J. DIFFERENTIAL GEOMETRY
2 (1968) 1-7

A NOTE ON CURVATURE AND FUNDAMENTAL GROUP

J. MILNOR

Define the *growth function* γ associated with a finitely generated group and a specified choice of generators $\{g_1, \cdots, g_p\}$ for the group as follows (compare [9]). For each positive integer s let $\gamma(s)$ be the number of distinct group elements which can be expressed as words of length $\leq s$ in the specified generators and their inverses. (For example, if the group is free abelian of rank 2 with specified generators x and y, then $\gamma(s) = 2s^2 + 2s + 1$.) We will see that the asympotic behavior of $\gamma(s)$ as $s \to \infty$ is, to a certain extent, independent of the particular choice of generators (Lemma 1).

This note will make use of inequalities relating curvature and volume, due to R. L. Bishop [1], [2] and P. Günther [3], to prove two theorems.

Theorem 1. *If M is a complete n-dimensional Riemannian manifold whose mean curvature tensor R_{ij} is everywhere positive semidefinite, then the growth function $\gamma(s)$ associated with any finitely generated subgroup of the fundamental group $\pi_1 M$ must satisfy*

$$\gamma(s) \leq \text{constant} \cdot s^n .$$

It is conjectured that the group $\pi_1 M$ itself must be finitely generated.

The constant in this inequality will depend, of course, on the particular set of generators which is used to define $\gamma(s)$.

Theorem 2. *If M is compact Riemannian with all sectional curvatures less than zero, then the growth function of the fundamental group $\pi_1 M$ is at least exponential:*

$$\gamma(s) \geq a^s$$

for some constant $a > 1$.

In both cases, any set of generators for the group may be used in defining $\gamma(s)$.

Remarks. Note that there is always an exponential upper bound for $\gamma(s)$. In fact the inequality

$$\gamma(s + t) \leq \gamma(s)\gamma(t)$$

Received February 12, 1968. This work was supported in part by National Science Foundation Grant #GP-7917.

implies that $\gamma(s) \leq \gamma(1)^s$. With only a little more work one can verify that the sequence $\gamma(s)^{1/s}$ necessarily converges to a finite limit as $s \to \infty$. (For each fixed t, setting $k = [s/t] + 1$, the inequality $\gamma(s) \leq \gamma(kt) \leq \gamma(t)^k \leq \gamma(t)^{1+s/t}$ implies that $\limsup \gamma(s)^{1/s} \leq \gamma(t)^{1/t}$. Therefore $\limsup \gamma(s)^{1/s} \leq \inf \{\gamma(t)^{1/t}\} \leq \liminf \gamma(s)^{1/s}$, and the sequence converges.) It will be convenient to say that a group has *exponential growth* whenever $\lim \gamma(s)^{1/s} > 1$.

The two theorems stated above may be contrasted with three classical theorems; namely Myers' theorem which asserts that if mean curvature is positive definite on a compact manifold then the fundamental group is finite; the Hadamard-Cartan theorem which asserts that if all sectional curvatures are ≤ 0 on a complete manifold then the higher homotopy groups $\pi_i M$, $i \geq 2$, are zero; and Preissmann's theorem which asserts that if all sectional curvatures are <0 on a compact manifold then every abelian subgroup of $\pi_1 M$ is cyclic.

The proofs of Theorems 1 and 2 follow. Some examples and further discussion will be given at the end.

Proof of Theorem 1. Choose a base point x_0 in the universal covering space $\tilde{M}$, and let $V(r)$ denote the volume of the neighborhood $N_r(x_0)$ consisting of all $y \in \tilde{M}$ with $d(x_0, y) \leq r$, using the Riemannian distance function and the Riemannian n-dimensional volume element in $\tilde{M}$.

If the mean curvature is everywhere ≥ 0, then according to Bishop:

$$V(r) \leq \omega_n r^n \ , \qquad (1)$$

where ω_n denotes the volume of the unit disk in euclidean n-space.

We will identify $\pi_1 M$ with the group of all covering transformations of $\tilde{M}$ over M. Let $g_1, \cdots, g_p : \tilde{M} \to \tilde{M}$ be the preferred generators for some subgroup of $\pi_1 M$, and let μ denote the maximum of the numbers $d(x_0, g_i(x_0))$ for $i = 1, 2, \cdots, p$. Note that the neighborhood $N_{\mu s}(x_0)$ contains at least γ(s) distinct points of the form $g(x_0)$ with $g \in \pi_1 M$.

Choose a number $\varepsilon > 0$ which is small enough so that the neighborhood $N_\varepsilon(x_0)$ in $\tilde{M}$ is disjoint from all of the translates $g(N_\varepsilon(x_0))$, with $g \neq 1$. Then the neighborhood $N_{\mu s+\varepsilon}(x_0)$ contains at least $\gamma(s)$ disjoint sets of the form $g(N_\varepsilon(x_0))$. This proves that

$$\gamma(s)V(\varepsilon) \leq V(\mu s + \varepsilon) \ . \qquad (2)$$

Combining (1) and (2) we obtain the required inequality

$$\gamma(s) \leq cs^n \ , \qquad \text{for} \qquad s \geq 1 \ ,$$

where $c = \omega_n(\mu + \varepsilon)^n / V(\varepsilon)$. This proves Theorem 1.

Before proving Theorem 2 we need to investigate the dependence of the growth function $\gamma(s)$ of a group on the particular choice of generators $\{g_1, \cdots, g_p\}$ for the group.

Lemma 1. *Let $\{g_1, \cdots, g_p\}$ and $\{h_1, \cdots, h_q\}$ be two different sets of generators for the same group, and let $\gamma(s)$ and $\gamma'(t)$ be the corresponding growth functions. Then there exist positive constants k and k' so that*

$$\gamma'(t) \leq \gamma(kt)$$

for all t and $\gamma(s) \leq \gamma'(k's)$ for all s.

Proof. If k is large enough so that each h_j can be expressed as a word of length $\leq k$ in the generators g_i and g_i^{-1}, then the required inequality $\gamma'(\mathrm{t}) \leq \gamma(kt)$ is clearly satisfied.

Proof of Theorem 2. Let $-\alpha^2 < 0$ be an upper bound for the sectional curvature of M. Then according to Günther the volume $V(r)$ is greater than or equal to the volume

$$n\omega_n \int_0^r (\alpha^{-1} \sinh \alpha x)^{n-1} dx$$

of a corresponding ball in the space of constant negative curvature. This expression is asymptotically equal to $2c \exp(\lambda r)$ as $r \to \infty$ (where $c = n\omega_n/(n-1)(2\alpha)^n$ and $\lambda = (n-1)\alpha$ are positive constants). Hence

$$V(r) > c \exp(\lambda r) \tag{3}$$

for r sufficiently large.

Let δ be the diameter of M. The projection $\tilde{M} \to M$ clearly maps the compact neighborhood $N = N_\delta(x_0)$ onto M. Hence the set of all translates gN, with $g \in \pi_1 M$, forms a covering of $\tilde{M}$. Note that this covering is locally finite. (For otherwise, for some finite r the neighborhood $N_r(x_0)$ would intersect infinitely many of the gN, and hence $N_{r+\delta+\epsilon}(x_0)$ would contain infinitely many disjoint sets $gN_\epsilon(x_0)$, each of volume $V(\epsilon) > 0$. This is impossible since $V(r + \delta + \epsilon) < \infty$.)

Let F be the finite set consisting of all elements g in $\pi_1 M$ such that the translate gN intersects N. Let $\nu > 0$ be the minimum of $d(gN, N)$ as gN ranges over all translates of N which do not intersect N.

Lemma 2. *If $d(x_0, gN) < \nu t + \delta$ for some positive integer t, then g can be expressed as a t-fold product, $g = f_1 f_2 \cdots f_t$, with $f_1, \cdots, f_t \in F$.*

Proof. Choose $y \in gN$ with $d(x_0, y) < \nu t + \delta$, and choose points $y_1, y_2, \cdots, y_{t+1} = y$ along the minimal geodesic from x to y so that

$$d(x_0, y_1) \leq \delta \qquad \text{and} \qquad d(y_i, y_{i+1}) < \nu \ .$$

Each y_i belongs to some translate $h_i N$, where we can choose h_1 to be 1 and h_{t+1} to be g. Let $f_i = h_i^{-1} h_{i+1}$ so that $f_1 f_2 \cdots f_t = g$. Since the two points $h_i^{-1} y_i$ and $h_i^{-1} y_{i+1}$ have distance $< \nu$, and belong to N and to $f_i N$ respectively, it follows from the definitions of ν and F that $f_i \in F$. This proves Lemma 2.

In particular it follows that the elements of F generate the group $\pi_1 M$, Let $\gamma'(t)$ denote the growth function of $\pi_1 M$, computed using F as the set of preferred generators.

According to Lemma 2, the interior of the neighborhood $N_{\nu t+\delta}(x_0)$ is completely covered by the translates $f_1 f_2 \cdots f_t N$ with $f_i \in F$. Since precisely $\gamma'(t)$ of these translates are distinct, this proves that

$$(4) \qquad \gamma'(t)V(\delta) \geq V(\nu t + \delta) .$$

By Lemma 1 there exists a positive integer k so that

$$\gamma(kt) \geq \gamma'(t)$$

or in other words

$$(5) \qquad \gamma(s) \geq \gamma'([s/k]) , \qquad \text{for} \qquad s \geq k .$$

Combining (3), (4) and (5) we clearly obtain

$$(6) \qquad \gamma(s) \geq c_1 \exp(\lambda_1 s)$$

for all sufficiently large s, where the constants $c_1 = c \exp(\lambda\delta - \lambda\nu)/(V(\delta)$ and $\lambda_1 = \lambda\nu/k$ are both positive. Since $\gamma(s) > 1$ for all $s \geq 1$, it is now easy to choose a constant $a > 1$ so that

$$\gamma(s) \geq a^s$$

for all s. This completes the proof of Theorem 2.

Remark. It may be conjectured that some weaker hypothesis, in which some zero sectional curvatures were allowed, would still be sufficient to imply exponential growth. Perhaps the hypothesis of negative definite mean curvature would already suffice?

Examples. First consider the 2-dimensional case. On a sphere or projective plane or torus or Klein bottle we can impose a metric with curvature ≥ 0, so that Theorem 1 applies. Choosing the most familiar generators for $\pi_1 M$, the growth function is given by

$$\gamma(s) \equiv 1 \quad \text{for the sphere,}$$
$$\gamma(s) \equiv 2 \quad \text{for the projective plane } P^2,$$
$$\gamma(s) = 2s^2 + 2s + 1 \quad \text{for the torus or Klein bottle.}$$

Thus $5s^2$ is an upper bound in each case.

Similarly for an n-dimensional torus the growth function of π_1 is a polynomical of degree n, providing that the standard generators are used (compare [6]).

Now consider the non-orientable surface $P^2 \# P^2 \# \cdots \# P^2$ which is obtained from the 2-sphere by attaching p "cross-caps". If $p \geq 3$ it is known that this surface admits a metric of (constant) negative curvature. The fundamental group has generators, say $x_1, x_2, \cdots, x_p$, subject to the single relation $x_1^2 x_2^2 \cdots x_p^2 = 1$. To estimate the growth function we will use the following.

Lemma 3. *For a free group, which is freely generated by the preferred set of generators* $\{g_1, \cdots, g_p\}$, *the growth function is given by*

$$\gamma(s) = (p(2p-1)^s - 1)/(p-1) .$$

Hence

$$(2p-1)^s \leq \gamma(s) \leq (2p+1)^s .$$

The proof is not difficult.

Now for the required group, with generators $x_1, \cdots, x_p$ and relation $x_1^2 \cdots x_p^2 = 1$, note that the elements $x_1, \cdots, x_{p-1}$ generate a free subgroup (see [7]). Thus we obtain the estimate

$$(2p-3)^s \leq \gamma(s) \leq (2p+1)^s ,$$

which supports Theorem 2.

In the case of an orientable surface of genus $q \geq 2$, a similar argument shows that $(4q-3)^s \leq \gamma(s) \leq (4q+1)^s$. (For examples of higher dimensional manifolds of negative curvature, see Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963) 111–122.)

As a final example, let G denote the nilpotent Lie group consisting of all 3×3 triangular real matrices with 1's on the diagonal, and let H be the subgroup consisting of all integer matrices of the same form. Then the coset space G/H is a compact 3-dimensional manifold with fundamental group H.

Lemma 4. *The growth function of H is quartic:*

$$c_1 s^4 \leq \gamma(s) \leq c_2 s^4 , \qquad \text{with} \qquad c_1 > 0.$$

Outline of proof. As preferred generators take the matrices $x = I + e_{23}$ and $y = I + e_{12}$. Let c denote the central element $yxy^{-1}x^{-1} = I + e_{13}$. Note that every expression of the form $x^t y^t c^k$ with $0 \leq k \leq t^2$ can be expressed as a word of length $2t$ in x and y. Hence every $x^i y^j c^k$ with $1 \leq i \leq t$, $1 \leq j \leq t$, and $1 \leq k \leq t^2$ has length $\leq 4t$, which proves that

$$\gamma(4t) \geq t^4 .$$

Conversely, if $x^i y^j c^k$ can be expressed as a word of length $\leq s$ then it is easily verified that $|i| \leq s$, $|j| \leq s$, and $|k| \leq (s/2)^2$, which yields a quartic upper bound for $\gamma(s)$.

Corollary.[1] *No Riemannian metric on G/H can satisfy either the hypothesis of Theorem 1 or the hypothesis of Theorem 2.*

For Lemma 4 shows that $\gamma(s)$ is larger than any constant times s^3, but smaller than any a^s, $a > 1$, for sufficiently large s.

If we impose the most natural metric, which comes from a left invariant metic on G, computation shows that the mean curvature tensor of G/H is indefinite, with signature $(+, -, -)$. The scalar curvature $\sum R^j{}_j$ turns out to be negative. Thus: *The fundamental group of a compact manifold of negative scalar curvature need not have exponential growth.* On the other hand the product of a surface of genus 2 with a small sphere provides an example of a manifold of positive scalar curvature whose fundamental group does have exponential growth. Perhaps the scalar curvature of a manifold of dimension ≥ 3 has no influence at all on the fundamental group?

(Here is an alternative example of a 3-manifold whose fundamental group is the nilpotent group H of Lemma 4. Let M^3 be the intersection of the algebraic variety $z_1^2 + z_2^3 + z_3^6 = 0$ in complex 3-space with the unit sphere. Compare [4], [8, § 9]. This manifold can also be described as the 6-fold cyclic branched covering of the 3-sphere $|z_1|^2 + |z_2|^2 = 1$, branched along the trefoil knot $z_1^2 + z_2^3. = 0$. I do not know whether or not this manifold M^3 is diffeomorphic to G/H.)

A concluding problem. It is interesting to compare the growth function $\gamma(s)$ of a group with the behavior of left-invariant, symmetric random walks on the group, as studied by[2] Kesten [5]. Each such random walk is described by a symmetric matrix m_{gh} whose spectrum has largest element

$$\lambda = \limsup\, (m_{11}^{(s)})^{1/s} \,.$$

Here $m_{gh}^{(s)}$ denotes the probability of passing from g to h in s steps of the random walk. We will assume that every pair g, h of group elements "communicate" (that is $m_{gh}^{(s)} > 0$ for some s).

This number λ is equal to 1 if the group is solvable, but is less than 1 if there exists a free non-cyclic subgroup. The distinction $\lambda = 1$ or $\lambda < 1$ depends only on the group and not on the particular probability distribution used to define the random walk.

Lemma 5. *If* $\lambda < 1$ *then the growth function satisfies* $\lim \gamma(s)^{1/s} > 1$ *(exponential growth).*

Proof. Choose the random walk so that only right multiplication by the preferred generators or their inverses has positive probability at each step. Applying Schwarz's inequality to the equation

$$m_{1g}^{(2t)} = \sum_h m_{1h}^{(t)} m_{hg}^{(t)}$$

[1] The second half of this corollary could equally well be derived from Preissmann's theorem.

[2] I am indebted to R. Lyndon for pointing this out.

we obtain $m_{1g}^{(2t)} \leq m_{11}^{(2t)}$. Averaging this inequality over the $\gamma(2t)$ elements g represented by words of length $\leq 2t$, we obtain

$$\frac{1}{\gamma(2t)} \leq m_{11}^{(2t)} ;$$

hence

$$\frac{1}{\lim \gamma(s)^{1/s}} \leq \limsup \, (m_{11}^{(s)})^{1/s} < 1 ,$$

which completes the proof.

Surprisingly, the converse statement is false. For example the group with generators x, y and defining relation $xyx^{-1} = y^2$ is solvable, so that $\lambda = 1$. But $\gamma(2t) \geq 2^t$, since the words

$$xy^{\epsilon_1}xy^{\epsilon_2} \cdots xy^{\epsilon_t}$$

with $\epsilon_i = 0$ or 1 all represent distinct elements of this group.

Problem. Consider such a random walk on the fundamental group of a compact manifold of negative curvature. Is λ necessarily less than 1?

References

[1] R. L. Bishop, *A relation between volume, mean curvature, and diameter,* Notices Amer. Math. Soc. **10** (1963) 364.
[2] —— & R. J. Crittenden, *Geometry of manifolds,* Academic Press, New York, 1964.
[3] P. Günther, *Einige Sätze über das Volumenelement eines Riemannschen Raumes,* Publ. Math. Debrecen **7** (1960) 78–93.
[4] F. Hirzebruch, *The topology of the normal singularities of an algebraic surface* (d'après Mumford), Séminaire Bourbaki, 15e année 1962–63, N° 250.
[5] H. Kesten, *Symmetric random walks on groups,* Trans. Amer. Math. Soc. **92** (1959) 336–354.
[6] H. U. Krause, *Gruppenstruktur und Gruppenbild,* Thesis, Eidg. Techn. Hochs. Zürich 1953, 45 pp.
[7] W. Magnus, *Über diskontinuierliche Gruppen mit einer definierenden Relation* (*Der Freiheitssatz*), J. reine angew. Math. **163** (1930) 141–165.
[8] J. Milnor, *Singular points of complex hypersurfaces,* Annals of Math. Studies, Princeton University Press, Princeton, to appear.
[9] ——, *Problem,* Amer. Math. Monthly, to appear.
[10] S. Myers, *Riemannian manifolds with positive mean curvature,* Duke Math. J. **8** (1941) 401–404.
[11] A. Preissmann, *Quelques propriétés globales des espaces de Riemann,* Comment. Math. Helv. **15** (1942–43) 175–216.
[12] J. L. Synge, *On the connectivity of spaces of positive curvature,* Quart. J. Math. Oxford, Ser. (1), 7 (1936) 316–320.

UNIVERSITY OF CALIFORNIA, LOS ANGELES