ℓ^2-homology of right-angled Coxeter groups based on barycentric subdivisions

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Abstract Associated to any finite flag complex L there is a right-angled Coxeter group W_L and a contractible cubical complex Σ_L on which W_L acts properly and cocompactly, and such that the link of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, then Σ_L is a contractible n-manifold. We establish vanishing (in a certain range) of the reduced ℓ^2-homology of Σ_L in the case where L is the barycentric subdivison of a cellulation of a manifold. In particular, we prove the Singer Conjecture (on the vanishing of the reduced ℓ^2-homology except in the middle dimension) in the case of Σ_L where L is the barycentric subdivison of a cellulation of $\mathbb{S}^{n-1}$, n=6,8.

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0 Introduction

This paper contains some new results about vanishing of the reduced ℓ^2-homology (which we denote by $\mathcal{H}_*$) for a certain class of right-angled Coxeter groups. Our main motivation comes from the following conjecture.

Singer Conjecture If M^n is a closed aspherical manifold, then $\mathcal{H}_i(\widetilde{M}^n) = 0$ for all $i \neq n/2$.

We refer the reader to [5] for the current status of this conjecture and further references. A construction of the first author ([1], [2], [4]), associates to any finite flag complex L, a "right-angled" Coxeter group W_L and a contractible cubical cell complex Σ_L on which W_L acts properly and cocompactly. The most important feature of this construction is that the link of each vertex of Σ_L is isomorphic to L. Therefore, if L is homeomorphic to the $(n-1)$-sphere, then Σ_L is an n-manifold.

If Γ is a torsion-free subgroup of finite index in W_L (and such subgroups always exist and are easy to construct), then Γ acts freely on Σ_L and Σ_L/Γ is a finite complex. Since Σ_L is contractible, Σ_L/Γ is aspherical. Hence, this construction gives many examples of closed aspherical manifolds. The Singer Conjecture for such manifolds becomes the following.

Conjecture 0.1 *Suppose L is a triangulation of the $(n-1)$-sphere as a flag complex. Then $\mathcal{H}_i(\Sigma_L) = 0$ for all $i \neq n/2$.*

The following theorem was proved in [5].

Theorem 0.2 *Conjecture 0.1 holds true for $n \leq 4$.*

The main source of flag complexes are barycentric subdivisions. The purpose of this paper is to extend the above theorem to dimensions 6 and 8 assuming that L is the barycentric subdivision of a cellulation of the sphere. In fact, we obtain this result as a corollary of a more general theorem asserting the vanishing of the reduced ℓ^2-homology above middle dimensions for barycentric subdivisions of cellulations of manifolds.

This paper is organized as follows. In Section 1 we recall some notation and results about ℓ^2-homology of right-angled Coxeter groups. Section 2 contains results about ℓ^2-acyclicity of (the Coxeter groups associated to) links of vertices in barycentric subdivisions. The main result is proved in Section 3. Finally, in the last section, we show that validity of the Singer Conjecture for odd-dimensional Tomei manifolds implies a much stronger vanishing theorem for the barycentric subdivision of simplicial complexes.

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1 Notation and preliminary results

We will use notation from [5, Section 7.1].

$$\mathfrak{h}_i(L) = \mathcal{H}_i(\Sigma_L)$$

$$\beta_i(L) = \dim_{W_L}(\mathfrak{h}_i(L))$$

The dimension of Σ_L is one greater than the dimension of L. Hence, $\beta_i(L) = 0$ for $i > \dim L + 1$.

We will use the following 4 properties of ℓ^2-homology, proved in [5, Sections 7.2 and 7.4].

Lemma 1.1 (The Mayer-Vietoris sequence) *If $L = L_1 \cup L_2$ and $A = L_1 \cap L_2$, where L_1 and L_2 (and therefore, A) are full subcomplexes of L, then*

$$\to \mathfrak{h}_i(A) \to \mathfrak{h}_i(L_1) \oplus \mathfrak{h}_i(L_2) \to \mathfrak{h}_i(L) \to$$

is weakly exact.

Lemma 1.2 (The Künneth Formula) *The Betti numbers of the join of two complexes are given by:*

$$\beta_k(L_1 * L_2) = \sum_{i+j=k} \beta_i(L_1)\beta_j(L_2).$$

Lemma 1.3 (The Betti numbers of a cone) *The Betti numbers of the simplicial cone CL on a complex L are given by:*

$$\beta_i(CL) = \frac{1}{2}\beta_i(L).$$

Lemma 1.4 (Poincaré Duality) *If L is a triangulation of the sphere $\mathbb{S}^{n-1}$, then $\beta_i(L) = \beta_{n-i}(L)$.*

If L is a flag simplicial complex, and v is its vertex, we let $L - v$ denote the flag simplicial complex obtained by removing (the open star of) v from L.

Lemma 1.5 *Let L be a flag simplicial complex, and let v be its vertex. Suppose that the link $\mathrm{Link}_L(v)$ is ℓ^2-acyclic, i.e. $\beta_*(\mathrm{Link}_L(v)) = 0$. Then $\beta_*(L - v) = \beta_*(L)$.*

Proof Apply the Mayer-Vietoris sequence and the cone formula to the decomposition $L = (L - v) \cup_{\mathrm{Link}_L(v)} C\,\mathrm{Link}_L(v)$. □

2 Acyclicity of links

If L is a regular CW-complex, bL will denote its barycentric subdivision. bL is a flag simplicial complex. Its vertices are naturally graded by "dimension": each vertex v of bL is the barycenter of a unique cell σ_v of the complex L, and we call the dimension of σ_v the *dimension* of the vertex v. Given a cell σ^k in L, its boundary as a subcomplex of L is denoted by $\partial\sigma^k$. By a *cellulation* of a space X we mean a homeomorphism from a regular CW-complex L onto X. If X is a manifold, by *PL-cellulation* we mean a PL-homeomorhism.

The following lemma is well-known.

Lemma 2.1 *Let bL be the barycentric subdivision of a regular CW-complex L, and let v be its vertex of dimension k. Then*

$$\mathrm{Link}_{bL}(v) = b\partial\sigma_v^k * b\,\mathrm{Link}_L\,\sigma_v^k$$

.

Applying this lemma to a PL-cellulation of a manifold, we obtain the following.

Lemma 2.2 *Let bL be the barycentric subdivision of a PL-cellulation L of $(n-1)$-manifold, and let v be a vertex of dimension k. Then*

$$\mathrm{Link}_{bL}(v) = b\partial\sigma_v^k * S^{n-2-k}$$

where S^{n-2-k} is a flag triangulation of the $(n-2-k)$-sphere.

Lemma 2.3 *Let bL be the barycentric subdivision of a cell complex L, and let v be its vertex. If the dimension of v is 1 or 3, then the $\mathrm{Link}_{bL}(v)$ is ℓ^2-acyclic.*

Proof It follows from Theorem 0.2 that $\beta_*(b\partial\sigma_v^1) = \beta_*(b\partial\sigma_v^3) = 0$. Therefore, in the decomposition of Lemma 2.2 the first factor is ℓ^2-acyclic, and the result follows from the Künneth formula. □

Lemma 2.4 *Let bL be the barycentric subdivision of a PL-cellulation L of $(n-1)$-manifold, and let v be its vertex. If the codimension of v (i.e. $n-1-\dim(v)$) is 1 or 3, then the $\mathrm{Link}_{bL}(v)$ is ℓ^2-acyclic.*

Proof The codimension condition and Theorem 0.2 imply that the second factor in the decomposition of Lemma 2.2 is ℓ^2-acyclic, and the result follows from the Künneth formula. □

3 Vanishing theorem

Theorem 3.1 *Let bL be the barycentric subdivision of a PL-cellulation L of $(n-1)$-manifold, where $n \leq 9$. Then $\beta_i(bL) = 0$ for $i \geq n/2 + 1$.*

The idea of the proof is to remove (the open stars of) vertices of bL one by one without changing the ℓ^2-Betti numbers until the dimension of the remaining complex is less than $n/2$. Note that Lemmas 2.3 and 2.4, imply that vertices of bL of dimensions 1 and 3 and codimensions 1 and 3 have acyclic links and we can try to apply Lemma 1.5 to remove them. However, removing a vertex changes links of other vertices, so we have to choose the order of removal carefully to preserve the acyclicity of the links.

The following lemma allows us to do this.

Lemma 3.2 *Let bL be the barycentric subdivision of a regular CW-complex L, and let v be its vertex of dimension k. The removal of v from bL does not change the first factor in the decomposition of Lemma 2.1 of links for vertices of dimension less or equal than k, and it does not change the second factor for vertices of dimension greater or equal than k.*

Proof Suppose the removal of v changes $\mathrm{Link}_{bL}(u)$ for some vertex u. This implies that $v \in \mathrm{Link}_{bL}(u)$ and therefore either $\sigma_v \subset \sigma_u$ or $\sigma_u \subset \sigma_v$. In the first case the dimension of u is greater than the dimension of v and v belongs to the first factor, and therefore its removal does not change the second factor. In the second case the dimension of u is less than the dimension of v and v belongs to the second factor, and therefore its removal does not change the first factor. □

Proof of Theorem 3.1 The removal procedure consists of four steps, some of which can be vacuous. The first step is to remove all vertices of dimension 3. This requires the use of Lemma 2.3 (acyclicity) and Lemma 3.2 (the links of these vertices are independent). The second step is to remove all vertices of dimension 1, again using Lemma 2.3 and Lemma 3.2. The third step is to remove all vertices of codimension 3 if their dimension is greater than 3, using Lemma 2.4 and Lemma 3.2. Finally, the fourth step is to remove all vertices of codimension 1 if their dimension is greater than 3, again using Lemma 2.4 and Lemma 3.2. We note that Lemma 3.2 guaranties that after each step the links of vertices of the following steps are acyclic.

It is easy to check (and we leave this to the reader), that the cardinality of the set of the dimensions of the remaining vertices is less than $n/2+1$. Therefore, since a simplex of bL has vertices of pairwise different dimensions, the dimension of the remaining complex is less than $n/2$ and it follows that $\beta_i(bL) = 0$ for $i \geq n/2+1$.

As an example, if $n = 8$ this procedure removes all vertices of dimensions $3,1,4,6$, in this order. The remaining vertices have dimensions $0,2,5,7$ and they span 3-dimensional complex. Therefore, $\beta_i(bL^7) = 0$ for $i \geq 5$.

$\square$

Remark 3.3 If $n > 9$, then the above argument shows that $\beta_i(L) = 0$ for $i > n-4$.

If L is a PL-cellulation of a sphere, we can apply Poincaré duality to get the following.

Corollary 3.4 *Conjecture 0.1 holds true for the barycentric subdivision of a PL-cellulation of $\mathbb{S}^{n-1}$ for all even $n \leq 8$.*

Remark 3.5 If $n \leq 9$ is odd, it follows that $\beta_i(L) = 0$ outside of the two middle dimensions.

Remark 3.6 Using [5, Section 11.2] in place of Theorem 0.2, one can easily generalize the above arguments to arbitrary cellulations of rational homology manifolds and spheres.

4 On the Singer Conjecture for Tomei manifolds

In this section we show that validity of Conjecture 0.1 for barycentric subdivisions of *triangulations* of odd-dimensional spheres would follow from a very special case of the Singer Conjecture.

Let $\partial\Delta^n$ denote the boundary complex of the standard n-dimensional simplex.

Conjecture 4.1 $\beta_i(b\partial\Delta^n) = 0$ *for all odd n.*

Remark 4.2 In fact, as shown in [3], a natural finite index manifold cover of $\Sigma_{b\partial\Delta^n}/W_{b\partial\Delta^n}$ is a well-known object. Let T_n denote the space of all symmetric tridiagonal $(n+1)\times(n+1)$-matrices with fixed generic eigenvalues, the so-called Tomei manifold. The spaces T_n were introduced in [6] and have been extensively studied in the context of dynamical systems (the Toda lattice) and numerical analysis (QR-factorization). Note that the natural fundamental domain of the action of $W_{b\partial\Delta^n}$ on $\Sigma_{b\partial\Delta^n}$ is the n-dimensional permutahedron. It follows that T_n is an orbifoldal cover of $\Sigma_{b\partial\Delta^n}/W_{b\partial\Delta^n}$.

Theorem 4.3 *Suppose Conjecture 4.1 holds true. Let bL be the barycentric subdivision of an $(n-1)$-dimensional simplicial complex L. Then $\beta_i(bL)=0$ for $i \geq n/2+1$.*

Proof The proof is similar to the proof of Theorem 3.1, but somewhat easier. Conjecture 4.1 implies that odd-dimensional vertices have ℓ^2-acyclic links (cf. Lemma 2.3). Therefore we can remove all odd-dimensional vertices from bL starting from the top using Lemmas 2.3 and 3.2. What is left is spanned by even-dimensional vertices, and since a simplex of bL has vertices of pair-wise different dimensions, the dimension of the remaining part is less than $n/2$. □

As before, if L is a triangulation of a sphere, we can apply Poincaré duality to get the following.

Corollary 4.4 *Suppose Conjecture 4.1 holds true. Then Conjecture 0.1 holds true for the barycentric subdivision of a triangulation of $\mathbb{S}^{n-1}$ for all even n.*

References

[1] **Michael W Davis**, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) 117 (1983) 293–324

[2] **Michael W Davis**, *Coxeter groups and aspherical manifolds*, from: "Algebraic topology, Aarhus 1982 (Aarhus, 1982)", Lecture Notes in Math. 1051, Springer, New York (1984) 197–221

[3] **Michael W Davis**, *Some aspherical manifolds*, Duke Math. J. 55 (1987) 105–139

[4] **Michael W Davis**, *Nonpositive curvature and reflection groups*, from: "Handbook of Geometric Topology", (R Daverman, R Sher, editors), Elsevier, Amsterdam (2002) 373–422

[5] **Michael W Davis**, **Boris L Okun**, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geom. Topol. 5 (2001) 7–74

[6] **C Tomei**, *The topology of isospectral manifolds of tridiagonal matrices*, Duke Math. J. 51 (1984) 981–996