^f$.
(d) The isotropy subgroup at a point $x \in C^f$ is $W_T$ where $C_T$ is the smallest face such that $x \in C_T$.

It follows from (a) that $\rho^* : W \to GL(\mathbb{R}^n)$ is injective and that its image is a discrete subgroup of $GL(\mathbb{R}^n)$. By (d) each isotropy subgroup is finite; hence, $W$ acts properly on $\Omega$.

**Theorem 4.4.** *The group $W$ acts on $\Omega$ with strict fundamental domain $C^f$. Thus, $\Omega$ is equivariantly homeomorphic to the basic construction $D(W, C^f)$ defined by formula* (3.8) *in Section 3.1. It follows that there is a $W$-equivariant, piecewise linear embedding $\Sigma(W,S) \hookrightarrow \Omega$ whose image is a spine for $\Omega$.*

So, $W \curvearrowright \Omega$ as a reflection group.

## 4.3 Artin groups

The material in this section comes from [64], [32, 33].

### *4.3.1 The complement of the complexification of a reflection arrangement*

Let $(W,S)$ be a spherical Coxeter system so that $W$ acts on $\mathbb{R}^n$ $(= \mathbb{R}^S)$ via the geometric representation. Let $\mathcal{A}$ be the corresponding reflection arrangement of hyperplanes in $\mathbb{R}^n$. Complexifying, we get $W \curvearrowright \mathbb{C}^n$ as well as a reflection arrangement of complexified hyperplanes $\mathcal{A} \otimes \mathbb{C}$ in $\mathbb{C}^n$. The complement of this arrangement is denoted by

$$M(\mathcal{A} \otimes \mathbb{C}) := \mathbb{C}^n - \bigcup_{H \in \mathcal{A} \otimes \mathbb{C}} H \,. \tag{4.14}$$

It is well-known that $\pi_1(M(\mathcal{A} \otimes \mathbb{C})) = PA_W$, where $PA_W$ is the *pure Artin group* is defined as the kernel of the canonical epimorphism $\varphi : A_W \to W$. The group $W$ acts freely on $M = M(\mathcal{A} \otimes \mathbb{C})$. So, $\pi_1(M/W)$ is the Artin group $A_W$. For example, if $W$ is the symmetric group $S_{n+1}$ acting on $\mathbb{R}^{n+1}$ by permuting the coordinates, then the complement $M$ is identified with the configuration space of $n+1$ distinct ordered points in $\mathbb{C}$ and $M/S_{n+1}$ is the unordered configuration space. It follows from the definition of the braid group, that $\pi_1(M)$ is the *pure braid group* on $n+1$ strands, $PB_{n+1}$, while $\pi_1(M/S_{n+1})$ is the *braid group*, $B_{n+1}$.

**Theorem 4.5.** (Deligne's Theorem [64]). *The arrangement complement $M$ is aspherical. Hence, $M$ is a model for $BPA_W$ and $M/W$ is a model for $BA_W$*

In fact, Deligne proved that whenever $\mathcal{A}$ is a real simplicial arrangement in $\mathbb{R}^n$, the complexified complement $M(\mathcal{A} \otimes \mathbb{C})$ is aspherical.

When $W$ is not required to be finite, we can realize it as a group generated by reflections on $\Omega < \mathbb{R}^n$ via the Tits representation $\rho^* : W \to GL(\mathbb{R}^n)$, as in Subsection 4.2.6. The group $W$ acts properly on the open subset $\mathbb{R}^n + i\Omega$ of $\mathbb{C}^n$ and freely on the arrangement complement, $M = M(\mathcal{A} \otimes \mathbb{C})$, defined by

$$M = (\mathbb{R}^n + i\Omega) - \bigcup_{H \in \mathcal{A} \otimes \mathbb{C}} H \,, \tag{4.15}$$

where $\mathcal{A}$ is the collection of all reflecting hyperplanes in $\mathbb{R}^n$.

The following conjecture, which was made independently by Arnold, Phan, and Thom, asserts that a generalization of Theorem 5.3 holds for any Artin group. It is the most important open question about Artin groups. the conjecture is explained in detail in [32, 33] and [96].

*Conjecture 4.1.* (*The $K(\pi,1)$-Conjecture for Artin groups*). The space $M/W$ is a model for $BA_W$.

There is an order relation on $W \times \mathcal{S}$ that is closely related to the one given in (4.9). it is defined by $(u,T) < (v,T')$ if and only if the following two conditions hold:

(a) $uW_T < vW_{T'}$ as in (4.9), i.e., $v^{-1}uW_T < W_{T'}$,
(b) $v^{-1}u$ is the shortest element in the coset $v^{-1}uW_T$.

After comparing (a) and (4.9), we see that the projection $p : W \times \mathcal{S} \to W\mathcal{S}$ defined by $(w,T) \mapsto wW_T$ is order-preserving.

**Theorem 4.6.** (Salvetti [122] and Charney-Davis [33]). *The arrangement complement $M$ is homotopy equivalent to $|W \times \mathcal{S}|$.*

The proof is based on the following standard result which combinatorialists call the "Nerve Lemma".

**Lemma 4.6.** (The Nerve Lemma). *Suppose $\mathcal{U}$ is a locally finite, open cover of a paracompact space $Y$ such that each $U \in \mathcal{U}$, as well as, each finite, nonempty intersection of such $U$ is contractible. Then $Y$ is homotopy equivalent to the nerve of the open cover,* Nerve($\mathcal{U}$).

Theorem 4.6 is then proved by applying the Nerve Lemma to a certain open cover $\mathcal{U}$ of $M$, the nerve of which is the order complex of $W \times \mathcal{S}$. Although $M$ is defined as a subset of $\mathbb{R}^n + i\Omega$, we can replace it by the homotopy equivalent space $M \cap (b\Sigma \times b\Sigma)$, where the barycentric subdivision $b\Sigma$ of $\Sigma$ is identified with the order complex of $W\mathcal{S}$. For each $(w,T) \in W \times \mathcal{S}$, define $\mathrm{Star}(w,T)$ to be the open star of vertex corresponding to $wW_T$ in $b\Sigma$, let $\mathrm{Sec}(w,T)$ be the open "sector" in $\Sigma$ which is bounded by the walls indexed by $wTw^{-1}$ and which contains the open chamber $\mathrm{Star}(w,\emptyset)$. Put

$$U(w,T) := \mathrm{Star}(w,T) \times \mathrm{Sec}(w,T).$$

It is clear that both $\mathrm{Star}(w,T)$ and $\mathrm{Sec}(w,T)$ are contractible; hence, so is $U(w,T)$ If $x \in \mathrm{Star}(w,T)$ lies in a wall, then that wall is indexed by a reflection in $wW_Tw^{-1}$. Since the open sector $\mathrm{Sec}(w,T)$ intersects no such wall, we see that $U(w,T) < M$. We see that $\mathcal{U} = \{U(w,T))\}_{(w,T) \in W \times \mathcal{S}}$ is an open cover of $M$. It is proved in [33, Lemma 1.5.2 (ii)] that $U(w_0,T_0) \cap \cdots \cap U(w_k,T_k)$ is nonempty precisely when $\{(w_0,T_0),\dots,(w_k,T_k)\}$ is a chain in $W \times \mathcal{S}$. Thus, $\mathrm{Nerve}(\mathcal{U}) = |W \times \mathcal{S}|$. Theorem 4.6 follows.

Combining Salvetti's Theorem 4.6 with Deligne's Theorem 5.3, we see that when $W$ is spherical, the finite CW complex $X = |W \times \mathcal{S}|$ is a model for $BPA_W$ (and so, $X/W$ is a model for $BA_W$). A corollary is that each spherical Artin group is type F.

### 4.3.2 The Salvetti complex and its universal cover

Given a locally finite arrangement $\mathcal{A}$ of affine hyperplanes in $\mathbb{R}^n$, Salvetti [122] constructed a cell complex $X(\mathcal{A})$ homotopy equivalent to the complement in $\mathbb{C}^n$ of the complexification of the arrangement. If there are finitely many hyperplanes in the arrangement, then $X(\mathcal{A})$ is a finite cell complex. The cells of $X(\mathcal{A})$ are zonotopes – each zonotope is the dual of a central arrangement normal to an intersection of hyperplanes from $\mathcal{A}$. When $\mathcal{A}$ is the arrangement associated to the action of $W$ on $\Omega$, the cell complex $X$ $(= X(\mathcal{A}))$ is homotopy equivalent to the complement $M$ defined in (5.11).

The barycentric subdivision $bX$ of the Salvetti complex $X$ can be defined as the geometric realization of the order complex of $W \times \mathcal{S}$. A $k$-simplex in the order complex is a chain, $(w_0, T_0) < \cdots < (w_k, T_k)$. We note that $(W \times \mathcal{S})_{\leq (w,T)} \cong W\mathcal{S}_{\leq wW_T} \cong W_T(\mathcal{S}_{\leq T})$. So, $(W \times \mathcal{S})_{\leq (w,T)}$ is isomorphic to the poset of faces of the Coxeter zonotope $Z(W_T, T)$. There is a cell structure on $X$ defined by identifying the union of all in simplices of $bX$ with maximum vertex $(w,T)$ with the cell $Z(W_T, T)$. Let us also use the notation $X(W,S)$ for $X$ $(= X(\mathcal{A}))$ $(= X)$ and call it the *Salvetti complex* of $(W,S)$.

Using Theorem 4.6, we get the following analog of Proposition 4.1.

**Proposition 4.4.** ( cf. [122], [33]). *Let $X(W,S)$ be the Salvetti complex as defined above.*

(1) *The poset of cells in $X(W,S)$ is equal to $W \times \mathcal{S}$. The cell corresponding to $(w,T)$ is the Coxeter zonotope $wZ(W_T, T)$; its dimension is equal to* $\operatorname{Card} T$.
(2) *The fundamental group of $X(W,S)$ is the pure Artin group $PA_W$.*
(3) *There is a free action $W \curvearrowright X(W,S)$. The quotient space $X(W,S)/W$ is a finite CW complex (provided the set of generators $S$ is finite).*

Here is another way to understand the cell structure on $X$ $(= X(W,S))$. Each cell of $X$ can be represented as a pair $(v,F)$, where $F$ is a zonotopal cell in $\Sigma$ $(= \Sigma(W,S))$ and $v \in \operatorname{Vert} F$. The partial order on the set of such $(v,F)$ is defined by $(v,E) < (v',F')$ if and only if $F < F'$ and the shortest edge path from $v'$ to $F$ terminates at $v'$. The geometric realization of the projection $p : W \times \mathcal{S} \to W\mathcal{S}$ is also denoted $p : X \to \Sigma$. If $F$ is a cell of $\Sigma$ of type $(W_T, T)$, then the set of connected components of $p^{-1}(\operatorname{int} F)$ is naturally bijective with $\operatorname{Vert} F$. The closure of such a component is isomorphic to $F$. Just as we did for $\Sigma(W,S)$ in Subsection **??**, let us spell out the cell structure on $X(W,S)$ in more detail.

(a) A vertex of $X$ has the form $(v,v)$ where $v \in \operatorname{Vert}\Sigma$. Hence, the 0-skeleton of $X$ is naturally isomorphic to the 0-skeleton of $\Sigma$ and both correspond bijectively to $W$.
(b) If $E$ is an edge of $\Sigma$ with end points $u$ and $v$, then there are two edges in $X$ lying above it: $(u,E)$ and $(v,E)$. The edge $(u,E)$ is directed so that its initial vertex is $(u,u)$ and its terminal vertex is $(v,v)$. If $E$ is labeled by the element $s$, then label the two directed edges lying above it by the Artin generator $\sigma_s$. Thus, each edge of $\Sigma$ is doubled to get a circuit consisting of two directed edges in $X$.
(c) Similarly, if $F$ is a $2m$-gon in $\Sigma$ corresponding to an edge $\{s,t\} \in \operatorname{Edge} L^1$ with $m(s,t) = m$, then $p^{-1}(F)$ consists of $2m$ copies of $F$. The face $(v,F)$ is glued to the 1-skeleton as follows. The two edge paths from $v$ to the antipodal vertex $-v$ in $F$ are labeled by alternating words of length $m$, $st\cdots$ and $ts\cdots$. If we direct these edge paths as traveling from $v$ to $-v$, then correspnding to an edge labeled by $s$ (or $t$) there is a directed doubled edge labeled by $\sigma_s$ (or $\sigma_t$). Then $(v,F)$ is glued onto $X^1$ so that the two edge paths of $F$ correspond to the positively oriented edge paths $\sigma_s\sigma_t\cdots$ and $\sigma_t\sigma_s\cdots$ in $X^1$.
(d) The projection $p : W \times \mathcal{S} \to W\mathcal{S}$ has a section $f : W\mathcal{S} \to W \times \mathcal{S}$ defined by $wW_T \mapsto (u,T)$ where $u$ is the shortest element in the coset $wW_T$. This induces a continuous map $f : \Sigma \to X$ that is a section of $p : X \to \Sigma$. THIS IS WRONG

The 1-skeleton of $X$ is isomorphic to the Cayley graph of $(W,S)$ except that each edge is doubled and each of the new edges is then assigned a direction. Each doubled edge corresponds to a conjugate of the square of an Artin generator by an element of $A$ and these conjugates give a set of generators for $PA_W$. By (c) the 2-cells in $X$ correspond to conjugates of the Artin relations in (4.1). It follows that $\pi_1(X^2) = PA_W$. (This also follows from the fact that $X$ is homotopy equivalent to the arrangement complement, $M(\mathcal{A} \otimes \mathbb{C})$.)

**Definition 4.4.** Let $\tilde{X}$ be the universal cover of $X$.

Here are some more properties of $X$ and $\tilde{X}$.

(e) The group $W$ acts freely on $X$ and $\pi_1(X/W) = A_W$ (cf. Theorem 4.6).
(f) There is an equivalence relation $\sim$ on $\Sigma$ so that $X/W = \Sigma/\sim$. (By (d) there is a surjection $\Sigma \hookrightarrow X \to X/W$ which induces the equivalence relation on $\Sigma$.
(g) The space $X/W$ is a finite CW complex (provided $S$ is finite) with a single vertex and with one cell of dimension $k$ for each spherical subset $T \in \mathcal{S}$ with $\operatorname{Card} T = k$.
(h) The 1-skeleton $\tilde{X}^1$ is the Cayley graph of $A_W$ with respect to the standard generating set $\{\sigma_s\}_{s\in S}$. Similarly, $\tilde{X}^2$ is the Cayley 2-complex of $A_W$.
(i) The cell complex $\tilde{X}$ satisfies the classical conditions for a convex cell complex in Section 2.1: the intersection of two cells is either empty or a common face of both.

The $K(\pi,1)$-Conjecture can be rewritten as follows.

*Conjecture 4.2.* (*The $K(\pi,1)$-Conjecture for Artin groups, second version*). The finite CW complex $X/W$ is a model for $BA_W$. In other words, $\tilde{X}$ is contractible.

### *4.3.3 The Deligne complex*

Given an Artin group $A = A_W$, let $\mathcal{G} = \mathcal{A}\mathcal{S}^{\mathrm{op}}$, the complex of spherical Artin groups defined in Example 5.2. The basic constructions of Theorem 5.1 give a poset $D(A,\mathcal{S}^{\mathrm{op}})$ and a space $D(A,K)$, with $K = |\mathcal{S}^{\mathrm{op}}|$, see (5.1). The poset $D(A,\mathcal{S}^{\mathrm{op}})$ is equal to the poset $A\mathcal{S}$ from (4.4) of spherical cosets in $A$. The space $D(A,K)$ is the *Deligne complex* for $A$; it will be denoted by $\Lambda$ $(= \Lambda(W,S))$. So, $\Lambda(W,S)$ is analogous to the definition of $\Sigma(W,S)$ given in Subsection 4.2.5 using the basic construction and fundamental chamber $K$ as of Subsection 4.2.5.

The facts which are summarized in the next paragraph can be found in [23] as well as in [32].

Since $K$ is simply connected, $\Lambda$ can be identified with the "universal cover of the complex of groups" $\mathcal{G}$ (see **??**). As is true for any complex of groups, there is a classifying space $B\mathcal{G}$ for $\mathcal{G}$, together with a projection map $p : B\mathcal{G} \to K = |\mathcal{S}^{\mathrm{op}}|$. In our case, $B\mathcal{G}$ can be identified with $\Lambda \times_A EA$ where $EA$ denotes the universal cover of the aspherical space $BA$ and where the projection map $p$ is induced by projection onto the first factor $\Lambda \times EA \to \Lambda$ after dividing by $A$ to get $\Lambda \times_A EA \to \Lambda/A = K$. The space $B\mathcal{G}$ is characterized by the fact that it is an "aspherical realization of the local groups". In our context this means that if $v_T \in |\mathcal{S}^{\mathrm{op}}|$ denotes the vertex corresponding to $T$, then $p^{-1}(v_T)$ is homotopy equivalent to $BA_T$.

**Proposition 4.5.** *The universal cover $\tilde{X}$ of the Salvetti complex $X(W,S)$ is contractible if and only if the Deligne complex $\Lambda(W,S)$ is contractible.*

*Proof.* As in Subsection 4.3.2, the natural projection $W \times \mathcal{S} \to W\mathcal{S}$ induces $X \to \Sigma$. Dividing out by $W$ we get a projection map $q : X/W \to K$. One checks that $q^{-1}(v_T) \cong X(W_T,T)/W_T$. By Deligne's Theorem 5.3 and Salvetti's Theorem 4.6, $X(W_T,T)/W_T$ is homotopy equivalent to $BA_T$. So, $X/W$ also is an aspherical realization of $\mathcal{G}$ and hence, $X/W \sim B\mathcal{G}$ (where the symbol $\sim$ stands for "homotopy equivalent". It follows that $\tilde{X} \sim (\Lambda \times EA)$. So, $\tilde{X}$ and $\Lambda$ are simultaneously contractible or not.

A corollary is that the $K(\pi,1)$-Conjecture holds for $A$ if and only if $\Lambda$ is contractible.

As explained in Subsection 4.2.5, each copy of the fundamental chamber $K$ is cellulated by Coxeter blocks, each of which is the intersection of a Coxeter zonotope and a sector in a fan for a reflection arrangement. This induces a natural piecewise Euclidean metric on $\Lambda$ analogous to the natural piecewise euclidean metric on the zonotopal complex $\Sigma$.

*Conjecture 4.3.* (cf. [32, Conjecture 4.4.4, p. 622]). With its natural piecewise euclidean metric $\Lambda$ is $\mathrm{CAT}(0)$.

Using Theorem 2.1 this conjecture implies that $\Lambda(W,S)$ is contractible, Hence, by Proposition 4.5, Conjecture 4.3 implies Conjecture 4.2. The issue in Conjecture 4.3 is showing that the link of each cell of $\Lambda$ satisfies the Link Condition of Definition 2.1.

When $(W,S)$ is spherical, we define its *spherical Deligne complex* $\Lambda'(W,S)$ by using $|\mathcal{S}_{<S}|$ as fundamental chamber instead of $|\mathcal{S}|$. In other words, the fundamental domain for $A_W \curvearrowright \Lambda'(W,S)$ is the fundamental simplex $\sigma(W,S)$ for the $W$-action on the unit sphere $\mathbb{S}^{n-1}$ in the reflection representation, $W \curvearrowright \mathbb{R}^n$. This induces the *natural piecewise spherical metric* on $\Lambda'(W,S)$. . As we will explain at the end of this subsection, when $(W,S)$ is spherical $\Lambda'(W,S)$ is a union of subcomplexes, called *apartments*, each of which is isometric to the round sphere $\mathbb{S}^{n-1}$. (When $(W,S)$ is spherical, Deligne's actual definition in [64] was of a complex $\Lambda''(W,S)$ obtained by filling in each apartment of $\Lambda'$ with a round ball.)

To prove Conjecture 4.3 there are three types of links of cells in $\Lambda$ which must be shown to be $\mathrm{CAT}(1)$.

(a) The link of a Coxeter block $B_T$. Such a link corresponds to the order complex of $\mathcal{S}_T$, i.e., it is a sublink of $L(W,S)$.
(b) A link isomorphic to $\Lambda'(W_T,T)$.
(c) A join of a link of type (a) with one of type (b)

Links of type (a) are metric flag complexes; hence, they are $\mathrm{CAT}(1)$ by Moussong's Lemma 4.2.4. The spherical join of two $\mathrm{CAT}(1)$ piecewise spherical complexes is $\mathrm{CAT}(1)$ (see [23, Proposition 5.15, p.,64]] or [?cd93, Appendix, Theorem A9]). So, it suffices to consider links of type (b). This leads to the following conjecture in [33, Conjecture 3, p.,6]).

*Conjecture 4.4.* For any spherical $(W,S)$, the spherical Deligne complex $\Lambda'(W,S)$ is $\mathrm{CAT}(1)$.

So, Conjecture 4.4 implies Conjecture 4.3. (When proving the natural metric on $\Sigma$ is $\mathrm{CAT}(0)$, the links of type (b) are replaced by Coxeter complexes of the form $D(W_T,\sigma(W_T,T)$ which are round spheres and hence, $\mathrm{CAT}(1)$. So, in the case of $\Sigma$, it was unnecessary to consider links of type (b) and (c).)

Nonpositive curvature and the Deligne complex was used successfully in [32] in proving the $K(\pi,1)$-Conjecture [32] in two special cases described below. First, Conjecture **??** is true for any $(W,S)$ with $\operatorname{Card} S \le 2$.

**Proposition 4.6.** (cf. [32, Proposition 4.4.5]). *The spherical Deligne complex* $\Lambda'(W,S)$ *is* $\mathrm{CAT}(1)$ *whenever* $\operatorname{Card} S \le 2$.

This is proved in [32] by observing that it follows from a lemma of Appel-Schupp that for $S = \{s,t\}$, the shortest loop in $\Lambda'(W,S)$ has length $2m(s,t)$ corresponding to an Artin relation. A corollary is the following.

**Corollary 4.1.** *The* $K(\pi,1)$*-Conjecture holds for* $A_W$ *whenever* $\dim \Lambda \le 2$.

The cubical structure on $K$ defined in Subsection 4.2.5 gives $\Lambda$ the structure of a cube complex denoted $\Lambda^{\square}$. Once one shows that links of type (b) are flag complex (by [32]), the proof of Proposition 4.3 shows that cube complex $\Lambda^{\square}$ is CAT(0) if and only if $L(W,S)$ is a flag complex.

**Proposition 4.7.** (cf. [32]). *The cube complex $\Lambda^{\square}$ is* CAT(0) *if and only if $(W,S)$ is type FC (see Definition 4.3).*

First, one could define its barycentric subdivision $b\Lambda$ to be the geometric realization of the order complex of the poset $A\mathcal{S}$ of spherical Artin cosets (cf. (4.3)):

$$A\mathcal{S} := \bigsqcup_{T \in \mathcal{S}} A/A_T \ . \tag{4.16}$$

In the same way as $\Sigma$ was defined in Subsection 4.2.5 we also could define $\Lambda$ by using the basic construction with fundamental chamber $K$ $(= |\mathcal{S}|)$ as a strict fundamental domain, i.e., $\Lambda = D(A,K) = (A \times K)/\sim$, where the local group at $x \in K$ is $A_{S(x)}$. When $(W,S)$ is spherical, one defines its *spherical Deligne complex* $\Lambda'$ by using $\mathcal{S}_{<S}$ in place of $\mathcal{S}$. (When $(W,S)$ is spherical, Deligne's actual definition in [64] was of a complex $\Lambda''(W,S)$ obtained by filling in certain round spheres in $\Lambda'$ with balls, see below.) The *natural piecewise euclidean metric on* $\Lambda$ is induced from the natural piecewise euclidean metric on $K$. This metric on $K$ also induces the natural metric on $\Sigma$. Similarly, since the fundamental spherical simplex $\sigma(W_T,T)$ is a fundamental domain for the $W_T$-action on the round sphere $\mathbb{S}^{k-1}$ $(= \mathbb{S}\mathrm{Fan}(W_T))$, the natural piecewise spherical metric on the spherical Coxeter complex $\mathbb{S}\mathrm{Fan}(W_T)$ is isometric to $\mathbb{S}^{k-1}$. In the same way, there is a *natural piecewise spherical metric on the spherical Deligne complex* $\Lambda'(W_T,T)$ induced from the fundamental spherical simplex $\sigma(W_T,T)$.

*Remark 4.1.* (*Apartments in $\Lambda(W,S)$*). If $s_1 \dots s_k$ is a reduced decomposition of an element $w \in W$, then the element $a_w = \sigma_{s_1} \dots \sigma_{s_k}$ is a well-defined element of $A$ depending only on $w$ and not on the choice of reduced decomposition for it. (This follows from the solution to the word problem for Coxeter groups by Matsumoto and Tits.) Hence, $q : w \mapsto a_w$ is a section of the canonical epimorphism $A \to W$ (The function $q$ is only a map of sets; it is not a homomorphism). The canonical epimorphism $A \to W$ induces a simplicial projection $b\Lambda \to b\Sigma$ and $q$ induces a simplicial section $q : b\Sigma \to b\Lambda$. Moreover, for any $a \in A$ by composing with translation with $a$ we get another section $aq : b\Sigma \to b\Lambda$. The image of such a section is called an *apartment* of $\Lambda$ (cf. [64]).

## 4.4 Buildings

date, reference

In the 1960s, J. Tits [133,134] developed the theory of buildings and their automorphism groups in connection with his work on incidence geometries and algebraic groups. Each building has an associated Coxeter system $(W,S)$. For buildings that arise from algebraic groups, only spherical and euclidean Coxeter groups play a role. Nevertheless, Tits was careful to develop the theory for arbitrary Coxeter systems. In fact, it seems that Tits motivation for introducing the notion of a Coxeter system was for its use in developing the theory of buildings. Another remarkable aspect of this work is that the symmetries of a building were not involved in its definition; in particular, a building need not have the structure of a complex of groups.

References for this material include [1], [133–136], [43,49], and [48, Ch. 18]

### 4.4.1 The combinatorial theory of buildings

A *chamber system* is a set $\mathcal{C}$ together with a family of equivalence relations on $\mathcal{C}$ indexed by another set $S$. Chambers $C, D \in \mathcal{C}$ are *s-adjacent* if they are $s$-equivalent and not equal. A *gallery* in $\mathcal{C}$ is a finite sequence of chambers $(C_0, \dots, C_k)$ such that $C_{j-1}$ is adjacent to $C_j$, for $1 \le j \le k$. The *type* of this gallery is the word $\mathbf{s} = (s_1, \dots, s_k)$ where $C_{j-1}$ is $s_j$-adjacent to $C_j$. If each $s_j$ belongs to a given subset $T$ of $S$, then the gallery is a *T-gallery*. Two chambers are in the *same T-connected component* if they can be connected by a $T$-gallery. The $T$-connected components of a chamber system $\mathcal{C}$ are its *residues* of *type T*. An $s$-equivalence class is the same thing as a residue of type $\{s\}$. If $C \in \mathcal{C}$ and $T \le S$, then $\mathrm{Res}_T(C)$ denotes the residue of type $T$ containing $C$.

*Example 4.7.* (*The chamber system associated to a family of subgroups*). Suppose that $G$ is group, that $B$ is a subgroup, and that $(G_s)_{s\in S}$ a family of subgroups of $G$ such that each $G_s$ contains $B$. Define a chamber system $\mathcal{C} = \mathcal{C}(G, B, (G_s)_{s\in S})$ as follows: $\mathcal{C} = G/B$; chambers $gB$ and $g'B$ are $s$-adjacent if they have the same image in $G/G_s$.

For each subset $T$ of $S$, let $G_T$ be the subgroup generated by $\{G_s\}_{s\in T}$. As we will see in Chapter 5, if $\mathcal{P}$ denotes the power set of $S$, then $\{G_T\}_{T\in\mathcal{P}^{\mathrm{op}}}$ is a simple complex of groups over the opposite poset to $\mathcal{P}$.

A building is a chamber system with some extra structure that depends on a Coxeter system $(W,S)$.

**Definition 4.5.** (cf. [1]). Suppose $(W,S)$ is a Coxeter system. A *building of type* $(W,S)$ is a pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$ (the elements of which are called *chambers*), and a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ (called the *Weyl distance*) so that the following conditions hold for all $C, D \in \mathcal{C}$

**(WD1)** $\delta(C,D) = 1$ if and only if $C = D$.
**(WD2)** If $\delta(C,D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C',C) = s \in S$, then $\delta(C',D) = sw$ or $w$. If, in addition, $l(sw) = l(w) + 1$, then $\delta(C',D) = sw$.
**(WD3)** If $\delta(C,D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathcal{C}$ such that $\delta(C',C) = s$ and $\delta(C',D) = sw$.

*Example 4.8.* The group $W$ itself has the structure of a building: $\delta : W \times W \to W$ is defined by $\delta(v,w) = v^{-1}w$. It is called the *thin building* of type $(W,S)$.

Now suppose $\mathcal{C}$ is a building of type $(W,S)$. Chambers $C, D \in \mathcal{C}$ are *s-adjacent* if $\delta(C,D) = s$; they are *s-equivalent* if they are either $s$-adjacent or equal. This gives $\mathcal{C}$ the structure of a chamber system. By **(WD3)** every $s$-equivalence class has at least two elements. A building $\mathcal{C}$ is *thick* if each $s$-equivalence class contains at least 3 elements.

When $\mathcal{C} = W$, a $T$-residue is just a left coset of $W_T$. A residue of type $T$ in a building is itself a building; its type is $(W_T, T)$. A building is *spherical* if its type is a spherical Coxeter system (cf. Subsection 4.2.1). It is a *right-angled* building (a RAB) if its type is a right-angled Coxeter system (cf. Subsection 3.3). Let $R_T(\mathcal{C})$ denote the set of all residues in $\mathcal{C}$ of type $(W_T, T)$. In practice the set of chambers $\mathcal{C}$ in a buiding will usually be associated to a family of subgroups $\mathcal{C}(G, B, (G_s)_{s\in S})$ as in Example 4.7. When this is the case, $\mathcal{C} = G/B$ and a residue of type $T$ is the image of a left coset $gG_T$ in $G/B$.

Let $R(\mathcal{C})$ denote the disjoint union of all $R_T(\mathcal{C})$ with $T \in \mathcal{S}$. The set $R(\mathcal{C})$ is called the poset of *spherical residues*. The order relation is inclusion of residues: $\mathrm{Res}_T(C) < \mathrm{Res}_{T'}(C')$ if $T < T'$ and the first residue is a subset of the second. The geometric realization of the order complex of $R(\mathcal{C})$ is the *standard realization* of $\mathcal{C}$. In this realization each apartment is isomorphic to $b\Sigma(W,S)$, the barycentric subdivision of the Davis-Moussong complex in Subsection 4.2.3.

**Definition 4.6.** An *apartment* in $\mathcal{C}$ is a subset which is $W$-isometric to the thin building $W$. In other words, if $\rho_C : \mathcal{C} \to W$ denotes the function $D \mapsto \delta(C,D)$, then a subset $\mathcal{A}$ of $\mathcal{C}$ is an apartment if $(\rho_C)|_{\mathcal{A}} : \mathcal{A} \to W$ is an isomorphism. The function $\mathcal{C} \to \mathcal{A}$, defined as the composition of $\rho_C$ with the inverse of $(\rho_C)|_{\mathcal{A}}$, is called the *retraction onto $\mathcal{A}$ centered at $C$*.

**Definition 4.7.** The *thickness* of a locally finite building $\mathcal{C}$ of type $(W,S)$ (at a chamber $C$) is the $s$-tuple $(q_s)_{s\in S}$, where each $\{s\}$-residue containing $C$ has exactly $q_s+1$ elements.

If $\mathcal{C}$ is the building for an algebraic group over a finite field $\mathbb{F}_q$ of order $q$, then each $\{s\}$-residue has $q+1$ elements since it can be identified with the projective line over $\mathbb{F}_q$. Hence, each $q_s$ is equal to $q$.

The *rank* of a building of type $(W,S)$ is $\mathrm{Card}(S)$.

*Example 4.9.* (*Rank one buildings*). The only Coxeter system of rank one is $(\langle s\rangle, \{s\})$, where $\langle s\rangle$ means the cyclic group of order two. So, a rank one building is any set with more than two elements (the chambers) and $\delta : \mathcal{C}\times\mathcal{C} \to \langle s\rangle$ is defined by

$$\delta(C,D) = \begin{cases} 1, & \text{if } C = D,\\ s, & \text{if } C \neq D.\end{cases}$$

*Example 4.10.* (*Rank two buildings*). If $(W,\{s,t\})$ is a Coxeter system of rank two, then $W$ is a dihedral group $D_m$ of order $2m$, where $m = m(s,t)$, and where $D_\infty$ means the infinite dihedral group. If $m < \infty$, then a building $\mathcal{C}$ of type $(D_m,\{s,t\})$ is a *generalized m-gon*. A generalized 3-gon is a *projective plane*. The spherical realization of $\mathcal{C}$ is a bipartite graph of girth $2m$ and diameter $m$, the edges of the graph are the chambers $\mathcal{C}$. Each vertex of an edge determines a residue of type $\{s\}$ or $\{t\}$. An apartment in $\mathcal{C}$ is a subgraph which is a circuit of length $2m$. A building $\mathcal{C}$ of type $(D_\infty,\{s,t\})$ is the same thing as a tree without terminal vertices, that is to say, $\mathcal{C}$ is the set of edges in such a tree. An apartment is a subtree isomorphic to the real line, that is, to the Coxeter complex of $(D_\infty,\{s,t\})$.

*Example 4.11.* (*Products*). For $i = 1,2$, suppose $(\mathcal{C}_i,\delta_i)$ is a building of type $(W_i,S_i)$. Then $(W,S) = (W_1\times W_2, S_1\sqcup S_2)$ is Coxeter system and $(\mathcal{C},\delta) = (\mathcal{C}_1\times\mathcal{C}_2, \delta_1\times\delta_2)$ is a building of type $(W,S)$. For example, if each $(\mathcal{C}_i,\delta_i)$ is the building corresponding to a tree $T_i$, then $(\mathcal{C}_1\times\mathcal{C}_2, \delta_1\times\delta_2)$ is a building corresponding to $T_1\times T_2$; its associated Coxeter group is $D_\infty\times D_\infty$. Each apartment in $T_1\times T_2$ is isomorphic the euclidean plane with its square tiling.

If $\mathbf{s} = (s_1,\dots,s_k)$ is a word in $S$, then its *value* $w(\mathbf{s}) = s_1\cdots s_k$ is the corresponding element of $W$. A gallery $(C_0,\dots,C_k)$ of type $\mathbf{s}$ is *reduced* if $\mathbf{s}$ is a reduced expression for $w(\mathbf{s})$.

It is proved in [1, Prop. 5.23] that the conditions **(WD1)**,**(WD2)**, **(WD3)** in Definition 4.5 are equivalent to the following two conditions on a chamber system $\mathcal{C}$, equipped with a function $\delta : \mathcal{C}\times\mathcal{C}\to W$.

- Each $s$-equivalence class has at least two elements.
- Given a reduced expression $\mathbf{s}$ for an element $w \in W$, there is a gallery of type $\mathbf{s}$ from $C$ to $D$ if and only if $\delta(C,D) = w$.

previous 3 paragraphs could be omitted or put later

(This is the definition in [117].)

An *automorphism* of $\mathcal{C}$ is a self-bijection which preserves $s$-equivalence classes for each $s\in S$. Equivalently, it is a self-bijection which preserves Weyl distance.

Given $C \in \mathcal{C}$, the *combinatorial ball of radius n about C* is the set $B_C(n) := \{D \in \mathcal{C} \mid l(\delta(C,D)) \leq n\}$. There is a natural topology on the group $\mathrm{Aut}(\mathcal{C})$ of automorphisms of $\mathcal{C}$: an open neighborhood of $1 \in \mathrm{Aut}(\mathcal{C})$ is the set of automorphisms which fix each element of $B_C(n)$ for some $n\in\mathbb{N}$ and $C\in\mathcal{C}$. (The neighborhood is small if $n$ is large.) Since $\mathcal{C}$ is locally finite, $\mathrm{Aut}(\mathcal{C})$ is a locally compact, totally disconnected topological group. As such, it has a Haar measure. A closed subgroup $G \subset \mathrm{Aut}(\mathcal{C})$ inherits a topology and a Haar measure. A subgroup $\Gamma \subset G$ is a *lattice* if it is discrete and $G/\Gamma$ has finite volume. It is a *uniform* lattice if $G/\Gamma$ is compact. A discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{C})$ is a uniform lattice if and only if $\mathcal{C}/\Gamma$ is finite. (See [?thomas] for a discussion of lattices in $\mathrm{Aut}(\mathcal{C})$, when $\mathcal{C}$ is a right angled building.)

**Definition 4.8.** A subgroup $G \subset \mathrm{Aut}(\mathcal{C})$ is *chamber-transitive* if it is transitive on $\mathcal{C}$. It is *strongly transitive* if it is transitive on the set of pairs $(\mathcal{A}, C)$, where $\mathcal{A}$ is an apartment in $\mathcal{C}$ and $C \in \mathcal{A}$ (cf. [1, §6.1.1]). (In fact, it is not necessary use all apartments in this definition, $\mathcal{A}$ need only belong to a certain "system of apartments" satisfying the classical axioms for a building, cf. [1, §6.1]).

It turns out that if $G$ is strongly transitive on a thick building, then it inherits the structure of a *BN* pair (also called a "Tits system"), cf. [1, Thm. 6.56].

### *4.4.2 Geometric realizations of buildings*

Given a Coxeter system $(W,S)$, let $K = |\mathcal{S}^{\mathrm{op}}|$ be its fundamental chamber. Even though $\mathcal{C}$ need not be a group, the definition of the basic construction can be modified to work for $\mathcal{C}$. The definition of $D(\mathcal{C},K)$ is essentially the same as that for the standard complex $\Sigma(W,S)$ in Subsection 4.2.3 or for the Deligne complex $\Lambda(W,S)$ in Subsection 4.3.2. To wit, $D(\mathcal{C},K) = (\mathcal{C} \times K)/\sim$, where the equivalence relation $\sim$ is given by

$$(C,x) \sim (C',x') \iff x = x', \text{and the chambers } C,\, C' \text{ belong to the same } S(x)\text{-residue.}$$

(Here, as before, $S(x) = \{s \in S | x \in K_s\}$).) The polyhedron $D(\mathcal{C},K)$ is the *standard realization of* $\mathcal{C}$. Since $K$ is the geometric realization of $\mathcal{S}$, $D(\mathcal{C},K)$ can be identified with $|R(\mathcal{C})|$, the geometric realization of the poset of spherical residues.

As in Subsection 4.2.5, $K$ is partitioned into Coxeter blocks $B_T$, with $T \in \mathcal{S}$, where $B_T$ is the union of all simplices in the order complex of the subposet $[T,\emptyset]$ of $\mathcal{S}^{\mathrm{op}}$; furthermore, each $B_T$ has a natural metric as a convex polytope in some euclidean space. (Recall $B_T$ is the intersection of a Coxeter zonotope with a $W_T$-sector.) This induces the natural piecewise euclidean metric on $K$, as well as, the *natural piecewise euclidean metric on* $D(\mathcal{C},K)$. Moreover, each apartment in $D(\mathcal{C},K)$ is isometric to $D(W,K) = \Sigma(W,S)$, with its $\mathrm{CAT}(0)$ metric. To simplify notation denote the standard realization $D(\mathcal{C},K)$ with its natural piecewise euclidean metric by $|\mathcal{C}|$.

**Theorem 4.7.** (cf. [43]). *For any building* $\mathcal{C}$*, the natural piecewise Euclidean metric on its standard realization is* $\mathrm{CAT}(0)$.

Suppose $(W_T,T)$ is a spherical Coxeter system as in Subsection 4.2.1. The group $W_T$ acts on the unit sphere $\mathbb{S}^{m-1}$ in the geometric representation on $\mathbb{R}^m$. A fundamental domain for the action is the fundamental spherical simplex $\sigma(T)$; so, $\mathbb{S}^{m-1} = D(W_T, \sigma(T)$ (see Subsection 4.2.1). Now suppose $\mathcal{C}(T)$ is a spherical building of type $(W_T,T)$. The *spherical realization* of $\mathcal{C}(T)$ is the basic construction applied to $\mathcal{C}(T)$ with fundamental domain $\sigma(T)$:

$$D(\mathcal{C}(T),\sigma(T)) = (\mathcal{C}(T) \times \sigma(T))/\sim\,,$$

with its natural piecewise spherical metric induced from $\sigma(T)$. We shall also write $\Delta(\mathcal{C}(T)$ for $D(\mathcal{C}(T),\sigma(T))$. In this spherical realization, each apartment is isometric to the round sphere $D(W_T,\sigma(T)) = \mathbb{S}^{m-1}$. More precisely, if $\mathcal{A}$ is an apartment in $\mathcal{C}(T)$, then the map induced by $(\rho_C)|_{\mathcal{A}}$ from Definition 4.6 induces an isometry from the spherical realization of $\mathcal{A}$ to $\mathbb{S}^{m-1}$. (N.B. The standard realization of $\mathcal{C}(T)$ uses for fundamental domain the Coxeter block $B_T$ so that each apartment becomes isometric to the Coxeter zonotope $Z(W_T,T)$.)

Consider the standard cellulation of $\Sigma(W,S)$ by Coxeter blocks. Let $v_T$ be a vertex corresponding to the center of a Coxeter zonotope $Z(W_T,T)$, then the link of $v_T$ in $\Sigma=\Sigma(W,S)$ decomposes as a spherical join:

$$\mathrm{Lk}(v_T,\Sigma)=\mathbb{S}^{m-1}*\mathrm{Lk}(T,|\mathcal{S}|). \tag{4.17}$$

Here $\mathbb{S}^{m-1}$ is the link of $v_T$ in the zonotope and $\mathrm{Lk}(T,|\mathcal{S}|)=|\mathcal{S}_{>T}|=\mathrm{Lk}(v_T,K_T)$ is the the link of $v_T$ in the stratum of a chamber corresponding to $T$. Since $|\mathcal{S}_{>T}|$ is a link in the metric flag $L(W,S)$, it is also a metric flag complex; so, by Moussong's Lemma 4.3, it is $\mathrm{CAT}(1)$. A similar analysis holds for the link of a vertex $v_T$ in the geometric realization $|\mathcal{C}|=D(\mathcal{C},K)$ except that in (4.17), the round sphere $\mathbb{S}^{m-1}$ must be replaced by the spherical realization of a building of the form $\mathrm{Res}_T(C)$. In other words, the formula in (4.17) should be replaced by

$$\mathrm{Lk}(v_T,|\mathcal{C}|)=\Delta(\mathcal{C}(T))*\mathrm{Lk}(T,|\mathcal{S}|), \tag{4.18}$$

where $\Delta(\mathcal{C}(T))=D(\mathrm{Res}_T(C),\sigma(T))$. So, to show that $\mathrm{Lk}(v_T,|\mathcal{C}|)$ is $\mathrm{CAT}(1)$ it suffices to prove the following lemma which states that the spherical realization of any spherical building is $\mathrm{CAT}(1)$.

**Lemma 4.7.** (cf. [43]). *If $\mathcal{C}(T)$ is a spherical building of type $(W_T,T)$, then its spherical realization $\Delta(\mathcal{C}(T))$ is* $\mathrm{CAT}(1)$.

Lemma 4.7 is a corollary of Theorem 4.7. Indeed, a neighborhood of the vertex $v_T$ in $|\mathcal{C}(T)|$ is isometric to a neighborhood of the cone point in $\mathrm{Cone}(\Delta(\mathcal{C}(T))$. By Lemma 2.2, if $v_T$ has a $\mathrm{CAT}(0)$ neighborhood, then $\Delta(\mathcal{C}(T))$ is $\mathrm{CAT}(1)$.

*Remark 4.2.* If Lemma 4.7 holds for all spherical building, then, by the Link Condition (Definition 2.1), $|\mathcal{C}|$ is NPC. Since it is also not hard to see that $|\mathcal{C}|$ is simply connected, it might seem that the way to prove Theorem 4.7 is to first establish Lemma 4.7. However, the proof in [43] is by a more direct argument which we shall explain below.

*Proof (Sketch of proof of Theorem 4.7).* The proof in [43] follows a standard argument in [`?brbook`, Ch. VI §3]. An important consequence of Definition 4.5 is one of the classical axioms for a building: any two chambers are contained in a common apartment. Passing to the geometric realization, this that any two points $x,y\in|\mathcal{C}|$ are contained in the realization of $|\mathcal{A}|$ of an apartment. If $C$ is a chamber in an apartment $\mathcal{A}$, we have $\rho_{C,\mathcal{A}}:\mathcal{C}\to\mathcal{A}$ the retraction onto $\mathcal{A}$ centered at $C$ (cf. Definition 4.6. Its geometric realization is denoted $\overline{\rho}_{C,\mathcal{A}}:|\mathcal{C}|\to|\mathcal{A}|$ (or simply by $\overline{\rho}$). If $x,y\in|\mathcal{A}|$, then $\overline{\rho}$ takes a geodesic segment connecting them in $|\mathcal{C}|$ to a geodesic segment in $|\mathcal{A}|$. Using the fact that $|\mathcal{A}|$ is $\mathrm{CAT}(0)$ one argues that the geodesic segment in $|\mathcal{C}|$ from $x$ to $y$ is equal to the geodesic segment in $|\mathcal{A}|$. Moreover, $\overline{\rho}$ is distance decreasing.

If $x,y,z$ are vertices of a triangle in $\mathbb{E}^2$ and $p_t=tx+(1-t)y$ is a point on the segment from $x$ to $y$, then it is an easy exercise to see that $d^2(z,p_t)=(1-t)d^2(z,x)+td^2(z,y)-t(1-t)d^2(x,y)$, where $d^2(x,y)$ denotes the square of the euclidean distance. It follows that the $\mathrm{CAT}(0)$ inequality for a triangle in a geodesic space with vertices $x,y,z$ is equivalent to the same formula with the equality replaced by an inequality:

$$d^2(z,p_t)\le(1-t)d^2(z,x)+td^2(z,y)-t(1-t)d^2(x,y) \tag{4.19}$$

Now choose an apartment $\mathcal{A}$ so that $x,y\in|\mathcal{A}|$ and a chamber $C$ so that $p_t\in|C|$ and let $\overline{\rho}:|\mathcal{C}|\to|\mathcal{A}|$ be the geometric realization of $\rho_{C,\mathcal{A}}$. Then

$$\begin{aligned} d^2(z,p_t) &= d^2(\overline{\rho}(z),p_t) \\ &\le (1-t)d^2(\overline{\rho}(z),x)+td^2(\overline{\rho}(z),y)-t(1-t)d^2(x,y) \\ &\le (1-t)d^2(z,x)+td^2(z,y)-t(1-t)d^2(x,y)\ . \end{aligned}$$

Hence, the CAT$(0)$-inequality holds for the triangle $[x,y,z]$.

One can also use this argument to decide when the standard realization of a building has a piecewise hyperbolic, CAT$(-1)$ metric. This gives the following analog of Theorem 4.3.

**Theorem 4.8.** (cf. [43, Remark 11.8]). *Suppose $\mathcal{C}$ is a building of type $(W,S)$, wtih $W$ word hyperbolic. Then its standard realization $D(\mathcal{C},K)$ can be given a piecewise hyperbolic,* CAT$(-1)$ *metric.*

For example, if $W$ is a reflection group on $\mathbb{H}^2$ generated by reflections across the faces of a hyperbolic polygon, then $\mathcal{C}$ is a *Fuchsian bulding*. In the right-angled case these have been constructed and studied by Bourdon [20, 21].

### *4.4.3 Using covering spaces to construct buildings*

In Section 3.1 we gave a technique for constructing RACGs, RAAGs and RAB, that involved first taking a polyhedral product and then passing to the universal cover. Here we discuss a generalization of [49] that works for arbitrary Coxeter systems not just the right-angled ones. Suppose $(W',S')$ is a Coxeter system with nerve $L'$ and that $L$ is another simplicial complex with vertex set $S$. Suppose that $f:S\to S'$ defines a simplicial map $L\to L'$ whose restriction to each simplex is injective. Let $(m'(s',t'))$ denote the Coxeter matrix of $(W',S')$. Define a new $(S\times S)$ Coxeter matrix $(m(s,t))$ by

$$m(s,t) := \begin{cases} 1 & \text{if } s=t, \\ m'(f(s),f(t)) & \text{if } \{s,t\}\in \mathrm{Edge}(L), \\ \infty & \text{otherwise,} \end{cases} \tag{4.20}$$

and let $(W,S)$ be the corresponding Coxeter system. in other words, if $m':\mathrm{Edge}(L')\to\{2.3,\dots\}$ is the edge labeling defining $(W',S')$, then $(W,S)$ is defined by the edge labeling $m$ which is the composition $m$ of $m'$ with $f|_{\mathrm{Edge}(L')}$, i.e., $m=m'f:\mathrm{Edge}(L)\to\{2,3,\dots\}$. The map of generating sets $S\to S'$ extends to a homomorphism $\varphi_f:W\to W'$; moreover, $\varphi_f$ is surjective if and only if $f$ is surjective.

For example, $f:L\to L'$ could be

(a) a covering projection,
(b) an embedding onto a subcomplex, or
(c) a covering space of an embedded subcomplex.

Let $K(L')$ denote the fundamental chamber for $(W',S')$, i.e., $K(L')$ is the geometric realization of the order complex of $\mathcal{S}(L')$. Similarly, if $\mathcal{S}(L)$ denotes the poset of simplices of $L$, then $K(L)=|\mathcal{S}(L)|$ is the fundamental chamber for a $W$-action on $D(W,K(L))$. The space $D(W,K(L))$ need not be contractible: a necessary and sufficient condition for this to be true is that for every spherical subset $T\in\mathcal{S}(W,S)$, $T$ is the vertex set of a simplex of $L$. The map $f$ can be used to get a new stratification of $K(L)$ indexed by $\mathcal{S}(L')$:

$$K(L)_{s'} := \bigsqcup_{s \in f^{-1}(s')} K(L)_s \quad \text{and} \quad K(L)_{T'} = \bigsqcup_{T \in f^{-1}(T')} K(L)_T. \tag{4.21}$$

Let $D(W', K(L))$ be the result of applying the basic construction to this new stratification of $K(L)$. If $W \neq W'$, then $D(W', K(L))$ will not be simply connected. Indeed, a necessary condition for it to be so is that $K(L)_{s'}$ is connected for each $s' \in S'$ and that $K(L)_{\{s',t'\}} \neq \emptyset$ whenever $m'(s',t') \neq \infty$ (see [48, Prop. 8.2.11]). However, the natural $\varphi_f$-equivariant map, $D(W, K(L)) \to D(W', K(L)$, is projection map of the universal covering . Similarly, for $A'$ the Artin group associated to $(W', S')$, we get a version of the Deligne complex $D(A', K(L))$. More interesting is the case of buildings. Given a building $\mathcal{C}'$ we get a building of type $(W', S')$, we get a nonstandard realization $D(\mathcal{C}', K(L))$. The main result of [49] is the following.

**Theorem 4.9.** ([49]). *With notation as above, the universal cover of $D(\mathcal{C}', K(L))$ is the geometric realization of a building $\mathcal{C}$ of type $(W, S)$.*

*Example 4.12.* (RACG*s,* RAAG*s,* RAB*s*). Here are the examples of type (b) which were considered previously in Section 3.1. Suppose that $L$ is a flag complex with vertex set $S$, that $S' = S$ and that $\Delta(S)$ is the simplex on $S$. Then $L$ is a subcomplex of $\Delta(S)$ and $K(L)$ is a cubical subcomplex of the cube $[0,1]^S$. Let $W' = (\mathbf{C}_2)^S$ be the sum of cyclic 2-groups. Then $D((\mathbf{C}_2)^S, K(L))$ is the polyhedral product $P_L$ of (3.3), $W = \prod_{L^1} \mathbf{C}_2$ is the RACG associated to $L^1$ in Example 3.3 and $D(W, K(L)) = \tilde{P}_L = \Sigma(W,S)$ is the Davis-Moussong complex of Subsection 4.2.3. If $A' = \sum_S \mathbb{Z}$ $(= (\mathbb{Z})^S)$, then $D(\sum \mathbb{Z}, K(L)$ is the polyhedral product $Z_L = (\operatorname{Cone}\mathbb{Z}, \mathbb{Z})^L$ of (3.4) and if $A = \prod_{L^1} \mathbb{Z}$ denotes the associated RAAG of Example 3.3, then $D(A, K(L)) = \tilde{Z}_L = \Lambda(W,S)$, the Deligne complex of Subsection 4.3.3 ($\Lambda(W,S)$ is also the RAB for $A$). Finally, if $\mathbf{E} = \{E_s\}_{s \in S}$ is a family of discrete sets, each having more than two elements, then $\mathcal{C}_s = E_s$ is a rank one building and the product $\mathcal{C}' = \prod \mathcal{C}_s$ is a building of type $((\mathbf{C}_2)^S, S)$. The thickness $(q_s)$ is the $S$-tuple defined by $q_s + 1 = \operatorname{Card}(E_s)$. Moreover, $D(\mathcal{C}', K(L))$ is the polyhedral product $(\mathbf{Cone}\,\mathbf{E}, \mathbf{E})^L$ of (3.4) and its universal cover is the geometric realization a RAB $\mathcal{C}$ of type $(W, S)$.

**Definition 4.9.** A *coloring* of a simplicial complex $L$ is a simplicial map $f : L \to \Delta(S')$ whose restriction to each simplex is injective. The elements of $S'$ are the *colors*. (The map $f$ gives a coloring of the vertices of $L$ by elements of $S'$.)

*Example 4.13.* (*Coxeter systems of type FC*). Here is a generalization of the previous example. Suppose $(W', S')$ is spherical so that $L' = \Delta(S')$ is the simplex on $S'$. Suppose $L$ is a flag complex with vertex set $S$ and that $f : L \to \Delta(S")$ is a coloring. The resulting Coxeter system $(W, S)$ is type FC (see Definition 4.3). Conversely, if $(W, S)$ is type FC, then there is an inclusion $f : L(W,S) \hookrightarrow \Delta(S)$ (in particular, $f$ is a coloring) and one can realize $\Delta(S)$ as the nerve of a spherical Coxeter system $(W', S)$ by the simple expedient of labeling by 2 all edges of $\Delta(S)$ which are missing from $L$, i.e., if $m(s,t) = \infty$, then change it to $m'(s,t) = 2$. Thus, every Coxeter system of type FC is the pullback of a spherical Coxeter system via some coloring. The complex $D(W', K(L))$ plays the role of the polyhedral product $P_L$ and the natural map $D(W, K(L)) \to D(W', K(L))$ is the universal cover. Thus, $\pi_1(D(W', K(L))$ is a torsion-free subgroup of finite index in $W$. As before, $(W', S')$ and $(W, S)$ have respective Artin groups $A'$ and $A$. Not all Coxeter systems occur as the type of a locally finite, thick spherical building; however, the irreducible Coxeter systems with diagrams $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$ all can occur. For spherical buildings of rank two whose Coxeter group is a dihedral group of order $2m$ (= a generalized $m$-gon), by a theorem of Feit-Higman [?fh] there are locally finite examples only for $m \in \{2,3,4,6,8\}$

*Example 4.14.* (*The case* $\dim L = 1$*, Fuchsian buildings*). If $\mathcal{C}'$ is a generalized $m$-gon, then $(W', S') = (D_m, \{s', t'\})$ and $L'$ is the 1-simplex $\Delta(S')$. If $L$ is any bipartite simplicial graph, then it admits a coloring

$f: L \to \Delta(S')$. For example, $L$ could be a $2k$-gon or a finite tree. So, for any bipartite graph $L$, there is a labeling of $\mathrm{Edge}(L)$ where each edge is labeled by the same integer $m$. If $m \in \{2,3,4,6,8\}$, we can choose a thick rank two spherical building $\mathcal{C}'$ of this type. Hence, if $L$ is any bipartite graph, there is a building $\mathcal{C}$ such that each rank two spherical residue is isomorphic to $\mathcal{C}'$. In other words, the link of each vertex of $K(L)$ is isomorphic to the 1-dimensional piecewise spherical complex $D(\mathcal{C}', \sigma(S'))$. (A vertex of $K(L)$ corresponds to the midpoint of an edge of $L$.) If $L$ is a $2k$-gon and if $m \geq 3$ when $2k = 4$, then $K(L)$ can be realized as a $2k$-gon in $\mathbb{H}^2$; so, $\mathcal{C}$ is a Fuchsian building.

We can get many more examples with $\dim L = 1$ by starting with $(W', S')$ a spherical Coxeter system (in which case $L' = \Delta(S')$) or possibly a euclidean Coxeter system (in which case $L' = \partial\Delta(S')$). To simplify the discussion let us suppose $L$ is an $n$-gon. The simplicial map $f: L \to \Delta(S')$ is just a closed edge path, possibly with backtracking. If $(W', S')$ has diagram $\widetilde{\mathbf{A}}_{n-1}$, then $\partial\Delta(S')$ contains an $n$-circuit with each edge labeled 3. Let $f: L \to \partial\Delta(S')$ be an embedding with image this $n$-circuit. If $\mathcal{C}'$ is a building of type $(W', S')$, then over each edge of $f(L)$ we have a rank two spherical residue with diagram $\mathbf{A}_2$, i.e., a projective plane. In particular, $L$ can be an $n$-gon with $n$ odd and each edge labeled 3. The resulting building $\mathcal{C}$ over $K(L)$ will be Fuchsian provided $n > 3$. In this fashion we can find examples of buildings with $L$ an $n$-gon whose edges are labeled fairly randomly by elements of $\{2,3,4\}$. We note that when $\mathcal{C}'$ is an irreducible spherical building corresponding to an algebraic group over a finite field $\mathbb{F}_q$ or a euclidean building corresponding to discrete valuation ring with residue field $\mathbb{F}_q$, then $\mathcal{C}'$ as well as the resulting building $\mathcal{C}$ will have constant thickness $q$. I doubt it

*Example 4.15.* (*Branched covers*). Suppose $L'$ is the nerve of a Coxeter system $(W', S')$ and that $f: L \to L'$ is a regular covering space with group of deck transformations $\Gamma$. The simplicial structure on $L'$ lifts to a simplicial structure on $L$ – a simplex of $L$ is a component of the inverse image of a simplex of $L'$. We can pull back the labeling on $\mathrm{Edge}(L')$ to get a labeling on $\mathrm{Edge}(L)$ and hence, a Coxeter system $(W, S)$ where $S = f^{-1}(S')$. By construction, $L \leq L(W,S)$ and an easy argument shows that $L(W,S) \leq L$ (use the fact that each triangle in $L$ projects to a triangle in $L'$). Since $K(L') = \mathrm{Cone}\, L'$ and $K(L) = \mathrm{Cone}\, L$, the natural projection $K(L) \to K(L')$ is a branched cover (branched at the cone point). The stratification of $K(L')$ indexed by $\mathcal{S}(W', S')$ lifts to one on $K(L)$ with the same index set; $K(L)$ also has a natural stratification indexed by $\mathcal{S}(W, S)$.

*Example 4.16.* (*Fuchsian buildings*).

*Example 4.17.* (*Hyperbolic buildings*).

### 4.4.4 Constructions, lattices

automorphism groups, lattices, Algebraic groups, Simple complexes of groups, $(B, N)$ pairs

# Chapter 5
# Simple complexes of groups

Suppose a group $G$ acts on a space $X$. Two points of $X$ belong to the same *orbit type* if their isotropy groups are conjugate subgroups of $G$. The points of a given orbit type form a *pure stratum* of $X$. (In fact we will want to define a pure stratum to be a connected component of a set of points of a given orbit type.) The closure of a pure stratum is a *stratum*. So, $X$ is a stratified space as in [23, II.12.1].

A *strict fundamental domain* for $G \curvearrowright X$ is a closed subspace $C \subset X$ such that $C$ intersects each orbit in exactly one point. It follows that restriction of the orbit projection $p : X \to X/G$ to the subspace $C$ is a homeomorphism and hence, that $C$ provides a section $s : X/G \to X$ of $p$ defined by $s = (p|_C)^{-1}$. The stratification of $X$ induces a stratification of $C$. This means that two points of $C$ belong to the same pure stratum if and only if their isotropy subgroups are equal. Similarly, if $G$ acts on a poset $\mathcal{P}$ by order-preserving automorphisms, then a *strict fundamental domain* for $G$ on $\mathcal{P}$ is a subposet $\mathcal{Q}$ which intersects each $G$-orbit in exactly one element. Simple complexes of groups are designed to deal with group actions on cell complexes which admit a strict fundamental domain.

The basic reference for the following material is [23]. Other references include [32], [60], [68], [116]. [48]?

The theory of simple complexes of groups is developed in [23]. Basic examples of simple complexes of groups are provided by Coxeter groups, Artin groups and buildings. So, we will make a few comments concerning the general theory here and postpone a fuller discussion to Appendix **??**

Let $\mathcal{P}$ be a poset. A *chain* in $\mathcal{P}$ is a totally ordered subset. In other words, a finite subset $\{p_0, \dots p_k\}$ is a chain if, possibly after reordering, $p_0 < \cdots < p_k$. The *order complex* of $\mathcal{P}$ is the set of all chains in $\mathcal{P}$. It is an abstract simplicial complex; a $k$ simplex is a chain of length $k+1$. Denote the geometric realization of the order complex by $|\mathcal{P}|$. Reversing the order relation on $\mathcal{P}$ one obtains the *opposite poset* $\mathcal{P}^{\mathrm{op}}$. Note that the order complexes of $\mathcal{P}$ and $\mathcal{P}^{\mathrm{op}}$ are identical

Simple complexes of groups arise from group actions with strict fundamental domains. Suppose $G \curvearrowright Y$ with strict fundamental domain $|\mathcal{Q}|$. The space $|\mathcal{Q}|$ has a stratification with strata indexed by $\mathcal{Q}$, where $|\mathcal{Q}|_\sigma = |\mathcal{Q}_{\leq\sigma}|$. (For example if $\mathcal{Q}$ is the poset of cells in a cell complex, then each stratum is a cell.) The local groups are the stabilizers of the strata (one assumes the $G$-action is without inversions meaning that the setwise stabilizer of a stratum fixes each point of the stratum. So, $G_\sigma$ is the isotropy subgroup of $|\mathcal{Q}|_\sigma$. (The reason a simple complex of groups is defined as a cofunctor rather than a functor is that for group actions, a smaller stratum should correspond to a larger isotropy subgroup.) The simple complex of groups is *developable* if it is induced from a $G$-action.

The next result uses the basic construction of Subsection 3.1.1 to show that every developable $G\mathcal{Q}$ arises from an action with strict fundamental on the basic construction.

**Theorem 5.1.** (The basic construction, cf. [23]). *Suppose that $G\mathcal{Q}$ is developable and $G = \lim G\mathcal{Q}$. Then there is a poset $D(G,\mathcal{Q})$ such that $G \curvearrowright D(G,\mathcal{Q})$ with strict fundamental poset $\mathcal{Q}$. Taking geometric realizations, we get a polyhedron $D(G,|\mathcal{Q}|)$ such that $G \curvearrowright D(G,|\mathcal{Q}|)$ with strict fundamental domain $|\mathcal{Q}|$.*

The basic construction $D(G,|\mathcal{Q}|)$,

$$D(G,|\mathcal{Q}|) = (G\times|\mathcal{Q}|)/\sim \tag{5.1}$$

is defined as in Subsection 3.1.1. Given $x \in |\mathcal{Q}|$, let $\sigma(x)$ be the index the smallest stratum $|\mathcal{Q}|_\sigma$ such that $x \in |\mathcal{Q}|_\sigma$. the equivalence relation $\sim$ is then defined the same way as in (3.8):

$$(g,x)\sim(g',x') \iff x = x' \text{ and } g^{-1}g' \in G_{\sigma(x)}\ , \tag{5.2}$$

in other words, $gG_{\sigma(x)}$ and $g'G_{\sigma(x)}$ are the same coset in $G/G_{\sigma(x)}$.

It also turns out that when $|\mathcal{Q}|$ is simply connected, then so is $D(G,|\mathcal{Q}|)$ (see **??**). So, when $|\mathcal{Q}|$ is simply connected, $G = \lim G\mathcal{Q}$ should be regarded as the "fundamental group of $G\mathcal{Q}$".

Given a Coxeter system $(W,S)$, we define two simple complexes of groups over $\mathcal{S}^{\mathrm{op}}$, where $\mathcal{S} = \mathcal{S}(W,S)$, the poset of spherical subsets of $S$. Denote the order relation on $\mathcal{S}^{\mathrm{op}}$ by $\prec$, so that $T' \prec T$ if $T$ is a proper subset of $T'$. The *fundamental chamber* $K$ is defined as $|\mathcal{S}^{\mathrm{op}}|$. The stratum $K_T$ is the union of all simplices in $|\mathcal{S}^{\mathrm{op}}|$ with maximum vertex $= T$.

*Example 5.1.* (*The complex of spherical Coxeter subgroups of $W$.*) There is a simple complex of groups $\mathcal{W}\mathcal{S}^{\mathrm{op}}$ over $\mathcal{S}^{\mathrm{op}}$ called the *complex of spherical subgroups of $W$*. It is defined as the cofunctor $T \mapsto W_T$, so that if $T' \prec T$, then $W_T$ is a subgroup of $W_{T'}$. It is obvious that the direct limit, $\lim \mathcal{W}\mathcal{S}^{\mathrm{op}}$, is equal to$W$. On the level of posets the basic construction $D(W,\mathcal{S}^{\mathrm{op}}$ is equal to the poset of spherical cosets $W\mathcal{S}$ defined in (4.3). On the level of spaces it is the cell complex $\Sigma(W,S)$ of Subsection 4.2.3 (actually, the strata of $D(W,K)$ are dual cones to the cells in $\Sigma(W,S)$).

*Example 5.2.* (*The complex of spherical Artin subgroups of $A$.*) The *complex of spherical subgroups of $A$* is the simple complex of groups $\mathcal{A}\mathcal{S}^{\mathrm{op}}$ over $\mathcal{S}^{\mathrm{op}}$ defined by $T \mapsto A_T$. If $T' \prec T$, then $A_T$ is a subgroup of $A_{T'}$. We also have $\lim \mathcal{A}\mathcal{S}^{\mathrm{op}} = A$. On the level of posets, $D(A,\mathcal{A}\mathcal{S}^{\mathrm{op}}) = A\mathcal{S}$ where $A\mathcal{S}$ is poset of spherical cosets from (4.4) and on the level of spaces, $D(A,K)$ is the Deligne complex $\Lambda(W,S)$ of Subsection 4.3.3 below.

## 5.1 Definitions

**Definition 5.1.** (cf. Bridson-Haefliger [23, II.12.11, p. 375]). A *simple complex of groups* $G\mathcal{Q}$ over a poset $\mathcal{Q}$ is a collection of groups $\{G_\sigma\}_{\sigma\in\mathcal{Q}}$ and monomorphisms $\phi_{\tau\sigma} : G_\sigma \to G_\tau$ defined whenever $\tau < \sigma$. The $G_\sigma$ are the *local groups*. Furthermore, $G\mathcal{Q}$ must be a cofunctor from $\mathcal{Q}$ to the category of groups and monomorphisms in the sense that $\phi_{\mu\tau}\phi_{\tau\sigma} = \phi_{\mu\sigma}$ whenever $\mu < \tau < \sigma$. (A "cofunctor" means a contravariant functor i.e., a functor on $\mathcal{Q}^{\mathrm{op}}$.)

A *simple morphism* $\psi = (\psi_\sigma)$ from $G\mathcal{Q}$ to a group $H$ is a function which assigns to each $\sigma \in \mathcal{Q}$ a homomorphism $\psi_\sigma : G_\sigma \to H$ such that $\psi_\sigma = \psi_\tau\phi_{\tau\sigma}$ whenever $\tau < \sigma$. The simple morphism $\psi$ is *injective on local groups* if each $\psi_\sigma$ is injective. If $H$ acts on a poset $\mathcal{P}$ with strict fundamental domain $\mathcal{Q}$, we get a simple complex of groups $G\mathcal{Q}$ by setting $G_\sigma$ equal to the isotropy subgroup at $\sigma$ for each $\sigma \in \mathcal{Q}$as well as, a simple morphism $\psi' : G\mathcal{Q} \to H$ such that $\psi' : G_\sigma \to H$ is the inclusion. Similarly, if $H$ acts on a polyhedron $D$ with strict fundamental domain the geometric realization of $\mathcal{Q}$, we get a simple complex of groups $G\mathcal{Q}$

where $G_\sigma$ is the isotropy subgroup of $H$ at a point in the pure stratum of the fundamental domain $|\mathcal{Q}|$. If the simple complex of groups $G\mathcal{Q}$ arises from an action on a poset $\mathcal{P}$ or a polyhedron $D$, then we say that $G\mathcal{Q}$ is *developable* and that $D$ is a *development*. Conversely, as we shall see below in Theorem 5.2 that $G\mathcal{Q}$ is developable if there is a simple morphism $G\mathcal{Q}$ to some group $H$ that is injective on local groups (in fact we can restrict $H$ to being the direct limit of the system of groups $G\mathcal{Q}$, as defined below.

The *direct limit* of $G\mathcal{Q}$ is a group $G$ ($= \lim G\mathcal{Q}$) together with a simple morphism $\psi : G\mathcal{Q} \to G$ with the following universal property: if $\psi' : G\mathcal{Q} \to H$ is any simple morphism, then there exists a unique homomorphism $\theta : G \to H$ such that $\theta\psi_\sigma = \psi' : G_\sigma \to H$ for all $\sigma \in \mathcal{Q}$. It follows that if it exists, the direct limit is unique up to a canonical isomorphism. To establish existence we note that he direct limit can be constructed in a standard fashion by taking the free product of the $G_\sigma$ and then quotienting by relations to insure that $G_\sigma$ is identified with its image under $\phi_\tau\sigma$ in $G_\tau$ and that $\phi_{\mu\tau}\phi_{\tau\sigma}(g) = \phi_{\mu\sigma}(g)$ whenever $\mu < \tau < \sigma$, for all $g \in G_\sigma$ (cf. [128, p. ?].

*Remark 5.1.* The usual way to regard a poset $\mathcal{Q}$ as a category is have an arrow, $\tau \to \sigma$, whenever $\tau \le \sigma$. before simple morphism?
When $\mathcal{Q}$ is a set of subsets (e.g. a set of cells in a cell complex), it also is usual for the partial order to be given by inclusion. Suppose $G$ acts on a cell complex $X$, that $\mathcal{P}$ is the poset of cells in $X$, and that for each $\sigma \in \mathcal{P}$, $G_\sigma$ is the (pointwise) stabilizer of $\sigma$. If $\tau < \sigma$ is a face of $\sigma$, then $G_\sigma < G_\tau$, i.e., "smaller cells have larger isotropy groups." For example, in the theory of graphs of groups [128], edge stabilizers are included in vertex stabilizers. If a cell complex $Y$ is a strict fundamental domain for the $G$-action on $X$ and $\mathcal{Q}$ is its poset of cells, then we get a simple complex of groups $G\mathcal{Q} = \{G_\sigma\}_{\sigma\in\mathcal{Q}}$ where the $\phi_{\sigma\tau} : G_\sigma \to G_\tau$ are inclusions. This explains why $G\mathcal{Q}$ is defined to be a cofunctor.

On the other hand, in some earlier papers, e.g., [32, 60], the author has used the reverse convention that a simple complex of groups is a functor from a poset to groups. The reason for this reversed convention is that for the complexes of groups associated to Coxeter groups and Artin groups in Section 5.3 the poset which naturally indexes the local groups is the poset of simplices in a certain simplicial complex $L$ (so that smaller simplices correspond to smaller local groups). Here we will dispense with this reversed convention by using as index set, $\mathcal{S}(L)^{\mathrm{op}}$, the opposite poset to the face poset of $L$.

A *simple morphism* $\psi = (\psi_\sigma)$ from $G\mathcal{Q}$ to a group $G$ is a function which assigns to each $\sigma \in \mathcal{Q}$ a homomorphism $\psi_\sigma : G_\sigma \to G$ such that $\psi_\tau = \psi_\sigma\phi_{\sigma\tau}$ whenever $\tau < \sigma$. The simple morphism $\psi$ is *injective on local groups* if each $\psi_\sigma$ is injective. If there is a simple morphism $\psi$ from $G\mathcal{Q}$ to $G$, which is injective on local groups, then, as we shall see in Section 5.2, we get a $G$-action on a polyhedron $D$ with $G\mathcal{Q}$ the associated complex of groups. If this is the case, $G\mathcal{Q}$ is said to be "developable" and $D$ is a "development" of it. As in [128, p.?], one can define the direct limit of any system of groups and homomorphisms. So, given a simple complex of groups $G\mathcal{Q}$ we get a group, $\lim G\mathcal{Q}$, called its *direct limit*. It comes equipped with a canonical morphism $\psi : G\mathcal{Q} \to \lim G\mathcal{Q}$. The direct limit is characterized by a universal property: if $\psi : G\mathcal{Q} \to H$ is any simple morphism to a group $H$, then there is a unique homomorphism $\lim G\mathcal{Q} \to H$ compatible with the canonical simple morphism. The simple complex of groups $G\mathcal{Q}$ is *developable* if the canonical homomorphism $G\mathcal{Q} \to \lim G\mathcal{Q}$ is injective on local groups.

The *order complex* of a poset $\mathcal{P}$ is the simplicial complex whose simplices are the totally ordered finite Put some place else? Change (5.3)
subsets $\{\tau_0, \dots, \tau_k\}$ in $\mathcal{P}$ (where $\tau_0 < \cdots < \tau_k$). The order complex of a poset $\mathcal{P}$ is a simplicial complex whose underlying topological space is denoted $|\mathcal{P}|$ and is called the *geometric realization of* $\mathcal{P}$. If $\mathcal{P}^{opp}$ denotes the opposite poset where the order relations are reversed, then $|\mathcal{P}^{opp}|$ is isomorphic to $|\mathcal{P}|$. There are two natural stratifications of $|\mathcal{P}|$ both indexed by $\mathcal{P}$:

$$|\mathcal{P}|_\sigma := |\mathcal{P}_{\ge\sigma}| \quad \text{and} \quad |\mathcal{P}|^\sigma := |\mathcal{P}_{\le\sigma}|. \tag{5.3}$$

In the second case, inclusion of one stratum into another corresponds to the original order relation on $\mathcal{P}$; in the first case, the order relation is reversed. So, we should regard $\{|\mathcal{P}|_\sigma\}$ as being indexed by $\mathcal{P}^{op}$. Given $x \in |\mathcal{P}|$, let $\sigma(x)$ be the index of the smallest stratum $|\mathcal{P}|_\sigma$ containing $x$.

Did I reverse something?

If $L$ is a simplicial complex, then $\mathcal{S}(L)$ denotes its poset of simplices together with $\emptyset$ (the empty simplex).

The dual of a simplicial polytope is a simple polytope.

This seems wrong. Maybe ok

**Definition 5.2.** (*Simplices of groups*). Suppose $\sigma$ is a simplex and $\partial\sigma$ is its boundary complex. Put $\mathcal{Q} = \mathcal{S}(\partial\sigma)$. The geometric realization of $\mathcal{Q}$ can then be identified with the dual simplex $\sigma^*$. A simple complex of groups $G\mathcal{Q}$ over $\mathcal{Q}$ is called a *simplex of groups*. If $\dim\sigma = 1$, then $\sigma^*$ is a 1-simplex and $G\mathcal{Q}$ is an *edge of groups*. If $\dim\sigma = 2$, then $G\mathcal{Q}$ is a *triangle of groups*.

switch $\sigma$, $\sigma^*$ and $P$ , $P^*$ ?

If $G\mathcal{Q}$ is an edge of groups, then $\lim G\mathcal{Q}$ is the amalgamated product of $G_{v_0}$ and $G_{v_1}$ along $G_\emptyset$. Here $v_0$ and $v_1$ are the vertices of $\partial\sigma$ (= the vertices of $\sigma^*$) and $G_\emptyset$ corresponds to the 1-simplex $\sigma^*$.

Nice examples of simplices of groups come from the theory Coxeter groups (see section 5.3). For example, when $\dim\sigma = 2$, the triangle of groups pictured below the direct limit is a Coxeter group $W$. (In this figure, $D_m$ stands for the dihedral group of order $2m$. The group $W$ is finite if and only if $1/p + 1/q + 1/r > 1$.) Other examples of simplices of groups come from spherical buildings with chamber transitive automorphism groups (see Example 5.6.

**Definition 5.3.** (*Polytopes of groups*). Suppose $P$ is a simplicial polytope and $\partial P$ is its boundary complex (so $\partial P$ is a simplicial complex). Put $\mathcal{Q} = \mathcal{S}(\partial P)$. Let $P^*$ be the dual simple polytope (meaning that the poset of nonempty faces of $P^*$ isomorphic to $\mathcal{S}(\partial P)^{op}$). A simple complex of groups $G\mathcal{Q}$ over $\mathcal{Q}$ (= $\mathcal{S}(\partial P)$) is a *polytope of groups*. If $\dim P = 2$, then $G\mathcal{Q}$ is a *polygon of groups*.

**Definition 5.4.** (*Trees of groups*). Given a tree $X$, let $\mathcal{Q} = \operatorname{Vert} X \bigsqcup \operatorname{Edge} X$ be its set of cells, partially ordered by reverse inclusion. A *tree of groups* is a simple complex of groups over $\mathcal{Q}$. In fact, exactly the same definition works for any connected graph $X$ without loop, yielding the notion of a *graph of groups*.

Suppose $G\mathcal{Q}$ is a graph of groups as in the above definition. Put $G = \lim G\mathcal{Q}$. It follows from a result in [128] on the structure of amalgams that the natural simple morphism from $G\mathcal{Q}$ to $G$ is injective on local groups, i.e., $G\mathcal{Q}$ is developable. (Even when $X$ is allowed to have loops, one can still define a "graph of groups" as in [128, 4.4]; it is still a complex of groups as in [23, ?]; however, it is not a simple complex of groups. The difference is that when $X$ is a loop there are two inclusions from the edge group into the vertex group and the local groups need not inject into the direct limit. However, there is still a notion of a fundamental group of a graph of groups and the local groups inject into it; so, any graph of groups is developable. For example, If $X$ is a single loop, then the fundamental group is an HNN extension, cf. Example 5.11 in section 5.6.

## 5.2 The basic construction

move earlier

We begin by reviewing material on simple complexes of groups from [23, II.12], [32], [60].

**Theorem 5.2.** (The basic construction, cf. [23]). *Suppose $\psi' = (\psi'_\sigma)$ is a simple morphism from $G\mathcal{Q}$ to a group $H$ that is injective on local groups. Also, let $G$ be the direct limit of $G\mathcal{Q}$ and $\psi : G\mathcal{Q} \to G$ the canonical simple morphism.*

(i) *There is a $H$-action on a poset $D(\psi', \mathcal{Q})$ with $\mathcal{Q}$ as strict fundamental domain. (The poset $D(\psi', \mathcal{Q})$ is called the* development *of $G\mathcal{Q}$ with respect to $\psi$.)*

(ii) *There is a $H$-action on a polyhedron $D(\psi',\mathcal{Q})$ with strict fundamental domain $|\mathcal{Q}|$.*
(iii) *(Compare* [?vinberg]*.) The polyhedron $D(\psi',\mathcal{Q})$ has the following universal property: suppose $H \curvearrowright Z$ and $f : |\mathcal{Q}| \to Z$ is a map such that for all $\sigma \in \mathcal{Q}$, $f(|\mathcal{Q}|_\sigma)$ is contained in* $\mathrm{Fix}(G_\sigma, Z)$*,the fixed point set of $G_\sigma$ on $Z$. Then there is a unique extension of $f$ to a $H$-equivariant map $\tilde{f} : D(\psi',\mathcal{Q}) \to Z$.*
(iv) *The orbit projection $p : D(\psi',\mathcal{Q}) \to |\mathcal{Q}|$ is a retraction.*
(v) *The space $D(\psi',\mathcal{Q})$ is connected if and only if $|\mathcal{Q}|$ is connected and the subgroups $\{G_\sigma\}_{\sigma\in\mathcal{Q}}$ generate $H$.*
(vi) *The space $D(\psi',\mathcal{Q})$ is simply connected if and only if $|\mathcal{Q}|$ is simply connected and $H = G$.* vi is key
(vii) *If $D(\psi,\mathcal{Q})$ is contractible, then so is $|\mathcal{Q}|$.*

*Proof.* Since $\psi'$ is injective on local groups, we may identify each local group $G_\sigma$ with its image $\psi'_\sigma(G_\sigma)$.

(i) The poset $D(\psi',\mathcal{Q})$ is defined to be the disjoint union $\bigsqcup_{\sigma\in\mathcal{Q}} H/G_\sigma$ consisting of all pairs $(gG_\sigma,\sigma)$, with $g \in H$ and $\sigma \in \mathcal{Q}$. The partial order is the natural one, defined by $(hG_\tau,\tau) < (gG_\sigma,\sigma) \iff \tau < \sigma$ and $h^{-1}g \in G_\tau$.

(ii) For any $x \in |\mathcal{Q}|$, let $\sigma(x)$ be the least element $\sigma \in \mathcal{Q}$ such that $x$ belongs to the stratum $|\mathcal{Q}|_\sigma$. The space $D(\psi',|\mathcal{Q}|)$ is defined by

$$D(\psi',|\mathcal{Q}|) := (H \times |\mathcal{Q}|)/\sim, \tag{5.4}$$

where the equivalence relation $\sim$ is defined by $(g,x) \sim (g',x') \iff x = x'$and $gG_{\sigma(x)} = g'G_{\sigma(x)}$.

(iii) Suppose $[g,x]$ denotes the equivalence class of $(g,x)$. The map $\tilde{f} : D(\psi',|\mathcal{Q}|) \to Z$ is then defined by $\tilde{f}([g,x]) = gf(x)$.

(iv) This is precisely what is meant by the statement that $|\mathcal{Q}|$ is a strict fundamental domain.

(v) and (vii) Since $p : D(\psi',|\mathcal{Q}|) \to |\mathcal{Q}|$ is a retraction, it induces a surjection on $\pi_0$ (in fact, on the homotopy groups $\pi_i$ for all $i$). So, if $D(\psi',|\mathcal{Q}|)$ is path connected, then so is $|\mathcal{Q}|$. Let $H'$ be the subgroup of $H$ generated by $\{G_\sigma\}$. Since $\mathrm{Im}\,\psi'$ is contained in $H'$, we have a simple morphism $\psi'' : G\mathcal{Q} \to H'$ and an embedding $D(\psi'',|\mathcal{Q}|) \hookrightarrow D(\psi',|\mathcal{Q}|)$. Clearly, $D(\psi'',|\mathcal{Q}|)$ is both open and closed in $D(\psi',|G\mathcal{Q}|)$. Statement (v) follows. Similarly, (vii) is true.

(vi) The universal property of the direct limits gives a homomorphism $\theta : G \to H$. Put $D = D(\psi,|\mathcal{Q}|)$ and $D' = D(\psi',|\mathcal{Q}|,\psi')$, where $\psi : G\mathcal{Q} \to H$ is the canonical simple morphism. By (iii) there is a $\theta$-equivariant map $\pi : D \to D'$ inducing the identity on $|\mathcal{Q}|$; the map $\pi$ is easily seen to be a covering projection. First suppose that $D'$ is simply connected. Then $\pi$ is a homeomorphism, $\theta$ is an isomorphism and by (iv), $|\mathcal{Q}|$ is simply connected. Conversely, suppose that $|\mathcal{Q}|$ is simply connected and that $G = H$. Then we can assume $D = D'$. Let $E \to D$ be any connected covering space. We will show that $D$ is simply connected by showing that $E \to D$ is a homeomorphism. Since $|\mathcal{Q}|$ is simply connected the inclusion $|\mathcal{Q}| \hookrightarrow D$ lifts to $E$ and gives a fundamental domain for the group $\tilde{G}$ of all lifts to $E$ of $G$-action on $D$ (this is the key point). Each local group $G_\sigma$ lifts isomorphically to the isotropy subgroup of $\tilde{G}$ at a point of $|\mathcal{Q}|$. Since $E$ is connected, these local groups generate $\tilde{G}$; hence, $\tilde{G} \to G$ is onto. By (iii), the lift of the fundamental domain defines a section $D \to E$ of the covering projection $E \to D$. Therefore, $E \to D$ is a homeomorphism.

*Remark 5.2.* In view of this theorem from now on we shall interpret the statement that "$G\mathcal{Q}$ is developable" as being synonymous to the statement that the canonical morphism $\psi : \mathcal{Q} \to \lim G\mathcal{Q}$ is injective on local groups.

*Remark 5.3.* When $G = \lim G\mathcal{Q}$, we write $D(G,|\mathcal{Q}|)$ instead of $D(\psi,|\mathcal{Q}|)$. By part (vi) of the above theorem, remarks
if $|\mathcal{Q}|$ is simply connected, then so is $D(|\mathcal{Q}|,G)$. So, when this holds, the direct limit should be thought of as "fundamental group of $G\mathcal{Q}$" and $D(|\mathcal{Q}|,G)$ should be thought of as the "universal cover of $G\mathcal{Q}$." This means that in some sense the notion of a simple complex of groups should only be considered when $|\mathcal{Q}|$ is simply connected. Indeed, it is only in this case that the $G$-action on the universal cover has $|\mathcal{Q}|$ as a strict fundamental domain.

The polyhedron $D(|\mathcal{Q}|,\psi)$ is locally finite if and only if each local group $G_\sigma$ is finite. When this is the case the $G$-action on $D(|\mathcal{Q}|,\psi)$ is proper.

omit

*Example 5.3.* (*Graphs of groups*). Suppose $Y$ is a connected graph without loop. (This means that each edge of $Y$ has two distinct endpoints; in other words $Y$ is a regular 1-dimensional CW complex.) its vertex set and edge set are denoted $\operatorname{Vert} Y$ and $\operatorname{Edge} Y$. Let $\mathcal{Q} = \operatorname{Vert} Y \bigsqcup \operatorname{Edge} Y$ be its set of cells, partially ordered by reverse inclusion. A *graph of groups* on $Y$ is a simple complex of groups over $\mathcal{Q}$. (The standard definition of a graph of groups in [128, 4.4] allows the possibility of loops: one must use directed edges and for each edge $\tau$ one must give monomorphisms from $G_\tau$ to the local groups at its initial and terminal vertices.) If $Y$ is a tree, then $G\mathcal{Q}$ is a *tree of groups*.

omit

For any system of groups and homomorphisms $\{G_\sigma, \phi_{\sigma\tau}\}$ one can form the direct limit (cf. [23, p.]), denoted by $\varinjlim_\sigma G_\sigma$ In particular, one can take the direct limit of the system formed by $G\mathcal{Q}$ denoted $\lim G\mathcal{Q}$. The direct limit has the universal property that for any group $H$ and simple morphism $\psi : G\mathcal{Q} \to H$, there is a unique homomorphism $\hat{\psi} : \lim G\mathcal{Q} \to H$ such that $\psi_\sigma = \hat{\psi}\iota_\sigma$ (see [23, II.12.13, p. 376]). A *simple morphism* $\psi = (\psi_\sigma)$ from $G\mathcal{Q}$ to a group $G$ is a function which assigns to each $\sigma \in \mathcal{Q}$ a homomorphism $\psi_\sigma : G_\sigma \to G$ such that $\psi_\tau = \psi_\sigma\phi_{\sigma\tau}$ whenever $\tau < \sigma$. For each $\sigma \in \mathcal{Q}$, there is a canonical homomorphism $\iota_\sigma : G_\sigma \to \lim G\mathcal{Q}$, hence, a *canonical simple morphism* $\iota : G\mathcal{Q} \to \lim G\mathcal{Q}$. The simple complex of groups $G\mathcal{Q}$ is *developable* if $\iota$ is injective on local groups.

omit these paragraphs

A *strict fundamental domain* for the action of a group $G$ on a space $Y$ is a closed subspace $C \subset Y$ such that $C$ intersects each orbit in exactly one point. It follows that restriction of the orbit projection $p : Y \to Y/G$ to $C$ is a homeomorphism. Note that $C$ provides a section $s : Y/G \to Y$ of the orbit projection $p$ defined by $s = (p|_C)^{-1}$. Similarly, if $G$ acts on a poset $\mathcal{P}$ by order-preserving automorphisms, then a *strict funadamental domain* for $G$ on $\mathcal{P}$ is a subposet $\mathcal{Q}$ which intersects each $G$-orbit in exactly one element. Simple complexes of groups are designed to deal with group actions on cell complexes which admit a strict fundamental domain.

Suppose for example, that $G\mathcal{Q}$ is a tree of groups, where $\mathcal{Q}$ is the poset of cells in the tree $Y$. Put $G = \lim G\mathcal{Q}$. It follows from a result in [128] on the structure of amalgams that that natural maps from the local groups to $G$ are injective, i.e., $G\mathcal{Q}$ is developable. As in [128], one can construct a $G$-action on a tree $T$ with fundamental domain $Y$. The $T$ tree is defined by

$$\operatorname{Vert} T = \bigsqcup_{v \in \operatorname{Vert} Y} G/G_v \quad \text{and} \quad \operatorname{Edge} T = \bigsqcup_{\sigma \in \operatorname{Edge} Y} G/G_\sigma.$$

The monomorhisms $\phi_{v\sigma} : G_\sigma \to G_v$ induce maps $G/G_\sigma \to G/G_v$, hence, the map $\operatorname{Edge} T \to \operatorname{Vert} T$ which defines the order relation on the 0- and 1-cells of $T$. The group $G$ acts on $T$ and one can then check that $T$ is a tree. The isotropy subgroup at any vertex of $T$ is conjugate to one of $G_v$ and an isotropy subgroup at an edge is conjugate to one of the $G_\sigma$. Moreover, there is a natural inclusion $Y \hookrightarrow T$ under which $Y$ becomes a strict fundamental domain.

Conversely, suppose a group $G$ acts on a tree $T$ with strict fundamental domain a subtree $Y$. By associating to each vertex or edge of $Y$ the corresponding isotropy subgroup of $G$ one gives $Y$ the structure of a tree of groups.

## 5.3 Examples where $\mathcal{Q}$ is the dual face poset of a simplicial complex

Given a simplicial complex $L$, its poset of simplices (including the empty simplex) is denoted by $\mathcal{S}(L)$. This section concerns simple complexes of groups over posets of the form $\mathcal{S}(L)$. The geometric realization $|\mathcal{S}(L)|$ of this poset is the cone on the barycentric subdivision of $L$. (The vertex corresponding to the empty simplex is the cone point.) Given a simple complex of groups $G\mathcal{S}(L)$ and a morphism to a group $G$, the fundamental domain for the $G$-action on on its development $D$ is $|\mathcal{S}(L)|$. To simplify notation, put

$$K = |\mathcal{S}(L)| \tag{5.5}$$

and call it a *fundamental chamber* (for the $G$-action on $D$). As in (5.3), for each $\sigma \in \mathcal{S}(L)$, put $K_\sigma = |\mathcal{S}(L)|_\sigma := |\mathcal{S}(L)_{\geq \sigma}|$.

*Example 5.4.* If $L$ is a PL triangulation of $S^{n-1}$, then $K$ is an $n$-disk and $K_\sigma$ is the dual cell to the simplex $\sigma$. shorten
In particular, if $L$ is the boundary complex of a simplicial convex polytope, then $K$ is the dual simple convex polytope. To be even more particular, $L$ is the barycentric subdivision of $\partial\Delta$, where $\Delta$ is an $n$-simplex, then $K$ is an $n$-dimensional permutohedron.

Let $S$ be a set. A *Coxeter matrix* on $S$ is a symmetric $(S \times S)$-matrix $M = (m_{st})_{(s,t)\in S\times S}$ with entries in $\mathbb{N} \cup \{\infty\}$, where the diagonal entries all equal 1 and each off-diagonal entry is an integer $\geq 2$ or the symbol $\infty$. The Coxeter matrix is *right angled* if each of its off-diagonal entries is 2 or the symbol $\infty$. There is a *Coxeter group* $W$ associated to the Coxeter matrix $M$; its set of generators is $S$ and its relations are: $(st)^{m_{st}} = 1$, where $(s,t) \in S \times S$. Since $m_{ss} = 1$, we have $s^2 = 1$; so, each $s$ has order $\leq 2$ (in fact, it must $= 2$). The *cosine matrix* of $M$ is the matrix $\cos M = (c_{st})_{(s,t)\in S\times S}$ defined by

$$c_{st} = -\cos(\pi/m_{st}), \tag{5.6}$$

where when $m_{st} = \infty$, $c_{st}$ is interpreted to be $-\cos(\pi/\infty) = -cos(0) = -1$. There is an extensive literature on Coxeter groups, for example, [15], [48]. The pair $(W,S)$ is a *Coxeter system*. Given a subset $J \leq S$, let $M_J = (m_{st})_{(s,t)\in J\times J}$ denote the restriction of $M$ to $J$. The subgroup generated by $J$ is denoted $W_J$ and called the *special subgroup* corresponding to $J$. It is a standard fact that $(W_J, J)$ also is a Coxeter system (cf. [15, pp.12-13]). It turns out that the subgroup $W_J$ generated by $J$ is finite if and only if $\cos M_J$ is positive definite. If $W_J$ is finite, then it can be represented as a group generated by orthogonal linear reflection groups on $\mathbb{R}^n$, where $n = \mathrm{Card}(J)$. So, $W_J$ is a finite reflection group on the unit sphere $S^{n-1} < \mathbb{R}^n$. For this reason a finite Coxeter group $W_J$ is said to be *spherical*; its set of fundamental generators $J \leq S$ is said to be a *spherical subset* of $S$. It turns out that a fundamental chamber for $W_J$ on $S^{n-1}$ is a spherical $(n-1)$-simplex. The poset of nonempty spherical subsets of $S$ is an abstract simplicial complex $L(W,S)$ (or simply $L$) called the *nerve* of $(W,S)$. Thus, the vertex set of $L$ is $S$ and a nonempty subset of $S$ is a simplex of $L$ if and only if it is spherical. If we include the empty simplex in the face poset of $L$, we get $\mathcal{S}(L)$, the poset of all spherical subsets of $S$.

Suppose $\sigma(S)$ is the full simplex on $S$. The face poset of the dual simplex $\sigma^*(S)$ is isomorphic to $\mathcal{S}(\partial\sigma(S))^{op}$. If $(W,S)$ is a spherical Coxeter system, then $L(W,S) = \sigma(S)$. The poset of subgroups $\{W_J\}_{J<S}$ generated by the proper subsets of $S$ is a simplex of groups over $\sigma^*$ $(= \sigma^*(S))$; the development $D(\sigma^*, W)$ is a sphere; and $W$ acts properly of $D(\sigma^*, W)$ with strict fundamental domain $\sigma^*$. Most of this works for an arbitrary Coxeter system $(W,S)$: we get a simple complex of groups over $\mathcal{S}(\partial\sigma(S)$, i.e., a simplex of groups over $\sigma*$, and $W$ is the direct limit (provided $\dim\sigma(S) \geq 2$). this is the classical definition of the "Coxeter complex". The trouble with this definition is that, in general, the $W$-action on $D$ need not be proper, in general. (Also, $D$ will not be a sphere.) The remedy is explained in Example 5.5 below: one uses instead the natural simple complex of groups over the poset of spherical subsets of $S$.

*Remark 5.4.* (*Simplicial Coxeter groups*). A Coxeter system $(W,S)$ is *simplicial* if $W_J$ is finite for each proper subset $J < S$. If, in addition, $W$ is infinite, then it is called a *Lannér group*. So, if $(W,S)$ is Lannér, then its nerve is $\partial\sigma(S)$ and we get a simple complex of groups over $\mathcal{S}(\partial\sigma(S))$. Thus, the $W$-action on $D(W,\sigma^*)$ is proper exactly in the case when $(W,S)$ is simplicial. When $W$ is Lannér it turns out that $D(W,\sigma^*)$ can be identified with either with euclidean space $\mathbb{E}^{n-1}$ or hyperbolic space $\mathbb{H}^{n-1}$ depending on whether the matrix $\cos M$ defined by (5.6) is positive semidefinite or indefinite of signature $(n-1,1)$, , cf. [48, §6.9].

There is another way to associate a group to a Coxeter matrix $(m_{st})$. For each $s \in S$ introduce the symbol

maybe later?

$a_s$. The *Artin group* $A$ associated to $M$ has generating set $\{a_s\}_{s\in S}$ and relations:

$$\underbrace{a_s a_t \cdots}_{m_{st}\text{ terms}} = \underbrace{a_t a_s \cdots}_{m_{st}\text{ terms}}, \tag{5.7}$$

where both sides of the equation are alternating words in $a_s$ and $a_t$, where $\{s,t\}$ ranges over the edges of the nerve $L$. Notice that the relation (5.7) can be rewritten as $(a_s a_t \cdots)(\cdots (a_s)^{-1}(a_t)^{-1}) = 1$. In the Coxeter group $s = s^{-1}$, so after replacing $a_s$ by $s$, (5.7) becomes $(st)^{m_{st}} = 1$. Hence, the function $a_s \mapsto s$ extends to an epimorphism $p : A \to W$. The kernel of $p$ is denoted by $PA$ and called the *pure Artin group*.

Given a subset $J \le S$, let $A_J$ denote the subgroup generated by $\{a_s\}_{s\in J}$. It is known that $A_J$ is the Artin group associated to $(W_J, J)$. Moreover, $A$ is the direct limit of the $A_J$ where $J$ ranges over the simplices in the 1-skeleton of $L$. The subgroup $A_J$ is a *spherical Artin group* if $J$ is a spherical subset of $S$.

*Example 5.5.* (*The complex $W\mathcal{S}(L)$ of spherical subgroups in a Coxeter group*). Suppose $M$ is a Coxeter matrix, $(W,S)$ is the associated Coxeter system, $L$ is its nerve and $\mathcal{S}(L)$ is the poset of spherical subsets. For each $\sigma \in \mathcal{S}(L)$, let $M_\sigma$ be the restriction of $M$ to $\sigma$ and let $(W_\sigma, \sigma)$ be the associated Coxeter system.

(i) For each $\sigma \in \mathcal{S}(L)$, there is a natural homomorphism $\psi_\sigma : W_\sigma \to W$ induced by the inclusion $\sigma \hookrightarrow S$ and similarly, for each $\tau < \sigma$, there is $\phi_{\sigma\tau} : W_\tau \to W_\sigma$. It is immediate from the definition that $W$ is the direct limit of the family $\{W_\sigma\}_{\sigma\in\mathcal{S}(L)}$.
(ii) It is proved in [15] that the homomorphisms $\phi_{\sigma\tau}$ and $\psi_\sigma$ are injective. Since each of the $\phi_{\sigma\tau}$ is injective, the functor $\sigma \mapsto W_\sigma$ defines a simple complex of groups over $\mathcal{S}(L)$. We call this simple complex of groups the *Coxeter complex* and denote it $W\mathcal{S}(L)$. Since $\psi_\sigma : W_\sigma \to W$ is injective, $W\mathcal{S}(L)$ is developable.

Let $K$ be the fundamental domain defined by (5.5) and let $D = D(K,W) = (W \times K)/\sim$ be the result of applying the basic construction (5.4). Often, $D$ is called the *Davis complex*. The focus of the book [48] is the study of $D$.

(iii) It is shown in [48] and [102] that $D(K,W)$ admits a cell structure so that the resulting piecewise euclidean metric is CAT(0); in particular, $D(K,W)$ is contractible. In this cell structure the cells are the "Coxeter zonotopes" described in **??**. With this cell structure it is more appropriate to call $D(K,W)$ the *Davis-Moussong complex* (Moussong proved in his PhD thesis [102] that this cell structure is CAT(0)). In the right-angled case this is the cubical structure described in Section **??**.
(iv) Since each of the local groups $W_\sigma$ is finite, the complex $D(K,W)$ is locally finite and the $W$-action on it is proper. Since $D(K,W)$ has a CAT(0)-metric, it is a model for $\underline{E}W$ (the classifying space for proper $W$-actions).

Reflection groups on constant curvature space

RACGs

contractible subcomplexes of the Davis complex

Coxeter groups first arose in the study of discrete groups generated by reflections on the constant curvature spaces $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. Let $\mathbb{X}^n$ stand for one of these. The following facts are true (e.g. see [48]).

(a) Any such reflection group is a Coxeter group $W$.

(b) The $W$-action on $\mathbb{X}^n$ has a strict fundamental domain $P$, which is geodesically convex in $\mathbb{X}^n$. (The interior of $P$ is a component of the complement of the union of the reflecting hyperplanes in $\mathbb{X}^n$.)
(c) Let $D(P,W) = W \times P)/\sim$ be defined as in (5.4). The natural map $D(P,W) \to \mathbb{X}^n$ (defined by Theorem 5.2 (iii)) is a homeomorphism. Thus, $\mathbb{X}^n$ is tessellated by the $W$-translates of $P$.
(d) A generating set $S$ for $W$ can be taken to be the set of reflections across the codimension one faces of $P$. Then $(W,S)$ is a Coxeter system and the nerve $L(W,S)$ is dual to the boundary complex, $\partial P$. The set of reflections of $\mathbb{X}^n$ generated by $S$ is precisely the set of conjugates of $S$ in $W$. (For this reason, in a general Coxeter system $(W,S)$ a *reflection* in $W$ is defined as any conjugate of an element of $S$ in $W$.)
(e) If $P$ is compact, then it is a compact polyhedron homeomorphic to an $n$-disk and $L(W,S)$ is the triangulation of $S^{n-1}$ dual to $\partial P$. Hence, the polyhedron $K$ (defined as the dual of Cone$L$) is equal to $P$.
(f) If $W$ is finite, then it has a representation as an orthogonal reflection group on $\mathbb{S}^n$ (where $n+1 = \mathrm{Card}(S)$). This extends to a linear representation on $\mathbb{R}^{n+1}$ in which each $s \in S$ acts as a linear reflection (cf. [48]). The fundamental chamber is a simplicial cone in $\mathbb{R}^{n+1}$, bounded by hyperplanes which are fixed point sets of elements of $S$.

These facts indicate the way in which one should think of the Coxeter complex $W\mathcal{S}(L)$: $K$ should be thought of as a simple convex polytope, $L$ as the dual triangulation of $S^{n-1}$, and the local group $W_\sigma$ is the isotropy group of the cell of $K$ which is dual to $\sigma \in \mathcal{S}(L)$. The general definitions of a Coxeter system, the fundamental chamber $K$ and the basic construction $D(K,W)$ gives a much broader class of examples than occur in constant curvature. Even when $K$ is required to combinatorially equivalent to a simple convex polytope very few of these examples examples are equivalent to tessellations of constant curvature space.

If $L$ is a flag triangulation of $S^{n-1}$, then $K$ is an $n$-disk and $D(K,W)$ is a contractible $n$-manifold. If $L$ is PL homeomorphic to $S^{n-1}$, then $D(K,W)$ is PL homeomorphic to euclidean $n$-space. (For further information see [48, Ch. 10]).)

There are many classical beautiful tessellations of constant curvature space.

?$K(L)$ is an $m$-gon. Haglund [23, pp. 393–394]

*Example 5.6.* (*Buildings*). The theory of buildings gives an important way to construct a simple complexes of groups with the same fundamental chamber as a Coxeter system. The *type* of a building is a Coxeter matrix $M$. (Any $M$ can occur at least when all $m_{st}$, with $s \neq t$ lie in $\{2,3,4,6\} \cup \{\infty\}$.) The underlying poset is the same as in Example 5.5: it is $\mathcal{S}(L)$ where $L$ is the nerve of the Coxeter system $(W,S)$ of type $M$. Actually, the notion of building can be defined without specifying a local group for each $\sigma \in \mathcal{S}(L)$. This goes as follows. A *building of type M* is a pair $(\mathcal{C},\delta)$ where $\mathcal{C}$ is a set (whose elements are called *chambers*) and where $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a function (called a *W-valued distance function*) where $\delta$ satisfies certain axioms listed in [1, p.]. Let $J$ be a subset of $S$. Two chambers $c,d \in \mathcal{C}$ *belong to the same J-residue* if $\delta(c,d) \in W_J$. (This is an equivalence relation on $\mathcal{C}$; a $J$-residue is an equivalence class.) A residue of type $J$ is spherical if $J$ is a spherical subset of $S$. For example, an $\emptyset$-residue is a singleton $\{c\}$. The Coxeter group $W$ itself is a building where $\delta : W \times W \to W$ is given by $\delta(v,w) = v^{-1}$. Thus, $v$ and $w$ belong to the same $J$-residue if and only if they determine the same coset $vW_J = wW_J$. An *apartment* in a building $\mathcal{C}$ is the image of an embedding $W \hookrightarrow \mathcal{C}$ which is isometric with respect to the $W$-valued distance functions. The *basic construction* on $\mathcal{C}$ is defined using the same fundamental chamber $K$ as for the Coxeter complex (cf. (5.5)), that is,

$$D(K,\mathcal{C}) = (\mathcal{C} \times K)/\sim, \tag{5.8}$$

where $(c,x) \sim (d,y)$ if and only if $x = y$ and $c$, $d$ belong to the same $\sigma(x)$-residue. Thus, if $p : D(K,\mathcal{C}) \to K$ is the natural projection, then the fiber $p^{-1}(x)$ can be identified with the set of $\sigma(x)$-residues. It follows that if $\mathcal{P}(\mathcal{C})$ denotes the poset of spherical residues in $\mathcal{C}$, then its geometric realization $|\mathcal{P}(\mathcal{C})|$ is naturally identified with $D(K,\mathcal{C})$. The polyhedron $D(K,\mathcal{C})$ defined by (5.8) is also called as the *standard realization of* $\mathcal{C}$. The building $\mathcal{C}$ has *finite thickness* if each $\{s\}$-residue is finite for each $s \in S$. This implies that each spherical residue is finite and hence, that the complex $D(K,\mathcal{C})$ is locally finite. One should think of the geometric realization $D(K,\mathcal{C})$ as follows. For each $s \in S$, let $K_s$ denote the codimension one face $|\mathcal{S}(L)_{\geq\{s\}}|$. For a given copy of $K_s$, the set of chambers containing $K_s$ is an $\{s\}$-residue, so the number of chambers meeting along a codimension one face is $\geq 2$ (since in any building each $\{s\}$-residue must contain at least two elements). On the other hand, each apartment is a Coxeter complex in which exactly two chambers meet along each codimension one face. If the fundamental chamber $K$ happens to be a convex polytope (e.g. a simplex) then $D(K,\mathcal{C})$ is tessellated by copies of $K$ and the link of each codimension one face is a finite set of points while the link a each codimension two face is the realization of rank two spherical building as a graph. To be even more specific, suppose $W$ is a cocompact reflection group on the hyperbolic plane with fundamental domain $K$ a convex polygon with all face angles equal to, say $\pi/3$. Let $\mathbb{F}$ be a field with $q$ elements. Then there is a building $\mathcal{C}$ with realization $D(K,\mathcal{C})$, so that $q+1$ copies of $K$ meet along each edge. (Here $q+1$ is the number of points in the projective line over $\mathbb{F}$) and the link at each vertex is the projective plane over $\mathbb{F}$ (a rank two building).

By using the piecewise Euclidean metric on the geometric realizations of its apartments, it is proved in [43] that $D(K,\mathcal{C})$ also has a CAT$(0)$ cell structure; in particular, it is contractible.

A building $\mathcal{C}$ is *spherical* if the corresponding Coxeter system $(W,S)$ is spherical. The *spherical realization* of $\mathcal{C}$ is $D(\mathcal{C},\sigma^*)$ where $\sigma^*$ is the simplex dual to the full simplex on $S$. Each apartment in $D(\mathcal{C},\sigma^*)$ is a round sphere, namely, the spherical realization of $D(\mathcal{C},\sigma^*)$. So, the spherical realization of any spherical building is a piecewise spherical complex. The standard realization, $D(\mathcal{C},K)$, of the spherical building $\mathcal{C}$ is isomorphic to the cone on $D(\mathcal{C},\sigma^*)$. It follows that its spherical realization $D(\mathcal{C},\sigma^*$ is CAT$(1)$. (See [43].)

A residue in a building is itself a building. In particular, a spherical residue is a spherical building. This observation provides insight about the local structure of the standard realization of any building $\mathcal{C}$. Suppose $\mathcal{D} \in \mathcal{P}$ is a spherical residue of type $J$, where $J \leq S$ is the vertex set of a simplex $\tau \in \mathcal{S}(W,S)$. The geometric realization of any chamber $c \in \mathcal{D}$ is homeomorphic to $K$ and $\mathcal{D}$ corresponds to a stratum $K_\tau$ of $K$. The normal link of $K_\tau$ in $K$ is isomorphic to the dual simplex $\tau^*$. So, the normal link of the stratum corresponding to $\mathcal{D}$ is isomorphic to $D(\mathcal{D},\tau^*)$, which is the the spherical realization of the spherical building $\mathcal{D}$. Hence, the link of each stratum in the standard realization of a building is the speherical realization of a spherrical building.

A building $\mathcal{C}$ is *right-angled* (a RAB) if its associated Coxeter system is right-angled. For example, the edge set of any tree without terminal vertices is the set of chambers of a RAB with associated Coxeter group the infinite dihedral group on two generators $S=\{s,t\}$. Since any tree is bipartite the vertices can be assigned type $\{s\}$ or type $\{t\}$. Given a vertex of type $\{s\}$ (or $\{t\}$) the corresponding residue is the set of edges which share that vertex. One need not require any regularity of the tree: two residues of type $\{s\}$ can have different cardinality. An apartment in a tree is an embedded copy of the real line with its usual subdivision into edges. Similarly, a product of trees is a RAB. In general, a RAB $\mathcal{C}$ is *regular* if for each $s \in S$ each residue of type $\{s\}$ has the same cardinality.

terminology

Suppose a group $G$ acts on $\mathcal{C}$ through isometries of the $W$-valued distance function. Then $G$ acts on the poset $\mathcal{P}$ of all spherical residues in $\mathcal{C}$. If the $G$-action on $\mathcal{C}$ is transitive, then by choosing an element $c \in \mathcal{C}$ as a fundamental chamber, we get a strict fundamental poset for the $G$-action and this poset can be identified with $\mathcal{S}(L)$. This gives the data for a simple complex of groups over $\mathcal{S}(L)$: for each $\sigma \in \mathcal{S}(L)$, $G_\sigma$ is the isotropy subgroup of the residue of type $\sigma$ which contains $c$. Thus, $\mathcal{P}$ is the disjoint union of all cosets of the form $G/G_\sigma$, where $\sigma$ ranges over $\mathcal{S}(L)$. In other words, *every chamber-transitive action of a group on a building* $\mathcal{C}$ *of type* $(W,S)$ *gives rise to a simple complex of groups over* $K$. When $G$ is chamber transitive, the standard realization of the building, $D(K,\mathcal{C})$, is the same complex as $D(K,G)$, the development of the complex of groups $G\mathcal{S}(L)$. A good example to keep in mind is where $\mathcal{C}$ is a spherical building over a spherical Coxeter system $(W,S)$. If $G$ is chamber transitive on $\mathcal{C}$ we get a simplex of groups $G\mathcal{S}(\partial\sigma)$ with fundamental domain $\sigma^*$. In other words, any spherical building with a chamber transitive automorphism group gives a simplex of groups (cf. Definition 5.2). More generally, we get a simplex of groups whenever the type of the building is a simplicial Coxeter system. If $\mathcal{C}$ has finite thickness, then $G\mathcal{S}(\partial\sigma)$ is a simplex of finite groups and its development $D(\sigma^*,G)$ is a locally finite simplicial complex.

The building $\mathcal{C}$ has *finite thickness* if each rank one residue is finite (i.e., if for each $s\in S$ each $\{s\}$-residue is finite). It can be shown that this condition is equivalent to the condition that the complex $D(K,\mathcal{C})$ is locally finite. Suppose $\mathcal{C}$ is locally finite. Then its automorphism group $\mathrm{Aut}(\mathcal{C})$ is a locally compact topological group. The group $\mathrm{Aut}(\mathcal{C})$ is usually not discrete. (For example, in the the full automorphism group of a regular tree, the stabilizer of a chamber is an inverse limit of finite groups of increasing size.) In addition, suppose that the $\mathrm{Aut}(\mathcal{C})$-action on the building is chamber transitive. A *lattice* in $\mathrm{Aut}(\mathcal{C})$ is a discrete subgroup $G$ of finite covolume. The lattice is *uniform* if it is cocompact (i.e., if $D(K,\mathcal{C})/G$ is compact). Should it be Aut $\mathcal{C}$?

If $\mathcal{C}$ is locally finite, then any discrete subgroup $G\le \mathrm{Aut}(\mathcal{C})$ acts properly on $D(K,\mathcal{C})$. Since $D(K,\mathcal{C})$ is CAT(0), it is a model for $\underline{E}G$. Hence, if $G$ uniform lattice in $\mathrm{Aut}(\mathcal{C})$, we get a cocompact model for $\underline{E}G$. Maybe different spot

*Example 5.7.* (*The graph product complex*). We recall the construction of RABs in 3.3. Let $\Delta$ be the simplex on a set $S$. The RACG with nerve $\Delta$ is $(\mathbf{C}_2)^S$. A building of type $(W_s,\{s\})$ is just a discrete set $E_s$ of cardinality at least two. Its standard realization is $\mathrm{Cone}\,E_s$. The product building $\prod_{s\in S}E_s$ has type $((\mathbf{C}_2)^S,S)$ with standard realization $\prod_{s\in S}\mathrm{Cone}\,E_s$. The fundamental chamber for the product building is $K(\Delta)$ (as in **??**, $K(\Delta)$ can be identified the cube $[0,1]^S$). Next suppose that $L$ is a flag complex with the same vertex set $S$ andd with 1-skeleton $L^1$. Define a right-angled Coxeter matrix $(m_{st})$ by

$$m_{st}=\begin{cases}1, & \text{if } s=t,\\ 2, & \text{if } \{s,t\}\in \mathrm{Edge}\,L^1,\\ \infty, & \text{if } \{s,t\}\notin \mathrm{Edge}\,L^1,\end{cases} \tag{5.9}$$

Let $(W_L,S)$ be the associated RACS with nerve $L$. (In other words, $W_L$ is the graph product of copies of $\mathbf{C}_2$ over $L^1$.) Since $L$ is a subcomplex of $\Delta$, $K(L)$ is a subcomplex of $K(\Delta)$. Let $p:\prod_{s\in S}\mathrm{Cone}\,E_s\to K(\Delta)$ be the natural projection. As in Definition 3.4, put $Z_L=p^{-1}(K(L))$. (The space $Z_L$ can be identified with the polyhedral product $\mathrm{Cone}\,\mathbf{E}^L$.) When $L\neq\Delta$, the space $Z_L$ will not be simply connected. It is shown in [49] that the universal cover $\tilde{Z}_L$ is the standard realization of a RAB $\mathcal{C}$ of type $(W_L,S)$. If $c\in K(L)$ denotes the cone point corresponding to $\emptyset\in\mathcal{S}(L)$, then $\mathcal{C}$ can be identified with the inverse image of $c$ in $\tilde{Z}_L$. Since any residue of type $\{s\}$ in $\mathcal{C}$ is identified with $E_s$, the building $\mathcal{C}$ is regular. Moreover, any regular RAB comes from this construction. The building $\mathcal{C}$ is locally finite if and only if each $E_s$ is a finite set. For example, if $\mathrm{Card}(S)=2$ and $L^1=S^0$, then $W_L$ is the infinite dihedral group and the realization of $\mathcal{C}$ is a biregular tree.

Next suppose that each $E_s$ admits a transitive action of a group $G_s$. So, $E_s$ can be identified with $G_s/B_s$, where $B_s$ is the isotropy subgroup at a basepoint. The $G_s$-action is effective if and only if the subgroup $B_s$ is *malnormal*. (This means that $gB_sg^{-1}\cap B_s=\{1\}$ if $g\notin B_s$.) We assume each $B_s$ is malnormal. Then $\prod G_s\curvearrowright\prod\mathrm{Cone}\,E_s$ and the action is chamber transitive. The subcomplex $Z_L$ is stable under the $\prod G_s$-action. The group $G$ of all lifts of this action to the universal cover $\tilde{Z}_L$ is called the *generalized graph product* of the family $\{(G_s,B_s)\}$ with respect to $L^1$. If each $B_s$ is trivial, then $G$ is the ordinary graph product as defined in subsection 3.1.3.

The simple complex of groups $G\mathcal{S}(L)$ associated to the $G$-action on this RAB is called the *graph product complex*. Thus, the local group $G_\sigma$ associated to the simplex $\sigma\in\mathcal{S}(L)$ is the direct sum, $G_\sigma=\prod_{s\in\sigma}G_s$. If $\tau<\sigma$, then $\phi_{\sigma\tau}:G_\tau\to G_\sigma$ is the natural inclusion.

*Example 5.8.* (*Constructing buildings from spherical buildings*). We recall the construction of RABs in 3.3. Let $\Delta$ be the simplex on a set $S$. The RACG with nerve $\Delta$ is $(\mathbf{C}_2)^S$. For each $s \in S$, let $E_s$ be a discrete set with $\operatorname{Card}(E_s) \geq 2$. Then $E_s$ is a building of type $(\mathbf{C}_2, \{s\})$; its standard realization is $\operatorname{Cone} E_s$. Similarly, the product $\prod_{s\in S} \operatorname{Cone} E_s$ is a building of type $((\mathbf{C}_2)^S, S)$; its realization is $\prod_{s\in S} \operatorname{Cone} E_s$. The fundamental chamber for $\prod_{s\in S} \operatorname{Cone} E_s$ is $K(\Delta)$ (as in **??**, $K(\Delta)$ can be identified the cube $[0,1]^S$). Next suppose that $L$ is a flag complex with the same vertex set $S$. Define a right-angled Coxeter matrix $(m_{st})$ by

$$m_{st} = \begin{cases} 1, & \text{if } s = t, \\ 2, & \text{if } \{s,t\} \in \operatorname{Edge} L, \\ \infty, & \text{if } \{s,t\} \notin \operatorname{Edge} L, \end{cases} \tag{5.10}$$

and let $(W,S)$ be the associated RACS with nerve $L$. Since $L$ is a subcomplex of $\Delta$, $K(L)$ is a subcomplex of $K(\Delta)$. Let $p : \prod_{s\in S} \operatorname{Cone} E_s \to K(\Delta)$ be the natural projection. As in Definition 3.4, put $Z_L = p^{-1}(K(L))$. (The space $Z_L$ can be identified with the polyhedral product $\operatorname{Cone} \mathbf{E}^L$.) When $L \neq \Delta$, the space $Z_L$ will not be simply connected. It is shown in [49] that the universal cover $\tilde{Z}_L$ is the standard realization of a RAB. If $c \in K(L)$ denotes the cone point corresponding to $\emptyset \in \mathcal{S}(L)$, then the set of chambers $\mathcal{C}$ in this RAB can be identified with the inverse image of $c$ in $\tilde{Z}_L$. Any regular right-angled building comes from the above construction. More precisely, if $\mathcal{C}$ is a regular RAB with associated RACS $(W,S)$ each residue of type $\{s\}$ has cardinality $q_s + 1$, then taking $E_s$ to be a set of cardinality $q_s + 1$ and applying the above construction we recover $\mathcal{C}$.

?$K(L)$ is an $m$-gon

In [49] it is shown that the process of starting from a product building and constructing a RAB can be generalized to a process for constructing new buildings starting from an arbitrary spherical building $\mathcal{C}'$ of type $(W',S)$. As before, the nerve of $(W',S)$ is the simplex $\Delta$ on $S$. The standard realization of this building is $D(K(\Delta),\mathcal{C}')$ Let $(m'_{st})$ be the Coxeter matrix of $(W',S)$. Next choose a collection of edges in $\Delta$ and for each edge, $\{s,t\}$, in this collection change $m'_{st}$ to the symbol $\infty$. This produces a new Coxeter matrix $(m_{st})$ with associated Coxeter system $(W,S)$ with nerve $L$. As before, $K(L) \leq K(\Delta)$. By taking the inverse image of $K(L)$ in $D(K(\Delta),\mathcal{C}')$, we get $D(K(L),\mathcal{C}') \leq D(K(\Delta),\mathcal{C}')$. Finally, let $\tilde{D}$ denote the universal cover of $D(K(L),\mathcal{C}')$. This is the standard realization of a building whose set of chambers $\mathcal{C}$ can be identified with the inverse image of the cone point $c$ in the universal cover.

Variations of this constructions are given in [49].

Questions: Tits [135] covering paper: Simply connected local building $\implies$ building?

*Example 5.9.* (*The complex $A\mathcal{S}(L)$ of spherical Artin subgroups*). Given a Coxeter matrix $M$ there is another way to get a complex of groups over $\mathcal{S}(L)$. Let $A$ be the associated Artin group with generating set $X = \{a_s\}_{s\in S}$ and relations given by (5.7). For each $\sigma \in \mathcal{S}(L)$, let $A_\sigma$ be the spherical Artin group corresponding to $M_\sigma$. As for Coxeter groups as described in Example 5.5 we have the following.

(i) For each $\sigma \in \mathcal{S}(L)$, there is a natural homomorphism $\psi_\sigma : A_\sigma \to W$ induced by the inclusion $\sigma \hookrightarrow S$ and similarly, for each $\tau < \sigma$, there is $\phi_{\sigma\tau} : A_\tau \to A_\sigma$. It is immediate from the definition that $A$ is the direct limit of the family $\{A_\sigma\}_{\sigma\in\mathcal{S}(L)}$.
(ii) The homomorphisms $\phi_{\sigma\tau}$ and $\psi_\sigma$ are injective (cf. [32], [96]). Since each of the $\phi_{\sigma\tau}$ is injective, the functor $\sigma \mapsto A_\sigma$ defines a simple complex of groups over $\mathcal{S}(L)$. This simple complex of groups is called the *Artin complex* and denoted by $A\mathcal{S}(L)$. Since $\psi_\sigma : A_\sigma \to A$ is injective, $A\mathcal{S}(L)$ is developable.

Let $K$ be defined by (5.5) and let $D = D(K,A) = (A \times K)/\sim$ be the result of applying the basic construction (5.4).

(iii) The complex $D(K,A)$ is not locally finite since each nontrivial local group is infinite. For example, the subgroup $A_{\{s\}}$ generated by the Artin generator $a_s$ is infinite cyclic.
(iv) If $M$ is right-angled, then $A$ is a RAAG and $D(K,A)$ is a RAB of the type considered in Example **??**. If $A$ is not a RAAG, then $D(K,A)$ is not a building. It is probably never a building.

However, the complex $D(K,A)$ is "building-like" in that it has many properties similar to those of a building with a chamber transitive group action. First, the fundamental chamber for $A$ on $D(K,A)$ isomorphic to that of the associated Coxeter system $(W,S)$. Second, $D(K,A)$ has a large collection of subcomplexes each isomorphic to the Davis complex $D(K,W)$. These subcomplexes are natural candidates for apartments. (To see these subcomplexes use the fact that, as in [64], there is a set theoretic section of the canonical epimorphism $p : A \to W$.) Third, by Theorem 5.2 (vi), $D(K,A)$ is simply connected. It is not known whether or not $D(K,A)$ is contractible. This is the main unanswered question about Artin groups. (As explained below in section **??**, it is equivalent to answering the $K(\pi,1)$-Question for Artin groups.)

*Example 5.10.* (*Spherical Artin groups and complements of hyperplane arrangements*). An Artin group can be understood in terms of the fundamental group of a complement of an arrangement of complex hyperplanes. Suppose $A$ is a spherical Artin group associated to a spherical Coxeter system $(W,S)$. As was pointed out in Example 5.5 (f), there is a canonical representation of $W$ as a group generated by linear reflections on $\mathbb{R}^n$. Complexifying we get a $W$-action on $\mathbb{C}^n$. Let $\mathcal{A}$ denote the collection of linear hyperplanes which are fixed by reflections in $W$. Let

$$M(\mathcal{A}) = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}} H, \tag{5.11}$$

be the complement of the union of the reflecting hyperplanes. It is straightforward to see that $\pi_1(M(\mathcal{A})$ can be identified with the pure Artin group $PA$. Since $W$ acts freely on $M(\mathcal{A})$, $M(\mathcal{A}) \to M(\mathcal{A})/W$ is a covering space and so,

$$A = \pi_1(M(\mathcal{A})/W). \tag{5.12}$$

The prototypical example of a spherical Artin group is Artin's braid group, [6]. The braid group $B_{n+1}$ on $n+1$ strands has several different incarnations. For example, it is the fundamental group of the configuration space of $n+1$ points in the plane (cf. subsection 2.4.4). It can also be described as the fundamental group of a hyperplane arrangement. The associated Coxeter group is $S_{n+1}$ the symmetric group on $n+1$ letters. It acts as a reflection group on $\mathbb{R}^{n+1}$ by permuting the coordinates, as well as on the subspace ($\cong \mathbb{R}^n$) orthogonal to the diagonal. Complexifying we get a $S_{n+1}$-action on $\mathbb{C}^n$ with reflecting hyperplanes defined by the equations $x_i = x_j$ giving the arrangement $\mathcal{A}$. Then $B_{n+1} = \pi_1(M(\mathcal{A})/S_{n+1})$.

The most important result about spherical Artin groups is the following.

**Theorem 5.3.** (Deligne's Theorem, [64]). *Suppose $\mathcal{A}$ is the arrangement of reflecting hyperplanes for a spherical Artin group $A$. Then $M(\mathcal{A})$ is aspherical. Hence, $M(\mathcal{A})/W$ is a model for the classifying space $BA$.*

*Remark 5.5.* General Artin groups also have an interpretation as the fundamental groups as complements of complex hyperplane arrangement, cf. [96], [32]. Tits showed that for any Coxeter system $(W,S)$ there is a linear representation on $\mathbb{R}^n$ so that the elements of $S$ act as reflections across the facets of a simplicial cone (the fundamental chamber), that the union of all $W$-translates of this chamber is a convex cone, and that the $W$-action is proper on the interior $\mathcal{I}$ of this cone. (See [15] or [48].) Moreover, $W$ acts properly on $\mathbb{R}^n + \sqrt{-1}\mathcal{I}$. The reflecting hyperplanes give an arrangement $\mathcal{A}$ of codimension two subspaces in $\mathbb{R}^n + \sqrt{-1}\mathcal{I}$ and the quotient by $W$ of the complement of their union has fundamental group $A$. It is conjectured that Deligne's Theorem holds in this more general situation, i.e., that $M(\mathcal{A})/W$ is a model for for $BA$.

Given a Coxeter system $(W,S)$ there is an associated *Artin group* $A$. This group has one generator $x_s$ for each $s \in S$; its relations are the *braid relations*: and where $m_{st}$ denotes the order of $st$ in $W$. The matrix $(m_{st})$ is called the *Coxeter matrix*. For any $T \leq S$, let $A_T$ denote the Artin group corresponding to the Coxeter system $(W_T, T)$. As with Coxeter groups, $A_T$ can be identified with the special subgroup of $A$ generated by $\{x_t\}_{t \in T}$. (When $T$ is a spherical subset of $S$, this is proved in [64, Thèoréme 4.13 (iii)] and in general in [96, Theorem 4.14].) The Artin group $A_T$ is *spherical* if $T$ is a spherical subset of $S$. As with Coxeter groups, let $\mathcal{S}(L)$ be the poset of spherical subsets of $S$. The spherical special subgroups of $A$ give a simple complex of groups $A\mathcal{S}(L)$ called the *Artin complex*. When $\sigma$ is a simplex of $L$ and $T = \mathrm{Vert}\,\sigma$, we shall often write $A_\sigma$ instead of $A_T$. It is clear that the direct limit $\lim A\mathcal{S}(L)$ is $A$. Since $A_\sigma$ is isomorphic to a subgroup of $A$, $A\mathcal{S}(L)$ is developable. Note that if $L'$ is any subcomplex of the full simplex on $S$ with the same 1-skeleton as $L$, then $\lim A\mathcal{S}(L') = A$.

By Deligne's Theorem in [64], any spherical Artin group $A_\sigma$ has a classifying space $BA_\sigma$ which is a finite CW complex of dimension one greater than $\dim\sigma$. (There is a specific model for $BA_\sigma$ called the *Salvetti complex*, see [33] or Subsection 2.2.2). The "$K(\pi,1)$-Conjecture" for Artin groups is the conjecture that the $K(\pi,1)$-Question has a positive answer for any Artin poset $A\mathcal{S}(L)$. By Proposition 5.1 this is equivalent to the conjecture that $D(|A\mathcal{S}(L)|, A)$ is contractible. It is the most important unsolved problem concerning general Artin groups. A detailed discussion can be found in [32], where the conjecture is proved whenever $L$ is a flag complex (we generalize this in Theorem **??** below).

## 5.4 Polygons of groups

## 5.5 Aspherical realizations and the $K(\pi,1)$-Question

The classifying space of a discrete group $H$ is denoted $BH$ and its universal cover by $EH$. A simple complex of groups $G\mathcal{Q}$ gives the data for a poset of spaces $\{BG_\sigma, \overline{\phi}_{\sigma\tau}\}$, where $\tau < \sigma \in \mathcal{Q}$ and where $\overline{\phi}_{\sigma\tau} : BG_\tau \to BG_\sigma$ is the map induced by the monomorphism $\phi_{\sigma\tau} : G_\tau \to G_\sigma$. Using this data we can glue together the disjoint union of spaces $\coprod |\mathcal{Q}|^\sigma \times BG_\sigma$ by using iterated mapping cylinders. For each $\tau < \sigma$, $|\mathcal{Q}|^\sigma$ is a subcomplex of $|\mathcal{Q}|^\tau$ and one glues the subspace $|\mathcal{Q}|^\sigma \times BG_\tau$ of $|\mathcal{Q}|^\tau \times BG_\tau$ to $|\mathcal{Q}|^\sigma \times BG_\sigma$ via the map:

$$\mathbb{I} \times \overline{\phi}_{\sigma\tau} : |\mathcal{Q}|^\sigma \times BG_\tau \to |\mathcal{Q}|^\sigma \times BG_\sigma,$$

where $\mathbb{I}$ denotes the identity map on $|\mathcal{Q}|^\sigma$. The resulting space $BG\mathcal{Q}$ is called the *aspherical realization* of $G\mathcal{Q}$. It is well-defined up to homotopy equivalence. If $|\mathcal{Q}|$ is connected and simply connected, then it follows from van Kampen's Theorem that $\pi_1(BG\mathcal{Q}) = \lim G\mathcal{Q}$. We note that

$$\dim BG\mathcal{Q} = \sup\{(\dim BG_\sigma + \dim |\mathcal{Q}|^\sigma) \mid \sigma \in \mathcal{Q}\} \tag{5.13}$$

**Proposition 5.1.** *Suppose $G\mathcal{Q}$ is a simple complex of groups over $\mathcal{Q}$ with $|\mathcal{Q}|$ simply connected. Let $G = \lim G\mathcal{Q}$ and let $D = D(|\mathcal{Q}|, \iota)$ be the basic construction. When $G\mathcal{Q}$ is developable, $BG\mathcal{Q}$ is homotopy equivalent to the Borel construction $EG \times_G D$.*

*Proof.* Projection on the second factor induces a projection $p : EG \times_G D \to D/G = |\mathcal{Q}|$ so that the inverse image of the vertex $\sigma$ is homotopy equivalent to $BG_\sigma$. This uses the fact that $G\mathcal{Q} \to G$ is injective on local groups, otherwise, $p^{-1}(\sigma)$ is homotopy equivalent to $B\overline{G}_\sigma$, where $\overline{G}_\sigma$ means the image of $G_\sigma$ in $G$. It follows that $EG \times_G D$ is an aspherical realization of $G\mathcal{Q}$.

**Corollary 5.1.** *Suppose $G\mathcal{Q}$ is a developable simple complex of groups with $|\mathcal{Q}|$ simply connected. Then the $K(\pi,1)$-Question for $G\mathcal{Q}$ has a positive answer if and only if $D$ is contractible.*

*Proof.* By Proposition 5.1, $BG\mathcal{Q} \sim EG \times_G D$. Hence, $BG\mathcal{Q}$ is aspherical if and only if the universal cover of $EG \times_G D$ is contractible. This universal cover is $EG \times D$, which yields the corollary.

*Remark 5.6.* Since $|\mathcal{Q}|$ is a retract of $D$, a necessary condition for $D$ to be contractible is that $|\mathcal{Q}|$ is contractible.

In the construction of $BG\mathcal{Q}$ we can assume that for each $\sigma$, $\dim BG_\sigma = \operatorname{gd} G_\sigma$. Next, we seek a condition which implies that the maximum value of the quantity $\dim BG_\sigma + \dim |\mathcal{Q}|^\sigma$ in (5.13) occurs when $\sigma$ is a maximal element of $\mathcal{Q}$, i.e., when $|\mathcal{Q}|^\sigma$ is a point. The condition is the following:

$$\operatorname{gd} G_\sigma > \operatorname{gd} G_\tau, \quad \text{whenever } \sigma > \tau. \tag{5.14}$$

Since a $k$-simplex in $|\mathcal{Q}|^\tau$ corresponds to a chain $\tau = \tau_0 < \cdots < \tau_\kappa$, we see that (5.14) implies $\operatorname{gd} G_{\tau_k} \geq \operatorname{gd} G_\tau + k \geq \operatorname{gd} G_\tau + \dim |\mathcal{Q}|^\tau$; so the maximum value occurs when $\sigma$ is maximal.

**Proposition 5.2.** *Suppose a simple complex of groups $G\mathcal{Q}$ is developable and that the $K(\pi,1)$-Question for $G\mathcal{Q}$ has a positive answer. Then*

$$\operatorname{gd} G \leq \dim BG\mathcal{Q} = \sup_{\sigma \in \mathcal{Q}} \{\operatorname{gd} G_\sigma + \dim |\mathcal{Q}|^\sigma\}.$$

*If* (5.14) *holds, then* $\operatorname{gd} G = \sup_{\sigma \in \mathcal{Q}} \{\operatorname{gd} G_\sigma\}$.

*Proof.* Since $\operatorname{gd} G \leq \dim BG\mathcal{Q}$, the first formula follows from (5.13). So, if condition (5.14) holds, $\operatorname{gd} G \leq \sup\{\operatorname{gd} G_\sigma \mid \sigma \in \mathcal{Q}\}$. Since $EG/G_\sigma$ is a model for $BG_\sigma$, $\operatorname{gd} G \geq \operatorname{gd} G_\sigma$, so the previous inequality must be an equality.

## 5.6 The semidirect product construction

A simple complex of groups $G\mathcal{Q}$ is a functor from a poset $\mathcal{Q}$ to the category of groups. To say that $\Gamma$ *acts* on $G\mathcal{Q}$ means that $\Gamma$ is a group of natural automorphisms of this functor. In other words,

(a) $\Gamma \curvearrowright \mathcal{Q}$ through order-preserving bijections (for $(\gamma,\sigma) \in \Gamma \times \mathcal{Q}$, the action is denoted $(\gamma,\sigma) \mapsto \gamma\sigma$),
(b) for each $\sigma \in \mathcal{Q}$, there is an isomorphism $f_\sigma : G_\sigma \to G_{\gamma\sigma}$, so that
(c) whenever $\tau, \sigma \in \mathcal{Q}$ with $\tau < \sigma$, the following diagram commutes

$$\begin{array}{ccc} G_\tau & \xrightarrow{f_\tau} & G_{\gamma\tau} \\ {\scriptstyle \phi_{\tau,\sigma}}\downarrow & & \downarrow{\scriptstyle \phi_{\gamma\tau,\gamma\sigma}} \\ G_\sigma & \xrightarrow{f_\sigma} & G_{\gamma\sigma} \end{array}.$$

Let $G$ be the direct limit, $G = \lim G\mathcal{Q}$. Recall that the development of $G\mathcal{Q}$ is defined to be the set of pairs $(hG_\sigma, \sigma)$ in $G/G_\sigma \times \mathcal{Q}$ (see the proof of (i) in Theorem 5.2). Suppose $\Gamma \curvearrowright G\mathcal{Q}$. Then the semidirect product $G \rtimes \Gamma$ acts on $D(\mathcal{Q}, G)$ by

$$(g,\gamma) \cdot (hG_{\gamma\sigma}, \sigma) = (g\gamma(h)G_\sigma, \gamma\sigma). \tag{5.15}$$

This a corresponding action of $G \rtimes \Gamma$ on the space $D(|\mathcal{Q}|, G)$ with quotient space $|\mathcal{Q}|$. When the $\Gamma$-action on $|\mathcal{Q}|$ does not have a strict fundamental domain, the $(G \rtimes \Gamma)$-action on $D(|\mathcal{Q}|, G)$ will not have a strict fundamental domain; so, the action will not give rise to a simple complex of groups. However, it still gives rise to a nonsimple complex of groups, which we denote by $G\mathcal{Q} \rtimes \Gamma$. If $|\mathcal{Q}|$ is simply connected, then the *fundamental group* of $G\mathcal{Q} \rtimes \Gamma$ is $G \rtimes \Gamma$.

*Example 5.11.* (*HNN extensions*). Suppose $A$ is a subgroup of a group $G$ and $\theta : A \to H$ is a monomorphism. (This is the data for a loop of groups: $H$ is the vertex group and $A$ is the edge group). In [128, Prop. 5, p. 8] Serre defines the HNN construction $*_\theta H$ as a semidirect product of $\mathbb{Z}$ with a tree of groups (cf. Definition 5.4). The tree is the real line subdivided into edges of the form $[n, n+1]$ so that the vertex set of the tree is $\mathbb{Z}$. The poset $\mathcal{Q}$ is the dual of the face poset of the tree. The group $H_n$ associated to a vertex $n$ is a copy of $H$. Each edge group is a copy of $A$. For the edge $[n, n+1]$ there are monomorphisms $A \to H_n$ and $A \to H_{n+1}$ corresponding to the left and right endpoints. The first is natural inclusion $i : A \to H$ and the second is $\theta$. This defines a simple complex of groups $G\mathcal{Q}$. Let $G = \lim G\mathcal{Q}$, in other words, $G$ is an amalgam of infinitely many copies of $H$. The $\mathbb{Z}$-action on $\mathcal{Q}$ induced by translation on $\mathbb{R}$ gives a $\mathbb{Z}$-action on $G\mathcal{Q}$ and represents $\mathbb{Z}$ as a group of automorphisms of $G$. The *HNN extension* is then defined by $*_\theta H := G \rtimes \mathbb{Z}$.

*Example 5.12.* (*Semidirect products with Coxeter groups and their relatives*). A *diagram automorphism* of a Coxeter system $(W, S)$ is a permutation $\gamma$ of $S$ which preserves the associated Coxeter matrix (i.e., $m_{\gamma s, \gamma t} = m_{s,t}$). Such a permutation defines an automorphism of the Coxeter group $W$ as well as an automorphim of the associated Artin group $A$. Similarly, $\gamma$ induces an automorphism on any building of type $(W, S)$. (The diagram automorphism can be regarded as a simplicial automorphism of the nerve $L(W, S)$ which preserves the labels $m_{st}$ on the edges.) Suppose $\Gamma$ is a group of diagram automorphisms $(W, S)$. We can then form the semidirect products $W \rtimes \Gamma$ and $A \rtimes \Gamma$ with corresponding complexes of groups $W\mathcal{S}(L) \rtimes \Gamma$ and $A\mathcal{S}(L) \rtimes \Gamma$ (cf. Examples 5.5 and 5.9). Similarly, if $G$ is a chamber transitive automorphism group of a building, we can form a corresponding semidirect product $G \rtimes \Gamma$.

If $S$ is a finite set, then $L$ is a finite simplicial complex and the group of diagram automorphisms $\Gamma$ is necessarily a finite groups. When $L$ is an $(n-1)$-sphere, the finite group $\Gamma$ acts on the contractible $n$-manifold $D(K, W)$ $(= \underline{E}(W \rtimes \Gamma))$. By choosing $\Gamma$ so that the fixed set of $\Gamma$ on $L$ is not equivalent to a linear subsphere one can produce examples where the fixed set of $\Gamma$ on $D(K, W)$ is either (a) knotted at infinity as in [70] or not a manifold as in [61], see [48, Thm. 11.7.3]

**The universal cover of a development**. Our principal use for the semidirect product construction is to understand the universal cover of a development of a simple complex of groups over a poset $\mathcal{Q}'$ where $|\mathcal{Q}'|$ need not be simply connected. Let $P$ be the universal cover of $|\mathcal{Q}'|$ and let $p : P \to |\mathcal{Q}'|$. The elements of $\mathcal{Q}'$ are the vertices of the geometric realization $|\mathcal{Q}'|$ of its order complex. The simplicial structure on $|\mathcal{Q}'|$ lifts to a simplicial structure on $P$. This defines a poset structure $\mathcal{Q}$ on the vertex set of $P$ so that $p : \mathcal{Q} \to \mathcal{Q}'$ is order-preserving and $|\mathcal{Q}'| = P$. Explicitly, if $\tau, \sigma \in \mathcal{Q}$, then $\tau < \sigma$ if and only if $p(\tau) < p(\sigma)$ and the edge from $p(\tau)$ to $p(\sigma)$ lifts to an edge from $\tau$ to $\sigma$. Next suppose $G'\mathcal{Q}'$ be a developable simple complex of groups over $\mathcal{Q}'$. Define a simple complex of groups $G\mathcal{Q}$ over $\mathcal{Q}$ by: $G_\sigma := G_{p(\sigma)}$. Put $\pi = \pi_1(|\mathcal{Q}|)$. Then $\pi$ acts on $|\mathcal{Q}|$ as deck transformations. This extends to $\pi \curvearrowright G\mathcal{Q}$, where for any $\gamma \in \pi$ each of the corresponding isomorphisms $f_\gamma : G_\sigma \to G_{\gamma\sigma}$ is the identity. By the universal property of direct limits there is obvious epimorphism $\psi : G \to G'$. Put $N = \ker \psi$. The natural map $D(|\mathcal{Q}|, G) \to D(|\mathcal{Q}'|, G')$ is a covering projection. Since $|\mathcal{Q}|$ is simply connected, it follows from Theorem 5.2 (vi), that $D(|\mathcal{Q}|, G)$ is simply connected. Hence, $D(|\mathcal{Q}|, G)$ is the universal cover. As in the first paragraph of this section, $G \rtimes \pi$ acts on $D(|\mathcal{Q}|, G)$ and $N \rtimes \pi$ is the group of deck transformations of $D(|\mathcal{Q}|, G) \to D(|\mathcal{Q}'|, G')$. So, we have proved the following generalization of Theorem 5.2 (vii).

**Proposition 5.3.** *With notation as above, the fundamental group of $D(|\mathcal{Q}'|, G')$ is $N \rtimes \pi$ and its universal cover is $D(|\mathcal{Q}|, G)$. So, if $D(|\mathcal{Q}'|, G')$ is aspherical, then so is $|\mathcal{Q}'|$.*

*Proof.* The last sentence follows from Theorem 5.2 (iv) and the fact that a retraction of an aspherical space is aspherical.

Suppose $(W,S)$ is a Coxeter system (cf. [15] or [48]). This means that $W$ is a group, that $S$ is a distinguished set of generators and that $W$ has a presentation of the form

$$W := \langle s_i \in S \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$$

For any subset $T$ of $S$, the subgroup generated by $T$ is denoted $W_T$ and called the *special subgroup* corresponding to $T$. It is a standard fact that $(W_T, T)$ also is a Coxeter system (cf. [15, pp.12-13]). The subset $T$ is *spherical* if $W_T$ is finite, in this case $W_T$ is a *spherical special subgroup*. Let $\mathcal{S}(W,S)$ denote the poset of spherical subsets of $S$. There is a simplicial complex $L$ $(= L(W,S))$, called the *nerve* of $(W,S)$. Its vertex set is $S$ and a subset $T \leq S$ spans a simplex of $L$ if and only if $T$ is spherical. Thus, $\mathcal{S}(L) = \mathcal{S}(W,S)$, the poset of spherical subsets. This gives a simple complex of groups over $\mathcal{S}(L)$, denoted $W\mathcal{S}(L)$, and called the *complex of spherical Coxeter groups* (or simply the *Coxeter complex*). The local group $W_\sigma$ at a simplex $\sigma \in \mathcal{S}(L)$ is the spherical subgroup generated by the vertices of $\sigma$. The direct limit of $W\mathcal{S}(L)$ is the Coxeter group $W$.

The discussion in Example 5.9 yields the following proposition.

**Proposition 5.4.** *Let $A$ be an Artin group such that the nerve $L$ of its associated Coxeter system is $d$-dimensional. If the $K(\pi,1)$-Question for the associated Artin poset has a positive answer, then $gdimA = d+1$.*

*Proof.* Since the aspherical realization $BA\mathcal{S}(L)$ is formed by gluing together the $BA_\sigma \times |\mathcal{S}(L)|^\sigma$ with $\sigma \in \mathcal{S}(L)$ and since $\dim BA_\sigma = \dim\sigma + 1$, we have $\dim BA\mathcal{S}(L) = d+1$. Since $BA\mathcal{S}(L)$ is a model for $BA$, $gdimA \leq d+1$. On the other hand, for any $d$-simplex $\sigma \in \mathcal{S}(L)$, the spherical Artin group $A_\sigma$ contains a free abelian subgroup of rank $d+1$ (e.g., see [?dh16]). So, $\mathrm{cd}A_L \geq d+1$. Hence, $d+1 \geq gdimA_L \geq \mathrm{cd}A_L \geq d+1$; so all inequalities are equalities.

**Definition 5.5.** A simplicial complex $L$ is a *flag complex* if it satisfies the following: if $T$ is any finite set of vertices of $L$ which are pairwise connected by edges, then $T$ spans a simplex of $L$. A simplicial graph $L^1$ determines a flag complex $L$: the simplices of $L$ are the cliques in $L^1$. (This is also called the "clique complex" of $L^1$.)

**Definition 5.6.** Suppose $L^1$ is a simplicial graph with vertex set $V$ and edge set $E$. Let $\{G_v\}_{v\in V}$ be a collection of groups indexed by $V$. The *graph product* of the $G_v$, denoted $\prod_{L^1} G_v$, is the quotient of the free product of the $G_v$, $v \in V$, by the normal subgroup generated by all commutators of the form, $[g_v, g_w]$, where $\{v,w\} \in E$, $g_v \in G_v$ and $g_w \in G_w$.

*Example 5.13.* (*The graph product complex*). Suppose $\prod_{L^1} G_v$ is a graph product and that $L$ is the flag complex determined by $L^1$. There is a simple complex of groups $G\mathcal{S}(L)$ over $\mathcal{S}(L)$ called the *graph product complex*. It is defined by putting $G_\sigma$ equal to the direct product,

$$\prod_{v\in \mathrm{Vert}\,\sigma} G_v,$$

for each simplex $\sigma \in L$ and letting $\phi_{\sigma\tau} : G_\tau \to G_\sigma$ be the natural inclusion whenever $\tau < \sigma$. It is immediate that $\prod_{L^1} G_v = \lim G\mathcal{S}(L)$.

*Remark 5.7.* Another approach to the graph product complex is given in [43] or [?davis12] using the notion of a "polyhedral product". Suppose $L$ is a simplicial complex with vertex set $V$ and that we are given a collection of pairs of spaces $(uy,uy) = \{(X_v,Y_v)\}_{v\in V}$ together with a choice of basepoint $*_v \in Y_v$. Let $\prod_{v\in V} X_v$ denote the subspace of the Cartesian product consisting of all $V$-tuples $(x_v)_{v\in V}$ such that $x_v = *_v$ for all but finitely many $v$. Given a $V$-tuple, $x = (x_v)$, put $\sigma(x) = \{v \in V \mid x_v \in X_v - Y_v\}$. The *polyhedral product* $(ux,uy)^L$ is the subspace of $\prod_{v\in V} X_v$ consisting of all $x$ such that $\sigma(x)$ is a simplex in $L$. (Usually, we only shall be concerned with the case where each $Y_v$ equals the basepoint $*_v$, in which case the notation will be simplified to $ux^L$.) For any $\tau \in \mathcal{S}(L)$, the set of all $x$ with $\sigma(x) \geq \tau$, can be identified with the product, $X^\tau := \prod_{v\in \mathrm{Vert}\,\tau} X_v$. Thus, $ux^L$ is the union of the $X^\tau$, with $\tau \in \mathcal{S}(L)$.

Next let $L^1$ be any simplicial graph and $L'$ any simplicial complex with 1-skeleton $= L^1$. Let $G$ be the graph product $\prod_{L^1} G_v$ and $H = \prod_{v\in V} G_v$ be the direct product. Let $\iota : G\mathcal{S}(L^1) \to G$ and $\psi : G\mathcal{S}(L^1) \to G$ be the natural simple morphisms. Consider the polyhedral product:

$$\mathcal{Z}(L') = (\underline{\mathrm{Cone}\, G_v}, \underline{G_v})^{L'}.$$

Then $\mathcal{Z}(L')$ is locally isomorphic to a product of discrete sets. Moreover, $\mathcal{Z}(L')$ can be identified with the development $D(|\mathcal{S}(L')|, \psi)$. The fundamental group of $\mathcal{Z}(L')$ can be identified with the kernel of the natural epimorphism $G \to H$. So, the $H$-action on $\mathcal{Z}(L')$ lifts to a $G$-action on the universal cover $\tilde{\mathcal{Z}}(L')$ with the same strict fundamental domain. It follows from Remark **??** that $\tilde{\mathcal{Z}}(L') = D(|\mathcal{S}(L')|, \iota)$.

**Proposition 5.5.** (cf. [63, Theorem 2.22] and [?dk]). *Suppose $G = \prod_{L^1} G_v$ is the graph product of nontrivial groups over a simplicial graph $L^1$, and let $L$ be the determined flag complex. Then the $K(\pi,1)$-Question for $G\mathcal{S}(L)$ has a positive answer.*

Since $BG_1 \times BG_2$ is a model for $B(G_1 \times G_2)$, it is obvious that

$$gdim(G_1 \times G_2) \leq gdimG_1 + gdimG_2.$$

On the other hand, by using certain torsion-free subgroups of RACGs, Dranishnikov [?dran] showed that there are groups $G_1$ and $G_2$ for which the inequality is strict (cf. [48, Example 8.5.9]). Hence, for $G_\sigma = \prod_{v\in\sigma} G_v$, we have that $gdimG_\sigma \leq \sum_{v\in\sigma} gdimG_v$ and the inequality can be strict.

A corollary to Proposition 5.5 is the following calculation of the geometric dimension of any graph product of groups.

**Corollary 5.2.** *Suppose $G = \prod_{L^1} G_v$ is the graph product of nontrivial groups over a simplicial graph $L^1$. Let $L$ be the flag complex determined by $L^1$. Then $gdimG = \sup\{gdimG_\sigma \mid \sigma \in \mathcal{S}(L)\}$.*

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