5. Topological methods in group theory

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Introduction

This article is a revised version of notes on an advanced course given in Liverpool from January to March 1977 in preparation for the symposium. The lectures given by Terry Wall at the symposium were mainly taken from Sections 3 and 4, and much of the material in John Stallings' lectures is in Sections 5 and 6. It seemed worth publishing the whole, as a rather full introduction to the area for those with a background in topology. Originality is not claimed for the results in the earlier sections (though full references have not always been given), but the uniqueness results in Section 7 and most of Section 8 are due to Peter Scott.

1. BASIC NOTIONS

The link between topology and group theory comes from the fundamental group. I shall make no attempt to present this: almost every introductory topology text does so. Particularly suitable for this course is Massey's book [18]. An equivalent account, from a different viewpoint, is given by Brown [2]. Let us recall the basic properties of the fundamental group.

(1) For every topological space $X$ and point $x \in X$ we have a group $\pi_1(X; x)$. This depends only on the path component of $X$ containing $x$. A path from $x$ to $y$ induces an isomorphism $\pi_1(X; x) \to \pi_1(X; y)$; a closed path induces an inner automorphism. A map $f : X \to Y$ with $f(x) = y$ induces a homomorphism $f_* : \pi_1(X; x) \to \pi_1(Y; y)$, and this assignment is functorial: in fact we have a homotopy functor.

(2) A map $p : E \to B$ is a covering if it is locally trivial, with discrete fibres - i.e. every $x \in X$ has a neighbourhood $U$, and a homeo-
morphism \( \tau^{-1}(U) \subseteq U \times D \) with \( D \) discrete, such that \( \text{pr}_1 \circ h = z \). For a covering \( \tau \), and \( z \in Z \), the map \( \tau_*: \tau_* (Z; z) \to \tau_* (X; f(z)) \) is injective.

If \( X \) is reasonably nice (the minimal technical conditions are path-connected, locally path-connected, and weakly locally \( 1 \)-connected), the correspondence between triples \((\sigma: Z \to X, \text{a connected covering, } z \in \sigma^{-1}(x))\) and subgroups \( \tau_* \tau_* (Z; z) \) of \( \tau_* (X; x) \) induces an isomorphism of categories. Hence, in particular, the coverings are isomorphic if and only if the corresponding subgroups are conjugate.

(3) The third basic fact we need to recall is the technique for calculating fundamental groups, due to van Kampen. First suppose \( X \), \( X \) are path-connected open subsets of \( X \), with path-connected intersection \( X \). Then for any base point \( x \in X \), we have the following commutative diagram in which all the maps are induced by inclusions of spaces.

\[
\begin{array}{ccc}
\tau_* (X_1; x) & \xrightarrow{f_1} & \tau_* (X_1; x) \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
\tau_* (X_1; x) & \xrightarrow{f_2} & \tau_* (X_1; x)
\end{array}
\]

Proposition 1.1. This is a pushout diagram in the category of groups. In other words, for any group \( G \) we have a bijection, induced by \( (f_1, f_2) \), \( \text{Hom}(\tau_* (X_1; x), G) \cong \{ (f_1, f_2) \in \text{Hom}(\tau_* (X_1; x), G) \times \text{Hom}(\tau_* (X_2; x), G) : f_1 f_2 = f_2 f_1 \} \).

The standard argument with universals shows that the pushout is uniquely determined; existence is provided by the proposition. The proof, involving breaking up a path in \( X \) into subpaths each lying in \( X_1 \) or \( X_2 \), is somewhat messy. The restriction that the \( X_i \) are open can be relaxed if each is a deformation retract of some neighbourhood, as is usually the case in practice.

The restriction that \( X \) be connected is less desirable. One may reformulate the proposition to cover this case by using groupoids [5]. More naively, suppose \( X \) has just two path-components \( Y \) and \( Z \); define \( \tilde{Z} \) by identifying \( X_1 \) and \( X_2 \) along \( Z \), and \( \tilde{X} \) by attaching \( Y \times I \) to \( \tilde{Z} \) by identifying \( Y \times 1 \) to the copy \( Y_{i+1} \) of \( Y \) in \( X_{i+1} \) \((i = 0, 1)\).

There is an obvious map \( \phi: \tilde{X} \to X \) (identify \( Y \times 1 \) to \( Y \)) which is usually a homotopy equivalence, and induces isomorphisms of \( \tau_* \). We can calculate \( \tau_* (\tilde{Z}) \) by Proposition 1.1.

Now choose a base point \( y \in Y \); let \( y \) be the corresponding point in \( Y \), and \( t \) a path in \( \tilde{Z} \) joining \( y \) to \( y \). We have homomorphisms

\[
\begin{align*}
\alpha_1: \tau_* (Y; y) & \cong \tau_* (Y; y) \\
\alpha_2: \tau_* (Y; y) & \cong \tau_* (Y; y)
\end{align*}
\]

Proposition 1.2. For any group \( G \), we have a bijection

\[
\text{Hom}(\tau_* (\tilde{X}; y), G) \cong \{ (f_1, t) \in \text{Hom}(\tau_* (\tilde{X}; y), G) \times \text{Hom}(\tau_* (Y; y), G) : f_1 t = f_1 t \text{ for all } p \in \tau_* (Y; y) \}.
\]

Here, \( f_1 \) is the composite \( \tau_* (\tilde{Z}; y) \to \tau_* (\tilde{X}; y) \to \tau_* (Y; y) \), and \( t \) is the image of the class in \( \tau_* (\tilde{X}; y) \) of the loop \( t u y \times 1 \).

We will show that Proposition 1.2 follows from Proposition 1.1. In order to start our study of fundamental groups, we need to know that the circle \( S^1 \) has infinite cyclic fundamental group. This is easy to prove using the covering of \( S^1 \) by the real line \( \mathbb{R} \). One might think of deducing this result from Proposition 1.2 by taking \( X \) to be \( S^1 \) and \( X_1 \) and \( X_2 \) to be open intervals. For then \( \text{Hom}(\tau_* (X), G) \cong G \) for any group \( G \). However our proof of Proposition 1.2 uses the fact that \( S^1 \) has infinite cyclic fundamental group. Here is a quick sketch of that proof.

First, the intermediate space \( W = Z \cup (Y \times 1) \) can be considered as the union of \( Z \) and the circle \( t u (Y \times 1) \) intersecting in the arc \( t \).
Hence \( \tau_1(\tilde{W}, \tau_1) \) is the pushout of \( Z \to \tau_1(\tilde{Z}, \tau_1) \), i.e. (see later) the fiber product \( Z = \tau_1(\tilde{Z}, \tau_1) \). Now \( \tilde{X} = \tilde{W} \cup (Y \times 1) \), and \( W \cap (Y \times 1) = (Y \times 0) \cup (Y \times 1) \). We deduce, after a little manipulation, a pushout diagram

\[
\begin{array}{ccc}
\tau_1(Y, y) -\tau_1(Y, y) & \to & Z -\tau_1(Z, y) \\
\downarrow & & \downarrow \\
\tau_1(X, y) & \to & \tau_1(x, y)
\end{array}
\]

where \( \phi \) is given by \( \alpha_1 \) on the first factor and by \( c \mapsto t \cdot \alpha_2(c) \cdot t^{-1} \) on the second. This is equivalent to Proposition 1.2. We have given the proposition independently, however, since it introduces a construction which will be important below.

Example 1.3. \( X = S^1 \vee S^1 \), the one-point union. Now apply Proposition 1.1, taking \( X, X \) as the two committees. Thus \( \text{Hom}(x, y), G) \cong G \times G \). The group \( \tau_1(X, y) \) called the free group on generators \( t, t \), \( t \), the classes of the circles.

To put some bones into this abstraction, we next give a concrete description of this free group on \( t, u \). A letter is any one of \( t, t, u, u \). A word is a finite (perhaps empty) sequence of letters. The word is reduced if none of \( t^t, t^t, u^u, u^u \) occurs as a pair of consecutive letters.

Theorem 1.4. There is a bijection between elements of the free group \( F \) on \( t, u \) and the set \( W \) of reduced words. Each word defines an element of \( G \) by forming the product of various of \( t, t, u, u \) in the indicated order.

Proof. (i) Observe that \( F \) contains the elements \( t, u \) and hence the set \( H \) of products of finite ordered sequences of elements \( t, t, u, u \). Clearly \( H \) is closed under products and inverses, hence is a subgroup. There is a homomorphism \( \phi : H \to F \) such that \( \phi(t) = t, \phi(u) = u \). The composite \( F \to H \subset F \), coincides with the identity on \( t, u \), hence is the identity. Thus \( F = H \).

By definition, each element of \( H \) is represented by a word. If the word is not reduced, we can cancel products \( t^{-1}, \) etc. Thus each element is represented by a reduced word and we have a surjection \( \alpha : W \to F \).

Thus it remains to prove \( \alpha \) is bijective.

(ii) Write \( S \) for the symmetric group on the set \( W \) of reduced words. Define permutations \( \tau, \sigma \) as follows: If the word \( w \) ends in \( t \) (resp. \( u \)), then \( w' \) (resp. \( w'' \)) is obtained from \( w \) by deleting the last letter. Otherwise, \( w' \) (resp. \( w'' \)) consists of \( w \) followed by \( t \) (resp. \( u \)). We see at once that these are permutations with inverses \( \tau^{-1}, \sigma^{-1} \) defined similarly but interchanging the roles of \( t \) and \( u \).

By definition of \( F \), there is a unique homomorphism \( \phi : F \to S \) such that \( \phi(t) = \sigma, \phi(u) = \tau \). We define a map \( \beta : F \to W \) by \( \beta(g) = \phi(g) (1) \).

For any reduced word \( w \), we see by induction on the length of \( w \) that \( \beta(\alpha(w)) = w \). Thus \( \alpha \) is injective, hence bijective.

We used the example of \( S^1 \vee S^1 \) to demonstrate the existence of a free group \( F \) of rank two. Note that the proof above does this quite independently for it shows that the set \( W \) has a natural group structure which makes it a free group of rank two.

There is an obvious analogue to the above for the free group \( F(X) \) on any set \( X \) of generators. If \( X \) is finite, existence is seen by induction. We observe that if \( X \subset X' \), the natural map \( F(X) \to F(X') \) is injective. Now for \( X \) infinite, define

\[ F(X) = \bigcup \{ F(Y) : Y \subset X, \ Y \ finite \} \]

If \( y_1 \in F(Y_1) \subset F(X) \) for \( i = 1, 2, 3 \) we define \( y_1y_2 \) to be the product in \( F(Y_1 \cup Y_2) \). Associativity follows by considering \( F(Y_1 \cup Y_2 \cup Y_3) \). It is immediately verified that for any \( G \), restriction to \( X \) yields a bijection \( \text{Hom}(F(X), G) = \text{Map}(X, G) \), so \( F(X) \) is the free group on \( X \).

Now consider any group \( G \), set \( X \), and map \( \phi : X \to G \). By the above, \( \phi \) has a unique extension \( \psi : F(X) \to G \) to a homomorphism whose image is then a subgroup \( X \). Any element of \( X \) can be written as a word in the elements \( \phi(x) \), and thus lies in any subgroup of \( G \) containing \( \phi(X) \). Thus \( X \) is the intersection of the subgroups containing \( \phi(X) \). It is called
the subgroup generated by \(\phi(X)\) (or by \(\phi\)). If \(X = G\), we say \(G\) is generated by \(\phi(X)\).

Now consider a finite CW-complex \(K\) with one vertex \(x\). By induction we see that the 1-skeleton \(K^1\) has free fundamental group generated by the classes \(g_i\) of the 1-cells. As \(K^1\) is a deformation retract of a neighbourhood, we can apply van Kampen’s theorem to calculate the effect on the fundamental group of attaching a 2-cell \(e^2\). Now \(e^2\) is contractible, and a suitable neighbourhood of \(K^1\) meets it in a copy of \(S^1 \times \mathbb{R}\). The map \(\alpha : S^1 \to N(K^2) \to K\) is homotopic to the attaching map of the cell, and determines \(\alpha_* : \pi_1(S^1) \to \pi_1(K^2)\) with \(\alpha_*(1) = r\), say.

Then \(\pi_1(K^2 \cup e^2)\) is the pushout of

\[
\begin{array}{c}
z \\
\downarrow \\
1
\end{array} \quad \begin{array}{c}
\alpha_* \quad \alpha_1 \\
\downarrow \\
\pi_1(K^1)
\end{array}
\]

Arguing similarly with the other 2-cells \(e_j\), we find that if their attaching maps yield classes \(r_j \in \pi_1(K^1)\), then the 2-skeleton \(K^2\) has fundamental group \(\pi_1(K^2)\) characterized by the property that for any group \(G\),

\[
\text{Hom}(\pi_1(K^2), G) = \{ f \in \text{Hom}(\pi_1(K^1), G) : f(r_j) = 1 \text{ for each } j \}.
\]

Since \(\pi_1(K^1)\) is free on \(\{g_i : i \in I\}\), \(f\) is determined by the \(f(g_i)\).

The sequence \(\{g_i | r_j\}\) where the \(g_i\) are abstract symbols and \(r_j \in \pi_1(K^1)\) is called a presentation of \(\pi\) if, for any \(G\), \(\text{Hom}(\pi, G)\) is given as above. The same argument as in (i) of the proof above shows that the images of the \(g_i\) are generators of \(\pi\). The \(r_j\) are called

Let \(N\) be the subgroup of \(F\{g_i\}\) generated by the \(r_j\) and \(r_j^N\) their conjugates: \(N\) is called the normal closure of the \(r_j\). Clearly \(N\) is the least normal subgroup of \(F\{g_i\}\) containing them all. Hence \(\pi_1(K)\) has the universal property defining \(\{g_i | r_j\}\); this yields a construction of this group. Again, the restriction to finite sets of generators and relations is easily seen to be irrelevant.

Of course you have all seen generators and relations before: here it is the relation with two-dimensional CW complexes that I wish to stress. (Incidentally, adding cells of dimension \(> 2\) does not affect \(x_1\), as we see in applying van Kampen’s theorem again.) For example, let \(X^2\) be any such 2-manifold having one vertex \(x\) and \(Y\) any space.

**Lemma 1.5.** For any homomorphism \(\phi : \pi_1(X^2, x) \to \pi_1(Y, y)\), there is a map \(\alpha : X^2 \to Y\) with \(\alpha_* = \phi\).

**Proof.** The 1-cells and 2-cells of \(X^2\) give a presentation \(\{x_i, r_j\}\). The image of \(g_i\) by \(\phi\) is an element of \(\pi_1(Y, y)\), represented by a map \((S^1, x) \to (Y, y)\). Use these maps to define \(\alpha : X^2 \to Y\), then we have a diagram

\[
\begin{array}{ccc}
x_1(x) & \to & \pi_1(X^2, x) \\
\alpha_* & \to & \phi \\
\downarrow & & \downarrow \\
\pi_1(Y, y) & & \pi_1(Y, y)
\end{array}
\]

This commutes, by construction of \(\alpha\). Hence \(\alpha_*(r_j) = 1\). For each \(x_i\) in \(X^2\) with characteristic map \(x_i : (S^1, x) \to (X^2, x_i, x)\),

\[
x_i = r_j, \quad (S^1, x) \to (X^2, x_i, x)
\]

Thus \(x_i | S^1\) is \(r_j\) (by definition) so the class of \(x_i | S^1\) is \(r_j\). Thus \(\alpha_*(x_i)\) is nullhomotopic, so there is a continuous section

\[
\beta : S^1 \to Y
\]

Then \(\beta * x_i : S^1 \to Y\). Now by definition of the topology of \(X\) as an identification space, the diagram

\[
\begin{array}{ccc}
x_i(x) & \to & \pi_1(X^2, x) \\
\alpha_* & \to & \phi \\
\downarrow & & \downarrow \\
\pi_1(Y, y) & & \pi_1(Y, y)
\end{array}
\]

implies
defines a map \( \alpha : X^2 \to Y \) such that \( \alpha \mid X^1 = \alpha \mid X^1 \) and \( \alpha \mid x_j = \psi_j \).

Since \( \pi_1(X^2, x) \) is a quotient of \( \pi_1(X^1, x) \) it follows that \( \alpha_\ast = \phi. \)

Remarks. The condition that \( X \) has dimension 2 is essential here. However, a particularly interesting case is when \( Y \) is such that \( \pi_1(Y, y) = \pi \) and for any \( (X, x) \) the map \( \alpha \to \alpha_\ast \) gives a bijection between homotopy classes of maps \( (X, x) \to (Y, y) \) and \( \text{Hom}(\pi_1(X, x), \pi) \).

Such \( (Y, y) \) is called a classifying space for \( \pi \) (or Eilenberg-MacLane space \( K(\pi, 1) \)). The usual argument with universal shows its uniqueness (up to homotopy); existence is also not hard to prove, for any \( \pi \). However, the existence of \( (Y, y) \) which is e.g. a manifold, or finite complex imposes an interesting and subtle condition on \( \pi \).

I now return to Propositions 1.1 and 1.2. We have obtained in some cases fairly explicit descriptions of the groups so defined. I will now give some useful generalizations of these.

The real beginning of our subject was the discovery that in the case when the maps are injective one can obtain structure theorems similar to 1.4. In fact a description of \( A \ast_C B \) in terms of reduced words was already given by Schreier in 1927 [36], but a more thorough account and the start of the recent work of the subject is contained in Haina Neumann's thesis [35]. The groups \( A \ast_C B \) go back to [13], though they are not actually defined in that paper. There have of course been many papers on the subject since; our account derives from those of Serre [22] and Cohen [5] and seems simpler and more natural than the original papers.

Propositions 1.1, resp. 1.2 concerned diagrams:

\[
\begin{array}{ccc}
\alpha_1 & \rightarrow & A \\
\downarrow & & \downarrow \\
\beta_2 & \rightarrow & G
\end{array}
\]

resp.

\[
\begin{array}{ccc}
\alpha_1 & \rightarrow & A \\
\downarrow & & \downarrow \\
\beta_2 & \rightarrow & G
\end{array}
\]

Definition. If the maps \( \alpha_1, \alpha_2 \) are injective, the universal group \( G \) is called the free product of \( A \) and \( B \) amalgamated along \( C \) (resp. amalgamated free product of \( A \) along \( C \)), and denoted by \( A \ast_C B \) resp. \( A \ast_C \). If \( C \) is the trivial group, then \( A \ast_C B \) is denoted by \( A \ast B \) and is called the free product of \( A \) and \( B \).

Note. There seems no reason except tradition for calling the first an amalgamated product but the second an HNN group.

We now present the traditional combinatorial arguments for analysing the structure of amalgamated products. Essentially equivalent results will be obtained below independently by geometrical reasoning.

In each of the above cases, one can give an explicit description using reduced words. The easy first half of the proof of Theorem 1.4 shows (with only slight changes) that any element of \( A \ast_C B \) can be written as a product

\[ a_1 b_1 a_2 b_2 \cdots a_n b_n \text{ with } a_i \in \beta_1(A), b_i \in \beta_2(B) \text{ (maybe } = 1) \]

and that any element of \( A \ast_C B \) can be written as

\[ a_1^{r_1} a_2^{r_2} b_1^{-r_1} a_3^{r_3} b_2^{-r_2} \cdots a_n^{r_n} \text{ with } a_i \in \beta(A), r_i \in \mathbb{Z} \]

Now we restrict our attention to \( A \ast_C B \). Again, it is clear that some reduction is possible - e.g. for \( c \in C \), \( \beta_1(a) \beta_2(\alpha(c)b) = \beta_1(\alpha \phi_1(c)) \beta_2(b) \).

We deal with this by pushing all the \( c \)'s to the right, as follows. To simplify notation, write \( \alpha_1, \alpha_2 \) as inclusions so; \( C \subseteq A, C \subseteq B \). Pick representatives \( a_1, b_1 \) for the right cosets of \( a_1, b_1 \) of \( C \) - thus giving a section of the projection \( A \to A/C \), or right transversal of \( C \) in \( A \); then do the same for \( B \). We impose the restriction that the identity coset \( C \) is represented by the identity element.

A reduced word is now a sequence

\[ a_1 b_1 \cdots a_n b_n \]

such that \( c \in C, a_1 \) belongs to the chosen transversal \( T_A \) for \( C \) in \( A \),

\( b_1 \) belongs to the chosen transversal \( T_B \) for \( C \) in \( B \) and
\[ a_1 = 1 \Rightarrow i = 1, \ b_1 = 1 \Rightarrow i = n. \]

Any element of \( A * C \) may be represented by a reduced word. For write the element as

\[ a_1 b_1 \ldots a_n b_n \]

and use induction on \( n \). For \( n = 0 \), the empty word may be represented by \( 1 \in C \): a reduced word. Otherwise, by inductive hypothesis, we may write

\[ a_1 b_1 \ldots a_{n-1} b_{n-1} = a_1' b_1' \ldots a_r' b_r' c', \]

a reduced word with \( r = n - 1 \).

If now \( b_r = 1 \) resp. \( a_n \in C \) then \((a_1' c a_r') \in A \) resp. \((b_r' c a_r b_n) \in B \), and we have a word of length \( r \), which may be reduced by inductive hypothesis. Otherwise write \( c a_n' = a_{n+1}' c^n \) with \( 1 \neq a_{n+1}' \in T_A \) and \( c^n b_n = b_{n+1}' c^m \) with \( b_{n+1}' \in T_B \) and we have a reduced word \( a_1' b_1' \ldots a_r' b_r' c^n b_n \).

**Theorem 1.6.** The maps \( A \rightarrow A * C \), \( B \rightarrow A * C \) are injective: every element may be represented by a unique reduced word.

**Proof.** Again write \( W \) for the set of reduced words.

Define an action of \( B \) on \( W \) by

\[ (a_1 b_1 \ldots a_n b_n) b = a_1 \ldots a_n b c' \]

if \( b \in c b' = b c' \) in \( B \) with \( b' \in T_B \).

To check that this is an action, observe that the part of the word up to (and including) \( a_n \) is left fixed; for the words \( a_1 \ldots a_n b c' \) which start so, it is equivalent to the right action of \( B \) on itself.

Define an action of \( A \) on \( W \) by

\[ (a_1 b_1 \ldots a_n b_n) a = \begin{cases} a_1 \ldots b_n a c' & \text{if } b_n \neq 1, \ ca = a c' \text{ with } a' \in T_A \\ a_1 \ldots b_n a c' & \text{if } b_n = 1, \ a_n = a c' \text{ with } a' \in T_A \end{cases} \]

To check that this is an action, observe that the part of the word up to \( b_n \) (if \( b_n \neq 1 \)) or \( b_{n-1} \) (if \( b_n = 1 \)) is fixed; the rest is the standard right action. This defines maps \( A \rightarrow S(W) \), \( B \rightarrow S(W) \) which clearly agree on \( C \), hence define \( A * C \rightarrow S(W) \). It is now immediate by induction again that \( (\cdot) w(w) = w \) for any reduced word \( w \), so these elements of the group are distinct.

For \( A * C \), we proceed similarly. Pick right transversals \( T_1 \) of \( \alpha_1(C) \) in \( A \). Now \( t \alpha_1(c) = \alpha_1(c) \) so \( t \alpha_1(c) t^{-1} = \alpha_1(c) t^{-1} \) and we define a reduced word to be one of the form

\[ a_1' \ldots a_r' a_{n+1}^i \]

where \( i = \pm 1 \), \( a_1 \in T_1 \) if \( i = +1 \), \( a_1 \in T_2 \) if \( i = -1 \) and moreover \( a_1 \neq 1 \) if \( i \neq i_1 \). We let \( a_{n+1} \) be arbitrary. The above relations allow us to bring any word to a reduced form.

**Theorem 1.7.** The map \( \beta : A * A * C \) is injective. Every element is represented by a unique reduced word.

**Proof.** Again we define an action. The element \( a \in A \) acts by sending the final \( a_{n+1} \) to \( a_{n+1}^n \); \( t \) corresponds to the permutation \( \tau \) defined by setting

\[ \begin{align*}
(a_1 \ldots a_n)^\tau &= \epsilon_1 \ldots a_n \epsilon_{n+1} \\
(a_1 \ldots a_n)^{\tau} a_{n+1} &= (a_1 a_{n+1}^{-1} a_{n+1} a_{n+1}^{-1} (a_{n+1}) \text{ if } \epsilon_n = 1 \text{ and } a_{n+1} \epsilon_{n+1} \alpha_1(C)) \\
(a_1 \ldots a_n)^{\tau} a_{n+1} &= (a_1 a_{n+1} \alpha_1(c^n) \text{ otherwise, where } a_{n+1} = a_{n+1}^n \alpha_1(c^n) \text{ with } a_{n+1}^n \epsilon T_1) \end{align*} \]

We see that this is a permutation by verifying that an inverse is given by

\[ \begin{align*}
(a_1 \ldots a_n)^{\tau^{-1}} &= \epsilon_1 \ldots a_n \\
(a_1 \ldots a_n)^{\tau^{-1}} a_{n+1} &= (a_1 a_{n+1}^{-1} a_{n+1} a_{n+1}^{-1} (a_{n+1}) \text{ if } \epsilon_n = 1 \text{ and } a_{n+1} \epsilon_{n+1} \alpha_1(C), \\
(a_1 \ldots a_n)^{\tau^{-1}} a_{n+1} &= (a_1 a_{n+1} \alpha_1(c^n) \text{ otherwise, where } a_{n+1} = a_{n+1}^n \alpha_1(c^n) \text{ with } a_{n+1}^n \epsilon T_2) \end{align*} \]

The proof now concludes as before.
GRUSKÓ’S THEOREM

Grusko’s Theorem. Let \( F \) be a finitely generated free group, \( G = G_1 \ast G_2 \) and let \( \phi : F \to G \) be an epimorphism. Then there are subgroups \( F_1 \) and \( F_2 \) of \( F \) such that \( F = F_1 \ast F_2 \) and \( \phi(F_1) = G_1 \).

This is a subtle result about generators of \( G \). It says that if \( G \) can be generated by \( n \) elements, then there exists a set of \( n \) generators for \( G \) with each element in \( G_1 \) or \( G_2 \). This gives us the inequality

\[
\mu(G) \geq \mu(G_1) + \mu(G_2)
\]

where \( \mu(G) \) denotes the minimal number of generators of a group. But the reverse inequality is obvious, so we deduce

**Corollary 2.1.** If \( G = G_1 \ast G_2 \), then \( \mu(G) = \mu(G_1) + \mu(G_2) \).

As only the trivial group can have \( \mu \) equal to zero, we see that \( \mu(G_1) < \mu(G) \) when \( G_1 \) and \( G_2 \) are nontrivial.

**Corollary 2.2.** If \( G \) is a finitely generated group, then \( G = G_1 \ast \cdots \ast G_n \) for some \( n \), where each \( G_i \) is indecomposable. (i.e., \( G_1 = A \ast B \) implies \( A \) or \( B \) is trivial.)

We now give Stallings’ proof [23] of Grusko’s Theorem. See [12] for a proof using groupoids and see [3] for a proof using Bass-Serre theory (Chapter 4 of these notes).

Pick two CW-complexes with fundamental groups \( G_1 \) and \( G_2 \) and construct a CW-complex \( X \) with fundamental group \( G_1 \ast G_2 \), by joining these two complexes with an interval \( I \). Let \( v \) denote the midpoint of \( E \) and subdivide \( E \) so that \( v \) is a vertex of \( E \). We will take \( v \) as the basepoint of \( X \). Let \( X_1 \) denote the closure of the component of \( X - \{v\} \) whose fundamental group is \( G_1 \).

Let \( K \) be a based space and let \( f : K \to X \) be a based map. We will say that \( f \) represents \( \phi \) if there is an isomorphism of \( \pi_1(K) \) with \( F \) such that the diagram below commutes.

\[
\begin{array}{ccc}
\pi_1(K) & \xrightarrow{f_*} & F \\
\downarrow \phi & & \downarrow \phi \\
G & = & \pi_1(X)
\end{array}
\]

We consider 2-dimensional CW-complexes \( K \) and cellular maps \( f : K \to X \) which represent \( \phi \). Such maps certainly exist for one can take \( K \) to be the wedge of \( n \) circles where \( n \) is the rank of \( F \). However, this particular choice of \( K \) may not be the correct one for our purposes. Our aim is to choose \( K \) and a map \( f : K \to X \) representing \( \phi \) so that \( f^{-1}(v) \) is a tree. Once this is achieved, the result follows easily. For let \( L_1 \) denote \( f^{-1}(X_1) \) and let \( F_1 \) denote \( \pi_1(L_1) \). As \( L_1 \cup L_2 = K \) and as \( L_1 \cap L_2 = \{v\} \) which is simply connected, we see that \( \pi_1(K) = F_1 \ast F_2 \). Also \( f_*(F_1) \subset G_1 \) as \( f(K) \subset X_1 \). Now the results of Theorem 1.5 on reduced words in a free product, show that we must have \( f_*(F_1) = G_1 \), because \( \phi \) is an epimorphism. This is now the conclusion of Grusko’s Theorem.

We find an appropriate choice for \( K \) by starting with a space \( K_0 \) and a map \( f_0 : K_0 \to X \) representing \( \phi \) and then performing a sequence of modifications to \( K_0 \) and \( f_0 \). We do this so as to obtain spaces \( K_1, K_2, \ldots \) and maps \( f_1, f_2, \ldots \) all representing \( \phi \), such that \( f_n^{-1}(v) \) is a forest (disjoint union of trees) with \( a_1 \) components and \( a_{n+1} < a_n \) for \( n \geq 0 \). After at most \( a_1 \) steps, we will obtain a space \( K_n \) and map \( f_n : K_n \to X \) representing \( \phi \) such that \( f_n^{-1}(v) \) is a tree, and the result will follow.

We take \( K_0 \) to be the wedge of \( n \) circles, where \( n \) is the rank of \( F \) and choose a cellular map \( f_0 : K_0 \to X \) representing \( \phi \) so that \( f_0^{-1}(v) \) is a finite number of 0-cells in \( K_0 \). In particular, \( f_0^{-1}(v) \) is a forest with \( a_1 \) components. If \( f_n^{-1}(v) \) is connected (and hence a single point), we already have the space \( K_n \) and map \( f_n \) which we want. Otherwise, we use the following lemma to construct the sequence of spaces already described and hence deduce the result.

**Lemma 2.3.** Let \( K \) be a based CW-complex and \( f : K \to X \) a map representing \( \phi : F \to G \) such that \( f^{-1}(v) \) is a forest with \( a \) components. If \( a \geq 2 \), then there is a based CW-complex \( K' \) and a map \( f' : K' \to X \)
representing \( \phi \) such that \( f'^{-1}(v) \) is a forest with \( \alpha - 1 \) components.

Proof. Let \( \ell \) be a path in \( K \) with endpoints in \( f^{-1}(v) \). (By a path, we mean simply that \( \ell \) is a map \( I \to K \).) Then \( f \circ \ell \) is a loop in \( X \) based at \( v \). Pick a path \( \ell' \) joining distinct components of \( f^{-1}(v) \). We construct a space \( K' \) from \( K \) by attaching a 1-cell \( e \) to \( K' \) and then attaching a 2-cell \( B \) to \( e \) \( \ell' \). We would like to extend \( f : K \to X \) to a map \( f' : K' \to X \) such that \( f'(e) = v \) and \( f'^{-1}(v) \) does not meet the interior of the 2-cell \( B \). If this can be done, then \( f'^{-1}(v) = f^{-1}(v) \cup e \) which is a forest with \( \alpha - 1 \) components. Also \( f' \) represents \( \phi \), as \( f' \) extends \( f \) and \( K' \) deformation retracts to \( K \). Hence \( f \) has all the required properties. We will be able to construct such an extension \( f' \) if the loop \( f \circ \ell \) has image in \( X_1 \) (or \( X_2 \)) and is contractible in \( X \). For then \( f \circ \ell \) is null homotopic in \( X_1 \) and \( f' \) restricted to \( B \) is essentially this null homotopy. Our aim is to show that such a \( \ell \) exists.

Choose two distinct components \( A \) and \( B \) of \( f^{-1}(v) \) and let \( L \) be a path in \( K \) from \( A \) to \( B \). As \( t_\gamma : x_1(K) \to x_1(X) \) is onto, there is a loop \( \gamma \) in \( K \) based at \( L(0) \) such that \( f \circ \gamma \) is homotopic to the loop \( f \circ L \). Let \( \ell' \) be the path \( \gamma^{-1}L \) in \( K \). This is a path in \( K \) joining \( A \) to \( B \) such that \( f \circ \ell' \) is a contractible loop in \( X \).

We can suppose that \( \ell' \) is a cellular map \( I \to K \) by subdividing \( \ell \) and by choice of \( \ell \). Thus we can express \( \ell \) as a union of subpaths \( \ell_1, \ldots, \ell_n \) such that the ends of \( \ell_1 \) lie in \( f^{-1}(v) \) and \( f \circ \ell_1 \) is a loop in \( X_1 \) or \( X_2 \). Further we can suppose that the maps \( f \circ \ell_1 \) alternate between \( X_1 \) and \( X_2 \). (Note that \( \ell_1 \) may meet components of \( f^{-1}(v) \) in its interior.) We say that \( \ell \) has length \( n \).

Let \( g_1 \) denote the homotopy class of \( f \circ \ell_1 \) in \( x_1(X, v) \). Suppose that some \( g_1 \) has the two properties that \( g_1 \) is trivial and that the endpoints of \( \ell_1 \) lie in one component of \( f^{-1}(v) \). Then we can alter \( \ell \) to \( \ell' \) by removing \( \ell_1 \) and replacing it with a path \( \ell_1' \) in \( f^{-1}(v) \) which joins the endpoints of \( \ell_1' \). Clearly \( \ell' \) has length less than \( n \). By repeating this process, we can arrange that \( \ell \) has no subpath \( \ell_1 \) with these two properties.

Now the equation \( \ell = \ell_1 \ldots \ell_n \) gives rise to the equation

\[ 1 = g_1 g_2 \ldots g_n \] \( \text{in } x_1(X) \). As \( x_1(X) = G_1 * G_2 \) and the \( g_i \)'s lie alter-

nately in \( G_1 \) and \( G_2 \), we deduce that some \( g_i \) is trivial. The corresponding \( \ell_i \) joins distinct components of \( f^{-1}(v) \) and has \( f \circ \ell_i \) contractible. We can now construct the required space \( K' \) and map \( f' : K' \to X \) as previously described.

\section{SUBGROUPS AND COVERING SPACES}

We will apply the theory of covering spaces to the problem of describing subgroups of amalgamated free products. First, we consider free products.

Suppose given a group \( G = G_1 * G_2 \). As before we construct a space \( X \) with fundamental group \( G \) by taking CW-complexes \( X_1, X_2 \) with \( x_1(X_1) = G_1 \) and joining the basepoints of \( X_1 \) and \( X_2 \) with an interval \( E \). We take the midpoint \( v \) of \( E \) as the basepoint of \( X \). Now suppose that \( H \) is a subgroup of \( G \). Then \( H \) is the fundamental group of some connected covering space \( \tilde{X} \) of \( X \), with projection map \( p : \tilde{X} \to X \). Inside \( X \), we have \( \tilde{p}^{-1}(X_1) \) which is a covering space of \( X_1 \) and so consists of various connected covering spaces of \( X_1 \). Also \( \tilde{p}^{-1}(X_2) \) is a union of connected covering spaces of \( X_2 \). Finally, as \( E \) is simply connected, \( \tilde{p}^{-1}(E) \) is a union of copies of \( E \). Thus \( \tilde{X} \) looks like (and agrees up to homotopy with) a graph \( \Gamma \) with a covering space of \( X_1 \) or \( X_2 \) at each vertex. If \( \Gamma \) were a tree, then \( H \) would be the free product of the fundamental groups \( H_1 \) of all the spaces at the vertices of \( \Gamma \). In general, \( \Gamma \) consists of a tree \( T \) with extra edges attached to \( T \), where \( T \) is a maximal tree in \( \Gamma \). Thus \( H \) will be the free product of all the groups \( H_1 \) and of a free group whose generators correspond to the edges of \( \Gamma - T \).

Let \( \tilde{v} \) denote the basepoint of \( \tilde{X} \) and recall that \( H = \tilde{p}(\gamma_1(\tilde{X}, \tilde{v})) \).

Let \( C \) be a component of \( \tilde{p}^{-1}(X_1) \) and join it to \( \tilde{v} \) by a path in \( \tilde{X} \). We see that \( \tilde{p}(\gamma(\tilde{C}, \tilde{v})) \) is a conjugate of some subgroup of \( G \). Thus the above description of a typical covering space of \( X \) leads at once to the following result.

\textbf{Theorem 3.1 (Kurosh subgroup theorem).} If \( H \) is a subgroup of \( G = G_1 * G_2 \), then \( H \) is the free product of a free group with subgroups of conjugates of \( G_1 \) or \( G_2 \).
Corollary 3.2. If \( H \) is a subgroup of a free group, then \( H \) is free.

Corollary 3.3. If \( H \) is indecomposable and not infinite cyclic, and if \( H \cong G_1 \times G_2 \), then \( H \) lies in a conjugate of \( G_1 \) or \( G_2 \).

Examples of an application of Corollary 3.3 would be when \( H \) is finite or abelian.

**Exercise.** Prove that a non-trivial direct product cannot be a non-trivial free product.

**Lemma 3.4.** If \( G = G_1 \times G_2 \) and if \( w^{-1} G_1 w \cap G_1 \) is non-trivial, then \( w^{-1} G_1 w \cap G_1 = G_1 \).

**Proof.** Clearly \( G_1 \) must be non-trivial, and we may as well suppose that \( G_1 \) is non-trivial. Now let \( g \) be a non-trivial element of \( G_1 \). Then \( w^{-1} g w \in G_1 \). We can write \( w = \alpha g \), where \( \alpha \in G_1 \) and \( w_1 \) is a reduced word in \( G_2 \) beginning in \( G_2 \). Thus \( w^{-1} g w = \alpha^{-1} g \alpha w_1 = \alpha^{-1} g \alpha w_1 = \alpha^{-1} g \alpha w_1 \) where \( g' \) is a non-trivial element of \( G_1 \). Thus \( w^{-1} g w \) is a reduced word. But this is an element of \( G_1 \), and so has length 1. Hence \( w_1 \) is trivial and so \( w \) lies in \( G_1 \). Hence \( w^{-1} G_1 w \cap G_1 = G_1 \), and we have \( G_1 \cap G_1 \) non-trivial. This can only happen when \( i = 1 \), which completes the proof of the lemma.

**Theorem 3.5.** If \( G \) is a finitely generated group, then \( G = G_1 \times \ldots \times G_n \), where each \( G_i \) is indecomposable. If also \( G = G_1 \times \ldots \times G_n \cap H_1 \times \ldots \times H_m \), where each \( G_i \) and \( H_j \) is non-trivial and indecomposable, then \( m = n \) and, by re-ordering, we have \( G_i \cong H_i \) for each \( i \). Further, for each \( i \) with \( G_i \) not infinite cyclic we have \( G_i \cong H_i \) conjugate to \( H_i \).

**Proof.** The first sentence is just Corollary 2.2 stated again.

Now suppose that \( G = G_1 \times \ldots \times G_n = H_1 \times \ldots \times H_m \), where each \( G_i \) and \( H_j \) is indecomposable. If each \( G_i \) is infinite cyclic, then \( G \) is free, and Corollary 3.2 tells us that each \( H_j \) is free. As \( H_j \) is indecomposable, it must be infinite cyclic and it now follows easily by abelianising and

using the basis theorem for f.g. abelian groups that \( m = n \). Otherwise we can re-order the \( G_i \)'s so that \( G_1, \ldots, G_r \) are not infinite cyclic and \( G_{r+1}, \ldots, G_n \) are infinite cyclic.

Corollary 3.5 applied to \( G \cong H_1 \times \ldots \times H_m \) shows that, by re-ordering the \( H_i \)'s, we have \( u^{-1} G_1 u \cong H_1 \) for some \( u \in G \). Hence \( H_1 \) is not infinite cyclic and Corollary 3.3 applied to \( H_1 \cong G_1 \times \ldots \times G_n \) shows that \( v^{-1} H_1 v \cong G_1 \) for some \( v \). Hence \( w^{-1} H_1 w \cong G_1 \) where \( w = uv \). Now Lemma 3.4 shows that \( w \in G_1 \) and \( i = 1 \). Thus we have

\[ G_1 = w^{-1} G_1 w \cong v^{-1} H_1 v \cong G_1. \]

It follows that \( H_1 \) is conjugate to \( G_1 \) in \( G \) and hence also that \( H_1 \) is isomorphic to \( G_1 \).

Repeat this process for \( G_2, \ldots, G_n \) to show that \( G_i \) is conjugate to \( H_i \) for \( i = 1, 2, \ldots, r \). (Note that we cannot find two different \( G_i \)'s conjugate to the same \( H_j \) as different \( G_i \)'s cannot be conjugate to each other, by Lemma 3.4.)

Consider \( G / \langle G_1 \times \ldots \times G_n \rangle \), where \( \langle X \rangle \) denotes the normal closure of a subset \( X \) of \( G \). As \( G_1 \) is conjugate to \( H_1 \) for \( i = 1, 2, \ldots, r \) we have

\[ G_{r+1} \times \ldots \times G_n \cong G / \langle G_1 \times \ldots \times G_n \rangle = G / \langle H_1 \times \ldots \times H_r \rangle = H_{r+1} \times \ldots \times H_m. \]

The left hand group is free, and so each \( H_i \), \( i = r + 1, \ldots, m \), must be infinite cyclic. It now follows that \( m = n \) and we have completed the proof of Theorem 3.5.

One might ask whether an analogue of Theorem 3.5 holds for non-f.g. groups. At the end of this chapter, we give an example which shows that the first part of the theorem fails for non-f.g. groups. However, the uniqueness result which is the second part of Theorem 3.5 clearly applies to all groups which can be expressed as a finite free product of indecomposables.

The next step is to consider the structure of subgroups of amalgamated free products. Let us consider a group \( G = A \ast_C B \). Let \( X_0 \) be a CW-complex with fundamental group \( A \), let \( X_1 \) be a CW-complex with fundamental group \( B \) and let \( X_0 \) be a 2-dimensional CW-complex with fundamental group \( C \). Lemma 1.5 tells us that there are maps
We define $\delta, e = a \circ e$ and say that $e$ joins $a \circ e$ to $b \circ e$.

An abstract graph $\Gamma$ has an obvious geometric realisation $|\Gamma|$ with vertices $V(\Gamma)$ and edges corresponding to pairs $(a, b)$. When we say that $\Gamma$ is connected or has some other topological property, we shall mean that the realisation of $\Gamma$ has the appropriate property. An orientation of an abstract graph is a choice of one edge out of each pair $(a, b)$.

A graph of groups consists of an abstract graph $\Gamma$ (which will always be tacitly assumed to be connected) together with a function $S$ assigning to each vertex $v$ of $\Gamma$ a group $G_v$ and to each edge $e$ a group $G_e$, with $G_v = G_e$, and an injective homomorphism $f_e : G_v \rightarrow G_e$.

One may think of $\Gamma$ as a partial category and $S$ as a sort of functor, similarly we may define a graph of topological spaces, or of spaces with preferred basepoint; here it is not necessary for the map $X_v \rightarrow X_e$ to be injective, as we can use the mapping cylinder construction to replace the maps by inclusions; and this does not alter the total space defined below.

But we will suppose for convenience that the spaces are CW complexes and maps cellular.

Given a graph $G$ of spaces, we can define a total space $X_G$ as the quotient of $\bigcup \{ X_v : v \in V(\Gamma) \} \cup \{ X_e \times I : e \in E(\Gamma) \}$ by the identifications

$$X_v \times I \rightarrow X_v \times 1$$
$$X_e \times 0 \rightarrow X_e \times e$$

$G$ is a graph of (connected) based spaces, then by taking fundamental groups we obtain a graph $S$ of groups (with the same underlying abstract graph $\Gamma$). The fundamental group $G_v$ of each group $S$ of groups is defined to be the fundamental group of the total space $X_G$. Observe that in the cases where $\Gamma$ has just one pair $(a, b)$ of edges

we obtain the products $A \ast_B B, A \ast_C C$ already discussed, as follows by van Kampen's theorem (1.1 and 1.2). The general case may be considered...
as derived by an iterated application of these constructions; however if it is treated in this way, the underlying geometry is liable to be obscured by computational complexities.

We now show that $G_\Gamma$ does not depend on the choice of $\Gamma$. First, for any $G$, we can choose (using presentations) connected 2-dimensional CW complexes with $x_0(X_v, x) = G_v$ and $x_0(X_e, x) = G_e$. By Lemma 1.5, the homomorphisms $f_0$ are induced by continuous maps $(X_v, x) \rightarrow (X_e, x)$. This defines a graph $\mathcal{X}$ of connected based spaces giving rise to $G$. Next for any $G$, we can attach cells of dimension $\geq 3$ to each $X_v, X_e$ to obtain aspherical spaces $K_v, K_e$ still with the same fundamental group. Now the map $f_0 : X_v \times X_e \rightarrow X_0$ extends to a map $k_0 : K_v \times K_e \rightarrow X_0$ (there is no obstruction) so we have a new graph $\mathcal{K}$ of spaces, still inducing $G$; its total space $K_\Gamma$ is obtained from $X_\Gamma$ by adding cells of dimension $\geq 3$, so has the same fundamental group. But $K_v$ is a space of type $(K_v, 1)$. Its homotopy type is entirely determined by $G_v$ (similarly $K_e, G_e$). Also the map $k_0$ is determined up to homotopy by $f_0 : G_v \rightarrow G_e$. Thus $K_\Gamma$ is determined up to homotopy, and its fundamental group is unique up to isomorphism.

In order to relate the topology more closely to the group theory, we have insisted above on preservation of base points. However if the attaching maps to $X_v \times X_e$ are altered by any homotopy (not necessarily base point preserving), the homotopy type and hence the fundamental group of $X_\Gamma$ are unchanged. If the base point is pulled round a loop defining $g \in x_0(X_e, x)$, the homomorphism $G_e \rightarrow G_e$ is changed by conjugation by $g$. Thus even such a change will not alter $G_\Gamma$.

Corresponding to and generalising (1.6) and (1.7) we now have the

**Proposition 3.6.** (i) If $G$ is a graph of groups as above, each map $G_v \rightarrow G_\Gamma$ is injective.

(ii) If $\mathcal{X}$ is a graph of aspherical spaces as above, the total space $K_\Gamma$ is aspherical.

**Proof.** We start from the graph $\mathcal{X}$ of spaces. Observe that for each vertex $v$ of $\Gamma$, the space $\bigcup_{v \in V} (K_v \times I)$ admits $K_v$ as deformation retract, so its universal cover $\tilde{L}_v$ is contractible. Moreover, as each map $G_v \rightarrow G_\Gamma$ is injective, $\tilde{L}_v$ is obtained from $\tilde{K}_v$ by attaching copies of $K_e \times I$ with $K_v$ the universal cover of $K_e$; hence also contractible.

Now construct a space $Y = \bigcup_{n} Y_n$ by induction. Choose any vertex $v_0$ of $\Gamma$ and set $Y_0 = \tilde{L}_{v_0}$. Next for any $n \geq 1$, in forming $Y_{n+1}$, a number of copies of $K_e \times I$ will have been attached (each along $\tilde{K}_e \times 0$), for various edges $e$. We define $Y_n$ to be the union of $Y_{n-1}$ with a copy of $L_{e, v}$ for each such copy of $K_e \times I$, identified along $\tilde{K}_e \times I$. Since we are attaching contractible sets along contractible subsets, each $Y_n$ is contractible.

Set $Y = \bigcup Y_n$ with the weak topology. Then $Y$ also is contractible. There is an evident projection $Y \rightarrow K_\Gamma$; by construction, $K_\Gamma$ is evenly covered by $Y$. This proves (ii), and (i) follows since for each $K_v \subset K_\Gamma$, the induced covering of $K_v$ contains the universal covering.

**Remark.** Assertion (ii) is equivalent to the exact sequences of Minnelli [30]. A normal form, in the style of Theorems 1.6 and 1.7, is given by Higgins [32].

We can now state our first result about subgroups of amalgamated free products.

**Theorem 3.7.** If $G = A \ast_C B$ or $A \ast_B C$, and if $H \subset G$, then $H$ is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of $A$ or $B$ and the edge groups are subgroups of conjugates of $C$.

**Remarks.** This result has an obvious generalization to the case where $G$ is the fundamental group of a graph $\Gamma$ of groups. The theorem covers the special case when the geometric realisation of $\Gamma$ has a single edge.

There is a corollary of this result which is analogous to Corollary 3 in the case of free products. We say that a group $G$ splits over $a$
subgroup $C$ if $G = A \cdot C$ or $G = A + C + B$ with $A \cdot C \cdot B$. If $G$ splits
over some subgroup, we say that $G$ is splitable. Note that $Z$ is splitable
as $Z = \{1\} \cdot \{1\}$.

Corollary 3.8. If $G = A \cdot C + B$ or $A \cdot C$ and if $H$ is a finitely
generated non-splitable subgroup of $G$, then $H$ lies in a conjugate of $A$
or $B$.

Proof. We know that $H$ is the fundamental group of a graph $\Gamma$
of groups. As $H$ is finitely generated, there is a finite subgraph $\Gamma'$
whose fundamental group equals $H$. Now the fact that $H$ is not splitable
implies that one of the vertex groups of $\Gamma'$ is equal to $H$. The result
follows.

Remark. The finite generation of $H$ allows one to deduce that
some vertex group of $\Gamma$ equals $H$. If we consider non-finitely generated
subgroups $H$, we see that $H$ need not lie in a conjugate of $A$ or $B$. For
example, consider $G = \mathbb{Z} \cdot \mathbb{Z}$ where the two inclusion maps are the
identity and multiplication by 2. Thus $G$ has presentation $(a, t : t^{-1}a = a)$.
The subgroup $H$ of $G$ generated by $\{a^n : n \in \mathbb{Z}\}$ is isomorphic
to the dyadic rationals and is therefore non-splitable, but of course $H$
cannot be contained in an infinite cyclic group.

We must now consider covering spaces more closely. Recall paragraph 2 on page 138 of these notes. Let $(X, x_0)$ be a based connected
space with fundamental group $G$. Let $H$ be a subgroup of $G$ and let
$(\tilde{X}, \tilde{x}_0)$ be the corresponding connected covering space, with projection
map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$. Thus $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

Lemma 3.9. There is a natural bijection $\tilde{\phi} : H \backslash G \to p^{-1}(x_0)$, where
$H \backslash G$ denotes the quotient of $G$ under the action of $H$ by left multiplication.

Remark. The path lifting property of covering maps can be used to
define bijections $p^{-1}(x_0) \to p^{-1}(x)$ for each $x \in X$.

Proof. First we define a map $\phi : G \to p^{-1}(x_0)$. Given $g \in G$,
choose a map $(1, \{g\}) \to (X, x_0)$ representing $g$. We will call such a map
a loop. Let $L$ be the lift of this map starting at $\bar{x}_0$ and define
$\phi(g) = (L(1)) \cdot p^{-1}(x_0)$. This definition is independent of the choice of the
loop chosen to represent $g$.

The map $\phi$ is a surjection. For given $y \in p^{-1}(x_0)$, choose a path
$\tilde{L}$ in $\tilde{X}$ from $\tilde{x}_0$ to $y$. Then $p \cdot \tilde{L}$ is a loop in $X$ representing some
element $g \in \pi_1(X, x_0)$ and $\phi(g) = y$.

If $\phi(g_1) = \phi(g_2)$, then $L_1 \cdot I_{\tilde{x}_0}^{-1}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$, where
$L_1$ is a lift of a loop in $X$ representing $g_1$. Thus $p \cdot L_1 \cdot I_{\tilde{x}_0}^{-1}$ represents
an element $h$ of $H$ and we have the equation $g = g_2 \cdot h$. Conversely if
$\phi(g_1) = e \in H$, then $g = g_2 \cdot h$ and it is clear that $\phi(g_1) = \phi(g_2)$. Thus
$\phi(g_1) = \phi(g_2)$ if and only if $g_1 \cdot g_2^{-1} \in H$ and so $\phi$ induces a bijection
$\tilde{\phi} : H \backslash G \to p^{-1}(x_0)$.

Lemma 3.10. If $H$ is a normal subgroup of $G$, then $G/H$ acts
on $X$ by covering homeomorphisms with quotient $X$.

Proof. Let $g \in G$ and let $y \in p^{-1}(x_0)$ be the point determined by $g$.
Then $p_*(\pi_1(\tilde{X}, y)) = g \cdot \pi_1(\tilde{X}, x_0)$ which equals $H$ as $H$ is normal in $G$.
The uniqueness of covering spaces corresponding to $H$ shows that there is
a unique covering homeomorphism $\psi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$.

I claim that this process defines a homomorphism of $G$ to the group of covering homeomorphisms of $\tilde{X}$. One need only show that
$\psi g_1 g_2 (x_0) = \psi g_1 (x_0) \cdot \psi g_2 (x_0)$, as two covering homeomorphisms which agree on $\tilde{x}_0$ must be equal by the
uniqueness result again. Let $L_1$ and $L_2$ be paths in $X$ starting at $x_0$, which
are lifts of loops in $X$ representing $g_1$ and $g_2$, respectively.

Then $\psi g_1 (x_0)$ is a path in $\tilde{X}$ starting at $\tilde{x}_0$ and still lifting a loop
in $X$ representing $g_1$. Thus $\psi g_1 = L_2$ begins where $L_1$ ends and we
deduce that $\psi g_1 g_2 (x_0) = \psi g_1 (x_0) \cdot L_2(1) = \psi g_1 (x_0) \cdot \psi g_2 (x_0)$ as required. It is
clear that the kernel of this homomorphism is $H$, so that we do have an
action of $G/H$ on $\tilde{X}$.

The quotient of $\tilde{X}$ by the action of $G/H$ has a natural projection
$\pi : \tilde{X} \to X$ and $\pi$ is a covering map. Also, for each $x \in X$, $\pi^{-1}(x)$ is a
single point as $\pi^{-1}(x_0)$ is a single point. Hence $\pi$ is a homeomorphism
and this completes the proof of the lemma.
Before going further, we give an application of this result.

**Theorem 3.11.** If \( G = A \ast B \) where \( A, B \) are non-trivial and \( H \) is a finitely generated, normal subgroup of \( G \), then \( H \) is trivial or has finite index in \( G \).

**Proof.** We suppose that \( H \) has infinite index in \( G \) and will prove that \( H \) must be trivial. We know that \( H \) is the fundamental group of a graph \( \Gamma \) of groups, where the edge groups are trivial and the vertex groups \( H_\alpha \) are subgroups of conjugates of \( A \) or \( B \). If \( T \) is a maximal tree in \( \Gamma \), then \( H = F \ast (\star H_\alpha) \) where \( F \) is a free group whose generators correspond to the edges of \( \Gamma - T \). The fact that \( H \) is normal in \( G \) tells us that \( G/H \) acts on \( \Gamma \) with quotient an interval.

As \( H \) has infinite index in \( G \), we deduce that \( \Gamma \) has finitely many edges. As \( H \) is finitely generated, we deduce that \( \Gamma - T \) is finite and only finitely many of the groups \( H_\alpha \) are non-trivial. Thus \( H \) is the fundamental group of some connected finite subgraph \( \Gamma' \) of \( \Gamma \). Let \( E \) be an edge of \( \Gamma - \Gamma' \). Then removing \( E \) from \( \Gamma \) gives two subgraphs \( \Gamma_1 \) and \( \Gamma_2 \), one of which has trivial fundamental group. As \( G/H \) acts transitively on the edges of \( \Gamma \), we deduce that every edge of \( \Gamma \) has these properties. Thus \( \Gamma \) must be a tree and at most one vertex group can be non-trivial. Thus \( H \) is contained in a conjugate of \( A \) or \( B \). As \( H \) is normal in \( G \), it must lie in the intersection of all conjugates of \( A \) (or of \( B \)). But \( A \cap B^{-1}A \) is trivial for any non-trivial \( A \) and \( B \). Hence \( H \) is trivial.

Q.E.D.

**Exercise.** Is there an analogous result when \( G = A \ast_C B \) or \( A \ast_C ? \)

We now return to covering spaces. The aim of our next result is to give a more precise structure theorem for subgroups of amalgamated free products.

Let \( H \) be a subgroup of \( G = \pi_1(X, x_0) \) and let \( \tilde{X}, \tilde{x}_0 \) be as before. Let \( Y \) be a subspace of \( X \) which contains \( x_0 \), such that inclusion of \( Y \) in \( X \) induces an injective map \( \pi_1(Y, y_0) \to \pi_1(X, x_0) \). We denote the image group by \( A \) and identity \( \pi_1(Y) \) with this subgroup of \( G \).

**Lemma 3.12.** There is a natural correspondence \( \theta \) between the double cosets \( H \backslash G / A \) and the components of \( \pi^{-1}(Y) \) in \( \tilde{X} \).

**Proof.** Let \( \iota \) denote the inclusion of \( \pi^{-1}(y_0) \) in \( \pi^{-1}(Y) \) and recall the map \( \phi : G \to \pi^{-1}(Y) \). We define \( \theta : G \to \pi^{-1}(Y) \) by \( \theta = \iota \ast \phi \).

Suppose that \( \theta(g) \) and \( \theta(g) \) lie in the same component of \( \pi^{-1}(Y) \). Then they can be joined by a path \( \lambda \) in \( \pi^{-1}(Y) \). Let \( \alpha \in A \) be the element of \( \pi_1(X, x_0) \) represented by the loop \( p \ast \lambda \). Then, by projecting into \( X \), we see that \( g \ast h^{-1} \in \pi_1(X, x_0) \), so that \( g = h \ast \alpha \) for some \( h \in H \). Conversely, if \( g = h \ast \alpha \) for some elements \( \alpha \in A \) and \( h \in H \), then lifting to \( \tilde{X} \) tells us that \( \theta(g) \) and \( \theta(g) \) can be joined by a path \( \iota \) in \( \pi^{-1}(Y) \), where \( \iota \) is a lift of \( \alpha \).

Hence \( \theta \) induces the required bijection \( \tilde{\theta} \).

**Lemma 3.13.** Let \( g \in G \), \( y = \phi(g) \in \pi^{-1}(y_0) \) and let \( C \) be the component of \( \pi^{-1}(Y) \) which contains \( y \). Let \( \lambda \) be a loop in \( X \) representing \( g \) and let \( \lambda \) be the lift of \( \lambda \) which goes from \( \tilde{x}_0 \) to \( y \). Then \( p_*(\pi_1(C, \tilde{x}_0)) = H \cap g \pi^{-1}(Y)g^{-1} \), where we define \( \pi_1(C, \tilde{x}_0) \) by using the path \( \lambda \).

**Proof.** We know that \( p_*(\pi_1(C, y)) \subseteq A \) and so \( p_*(\pi_1(C, \tilde{x}_0)) \subseteq g \pi_1^{-1}g^{-1} \). As \( p_*(\pi_1(C, \tilde{x}_0)) \subseteq H \), we have \( p_*(\pi_1(C, \tilde{x}_0)) \subseteq H \cap g \pi_1^{-1}g^{-1} \).

Consider an element \( \beta = g \pi_1^{-1} \) of \( H \cap g \pi_1^{-1}g^{-1} \), where \( \alpha \in A \). Let \( \lambda \) be a loop in \( Y \) representing \( \alpha \). Then \( \lambda \pi_1^{-1} \) is a loop in \( X \) representing \( \beta \). We know that \( \lambda \pi_1^{-1} \) lifts to \( \tilde{X} \) because \( \beta \in H \). Thus \( \lambda \) lifts to \( \tilde{\lambda} \) and \( \lambda \pi_1^{-1} \) lifts to \( \tilde{\lambda} \pi_1^{-1} \), where \( m \) is some loop in \( \pi^{-1}(Y) \) based at \( y \). Therefore \( \beta \) lies in \( p_*(\pi_1(C, \tilde{x}_0)) \) and we have shown that \( p_*(\pi_1(C, \tilde{x}_0)) = H \cap g \pi_1^{-1}g^{-1} \) as required.

We can now state a more precise version of the subgroup theorem. Similar statements can be found in [4], [15], [22], [28]. For convenience in the case \( G = A \ast_C \), where we have two injections \( i_1 \) and \( i_2 \) of \( C \) into \( A \), we identify \( C \) with \( i_1(C) \). Then \( i_1(C) = \pi^{-1}(C) \) and the subgroup \( C \ast i_1(C) \) of \( A \) is also a subgroup of \( \pi^{-1}(C) \). 

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Subgroup Theorem 3.14. If $H$ is a subgroup of $G = A \ast_{C} B$ (or $A \ast_{C} B$), then $H$ is the fundamental group of a graph $\Gamma$ of groups. The vertices of $\Gamma$ correspond to the double cosets $HgA$ (and $HgB$), and the corresponding groups are $H \cap gA^{-1}$ (and $H \cap gB^{-1}$). The edges of $\Gamma$ correspond to the double cosets $HgC$ and the corresponding groups are $H \cap gG^{-1}$.

If $G = A \ast_{C} B$, the two ends of the edge $HgC$ are the vertices $HgA$ and $HgB$, and the injections of the associated groups are simply the inclusion mappings.

If $G = A \ast_{C}$, the two ends of the edge $HgC$ are the vertices $HgA$ and $HgB$. The injections of the associated groups are simply the inclusion mappings.

A corresponding theorem for subgroups $H$ of $G$, where $G$ is the fundamental group of any graph of groups, follows from the same lemma. We leave the precise formulation to the reader.

As applications of the subgroup theorem, we prove the following results.

Lemma 3.15. If $G = G_{1} \ast_{C} G_{2}$, then either $gG_{1}g^{-1} \cap G_{1}$ is a subgroup of a conjugate of $C$, or $i = 1$ and $g \in G_{1}$, so that $gG_{1}g^{-1} \cap G_{1} = G_{1}$.

Proof. Let $H = gG_{1}g^{-1} \cap G_{1}$. Then $H$ is the fundamental group of a graph $\Gamma$ of groups. The vertices $v_{1}, v_{2}$ of $\Gamma$ corresponding to the double cosets $HgA$ and $HgB$ have associated groups $H \cap gG_{1}g^{-1} = H$ and $H \cap G = H$. If $v_{1}$ and $v_{2}$ are distinct vertices of $\Gamma$, choose a path in $\Gamma$ joining them. As the inclusion of each of the groups associated to $v_{1}, v_{2}$ is an isomorphism with $H$, we see that each vertex and edge of this path has associated group $H$. Thus $H$ is the group associated to some edge and so lies in some conjugate of $C$. The only other possibility is that $v_{1} = v_{2}$. This implies that $i = 1$ and $HgG_{1} = Hg_{1}$. Thus $g \in G_{1}$, and the result is proved.

Next we give the promised example of a non-f.g. group $G$ which fails to satisfy the conclusion of Theorem 3.5. This example is due to Kuroš [17].

Example. The group $G = \langle a_{1}, b_{1}, a_{1}^{-1}, a_{2}, \ldots, b_{1}, b_{2}, \ldots \rangle$ cannot be expressed as a free product of indecomposable subgroups.

First observe that $G = \langle a_{1}, b_{1}, \ast_{C} a_{2}, b_{2}, \ast_{C} a_{3}, b_{3}, \ldots \rangle$, where each factor group is free of rank 2, each $C_{i}$ is infinite cyclic, the inclusion map $C_{i} \rightarrow \langle a_{i}, b_{i} \rangle$ sends a generator to $a_{i}$ and the inclusion map $C_{i} \rightarrow \langle a_{i+1}, b_{i+1} \rangle$ sends the same generator to $a_{i+1}$, $b_{i+1}$.

Next observe that

$$G = \langle b_{1} \rangle \ast \langle a_{1}, a_{2}, \ldots, b_{1}, b_{2}, \ldots \rangle \langle a_{n-1} = [a_{n}, b_{n}] \rangle \,
\forall n \geq 2 \rangle \ast \mathbb{Z} \ast G.$$
C is non-trivial and so Corollary 2.1 of Grusko’s Theorem shows that each group is cyclic. Hence the abelianisation homomorphism \( H \rightarrow H/\text{ker} \) injects \( H \cap A \) and \( C \). This contradicts the fact that \( g \) is a non-trivial commutator in \( H \cap A \). Hence \( H \) must be contained in \( A \).

We finish this section by proving the famous embedding theorem of Higman and Neumann [13] which states that any countable group can be embedded in a 2-generator group. A nice example to consider is the subgroup \( K \) of \( F_2 \), the free group on \( a \) and \( b \), which is the kernel of the homomorphism \( F_2 \rightarrow \mathbb{Z} \), which sends \( b \) to a generator and \( a \) to the identity. Let \( X \) denote the wedge of 2 circles, so that \( \pi_1(X) \cong F_2 \). The covering space \( \tilde{X} \) of \( X \) corresponding to \( K \) consists of a copy of \( \mathbb{R} \) together with a circle attached at each integer point. Thus \( K \) has basis \( \{b^n a b^{-n} : n \in \mathbb{Z} \} \). If one started with a countably generated free group \( K = \langle x_n : n \in \mathbb{Z} \rangle \), one would embed it in a finitely generated group by adding an element \( b \) to \( K \) which makes all the \( x_i \)'s conjugate. More precisely, one has the shift automorphism of \( K \) sending \( x_i \) to \( x_{i+1} \) for each \( n \), and one takes the extension of \( K \) by \( \mathbb{Z} \) determined by this automorphism. The new group generated by \( x_0 \) and \( b \) is, of course, isomorphic to \( F_2 \). The idea of the proof of the embedding result is to do the same sort of thing in general, i.e. make lots of generators conjugate.

**Theorem 3.17.** If \( G \) is a countable group, then \( G \) can be embedded in a 2-generator group.

**Proof.** Let \( x_1, x_2, \ldots \) be a generating set for \( G \). We embed \( G \) in \( G_1 = G \times \mathbb{Z} \). Let \( t \) be a generator of \( \mathbb{Z} \), and write \( y_1 = x_1, y_2 = t \).

Then \( G_1 \) is generated by \( y_1, y_2, \ldots \) and each \( y_i \) has infinite order.

Now let \( G_2 = \langle G_1, t \rangle \). Then \( G_2 \) is obtained from \( G_1 \) by an infinite sequence of R.H. extensions and so \( G_2 \subset C_2 \).

A set of generators for \( G_2 \) is \( y_1, t \). The subgroup \( K \) of \( G_2 \) generated by \( t \) is free and has the \( t_i \)'s as a basis. To see this observe that by killing \( G_1 \) one obtains a homomorphism of \( G_2 \) to a free group \( F \) which maps the \( t_i \)'s to \(-1\). Thus we can embed \( K \) in \( F \) as in the discussion preceding the theorem and we let \( G = G_2 \times F \). Again \( G_2 \subset G_3 \) and \( G_3 \) is generated by \( y_0, a, b \) where \( a \) denotes one of the \( t_i \)'s. Finally we observe that the subgroup \( H \) of \( G_3 \) generated by \( y_0 \) and \( b \) is free of rank two. This can be seen by applying the Subgroup Theorem, as \( H \cap G_2 \) and \( H \cap F \) are infinite cyclic and \( H \cap K \) is trivial. Thus we can construct \( G_4 = G_3 \times F \) where our two inclusions of \( F \) in \( G_3 \) have images \( H \) and \( F \).

We choose \( G = \langle G_4, s \rangle \) where \( s = a \cdot b^{-1} \cdot b^{-1} \), and \( s \) which completes the proof of the embedding theorem.

4. GROUPS ACTING ON GRAPHS

The subject matter of this chapter is a reworking of the Bass-Serre theory [22]. We consider a (continuous) action of a group \( G \) on a (topological) graph \( \Gamma \); clearly this corresponds also to an action on the corresponding abstract graph. We say that \( G \) acts without inversions if whenever an element \( g \) of \( G \) fixes an edge \( e \) of \( \Gamma \), it fixes each point of \( e \). In the abstract setting, this means that \( g \cdot e = e \) is forbidden. Given any action, the process of subdividing each edge once by an extra vertex in the middle gives us an action without inversions.

The following simple example will be of use in Section 6. Let \( G \) be a group, \( S \subset G \) a subset (not subgroup), \( \Gamma = \Gamma(S, G) \) the (geometric) graph with vertex set \( G \) and, for each \( (g, s) \in G \times S \) a single edge \( e(g, s) \) joining \( g \) to \( gs \). There is an obvious action of \( G \) on \( \Gamma \), where \( h \in G \) takes the vertex \( g \) to the vertex \( hg \); the edge \( e(g, s) \) to the edge \( e(hg, hs) \). Only the identity element of \( G \) can leave a vertex or edge fixed, so \( G \) acts freely (without inversions). Note that even if \( s \cdot 1 = 1 \) we do not identify the edges \( e(g, s) \) and \( e(gs, s) \) with the same endpoints.

**Proposition 4.1.**

(i) \( \Gamma \) contains no loop \( \iff \Gamma \neq S \).

(ii) \( \Gamma \) is a simplicial complex \( \iff \Gamma \cap S = \emptyset \).

(iii) \( \Gamma \) is connected \( \iff S \) generates \( G \).

(iv) \( \Gamma \) is a tree \( \iff S \) freely generates \( G \).

**Proof.** (i) and (ii) are trivial.

If \( H \) denotes the subgroup generated by \( S \), and \( \Gamma' \) the full sub-
graph on $H$, then $\Gamma'$ is open (there are no edges with just one end in $\Gamma'$) and connected (any word $e_s e_s^{-1} e_s^{-1}$ in $S = S^{-1}$ defines, in an obvious way, a path in $\Gamma'$ joining 1 to the element in $H$ it represents). Now (iii) follows.

Finally if $\Gamma$ is a tree, it is contractible; as $G$ acts freely on $\Gamma$, we have an isomorphism of $G$ on $\pi_1(G, \Gamma, \star)$. But $G \Gamma$ is the graph with one vertex and edges labelled by $S$, so its fundamental group is the free group $F(S)$. Moreover the composed isomorphism $F(S) \to G$ takes $s \in F(S)$ to the class of the loop 's'; lifting this, we get the edge of $G$ from 1 to $s$, which corresponds to $s \in G$. This argument is reversible: if $S$ freely generates $G$, then $G \cong F(S) \cong \pi_1(G, \Gamma, \star)$, so $\Gamma$ is the universal cover of $G \Gamma$, hence contractible -- i.e. a tree.

Now suppose given a graph $G$ of groups, with vertices $v$ and edges $e$ corresponding to groups $G_v, G_e$ with $G_e = G_v$, and injections $a(e) : G_e \to G_v$. As before, we choose a corresponding graph of connected spaces $X_v, X_e$ with total space $X_{\Gamma}$ and fundamental group $G_{\Gamma}$. Since by (3, b) the natural maps $G \to G_{\Gamma}, G_e \to G_{\Gamma}$ are injective, the universal cover $\tilde{X}_{\Gamma}$ is a union of copies of the universal covers $\tilde{X}_v, \tilde{X}_e \times 1$.

In $\tilde{X}_{\Gamma}$, identify each copy of $\tilde{X}_v$ to a point, and each copy of $\tilde{X}_e \times 1$ to a copy of 1, giving a quotient space $Z$ with projection $\nu : \tilde{X}_{\Gamma} \to Z$. Clearly $Z$ is a graph. We define $J : Z \to X_{\Gamma}$ by first choosing for each vertex (edge) of $Z$ a point $V(E)$ in the corresponding copy of $\tilde{X}_v$ ($\tilde{X}_e$). Then divide each edge of $Z$ into three parts: $J$ maps the midle part to $E \times 1$; and the end parts to paths in the connected space $\tilde{X}_e \times 1$, joining the corresponding points $E, V$. Clearly $\nu = J$ is homotopic to the identity, so $Z$ is connected and simply-connected, and hence is a tree.

The map $\nu$ is compatible with the natural action of $G_{\Gamma}$ on $\tilde{X}_{\Gamma}$, so we inherit an action of $G_{\Gamma}$ on $Z$. This action has no inversions. The isotropy group of each vertex (obtained from collapsing a copy of $\tilde{X}_v$) is a conjugate of $G_v$, and of an edge (collapsed from $\tilde{X}_e \times 1$) is a conjugate of $G_e$. As $G_{\Gamma}$ acts without inversions, $G_{\Gamma} \times Z$ is also a graph, and in fact coincides with the geometric realization $|\Gamma|$ of the original graph $\Gamma$: there is an obvious map onto $|\Gamma|$ for each vertex (edge) of $|\Gamma|$. Determining $X_v(\tilde{X}_v \times 1)$ in $X_{\Gamma}$, hence a collection of copies of $\tilde{X}_v(\tilde{X}_v \times 1)$ in $X_{\Gamma}$, transitively permuted by $G_{\Gamma}$, hence a single vertex (edge) of $G_{\Gamma} \times Z$. Thus we recover the original graph $G$ of groups from the action of $G_{\Gamma}$ on $Z$. There is a slight problem here: how to choose the subgroups within their conjugacy classes to obtain the desired inclinations. This will be dealt with below.

We now start from an action of a group $G$ on a tree $Y$, having no inversions, and show how to construct a graph of groups $G_{\Gamma}$, $G_{\Gamma}$. Choose a connected CW complex $U$ with fundamental group $G$: then $U$ is simply connected and $G$ acts freely on it, hence also (diagonally) on $U \times Y$. Consider the quotient $X = U \times Y$ and the projection $X = G(\overline{U \times Y}) \to G(Y) = \Gamma$, say.

Since $G$ acts without inversions, $\Gamma$ is a graph and each vertex (edge) of $X$ projects isomorphically onto one of $\Gamma$. For a vertex $v$ (edge $e$) with isotropy group $G_v(G_e)$ in $G$, we see that $G(\overline{U \times v}) (G(\overline{U \times e}))$ has fundamental group $G_v(G_e)$. Thus $X$ has the structure of a graph $G$ of connected spaces realizing a graph $G$ of groups. Since $G$ acts freely on the 1-connected space $(U \times Y)$, we have $G \cong \pi_1(X)$ the fundamental group of $G$. Observe that $U$ plays no essential role in the construction of $G$, which could be expressed purely algebraically except for a certain vagueness about conjugates which will be considered below.

In order to deal with the points at the end of the two preceding paragraphs, and also to obtain a more precise formulation of the result which can be used for explicit calculation with words, we must now consider base points. The usual procedure with a CW-complex $K$ is to choose a maximal tree $T$ in the 1-skeleton $K^{(1)}$ (a graph). This is contractible and contains all the vertices. Hence $K = K(T)$ is a homotopy equivalence, and $K(T)$ a complex with only one vertex, which we take as base point. Thus the edges of $K$ not in $T$ give generators of $\pi_1(K)$.

Now let $G$ act without inversions on a (connected) graph $Y$, so that $X = G(Y)$ is also a graph; let $T$ be a tree in $X$ containing the vertex $v$, and let $\nu : Y \to \nu$ be over $v$. 

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Proposition 4.2. There is a lifting \( j : T \to Y \) of the inclusion of \( T \) in \( X \); moreover, we can take \( j(v) = \bar{v} \).

Proof. Applying Zorn's Lemma, we see that there is a maximal pair \((T', v') : T' \to Y\) over the inclusion (with \( j'(v') = \bar{v} \)). If \( T' = T \), let \( w \) be a vertex of \( T - T' \); since \( T \) is connected, we can join \( w \) to \( v \) by an edge path, and at least one edge in the path, say \( e \), has one vertex \( v_0 \) which is in \( T' \) and one which is not. Now \( e \) is the image of an edge \( \bar{e} \) of \( Y \), one of whose vertices \( \bar{v}_0 \) lies over \( v_0 \). As \( \bar{v}_0 \) and \( j(v_0) \) lie over \( v_0 \), they are equivalent under \( G \). If \( g_0 \bar{v} = j(v_0) \), we can extend \( j' \) over \( e \) by setting \( j(e) = g_0 \bar{e} \). This contradicts the supposed maximality and proves the result.

Returning now to the action of \( G \) on the tree \( Y \), we choose a maximal tree \( T = G \backslash Y \), a lifting \( j : T \to Y \) with \( j(T) = \bar{T} \), and use these as 'extended base points'. Over each vertex \( v \) of \( T \) there is just one vertex \( \bar{v} \) of \( T \); we define \( G_v \) as the stabiliser of \( \bar{v} \). If \( \bar{e} \) is an edge of \( \bar{T} \), we have the edge \( e = j(\bar{e}) \), and call its stabiliser \( G_e \). For each edge \( e \) of \( T \) we choose an edge \( \bar{e} \) of \( Y \) over \( e \) with \( \bar{e} = (\bar{e}, e') \), and an element \( g_e \in G \) such that \( \bar{e} = g_e \bar{e} = (\bar{e}, e') \), and define \( G_e \) to be the stabiliser of \( \bar{e} \). The map \( \alpha_e(\bar{e}) \) is the inclusion map; \( \alpha_e(e) \) is induced by conjugation by \( g_e \). Note that we have implicitly chosen \( e \) from the pair \( (\bar{e}, e) \) (one could set \( \bar{e} = g_e^{-1} e, \ e = e' \), but then would have \( g_e \neq G_e \)).

We have considered two constructions above. Given a graph \( G \) of groups, realised by a graph \( X \) of spaces, we defined a quotient \( Z \) of \( \tilde{X} \), proved it to a tree, and obtained an action of \( G_T \) on it. Conversely, given an action of \( G_T \) on a tree \( Y \) we defined (following (4.2)), using certain choices (maximal tree \( T \), liftings \( \bar{T} \), elements \( g_0 \)) a graph of groups over \( \Gamma = G \backslash Y \). The key result of the theory is

Theorem 4.3. These two constructions are mutually inverse up to isomorphism and (for graphs of groups) replacing the \( \alpha_e(e) \) by conjugate homomorphisms.
involving cancellation arguments.

We now note some special cases of the theorem.

**Corollary 4.4.** Let \( G \setminus Y \) consist of a single edge \( Y \). Then \( G \cong G_v \times_{H_v} G_Y \).

**Corollary 4.5.** Let \( G \setminus Y \) consist of a single loop. Then \( Y \) contains the edge \( \varepsilon \) and \( G \cong G_v \times_{H_v} G_Y \).

A somewhat different application arises from considering free actions. Of course, if \( G \) acts freely on a tree, it is free. The theorem allows us to write down a set of free generators. Suppose in particular that \( F \) is the free group on a set \( S \), and \( G \) a subgroup of \( F \). Write down the graph \( \Gamma_G \) for \( F \): by (4.1) it is a tree, and the action of \( F \) on it induces a free action of \( G \).

Identify the vertices of \( \Gamma_G \) with the corresponding elements of \( F \). Then the path in the tree joining 1 to the element with reduced word \( t_1 \cdots t_n \) (where each \( t_i \in S \cup S^{-1} \)) goes through the successive vertices \( t_1, t_1t_2, t_1t_2t_3, \ldots \). Thus if \( T \subset \Gamma_G \) contains 1, it contains the initial segments of the vertices of \( T \).

Since the action of \( G \) is free, the lift of an edge of \( G \setminus \Gamma_G = X \) is determined by its initial vertex. Thus the preferred lifts of the edges of \( X - T \) are just those edges of \( \Gamma_G \) whose initial vertex is in \( T \) but terminal vertex is not. Applying the theorem we have

**Proposition 4.6.** The left cosets of \( G \) in \( F \) are canonically represented by the set \( R \) of vertices of \( T \); if a reduced word \( w = uv \) belongs to \( R \), so does \( u \). If \( W = \{(t, s) \in R \times S : t \leq R \} \) and for each \( w = (t, s) \in W \) we write \( t = x_w^w u_w \) with \( g_w \in G, u_w \in R \) then \( \{g_w : w \in W\} \) is a free basis of \( G \).

We conclude this section with brief mentions of two alternative approaches. The first follows a paper of Serre [23]. We say that \( G \) has property (FA) if for any action of \( G \) on a tree \( Y \), there is a fixed point of \( G \) in \( Y \).

**Theorem 4.7.** \( G \) has (FA) if and only if \( G \) is (i) unsplittable, and (ii) not a union of an increasing sequence of subgroups.

Note that for countable \( G \), (ii) is equivalent to being finitely generated.

**Proof.** (FA) \( \Rightarrow \) (i) by Corollaries 4.4 and 4.5: any decomposition induces an action without a fixed point. As to (ii), if \( G = vG_n \) with \( G_n \subset G_{n+1} \), we form a graph with vertices \( v(G/G_n) \) and for each vertex \( g_n \) an edge joining it to \( g_n+1 \). It is immediate that this is a tree, and that the natural action of \( G \) on it has no fixed point.

Conversely if (i) and (ii) hold and \( G \) acts on \( Y \), \( G \) is the universal group of the graph of groups \( G \setminus Y \). By (ii), \( G \) is also the universal group of a finite subgroup. If this subgroup is not a tree, \( G \) is splittable; if the subgroup is a tree, we still have a splitting unless \( G \) coincides with one of the vertex groups - i.e., has a fixed point in \( Y \).

Some interesting examples of the above are given in Serre's paper [23] and several more in his monograph [22, Chapter 6].

We conclude this paragraph by mentioning length functions. These were introduced by Lyndon [34] to permit inductive arguments: they consist of an axiomatic generalisation of the length of a reduced word as in 1.4, 1.6 or 1.7 above. It was shown by Chiswell [3] however that every function satisfying the axioms defines an action on a tree and hence comes from a decomposition of \( G \) as fundamental group of a graph of groups.

Thus here we have a further equivalent concept.

5. ENDS

The definition of ends, and construction of the end point compactification (for a peripherally compact space) was achieved by Freudenthal in 1931 [9], and the application to group theory initiated by himself [10], [11], Hopf [14] and Specker [24]. We present a somewhat simplified version, adapted to the present applications.

Let \( X \) be a locally finite simplicial complex. For each finite subcomplex \( K \), the number of connected components of \( X - K \) is finite; denote by \( n(K) \) the number of infinite ones (equivalently, having noncompact closure in \( X \)). Now define the number of ends \( e(X) = \sup n(K) \). Clearly \( e(X) = 0 \iff X \) is finite; otherwise \( e(X) \) is a positive integer or \( + \infty \).
If \( X = \mathbb{R} \) and \( K \) is a point, clearly \( n(K) = 2 \). On the other hand, any compact \( K \) is contained in a closed interval \( J : \mathbb{R} - J \) has only two components, and a component of \( \mathbb{R} - K \) meeting neither is contained in \( J \), hence finite. Thus \( \epsilon(\mathbb{R}) = 2 \). Similarly, since the complement of a (large) disc is connected, \( \epsilon(\mathbb{R}^2) = 1 \) for any \( n \geq 2 \).

As \( X \) is locally finite, for any finite \( K \) the (open) subcomplex \( s(K) \) consisting of all simplices with a vertex in \( K \) is finite, and clearly \( n(s(K)) = n(K) \). Now any point of \( X - (st K) \) can be joined by a path avoiding \( st K \) to a vertex not in \( K \), and if two such vertices can be joined by a path avoiding \( K \), as none of the vertices of the simplices met by the path are in \( K \), we can find a path along edges not in \( st K \). It follows that in computing \( \epsilon(X) \) we may ignore all simplices of dimension \( > 1 \), and work in the 1-skeleton. This can now be formalized. The cochain complex \( C^*(X) \) of \( X \) (coefficients \( \mathbb{Z}_2 \) = integers mod 2 understood) contains a subcomplex \( C^*_f(X) \) of cochains with finite support. Note: the fact that \( C^*_f(X) \) is closed under the coboundary follows from local finiteness of \( X \). Write \( C^*_f(X) \) for the quotient complex, and \( H^*_f(X), \bar{H}_f(X) \) for the cohomology groups of \( C^*_f(X), C^*_f(X) \). Then the short exact sequence \( 0 \rightarrow C^*_f(X) \rightarrow C^*(X) \rightarrow C^*_f(X) \rightarrow 0 \) induces a long exact sequence of cohomology groups.

Our interest in these comes from

**Proposition 5.1.** \( \epsilon(X) \) is the dimension of \( H^*_f(X) \) over \( \mathbb{Z}_2 \).

**Proof.** Observe that \( H^*_f(X) = \delta^{-1}(C^*_f(X))/C^*_f(X) \) is the quotient of 0-cochains with finite coboundary by finite 0-cochains. Also, by the above, we may suppose \( X \) 1-dimensional.

Now if the 0-cochains \( c_1, \ldots, c_n \) define linearly independent elements of \( H^*_f(X) \), as each \( \delta c_i \) is finite we can choose a finite subcomplex \( K \) containing the supports of all \( \delta c_i \). But then for each edge \( e \) not in \( K \), each \( c_i \) takes the same value at both ends of \( e \). Thus for each connected component \( A \) of \( X - K \), each \( c_i \) takes a constant value \( c_i(A) \) on the vertices of \( A \). If there were only \( r \leq n \) infinite components \( A_i \), there would be a nontrivial linear relation \( \sum \lambda_i c_i(A) = 0 \) holding for all such \( A_i \). But then \( \sum \lambda_i c_i \) would be a finite cochain, contradicting our choice. Hence \( n = \dim H^*_f(X) \) implies \( \epsilon(X) = n \).

Conversely, if \( \epsilon(X) \geq n \), we choose \( K \) finite with \( n(K) = n \), and let \( A_1, \ldots, A_n \) be distinct infinite components of \( X - K \). Define the cochain \( c_i \) to take the value 1 on vertices of \( A_i \), 0 otherwise. Then if \( \delta c_i(e) = 1 \), \( e \) has one end in \( A_i \), the other not (and hence in \( K \)), so \( e \) is one of the finitely many edges of \( K \). So each \( \delta c_i \) is finite and, by construction, the \( c_i \) are independent modulo finite cochains. Hence \( \dim H^*_f(X) = n \).

We next construct another theory, analogous to the above. For any group \( G \), let \( PG \) be the power set of all subsets. Under Boolean addition ('symmetric difference') this is an additive group of exponent 2. Write \( FG \) for the additive subgroup of finite subsets. Now define

\[ \mathcal{QG} = \{ A \subseteq G : \forall g \in G, A + Ag \text{ is finite} \} \]

We refer to two sets \( A \) and \( B \) whose difference lies in \( FG \) as almost equal, and write \( A \approx B \). This amounts to equality in the quotient group \( PG/FG \). Moreover \( G \) acts by right translation on these groups, and \( QG/FG \) is the subgroup of elements invariant under this action. Elements of \( QG \) are said to be almost invariant. We define the number of ends of \( G \) to be

\[ e(G) = \dim_{\mathbb{Z}_2}(QG/FG) \]

If \( G \) is finite, all subsets are finite and clearly \( e(G) = 0 \). Otherwise, \( G \) is an infinite set which is invariant (not merely 'almost'), so \( e(G) = 1 \).

For finitely generated groups \( G \) we can identify these two definitions as follows. Choose a finite set \( S \) of generators, and form the Cayley graph \( \Gamma_S = \Gamma(S, G) \). Clearly this is locally finite.

**Proposition 5.2.** \( e(G) = e(\Gamma_G) \).

**Corollary 5.3.** \( e(\mathbb{Z}) = 2 \). For \( (1) \) generates \( \mathbb{Z} \), and the corresponding graph is homeomorphic to \( \mathbb{R} \).

**Proof.** We can identify the vertices of \( \Gamma_G \), with elements of \( G \), and hence \( C^*_f(\Gamma_G) \) with \( PG \) and \( C^*_f(\Gamma^G) \) with \( FG \). What we have to show,
then, is that if the 0-cochain $c$ corresponds to the subset $A$, then
\[ \delta c \text{ is finite } \iff A \subset QG. \]

Now $\delta c$ is supported by the set of edges $(g, gs)$ ($g \in G$, $s \in S$) with just one end in $A$. For fixed $s$, this means that $g$ belongs to just one of $A$, $A^{-1}$; i.e. $c \in A + A^{-1}$. If $A$ is almost invariant, for each $s$, we have finitely many $g$, hence a finite number of edges in total. Conversely if $\delta c$ is finite, the class of $A$ in $PG/\Gamma$ is invariant under each $s^{-1}$ ($s \in S$), hence under the group $G$, which they generate.

This connection can now be extended.

**Theorem 5.4.** Let $G$ act freely on the connected complex $X$, with finite quotient $K$ (equivalently, $X \twoheadrightarrow K$ is a connected regular covering, with group $G$). Then $e(G) = e(X)$.

**Proof.** As before, we may suppose $X$ a graph by ignoring cells of dimension $> 1$. Let $T$ be a maximal tree in $K$, $T$ a lift in $X$. The trees $gT$ ($g \in G$) are all disjoint; if we identify each to a point (obtaining $Y$, say) $H^0(X)$ is unaltered. For if $c$ has finite coboundary, it is constant on all but finitely many $gT$; hence almost equal to a $c'$ which is constant on each. The natural map $C^0Y \twoheadrightarrow C^0X$ preserves the subgroups of finite cochains and of cochains with finite coboundaries, hence induces $H^0(Y) \twoheadrightarrow H^0(X)$. This is clearly injective, and the above observation proves it surjective.

We may thus suppose $K/T$ has only one vertex. But now $Y$ can be identified with a suitable graph $\Gamma_G$, and the result follows from (5.2).

**Corollary 5.5.** If $G$ acts freely on $\mathbb{R}^n$ with compact quotient, e.g. if $G = Z^n$, we have $e(G) = 1$.

The connection with topology is valid only for finitely generated $G$. However, an interpretation in terms of group cohomology can always be given. For any $G$, $H^0(G; PG) \cong Z_2$ $(n = 0)$, $0$ $(n > 0)$. Moreover $FG$ can be identified with the group ring $Z_2G$. The invariant subgroup $QG/\Gamma_G = H^0(G; PG/\Gamma_G)$, and for $G$ infinite, since $H^0(G; Z_2G) = 0$, we deduce that $e(G) = 1 + \dim H^0(G; Z_2G)$.

We now begin work on calculating the number of ends of various groups. We start with lemmas noting the invariance of $e(G)$ under commensurability and isogeny.

**Lemma 5.6.** If $H$ is a subgroup of finite index in $G$, $e(G) = e(H)$.

**Proof.** If $G$ is a finitely generated, we may use (5.2) and observe that $H$ acts freely on $\Gamma_G$ with finite quotient (a covering of $G/\Gamma_G$ of degree $|G: H|$).

In general, if $A \subset G$ is almost invariant in $G$ so is $A \cap H$ in $H$. For $(A \cap H) + (A \cap H) = (A + Ah) \cap H$ is finite, for any $h \in H$. Thus intersection induces a homomorphism $QG/\Gamma_G \rightarrow QH/\Gamma_H$. Choose a left transversal $T$ for $H$ in $G$.

Now $\phi$ is injective, for if $A \cap H$ is finite so is each $Ag \cap H$, hence $A \cap BH^{-1}$. Letting $g$ run through the finitely many elements of $T^{-1}$, we deduce $A$ is finite.

And $\psi$ is surjective, for if $B \subset H$ is almost-invariant consider $A = BT$. Certainly $A \cap H = B$. For any $g \in G$, $t \in T$, write $tg = h \sigma$ ($s \in S$). Then $A + Ag = Z(BT + Bt) = Z(BT + B\sigma)$. But $B\sigma$ is almost equal to $B$, and $s$ runs through $T$ as $t$ does. Hence $A + Ag$ is finite.

**Lemma 5.7.** If $K$ is a finite normal subgroup of $G$,
\[ e(G) = e(G/K). \]

**Proof.** Write $p : G \to G/K$ for the natural map and $p \circ : PG \to PG/K$, $p^{-1} : P(G/K) \to PG$ for the direct and inverse image maps induced by $p$. Then $p \circ p^{-1} = B$ for any $B \subset G/K$, while for $A \subset G$, $p^{-1}pA = AK$. Trivially $B$ is almost-invariant $\iff p^{-1}B$ is, and if $A$ is almost-invariant it is almost equal to $AK$, so $pA$ is almost invariant. Hence $p^{-1}$ preserves the subgroups $Q$ and $F$ and the induced maps between $QG/\Gamma_G$ and $Q(G/K)/\Gamma(G/K)$ are two-sided inverses, hence isomorphisms.

We now come to the main result of this section.
Theorem 5.8. Suppose \( G \) is finitely generated, \( A \in \text{QG} \) such that both \( A \) and \( A^- (G - A) \) are infinite, and that \( H = \{ h \in G : hA = A \} \) is infinite. Then \( G \) has an infinite cyclic subgroup of finite index.

Since the left translations on \( PG \) commute with the right, there is an induced action of \( G \) on the left on \( \text{QG/FG} \). As \( H \) is the stabilizer of \( A \) for this action, it is a subgroup.

Corollary 5.9. If \( G \) is finitely generated, \( e(G) = 0, 1, 2 \) or \( \infty \).

For suppose \( e(G) \neq 0, 1 \) or \( \infty \). Then \( G \) is infinite (\( e(G) \neq 0 \)) and acts on the finite group \( e(G) = 1 \) group \( \text{QG/FG} \). As \( e(G) = 1 \), we can find an \( A \) as above; the isotropy group \( H \) has finite index in \( G \), so \( G \) is infinite. Then by (5.9) there is a subgroup \( Z \) of finite index which is infinite cyclic, by (5.6) \( e(G) = e(Z) \) and by (5.3), \( e(Z) = 2 \).

We also have a characterization of groups with \( 0 \) ends (finite) or \( 2 \) ends (finite extensions of \( Z \)). Our next main objective will be a study of groups with \( \alpha \) ends. The restriction to finitely generated groups is not essential: the result of Corollary 5.9 is proved in Cohen's book [5] for groups which are not locally finite: he also shows that a countable locally finite group has \( \alpha \) ends, and (see Godby [31]) an uncountable one also has \( 1 \) or \( \infty \).

We begin the proof of (5.8) with a lemma, which (together with the corollary) will also be repeatedly used in chapter 6.

Lemma 5.10. Let \( A, A^- \in \text{QG} \). For all \( g \in A \), either \( gA \subset A \) or \( gA^- \subset A \).

Proof. Choose a finite set \( S \) of generators of \( G \), and use (as in (5.2))) the action of \( G \) on \( \Gamma_G \). Pick connected finite subgraphs \( C_i \) of \( \Gamma_G \) containing \( A \).

For each vertex \( c \) of \( C_i \), \( gc \in A \) for all \( g \in A \). As \( C_i \) is finite, \( gC_i \cap C_i = \emptyset \) for all \( g \in G \). Hence for all \( g \in A \), we have \( gC_i \cap C_i = \emptyset \), and \( gc \in A \) for each vertex \( c \) of \( C_i \).

For any collection \( A \) of vertices of \( G \), let \( \hat{A} \) denote the maximal subgraph of \( G \) with vertex set equal to \( A \). Each component \( E \) of \( \hat{A} \) contains a vertex of \( C_i \), so \( gE \) meets \( A \); if it also meets \( A^- \), \( gE \) meets \( C_i \). But \( C_i \) is connected and disjoint from \( gC_i \), so lies in a single component \( gE \). Thus \( A^- \) cannot meet both \( gA \) and \( gA^- \).

Corollary 5.11. If \( A, A^- \in \text{QG} \), then for all \( g \in G \) one (at least) of \( gA \subset A \) or \( gA^- \subset A^- \) holds.

Proof of 5.8. Interchanging \( A, A^- \) if necessary, we may assume \( H \cap A \) is finite. We may also adjoint \( 1 \) to \( A \). By the lemma, for almost all \( g \in A \) either \( gA \cap A^- = \{ 1 \} \) or \( gA^- \subset A \). Hence we can choose \( c \in H \cap A \) satisfying one of these: necessarily \( cA \subset A^- \). We will show that \( c \) generates the required subgroup.

If \( n > 0 \), \( c^nA \cap cA = A \). Thus \( c^n = 1 \), so \( c \) has infinite order. As \( 1 \in A \), \( c^nA \) for all \( n > 0 \), and as \( c^nA \subset A \), \( c^nA \subset A \) for all \( n > 0 \), we have \( c^nA \subset A \) for all \( n > 0 \).

If \( d \in n \{ c^nA : n > 0 \} \), then \( c^nA \subset cA \subset A \), contradicting the fact that \( Ad^{-1} + A \) is finite, and all the \( c^nA \) distinct. Hence \( n \{ c^nA : n > 0 \} = \infty \).

\[ A = \bigcup \{ c^nA : n > 0 \} \]
\[ = \bigcup \{ c^n(A - cA) : n > 0 \} \]

is contained in the union of finitely many (right) cosets of \( \langle c \rangle \) in \( G \). Recall that \( c \in H \), so \( A - cA \) is finite. The same holds for \( A^- \) (replacing \( c \) by \( c^{-1} \)). Hence the infinite cyclic subgroup \( \langle c \rangle \) has finite index in \( G \).

Alternate proof of 5.8. As before, we may assume that \( H \cap A \) is infinite. Lemma 5.10 and its proof tell us that for almost all \( g \in A \), \( gA \cap A \) is empty and either \( gA \subset A \) or \( gA^- \subset A^- \). Hence there is an element \( c \) of \( H \cap A \), such that \( gA \cap A \) is empty and either \( gA \subset A \) or \( gA^- \subset A^- \). As \( cA \) is almost equal to \( A \), we must have \( cA \subset A \) and the inclusion must be strict, as \( cA \cap A = \emptyset \). Let \( B = A + cA \). Then \( B \) is non-empty, finite and \( B \subset A \). Further for any two integers \( r, s \), with \( r > s \), we have \( c^rB \cap c^sB = \emptyset \). For \( c^rB \subset c^sB \subset c^{r-s}B \subset B \), and \( c^rB \cap c^{r-s}B \subset c^{r-s}A \subset A \). Thus \( c^rB \cap c^{r-s}B = \emptyset \).
Now consider \( \sum_{n \in \mathbb{Z}} c^n \mathbb{B} \), which equals \( \bigcup_{n \in \mathbb{Z}} c^n \mathbb{B} \) by the above. As \( \delta \) is additive, we have \( \delta \left( \sum_{n \in \mathbb{Z}} c^n \mathbb{B} \right) = \sum_{n \in \mathbb{Z}} (c^{n+1} \alpha + c^{n+1} \alpha) = 0 \). (Note that these infinite sums make sense.) Thus \( \bigcup_{n \in \mathbb{Z}} c^n \mathbb{B} \) must equal \( c \mathbb{B} \), and so the cyclic subgroup of \( G \) generated by \( c \) has index equal to the order of \( B \). As \( G \) is infinite, \( c \) must have infinite order and the result follows.

One can also give a more direct proof of the result (5.9) that a finitely generated group must have \( 0, 1, 2 \) or \( \infty \) ends. Let \( G \) be a finitely generated, infinite group and suppose that \( s(G) \) is a positive integer \( n \). Choose a finite generating set for \( G \) and let \( G \) be the corresponding graph. \( G \) acts on \( \Gamma \) from the left with finite quotient. Let \( L \) be a finite connected subgraph of \( \Gamma \) such that \( G - L \) consists of \( n \) infinite components \( V_1, \ldots, V_n \). As \( G \) is infinite, there exists \( g \in G \) with \( gL \cap L = \emptyset \). Thus \( gL \) lies in one of the \( V_i \). \( V_i \) is an exact one of the components of \( G - L \) is infinite, for \( G - (L \cup gL) \) has only \( n \) infinite components. Now \( L \cup V_1 \cup \cdots \cup V_n \) is connected, so that \( G - L \) has at most two infinite components. As \( g \) is a homeomorphism, \( G - L \) must have at most two infinite components and this proves the required result.

We finish this section by giving some more information about groups with two ends.

**Theorem 5.12.** The following conditions on a finitely generated group \( G \) are equivalent:

(i) \( s(G) = 2 \),

(ii) \( G \) has an infinite cyclic subgroup of finite index,

(iii) \( G \) has a finite normal subgroup with quotient \( \mathbb{Z} \) or \( \mathbb{Z} \times \mathbb{Z} \),

(iv) \( G = F \ast F \) with \( F \) finite, or \( G = A \ast p B \) with \( F \) finite and \( |A : F| = |B : F| = 2 \).

**Proof.** (i) \( \Rightarrow \) (ii) by Theorem 5.8, for \( H \) will have index at most 2 in \( G \).

(ii) \( \Rightarrow \) (i) by Lemma 5.6 and the fact that \( s(\mathbb{Z}) = 2 \).

(iii) \( \Rightarrow \) (iv) If \( F \) is a finite normal subgroup of \( G \) with quotient \( G \), then \( G = F \ast F \). If the quotient is \( \mathbb{Z} \times \mathbb{Z} \), then \( G = A \ast_p B \), where \( A \) and \( B \) are the inverse images of the \( \mathbb{Z} \)-factors. Thus \( |A : F| = |B : F| = 2 \) as required.

(iv) \( \Rightarrow \) (iii) If \( G = F \ast F \) with \( F \) finite, then both inclusions of \( F \) in \( G \) must be isomorphisms and so \( F \) is normal in \( G \) with quotient \( \mathbb{Z} \).

Finally, we prove (ii) \( \Rightarrow \) (iii), to complete the theorem.

First, \( G \) must contain an infinite cyclic subgroup \( K \) of finite index which is also normal in \( G \). One takes for \( K \) the intersection of all the conjugates of the original infinite cyclic subgroup. Let \( H \) denote the centralizer of \( K \) in \( G \). Thus \( |G : H| = 2 \). \( H \) is finitely generated and its centre is a subgroup of finite index. A theorem of Schur (see e.g. W. R. Scott, Group Theory, Prentice-Hall, 1964, §15.1.13) tells us that \( H \), the commutator subgroup of \( G \), is finite. Now \( H/ \mathbb{Z} \) must have rank 1, and so there is an epimorphism \( \phi : H \to \mathbb{Z} \) with finite kernel \( L \). If \( G = H \), our result is proved. Otherwise observe that \( H \) is normal in \( G \) and \( L \) is characteristic in \( H \), as \( L \) is the torsion subgroup of \( H \). Thus \( L \) is normal in \( G \) and we have the exact sequence

\[ 1 \to H/L \to G/L \to \mathbb{Z} \to 1. \]

We know that \( G/L \) must be non-abelian, and therefore \( G/L \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). This completes the proof of Theorem 5.12.

**6. THE STRUCTURE THEOREM FOR GROUPS WITH INFINITELY MANY ENDS**

The aim of this section is to describe which finitely generated groups have infinitely many ends. The nearest formulation of the result includes the case of two ends.
Theorem 6.1. If \( G \) is a finitely generated group, then \( e(G) = 2 \) if and only if \( G \) splits over a finite subgroup.

Remark. Theorem 5.12 tells us that \( e(G) = 2 \) if and only if either \( G = F * F \) with \( F \) finite or \( G = A * B \) with \( F \) finite and \( |A : F| = |B : F| = 2 \).

This remarkable result is due to Stallings [37], [26], but our treatment of the proof is an amalgam of results of Cohen [5], Dunwoody [7] and Stallings [26].

There is a close connection between Theorem 6.1 and the Sphere Theorem. In fact, Stallings discovered the result by considering the proof of the Sphere Theorem due to Papakyriakopoulos [19] and Whitehead [29].

Sphere Theorem. If \( M \) is an orientable 3-manifold with \( \tau_2(M) \neq 0 \), there is an embedded 2-sphere \( S \) in \( M \) which represents a non-trivial element of \( \tau_2(M) \).

Let \( M \) be a closed orientable 3-manifold with fundamental group \( G \). One can show easily (see below) that the hypothesis that \( \tau_2(M) \) is non-zero is equivalent to asserting that \( e(G) = 2 \). Also the conclusion of the Sphere Theorem implies that \( G \) splits over the trivial subgroup. Thus the Sphere Theorem is extremely like Theorem 6.1, when \( M \) is a closed manifold. Further, it is possible [26] to give a proof of the Sphere Theorem which uses Theorem 6.1.

The reason why \( \tau_2(M) \neq 0 \) if and only if \( e(G) = 2 \) is as follows. Let \( M \) denote the universal covering space of \( M \). Then Theorem 5.4 tells us that \( e(G) = e(M) \). For these purposes it will be convenient to use coefficients \( Z \), not \( Z_2 \), when defining the groups \( H^n(M), H^n_1(M), H^n_e(M) \). The natural analogue of Proposition 5.1 is that \( e(M) \) equals the rank of \( H^n_e(M) \), where the rank of an abelian group is defined to be the maximal rank of all finitely generated free abelian subgroups. If this maximum does not exist, then consider the long exact sequence connecting the groups \( H^n(M), H^n_1(M), H^n_e(M) \). This begins

\[
H^n_e(M) \rightarrow H^n(M) \rightarrow H^n_0(M) \rightarrow H^n_1(M) \rightarrow H^n(M) \rightarrow \ldots
\]
subgroups of free groups are free.

We now come to the proof of Theorem 6.1 and we start with the easy half.

**Lemma 6.3.** If $G$ splits over a finite subgroup, then $e(G) \geq 2$.

**Remark.** For this result, $G$ need not be finitely generated.

**Proof.** It suffices to produce an almost invariant subset $E$ of $G$ such that $E$ and $E^*$ are infinite.

First suppose that $G = A \ast_C B$, where $C$ is finite, and recall the canonical form for elements of $G$ given by Theorem 1.6. One chooses based transversals $T_A$ and $T_B$ for $C$ in $A$ and $B$ and obtains the form $a_1 b_1 \ldots a_n b_n c$ for any element of $G$, where $c \in C$, $a_1 \in T_A$, $b_1 \in T_B$, and $a_1 = 1 	o i = 1$, $b_1 = 1 	o i = n$. Let $E$ be the subset of $G$ consisting of elements for which $a_1$ is non-trivial. Clearly $E$ and $E^*$ are infinite. (This uses the fact that $A \ast_C B$.) If $b \in B$, then $E_b = E$. If $a \in A$, then $E_a \subseteq E \cup C$ and so also $E_a^{-1} \subseteq E \cup C$. Hence

$$E \subseteq E_a \cup E_b \cup E \subseteq E \cup C$$

so that $E \cong E_a$. As $A$ and $B$ together generate $G$, we have $E_g \cong E$ for all $g \in G$.

Secondly suppose that $G = A \ast_C$, where $C$ is finite, and recall the canonical form for elements of $G$ given by Theorem 1.7. One chooses based transversals $T_1$ of $a_1(C)$ in $A$ and obtains the form $a_1 e_1 a_1^{-1} a_1 e_2 a_1^{-1} \ldots a_1 e_n a_1^{-1},$ where $a_{n+1} \in A$, $a_1 \in T_1$ if $e_1 = 1$, $a_1 \in T_2$ if $e_1 = -1$, and moreover $a_1 \not\in E$ if $e_{i-1} \not= e_i$. Let $E$ be the subset of elements of $G$ for which $a_1$ is trivial and $e_1 = 1$. If $a \in A$, then $E_a \subseteq E$. Also $E \subseteq E$ and $E_a^{-1} \subseteq E \cup a_1(C)$. Therefore, as before, $E$ is almost invariant in $G$. This completes the proof of Lemma 6.3.

The hard part of Theorem 6.1 is the result that a finitely generated group $G$ with infinitely many ends must split over a finite subgroup. Our aim, following Dunwoody [7], is to produce a tree $T$ on which $G$ acts with quotient a single edge. We start by considering graphs and trees in more detail than before. Let $\Gamma$ be an abstract graph.

**Definition.** An edge path in $\Gamma$ is a sequence $e_1, \ldots, e_n$ of edges such that $e_i e_{i+1} = e_{i+1} e_i$, for $i = 1, 2, \ldots, n-1$. If $e$, $f$ are edges of $\Gamma$, we will write $e \leq f$ if there is an edge path with $e_1 = e$ and $e_n = f$.

The relation $\leq$ has the following properties.

(A) For any graph $\Gamma$, the relation $\leq$ is reflexive and transitive.

(B) For any graph $\Gamma$ and any edges $e$, $f$ of $\Gamma$, if $e \leq f$ then $\overline{f} \leq \overline{e}$.

(C) The graph $\Gamma$ is connected if and only if for any pair $e$, $f$ of edges of $\Gamma$, at least one of $e \leq f$, $e \leq f$, $\overline{e} \leq \overline{f}$ holds.

(D) If $\Gamma$ has no circuits, then for any pair $e$, $f$ of edges can we have $e \leq f$ and $f \leq e$.

(E) If $\Gamma$ has no circuits, then for no pair $e$, $f$ of edges can we have $e \leq f$ and $\overline{f} \leq \overline{e}$.

(F) If $\Gamma$ has no circuits, then for any pair $e$, $f$ of edges there are only finitely many edges $g$ with $e \leq g \leq f$.

If a relation satisfies the conditions in (A) and (D), we shall call it a partial order. Thus if $\Gamma$ is a tree, the relation $\leq$ on $E(\Gamma)$ is a partial order and the following conditions hold.

(1) If $e \leq f$, then $\overline{f} \leq \overline{e}$.

(2) If $e$, $f \in E(\Gamma)$, there are only finitely many $g \in E(\Gamma)$ such that $e \leq g \leq f$.

(3) If $e$, $f \in E(\Gamma)$, at least one of $e \leq f$, $e \leq f$, $\overline{e} \leq \overline{f}$ holds.

(4) If $e$, $f \in E(\Gamma)$, we cannot have $e \leq f$ and $f \leq e$.

**Remark.** If two of the inequalities in (3) hold, then $e = f$ or $e \leq f$.

The next step is to show that if we start with a partially ordered set $E$ satisfying all the above conditions, then we can construct a tree out of $E$.

Let $E$ be a partially ordered set with an involution $e \mapsto \overline{e}$, where $e \mapsto e$, and suppose that conditions (1)-(4) hold. Write $e < f$ if $e \leq f$ and $e \not= f$. Write $e \leq \overline{f}$ if $e < f$, and $e \leq g \leq f$ implies $g = e$ or $g = f$. We need the following technical result.

**Lemma 6.4.** The relation $\leq$ on $E$, defined by $e \leq f$ if and only if $e = f$ or $e < f$, is an equivalence relation.
Proof. The relation is obviously reflexive and is symmetric because \( e \ll f \) implies \( f \ll e \).

We suppose \( e - f \) and \( e - h \) and will show \( f - h \). The first step is to show that \( f = h \) or \( f < h \). Condition (3) implies that one of \( f \leq h \), \( f = h \), \( h < f \) holds. If \( f \leq h \), then \( h = f \) and we would have \( e \ll f = h \) which implies \( h = f \) as \( e \ll f \). If \( f = h \), then \( e < f \ll h \) implies \( h = f \) as \( e \ll h \). If \( f < h \), we would have \( e < f \ll h \). As \( e \ll h \), this contradicts (4). The only remaining possibility is \( f < h \).

Hence either \( f = h \) or \( f < h \) as required.

The last step is to suppose \( f < h \) and \( f < g \ll h \) and prove \( g = f \) or \( g = h \). One of the inequalities \( g \leq e \), \( e \ll g \), \( e < g \), \( e \ll g \) must hold. If \( e \ll g \), then \( e = h \). As \( e \ll h \), this contradicts (4). If \( e \ll g \), then \( e \ll h \), which again contradicts (4). If \( e < g \), then \( e < g \ll f \) shows that \( g = h \) as \( e \ll f \). If \( e < g \ll f \), then \( e < g \ll f \) shows that \( g = f \) as \( e \ll f \). Hence \( g = f \) or \( g = h \) as required. This completes the proof of Lemma 6.4.

We construct a graph \( \Gamma \) out of \( E \) as follows. Let \( t(e) \) denote the equivalence class of \( e \) in \( E \) under the relation \( \sim \). Let \( V = \{t(e) : e \in E\} \) and let \( \delta_a e = t(e) \). Then the sets \( E, V \) and the map \( \delta \) form an abstract graph \( \Gamma \).

Theorem 6.5. \( \Gamma \) is a tree and the order relation which \( \Gamma \) induces on \( E \) is the same as the original relation.

Proof. We will prove the second part of the theorem. The fact that \( \Gamma \) is a tree will then follow from properties (C) and (D). Lemma 6.4 tells us that for distinct elements \( e, f \) of \( E \) we have \( t(e) \neq t(f) \) if and only if \( e \ll f \). Thus \( \delta_e e = \delta_f f \) if and only if \( e \ll f \). It follows that if \( e \) and \( f \) are joined by an edge path in \( \Gamma \), then \( e = f \), and condition (2) shows that if \( e \ll f \), then \( e \) and \( f \) can be joined by an edge path in \( \Gamma \). Hence \( \Gamma \) does induce the original order relation on \( E \), as required.

Consider the following examples of Theorem 6.5 in action.

Let \( G \) be the free group of rank 2 with generators \( a, b \) and let \( \Gamma \) be the corresponding graph for \( G \). If \( e \) is an edge of \( \Gamma \) it separates \( \Gamma \) into two components. Thus \( G = A \cup A^* \) where \( \Delta A \cap \Delta A^* = e \).

Let \( E \) be the set of all subsets \( A \) of \( G \) with \( \Delta A \) equal to a single edge of \( \Gamma \). We partially order \( E \) by inclusion. Then \( E \) has an involution \( A \rightarrow A^* \) and satisfies conditions (1)-(4). The tree constructed in Theorem 6.5 is the graph \( \Gamma \) again.

One can obtain an action of \( G \) on a different tree \( T \) as follows.

Let \( A \) be the subset of \( G \) which contains \( a \) and whose coboundary is the edge \( (e, a) \) of \( \Gamma \). Let \( F \) be the set of all translates \( gA, gA^* \) of \( A \) and \( A^* \), partially ordered by inclusion. Again \( F \) satisfies conditions (1)-(4). Hence we can construct a tree \( T \) from Theorem 6.5. The natural left action of \( G \) on \( F \) induces an action of \( G \) on \( T \) which has quotient a single edge. Let \( g \) be an element of \( G \) such that \( gA = A \) or \( gA^* \). Then \( g(\Delta A) = \Delta A \) so that the edges \( (gA, ga) \) and \( (e, a) \) are equal. This is only possible if \( g = e \) so that \( G \) acts on \( T \) without inversions and the stabilizer of any edge of \( T \) is trivial.

Recall that \( t(A) \) consists of \( A \) together with every element \( B \) of \( F \) such that \( A \ll B^* \). We will call an edge of \( \Gamma \) of the form \( (g, ga) \) an \( a \)-edge, and an edge of the form \( (g, gb) \) a \( b \)-edge. If \( A \ll B^* \), then \( \Delta B \) is an \( a \)-edge of \( \Gamma \) which has no vertices in \( A \) and such that the path from \( \Delta B \) to \( \Delta A \) consists only of \( b \)-edges. It is now easy to see that

\[
\{A \} = \{b^nA : n \in \mathbb{Z}\} \cup \{b^nA^* : n \in \mathbb{Z}\}
\]

Hence the stabilizer of \( t(A) \) is the infinite cyclic subgroup \( H \) of \( G \) generated by \( b \). One can also see that \( \delta_a A = \delta_A A^* = \delta_A A \). Hence \( G \) acts transitively on the vertices of \( \Gamma \), so that \( G \Gamma \) is a loop. This action of \( G \) on \( T \) corresponds to expressing \( G \) as \( H \times \{1\} \). The graph \( T \) is obtained from \( \Gamma \) by identifying each \( b \)-edge of \( \Gamma \) to a point.

We can now sketch the proof of the main part of Theorem 6.1, i.e., if \( G \) is a finitely generated group and \( \Delta(G) = e \), then \( G \) splits over a finite subgroup. Our idea is to proceed, as in the example above, to construct a tree \( T \) on which \( G \) acts so that the stabilizer of any edge is finite and the quotient \( G \Gamma \) is a single edge. However, we need to partially order our almost invariant sets by almost inclusion and not by strict inclusion in order to be able to prove that condition (3) holds.

For any almost invariant set \( B \) of \( G \), write \( [B] \) for the set of all almost invariant sets of \( G \) which are almost equal to \( B \). Define \([B] \leq [C]\)
if and only if $B \preceq C$. Fix a proper almost invariant set $A$ of $G$. Let $A$ and $A^*$ be infinite, and let $E$ be the set of $[gA]$ and $[gA^*]$ for all $g \in G$, partially ordered by $\preceq$. We have the involution $[A] \mapsto [A^*]$ on $E$. Our aim is to choose $A$ so that $E$ satisfies conditions (1)-(4). Assuming that we can do this, we can construct the required tree $T$. The stabilizer of the edge $[A]$, will be finite because $\{g \in G : gA \preceq A\}$ is finite by Theorem 5.6. $G$ may be divided but if so we simply subdivide. Hence, by Theorem 4.3, $G$ is an amalgamated free product over a finite subgroup $F$. Hence $G$ splits over $F$ so long as no vertex of $T$ has stabilizer equal to $G$. We show that this is possible.

Suppose that $G$ fixes a vertex $v$ of $T$. Then every edge of $T$ has $\{v\}$ as a vertex. In particular, $v$ must be one of the original vertices of $T$, and was not introduced by subdivision. Now we consider the original $T$. If $e$ and $f$ are edges of $T$ with $e < f$, we have $e < f$. Now Lemma 5.10 tells us that for almost all elements $x$ of $A$, $x^A C A$ or $x^A C A^*$. As $\{g \in G : gA \preceq A\}$ is finite, by Theorem 5.6, we deduce that there is an element $x$ of $A$ such that $x^A$ is not almost equal to $A$ and $A^*$ and either $x^A C A$ or $x^A C A^*$. Similarly, there is an element $y$ of $A^*$ such that $y^A$ is not almost equal to $A$ and $A^*$ and either $yA C A^*$ or $yA C A^*$. If $xA C A$, then we have $x^2 A C xA C A$ and this contradicts the fact that $e, f$ are edges of $T$ with $e < f$, then $e < f$. Similarly, if $yA < A^*$, we obtain a contradiction. If $xA C A$ and $yA C A^*$, then $xyA C A^* A$ and again we have a contradiction. Therefore $G$ cannot fix a vertex of $T$.

In order to complete the proof of Theorem 6.1, we must show how to find a proper almost invariant set $A$ in $G$ such that the partially ordered set $E$ satisfies conditions (1)-(4). Conditions (1) and (4) hold automatically for any choice of $A$. Our next result says that condition (2) also holds for any choice of $A$.

Lemma 6.5 Let $G$ be a finitely generated group with infinitely many ends. If $B, C, D$ are proper almost invariant subsets of $G$, then $[g \in G : B \preceq gC \preceq D]$ is finite.

Proof. The result is trivial if $B$ is not almost contained in $D$. If $B$ is strictly contained in $D$, we can add an element of $D^*$ to $B$ without altering the problem. Hence we suppose that $B \preceq D$ but $B \not\preceq D$.

If $B \preceq C \preceq D$, then either $gC \not\preceq D$ or $B \not\preceq gC$. We will show that $[g \in G : gC \preceq D]$ and $[g \in G : B \preceq gC, B \not\preceq C]$ are both finite. This will prove the required result. Now Corollary 5.11 states that for almost all elements $g \in G$ one of $gC \preceq D, gC < D^*$, $gC^* \preceq D$, $gC^* < D^*$ holds. If $gC \preceq D$ and $gC \not\preceq D$, none of these four inclusions can hold except for $gC^* \preceq D^*$. If $gC^* \preceq D$ and $gC^* < D^*$, then $gC \not\preceq D$. As $[g \in G : gC \preceq D]$ is finite, by Theorem 5.6, we deduce that $[g \in G : gC \preceq D, gC \not\preceq D]$ is finite. Similarly, $[g \in G : B \not\preceq gC, B \not\preceq C]$ is finite.

Finally, we show that it is possible to choose $A$ so that $E$ satisfies condition (2). Note that for almost all $g \in G$, we know that one of $gA \preceq A, gA \preceq A^*, gA \preceq A^*$, $gA \not\preceq A^*$ holds. We must arrange that this holds for every element of $G$, when we replace strict inclusion by almost inclusion.

We fix a finite generating set $S$ for $G$ and let $\Gamma = \Gamma(S, G)$ be the corresponding graph. If $A$ is an almost invariant set in $G$, we denote the number of edges in $\delta A$ by $|\delta A|$. Let $k$ be the smallest value taken by $|\delta A|$ as $A$ ranges over proper almost invariant sets in $G$. We say that a set $A$ in $G$ is narrow if $|\delta A| = k$.

Lemma 6.7. Let $A_1 \supset A_2 \supset \ldots$ be a sequence of narrow sets in $G$. If $B = \cap A_n$ is non-empty, then the sequence stabilizes, i.e., there is an integer $K$ such that $A_n = B$, when $n \geq K$.

Proof. Let $e$ be an edge of $\delta B$. Then $e$ has one vertex in every $A_n$ and the other vertex is outside every $A_n$ for which $n$ exceeds some integer $N$. Therefore $e$ is an edge of $\delta A_n$, when $n \geq N$. If $\delta B$ contains $k + 1$ edges, the above argument shows that $\delta A_n$ would also contain $k + 1$ edges for a suitably large value of $n$. It follows that $|\delta B| = k$ and that $\delta B \preceq \delta A_n$ for all suitably large $n$. In particular, $B$ is almost invariant in $G$. We have the equations $A_n = (A_n + B) + B$ and $\delta A_n = \delta (A_n + B) + \delta B$. As $\delta B \preceq \delta A_n$, we see that $\delta (A_n + B) \cap \delta B$ is empty. As $A_n$ is infinite, one of $B$ and $(A_n + B)$ must be infinite.
The infinite one, \( X \), must have \( |\delta X| = k \), as \( X \) is a proper almost invariant subset of \( G \). The other one must then have empty coboundary and so be empty. As \( B \) is non-empty, we deduce \( B = A_0 \), which completes the proof of the lemma.

Let \( g \) be any element of \( G \), and let \( A \) be a narrow set in \( G \). Then \( A^* \) is also narrow so that \( g \) must lie in a narrow set in \( G \). Lemma 6.7 tells us that the set of all narrow subsets of \( G \) which contain \( g \) has minimal elements, where we partially order narrow sets by inclusion.

**Lemma 6.8.** Let \( A \) be a narrow set, minimal with respect to containing some element \( g \) of \( G \). Then for any narrow set \( A_i \), one of

\[ A \subseteq A_i, A \supseteq A_i, A^* \subseteq A_i, A^* \supseteq A_i, \]

holds.

**Proof.** The required result is equivalent to proving that one of the sets \( A \cap A_i, A \cap A_i^*, A^* \cap A_i, A^* \cap A_i^* \) is finite. For convenience we call these sets \( X_i, X_i^* \). For each \( i, \delta X_i \subseteq \delta A \cup \delta A_i \). As the \( X_i \)'s are disjoint, any edge in \( \delta A \cup \delta A_i \) has its ends in exactly two of the \( X_i \)'s. Hence each edge in \( \delta A \cup \delta A_i \) lies in the coboundary of exactly two of the \( X_i \)'s. Hence

\[ |\delta X_1| + |\delta X_2| + |\delta X_3| + |\delta X_4| = 2|\delta A \cup \delta A_i| \leq 4k, \]

where \( |\delta A| = |\delta A_i| = k \).

If each \( X_i \) is infinite, then we must have \( |\delta X_i| = k \) for each \( i \), because each \( X_i \) is infinite. Hence \( |\delta X| = k \) for each \( i \). But one of \( A \cap A_i, A \cap A_i^* \) (say \( A \cap A_i \)) is then a narrow subset of \( G \) which contains \( g \). Hence \( A \cap A_i = A \) by the minimality of \( A \), and so \( A \cap A_i^* \) is empty — a contradiction. Therefore some \( X_i \) must be finite which completes the proof of Lemma 6.8.

In order to carry out the proof of Theorem 6.1 as sketched after Theorem 6.5, we simply need to choose a narrow set \( A \) in \( G \) which is minimal with respect to containing some element of \( G \).

**Applications and Examples**

Many of the most important applications of Stallings' structure theorem for groups with infinitely many ends have to do with the cohomology of groups. We will only consider more simple-minded examples. We start by discussing the problem of accessibility first posed by Wall [28].

Think of a splitting of a group over a finite subgroup as a kind of factorization. Stallings' theorem tells us that if \( G \) is finitely generated and \( e(G) \geq 2 \), then \( G \) has such a factorization. The first natural question to ask is whether one can go on factorizing \( G \) forever, or whether the process of factorization must stop.

We will say that a f.g. group with at most one end is 0-accessible, and that a group \( G \) is \( n \)-accessible if \( G \) splits over a finite subgroup with each of the factor groups \((n-1)\)-accessible. We will call a group accessible if it is \( n \)-accessible for some \( n \).

**Conjecture.** Any finitely generated group is accessible.

Stamford and Dunwoody [1] have shown that accessibility is equivalent to a certain condition on the cohomology of the group, but, in general, one has no proof that their condition is satisfied. However, it is easy to see that any f.g. torsion free group \( G \) is accessible. Corollary 2.2, which follows from Gruško's Theorem, tells us that \( G \) is a free product of indecomposables. Each factor in this decomposition has at most one end or is infinite cyclic and so \( G \) is accessible.

The following result seems to clarify the concept of accessibility.

**Lemma 7.1.** Let \( G \) be a finitely generated group. Then \( G \) is accessible if and only if \( G \) is the fundamental group of a finite graph \( \Gamma \) of groups, where each edge group is finite and each vertex group has at most one end.

**Proof.** If \( \Gamma \) exists, \( G \) is obviously accessible. We prove the converse by induction on \( n \), where \( G \) is \( n \)-accessible. If \( n = 0 \), we can take \( \Gamma \) to be a single vertex.

If \( G = G_1 \ast_C G_2 \) where \( G_1 \) and \( G_2 \) are already the fundamental...
groups of graphs $\Gamma_1$ and $\Gamma_2$ of groups, and $C$ is finite, we construct $\Gamma$ as follows. By Corollary 3.8, there are vertex groups $H_1, H_2$ of $\Gamma_1, \Gamma_2$ and elements $g_1, g_2$ of $G_1, G_2$ such that $C \subseteq g_1^{-1}H_1g_1$ for $i = 1, 2$. By replacing every vertex and edge group $H_i$ by $g_iH_ig_i^{-1}$ we can suppose that $C \subseteq H_1$, and similarly for $\Gamma_2$. Now $\Gamma$ consists of $\Gamma_1, \Gamma_2$ and an edge $e$ joining the vertices underlying $H_1$ and $H_2$, where $e$ has associated group $C$.

If $G = A \ast_C$ where $A$ is already the fundamental group of a graph $\Gamma_1$ of groups, and $C$ is finite, we proceed as follows. We have two inclusions $\alpha_1, \alpha_2$ of $C$ in $A$ and each $\alpha_i(C)$ must lie in a conjugate of some vertex group $H_i$ of $\Gamma_i$. As above, we can suppose that $\alpha_1(C) \subseteq H_1$, and $\alpha_2(C) \subseteq H_2$. Now $G$ has presentation \[ \langle A, t; c^{-1}\alpha_1(c)t = \alpha_1(c), \forall c \in C \rangle. \] Write $u = t^a$. Then $G$ also has presentation \[ \langle A, u; u^{-1}\alpha_1(c)u = s^{-1}\alpha_2(c)s, \forall c \in C \rangle. \] We replace $\alpha_2$ by $\beta_2$, where $\beta_2(c) = s^{-1}\alpha_2(c)s$. As $\beta_2(C) \subseteq H_2$, we can take $\Gamma$ to be $\Gamma_1$ together with an edge $e$ joining the vertices of $\Gamma_1$ which underlie $H_1$ and $H_2$, where $e$ has associated group $C$.

We can now re-define accessibility to allow for infinite factorization. A group is accessible if and only if it is the fundamental group of a graph of groups in which every edge group is finite and every vertex group has at most one end. For, e.g., groups, this is equivalent to the old definition. One can ask if all groups are accessible, but the Kurosh example in Section 3 shows that the answer is negative. For Kurosh's group is an infinite amalgamation product of free groups and hence is torsion free. Thus his group is accessible if and only if it can be expressed as a free product of indecomposable subgroups.

There is one other class of groups known to be accessible. That is groups with a free subgroup of finite index. We have already shown (Theorem 6.2) that if such a group is torsion free it must be free. We now state a general structure theorem for such groups. This was proved by Karrass, Pietrowski and Solitar in the f.g. case [16], Cohen in the countable case [6], and Cohen [6] and Scott [20] in the general case. See also Dunwoody [7] for a more recent proof.

Theorem 7.3. A group $G$ has a free subgroup of finite index if and only if $G$ is the fundamental group of a graph $\Gamma$ of groups in which every vertex group of $\Gamma$ is finite and the orders of all the vertex groups are bounded.

Remark. We will prove this theorem only in the case when $G$ is f.g. We then assume that $\Gamma$ is finite, so that the boundedness condition is redundant.

Proof. Suppose that $G$ is f.g. and has a finite subgroup of finite index. We will show that $\Gamma$ exists by induction on $r(G)$, where $r(G)$ is the rank of free subgroups of $G$. If $r(G) = 0$, then $G$ is finite and the result follows. If $r(G) > 0$, then $r(G) > 2$ so $G$ splits over a finite subgroup. If $G = A \ast_C B$, or $G = A \ast_C C$, then $r(A)$ and $r(B)$ are each less than $r(G)$, so that the result will follow by induction as in the proof of Lemma 7.1. Let $F$ be a free subgroup of $G$ of finite index and of minimal rank. As $C$ is finite, $F$ meets any conjugate of $C$ trivially. Hence the Subgroup Theorem applied to $F \ast_C C B$ or $A \ast_C C$ tells us that $F = (F \ast A) \ast (F \ast B) \ast K$ or $F = (F \ast A) \ast K$, for some subgroup $K$ of $F$. Hence the ranks of $F \ast A$ and $F \ast B$ are each less than that of $F$ unless one of them equals $F$. But then we would have $F$ contained in $A$ or $B$ which is impossible as $F$ has finite index in $G$, but $A$ and $B$ have infinite index in $G$.

Now suppose that $G$ is the fundamental group of a finite graph $\Gamma$ of finite groups. We use induction on the number $n$ of edges of $\Gamma$. If $n = 0$, then $G$ is finite and the result is obvious.

If $n \geq 1$, we pick an edge $e$ of $\Gamma$ with associated group $C$. Then $G = A \ast_C B$ or $A \ast_C C$, according to whether $e$ separates $\Gamma$ or not, where $A$ and $B$ are the fundamental groups of the subgraphs of $\Gamma$ obtained by removing $e$. Thus, by our induction hypothesis, each of $A$ and $B$ has a free subgroup of finite index and hence a normal free subgroup of finite index. Let $A_1, B_1$ denote the quotients of $A$ and $B$ by their normal free subgroups of finite index. As $C$ is finite, the natural map from $A$ and $B$ to $A_1$ and $B_1$ both inject $C$. Hence we have a natural map $A \ast_C B \to A_1 \ast_C B_1$ or $A \ast_C C \to A_1 \ast_C C$ which injects any finite subgroup
Lemma 7.4. If $G = A \ast_{C} B$ or $A \ast_{C} C$, with $A$, $B$, and $C$ finite, then $G$ has a free subgroup of finite index.

Proof. We construct a homomorphism from $G$ to a finite group which injects $A$ and $B$. The kernel must be free, by the Subgroup Theorem.

Case $G = A \ast_{C} B$

Let $X = A/C \times C \times B/C$, where $A/C$ denotes the set of all cosets $aC$ of $C$ in $A$. We will represent $A$ and $B$ faithfully as permutation groups of the finite set $X$, in such a way that $C$ acts on $X$ in the same way for each action. There will then be a homomorphism $G \to S(X)$, the group of permutations of $X$, which injects $A$ and $B$.

Choose a transversal $t : A/C \to A$. We have a bijection $A/C \times C \to A$ sending $(a, c)$ to $t(a)c$. The action of $A$ on itself by right multiplication gives an action of $A$ on $A/C \times C$, by using this bijection. We let $A$ act on $X$ by defining $(a, c, \beta)a = ((a, c)a, \beta)$. If $c' \in C$, then $(a, c, \beta)c' = (a, cc', \beta)$. Similarly we use a transversal of $C$ in $B$ to define an action of $B$ on $X$. For this action also, we have $(a, c, \beta)c' = (a, cc', \beta)$ for all $c' \in C$.

Case $G = A \ast_{C} C$

We have two injections of $C$ into $A$. We use one of them to identify $C$ with a subgroup of $A$. Thus we have a subgroup $C$ of $A$ and an injective map $\phi : C \to A$, whose image we denote by $C_{1}$.

Let $X = A$, and let $A$ act on $X$ by right multiplication. The two induced actions of $C$ are each multiples of the right regular representation so are equivalent. We can write down an equivalence as follows. Choose transversals $T : A/C \to A$, $T_{1} : A/C_{1} \to A$ and a bijection $\psi : A/C \to A/C_{1}$.

Then $T$, $T_{1}$ induce bijections $U : A/C \times C \to A$, $U_{1} : A/C_{1} \times C_{1} \to A$ (as above), and we define $\theta$ to be the composite

$$A \overset{U_{1}}\to A/C \times C \overset{\psi \otimes 1}\to A/C_{1} \times C_{1} \overset{U_{1}}\to A.$$ 

Then $\theta(ac) = \theta(a)\theta(c)$. Now we can define a homomorphism $r : G \to S(X)$ by letting $r|A$ be the right regular representation and $r(t) = \theta$.

Remark. These constructions - which do not depend on finiteness (except to suppose the existence of a bijection $\psi$) - give an alternative proof of the assertion (1.6, 1.7) that if $C \to A$, $C \to B$ are injective, so are $A \to A \ast_{C} B$, $A \to A \ast_{C} C$.

Having discussed the accessibility of groups, i.e. the existence of a factorization, the next question to consider is that of uniqueness of the factorization. One would like some analogue of Theorem 3.5 for free products. The first point is that given any graph $\Gamma$ of groups with fundamental group $G$, one can construct a larger graph $\Gamma'$, also with fundamental group $G$ by adding an edge $e$ to $\Gamma$ with only one vertex of $e$ in $\Gamma$ and an isomorphism at the other end of $e$. This corresponds to expressing $G$ as $G \ast_{C} C$ for some subgroup $C$.

We will say that an edge $e$ in a graph of groups $\Gamma$ is trivial if the two ends of $e$ are distinct vertices of $\Gamma$ and $e$ has an isomorphism at one end. If $\Gamma$ has such an edge, we can replace $\Gamma$ by a new graph $\Gamma'$ obtained from $\Gamma$ by identifying $e$ to a point, such that $\Gamma'$ has the same fundamental group as $\Gamma$. Hence if we start with a finite graph $\Gamma$, we can eliminate all the trivial edges. However, this is false for infinite graphs. For example, let $\Gamma$ be the graph with vertices $1, 2, \ldots$ and edges $e_{i}$ joining $i$ to $i+1$. We associate an infinite cyclic group $A_{i}$ to the vertex $i$ and an infinite cyclic group $B_{i}$ to the edge $e_{i}$. The map $B_{i} \to A_{i}$ is an isomorphism and the map $B_{i} \to A_{i+1}$ is multiplication by two. The fundamental group of $\Gamma$ is the dyadic rationals, but every edge of $\Gamma$ is trivial.

We will say that a graph of groups with no trivial edges is minimal. Then any finitely generated accessible group $G$ is the fundamental group of some minimal graph $\Gamma$, where each edge group is finite and each vertex group has at most one end. Even with minimal graphs, one still
Lemma 7.5. Let $G$ be the fundamental group of a finite graph $\Gamma$ of groups, such that each edge group is finite and each vertex group has at most one end.

1. If $A$ is a subgroup of $G$ with at most one end, then $A$ lies in a conjugate of a vertex group of $\Gamma$.

2. Let $v_1, v_2$ be vertices of $\Gamma$ with associated groups $G_1, G_2$. If $A = G_1 \cap G_2$, then either there is an edge path in $\Gamma$ from $v_1$ to $v_2$ such that each of the associated edge groups contains $A$ or $G_1 = G_2$ and $g \in G_1$.

Proof. (i) The Subgroup Theorem tells us that $A$ is the fundamental group of a graph $\Gamma'$ of groups where the associated groups are conjugates of subgroups of the groups associated to $\Gamma$. Our aim is to show that $A$ must be a vertex group of $\Gamma'$.

The fact that $e(A) = 1$ tells us that each edge of $\Gamma'$ is trivial and that $\Gamma'$ is a tree. Thus the vertex groups of $\Gamma'$ are partially ordered by inclusion. Suppose that $A_1 \subseteq A_2 \subseteq \ldots$ is an infinite ascending chain of vertex groups of $\Gamma'$. If all of the inclusions are strict, then each $A_i$ equals an edge group of $\Gamma'$. But, as $\Gamma$ is finite, there is an upper bound on the orders of the edge groups of $\Gamma'$. Hence, one cannot have an infinite strictly increasing chain of vertex groups of $\Gamma'$. Hence there is a maximal vertex group. This vertex group must equal $A$, which completes the proof of (i).

(ii) The proof of this is the same as the proof of Lemma 3.15.

Now we consider a finitely generated group $G$ and two minimal graphs $\Gamma$ and $\Gamma'$ each with fundamental group $G$, such that each edge group is finite and each vertex group has at most one end. Note that $\Gamma$ and $\Gamma'$ must be finite.

Lemma 7.6. (i) There is a bijection between the vertices of $\Gamma$ and $\Gamma'$ such that corresponding vertex groups are conjugate in $G$.

(ii) $\Gamma$ and $\Gamma'$ have the same number of edges.

(iii) If $\Gamma$ does not have distinct edges $e, f$ with $G_e$ lying in a conjugate of $G_f$, then $\Gamma$ and $\Gamma'$ are isomorphic as graphs and corresponding vertex or edge groups are conjugate in $G$.

Remarks. In (iii), the hypothesis implies that no edge group of $\Gamma$ is trivial, unless $\Gamma$ has only one edge.

It seems reasonable to suppose that the analogue of (i) for the edges of $\Gamma$ and $\Gamma'$ always holds, but we cannot prove it.

Proof. (i) Let $A$ be a vertex group of $\Gamma$. Then $e(A) = 1$. Lemma 7.5(i) tells us that $A$ lies in conjugate of a vertex group $B$ of $\Gamma'$. The same lemma shows that $B$ lies in conjugate of a vertex group $A_1$ of $\Gamma'$. Hence $A$ lies in conjugate $A_1^g$ of $A_1$ for some $g \in G$. Lemma 7.5(ii) tells us that either $A = A_1$ and $g \in A$ or there is a path from $A$ to $A_1$ in $\Gamma$ for which each edge group contains a conjugate of $A$. As $\Gamma$ is minimal, the second case can only occur when $A = A_1$ and the path consists of a single loop. Therefore $A = A_1^g$ and $A$ is conjugate to $B$. As the groups associated to distinct vertices of $\Gamma$ cannot be conjugate (because $\Gamma$ is minimal), assertion (i) follows.

(ii) Let $G$ denote the quotient of $G$ obtained by killing all the vertex groups of $\Gamma$. This quotient is a free group of rank $E - V + 1$, where $E$ and $V$ are the number of edges and vertices of $\Gamma$. Part (i) tells us we obtain a group isomorphic to $\overline{G}$ by killing all the vertex groups of $\Gamma'$. Hence $E' = V + 1 = E - V' + 1$. As $V = V'$, by (1), we have $E = E'$ as required.

(iii) Let $e$ be an edge of $\Gamma$ with vertices $v_1$ and $v_2$ which may be equal. Let $G_1$ and $G_2$ be the groups associated to $v_1$ and $v_2$ and let $A$ be the group associated to $e$. Then $A = G_1 \cap G_2$. If $G_1 \neq G_2$ or $G_1 = G_2$ and $g \notin G_1$, it follows from Lemma 7.5(ii) and from part (i) of this lemma, that $A$ lies in conjugate of an edge group $B$ of $\Gamma'$. Similarly, $B$ lies in conjugate of an edge group $A_1$ of $\Gamma$. Our hypothesis on $\Gamma$, in (iii), implies that $A = A_1$ so that $A$ is con-
jugate to \( B \). Thus for each edge group of \( \Gamma \), there is an edge group of \( \Gamma' \) conjugate to it in \( G \) and distinct edges of \( \Gamma \) correspond to distinct edges of \( \Gamma' \). As \( \Gamma \) and \( \Gamma' \) have the same number of edges, by part (ii), we have a bijection between the edges of \( \Gamma \) and \( \Gamma' \).

Suppose that \( A \) is an edge group of \( \Gamma \) and that \( A \) is contained in a conjugate of a vertex group \( H \) of \( \Gamma \). Our condition that \( A \) is not contained in a conjugate of any other edge group of \( \Gamma \) implies that \( H \) is one of the vertex groups at the end of the edge to which \( A \) is associated. The same holds for \( \Gamma' \), so that the bijection between the edges of \( \Gamma \) and \( \Gamma' \) must actually induce an isomorphism of the graphs \( \Gamma \) and \( \Gamma' \).

We turn now to another embedding result proved in [20]. The result and its proof are similar to those of Theorem 3.17, which told us that any countable group could be embedded in a 2-generator group. The result is an essential part of the proof of Theorem 7.3 for arbitrary cardinality of the groups involved.

Theorem 7.7. If \( G \) is the fundamental group of a countable graph \( \Gamma \) of finite groups, where the vertex groups have bounded order, then \( G \) can be embedded in a group \( H \), which is the fundamental group of a finite graph of finite groups.

Remarks. The natural homomorphism \( A \to \mathbb{Z} \), obtained by killing \( A \), has kernel \( K \) equal to \( \cdots \to A \to C \to \mathbb{Z} \cdots \). This can be seen most simply by constructing a space \( X \) whose fundamental group is \( K \) and observing that \( \mathbb{Z} \) acts freely on \( X \), with quotient a space with fundamental group \( A \). The graph \( \Gamma \) corresponding to \( K \) is a copy of the real line with integer points as vertices, and all the vertex groups are copies of \( A \), all the edge groups are copies of \( C \). Thus \( \mathbb{Z} \) acts on \( \Gamma \), as a group of groups. This is an example of how to embed a group which is the fundamental group of an infinite graph of groups into a group which is the fundamental group of a finite graph of groups. One needs a fairly uniform sort of graph \( \Gamma \) so that \( \Gamma \) admits a group action.

Proof. The aim of our proof is to work in steps so as to make \( \Gamma \) uniform. Since the vertex groups have bounded order we can choose a group \( H \) (for example, a symmetric group) in which they all embed.

Let \( G_1 \) be obtained from \( G \) by replacing each vertex group of \( \Gamma \) by a copy of \( H \). Note that \( G \subseteq G_1 \).

Let \( H_1, \ldots, H_n \) be groups, one from each isomorphism class of subgroups of \( H \). Let \( f_1, \ldots, f_n \) be the distinct embeddings of \( H_1, \ldots, H_n \) in \( H \). Then each edge of \( \Gamma \) has a pair of \( f_i \)'s associated to it. We will say that two edges of \( \Gamma \) are of the same type if they have the same unordered pair associated.

We enlarge the graph of groups \( \Gamma \) by adding countably many edges of each type joining each distinct pair of vertices of \( \Gamma \). This new graph \( \Gamma_2 \) is still countable, and so is its fundamental group \( G_2 \). We have \( G_1 \subseteq G_2 \). Choose a maximal tree \( T \) in \( \Gamma_2 \) consisting of edges with the identity map of \( H \) at each end. Let \( \Gamma_3 \) be the graph of groups obtained from \( \Gamma_2 \) by identifying \( T \) to a point. Thus \( \Gamma_3 \) has one vertex labelled \( H \) and countably many loops of each type. Its fundamental group is still \( G_2 \).

We now have a graph which clearly admits a group action. Suppose that \( \Gamma_3 \) has \( m \) types of loop. We label the edges of \( \Gamma_3 \) by \( a_{ij} \), where \( 1 \leq i \leq m \), and for fixed \( i \), the suffix \( j \) runs through all the integers, thus enumerating all the loops of one given type.

\[ G_2 \] has a presentation of the form
\[ \langle H, \{a_{ij}\} | a_{ij}^*b_{ij}a_{ij}^{-1} = c_{ij} \rangle \text{ where } k \text{ runs through some set } K_i \text{ and } b_{ik}c_{ik} \in H \rangle. \]

We define an isomorphism \( \phi : G_2 \to G_3 \) by \( \phi(h) = h \) for \( h \in H \) and \( \phi(a_{ij}) = a_{i,j+i} \). This determines an extension of \( G_2 \) by \( \mathbb{Z} \) which we call \( G_3 \). \( G_3 \) can be presented as
\[ \langle H, \{a_{ij}\}, t | t^{-1}ht = h \text{ for } h \in H, a_{ij}^*b_{ik}a_{ij}^{-1} = c_{ik} \text{ for } k \in K_i \rangle. \]

Hence \( G_3 \) is the fundamental group of a graph of groups which has one vertex labelled \( H \), one loop of each type and one extra loop which is associated to it the identity map of \( H \) at each end. Hence \( G_3 \) is the required group \( H \).
ENDS OF PAIRS OF GROUPS

The concept of the number of ends of a pair of groups \((G, C)\), where \(C\) is a subgroup of \(G\), is a generalization of the number of ends of a group. Recall (Section 6) the close relationship between the theory of ends of groups and the Sphere Theorem. One is also interested in conditions which will guarantee the existence of other surfaces in a 3-manifold—particularly when the fundamental group of the surface injects into the fundamental group of the 3-manifold. Thus one is interested in groups which split over infinite subgroups. The starting point of my work on ends of pairs of groups was the idea that there should be a generalization of Stallings' structure theorem to this situation. Thus one is looking for a natural definition of a number \(e(G, C)\), and one hopes to prove that \(e(G, C) \geq 2\) if and only if \(G\) splits over some subgroup closely related to \(C\).

The correct definition of \(e(G, C)\) is due to Houghton [3]. Recall the definition of \(e(G)\). One lets \(PG\) be the power set of \(G\), \(FG\) be the collection of finite subsets of \(G\), each with Boolean addition, and defines \(EG = PG/FG\). The right action of \(G\) on itself induces a right action of \(G\) on \(EG\). Let \((EG)^G\) denote the subset of elements left fixed by this action (this is the same as \(QG/FG\), where \(QG\) is as in §5). Then \(e(G)\) is the dimension, as \(Z_2\)-vector space, of \((EG)^G\).

Let \(C\) be a subgroup of \(G\) and let \(H = C : G\). Then we define \(e(G, C)\) to be the dimension of \((EH)^G\). Clearly if \(C\) is trivial, then \(e(G, C) = e(G)\). The following result justifies the claim that this is the correct definition of \(e(G, C)\).

Lemma 8.1. Let \(X\) be a finite CW-complex with a connected regular covering space \(\tilde{X}\) whose covering group is \(G\). If \(C\) is a subgroup of \(G\), then \(e(G, C) = e(C \setminus \tilde{X})\).

Remark. The hypothesis that \(X\) is finite implies that \(G\) is f.g.

One can summarize the basic properties of \(e(G, C)\) as follows.

Lemma 8.2. (i) \(e(G, C) = 0\) if and only if \(|G : C|\) is finite.
(ii) If \(G \supseteq C\), \(|G : C_1|\) finite, then \(e(G, C) = e(C_1, C)\).

(iii) If \(K\) is a normal subgroup of \(G\) with quotient \(G_1\) and \(K : K \cap C\) finite, then \(e(G, C) = e(G_1, pC)\), where \(p : G \rightarrow G_1\) is the natural projection.
(iv) If \(C_1 \subseteq C \subseteq G\) and \(|C : C_1| = n\), then \(e(G, C) \leq e(G, C_1) \leq n \cdot e(G, C)\).

These results are all the analogues of results about \(e(G)\), except for (iv). One can give examples showing that either equality can be achieved in this part. The final basic property of \(e(G, C)\) is

Lemma 8.3. If \(G\) splits over \(C\), then \(e(G, C) = 2\).

Proof. The proof is very similar to that of Lemma 6.3. We consider only the case when \(G = A \ast C \ast B\). Recall the set \(E\) which was the subset of \(G\) consisting of elements whose canonical form starts in \(A\). If \(b \in B\), then \(E_b = E\) and if \(a \in A\), then \(E = Ea \cup Ca \subseteq E \cup C \cup Ca\). The subset \(pE\) of \(C \forall G\) is left almost invariant by every element of \(A\) or \(B\), and so is almost invariant in \(C \forall G\). Clearly \(pE\) and \(pE\) are infinite.

This leads us to the first large difference between \(e(G)\) and \(e(G, C)\). We know that \(e(G)\) can only take the values \(0, 1, 2\) or \(\infty\), but \(e(G, C)\) can take any positive integer value. This is shown by the following example. Note that both \(G\) and \(C\) are f.g. in this example.

Example. Let \(F\) be a closed surface and let \(X\) be a compact subsurface so that no component of \(F - X\) has closure homeomorphic to a 3-disc. Then the natural map \(\pi_1(X) \rightarrow \pi_1(F)\) is injective, and we call the groups \(G\) and \(C\). Now \(e(G, C)\) equals the number of ends of \(FC\), the covering space of \(F\) with fundamental group \(C\). But one observes that \(X\) is a subtorus of \(F\), and it is easy to prove that \(FC\) consists of \(X\) together with all open annuli \(S^1 \times [0, \infty)\) attached to each boundary component of \(X\). The number \(e(G, C)\) equals the number of boundary components of \(X\). By choosing \(F\) to be of appropriately high genus, one can find pairs \((G, C)\) for which \(e(G, C)\) takes any specified value.

Originally I hoped to prove that \(e(G, C) \geq 2\) if and only if \(G\) splits over some subgroup closely related to \(C\). The following example shows
that no such result can hold.

Example. Let $M$ be a closed, orientable irreducible 3-manifold. It can be shown that $M$ is sufficiently large if and only if $\pi_1(M)$ splits over some subgroup. There exists such a 3-manifold which is not sufficiently large, but has a finite covering space which is sufficiently large. See [8] for a discussion of such examples. Thus we have an unsplitable group $G$ with a subgroup $G_1$ of finite index which splits over some subgroup $C$. Hence $e(G, C) = e(G_1, C) = 2$, but $G$ is unsplitable. (One can find a subgroup $C$ which is finitely generated, so there is nothing pathological about this example.)

This example suggests that one must be careful to prove that $e(G, C) = 2$ if and only if $G$ has a subgroup $G_1$ of finite index such that $G_1$ splits over some subgroup closely related to $C$. It then seems reasonable that one will need a residual finiteness condition on $G$.

We say that a group $G$ is residually finite if given $g \in G$, there is $G_1 \leq G$, such that $|G : G_1|$ is finite and $g \notin G_1$. If $C$ is a subgroup of $G$, we say that $G$ is residually finite if given $g \in G$ there is $G_1 \leq G$ such that $|G : G_1|$ is finite, $G_1 \to C$, and $g \notin G_1$. The natural result seems to be the following, which is proved in [21].

Theorem 8.4. If $G$ and $C$ are f.g. groups and $G$ is $C$-residually finite, then $e(G, C) = 2$ if and only if $G$ has a subgroup $G_1$ of finite index such that $G_1$ contains $C$ and splits over $C$.

The residual finiteness condition cannot be omitted.

Example. Let $G = A \ast C$, where $A$ and $C$ are infinite, simple, f.g. groups. Thus $G$ has no subgroups of finite index and $C$ has no subgroups or supergroups of finite index. Now for any non-trivial free product $A \ast C$ except for $\mathbb{Z}_2 \ast \mathbb{Z}_2$, it is easy to show that $e(G, C) = \infty$. But if $G$ had a subgroup $G_1$ of finite index which split over some subgroup $C_1$ closely related to $C$ one would be forced to have $G = G_1$ and $C_1 = C$. The example is completed by showing that $G$ cannot split over $C$.

Lemma 8.5. Let $G = A \ast C$, where $A$ is indecomposable and not infinite cyclic. Then $G$ cannot split over $C$.

Proof. Suppose $G = x_1 \ast y$ or $x \ast y$. As no conjugate of $A$ meets $C$, we see from the Subgroup Theorem that $A$ lies in a conjugate of $x$ or $y$. We can suppose $x$ is involved. Use $(A)$ to denote the normal closure of $A$ in a group containing $A$. We know that $G/A \neq C$. Hence we have the equations $C = x(A) \ast C$, $x \ast C = x(A) \ast C$. The second equation is impossible, and the first equation can only hold when $x(A) = C$. But the equation $x = y$ contradicts the hypothesis that $G$ splits over $C$.

REFERENCES


