REFLECTION GROUPS AND CAT(0) COMPLEXES WITH EXOTIC LOCAL STRUCTURES

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We show that, in contrast to the situation for the standard complex on which a right angled Coxeter group $W$ acts, there are cocompact $W$-actions on CAT(0) complexes such that the local topology of the complex is distinctly different from the end topology of $W$.

1. Introduction

If $\tilde{X}$ is a contractible $n$-manifold or homology $n$-manifold, then, since it satisfies Poincaré duality, its cohomology with compact supports, $H^*_c(\tilde{X})$, is concentrated in dimension $n$ and is isomorphic to $\mathbb{Z}$ in that dimension. It follows that the homology at infinity of $\tilde{X}$ is concentrated in dimension $n - 1$ and is isomorphic to $\mathbb{Z}$ in that dimension (see [5] for definitions and references for homology at infinity). More particularly, recall that a simplicial complex is a homology $n$-manifold if and only if the link of each vertex is a “generalized homology $(n-1)$-sphere” (i.e., a homology $(n-1)$-manifold with the same homology as $S^{n-1}$). So, the above argument shows that if the link of each vertex in an aspherical simplicial complex $X$ is

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a generalized homology \((n - 1)\)-sphere, then the homology at infinity of its universal cover \(\tilde{X}\) is concentrated in dimension \(n - 1\) and hence, \(\tilde{X}\) is \((n - 2)\)-acyclic at infinity. This shows that hypotheses concerning the local topology of an aspherical space can have implications for the end topology of its universal cover.

There are other local-to-asymptotic results for nonpositively curved complexes. If \(L\) is a simplicial complex, we let \(S(L)\) denote the set of all closed simplices of \(L\), including the empty simplex. Let \(\tilde{X}\) be a (locally finite) CAT(0) cubical complex, and for each vertex \(x \in \tilde{X}\), let \(L_x\) denote its link. If for each vertex \(x\) and for each closed simplex \(\sigma \in S(L_x)\), \(L_x - \sigma\) is \(m\)-connected (resp., \(m\)-acyclic), then \(\tilde{X}\) is \(m\)-connected (resp., \(m\)-acyclic) at infinity (see [2] and the references cited there). We call the complexes \(L_x - \sigma\) the punctured links of \(\tilde{X}\).

In recent work [5], we have shown there is a close connection between local topology and end topology of the standard complexes on which Coxeter groups act. The nerve of a Coxeter system \((W, S)\) is the simplicial complex \(L\) with one vertex for each element of the generating set \(S\) and one simplex for each subset of \(S\) which generates a finite subgroup of \(W\). As explained in [3, 4 or 5], associated to \((W, S)\) there is natural cell complex, here denoted \(|W|\), such that \(|W|\) is a model for \(\mathbb{E}W\) and such that the link of each of its vertices is isomorphic to \(L\) (see [7] for the definition of \(\mathbb{E}G\)). This implies, for example, that if \(L\) is a triangulation of an \((n - 1)\)-sphere, then \(|W|\) is a contractible \(n\)-manifold.

If \(S\) is finite (which we shall henceforth always assume), then \(L\) is a finite complex and the quotient space \(|W|/W\) is compact. It is proved in [6] and [8] that the natural piecewise Euclidean metric on \(|W|\) is CAT(0). In [5] the authors established a direct correspondence between the topological properties of \(L\) and the asymptotic topological properties of \(|W|\) (or of any locally finite building with associated Coxeter system \((W, S)\)). For example:

**Theorem 1.1.** (See 4.1, 4.2 and 4.3 in [5]) Let \(W\) be a finitely generated Coxeter group with associated nerve \(L\). Then

1. \(W\) is simply connected at infinity if and only if \(L - \sigma\) is simply connected for each \(\sigma \in S(L)\).
2. \(W\) is \(m\)-acyclic at infinity if and only if \(L - \sigma\) is \(m\)-acyclic for each \(\sigma \in S(L)\).
3. \(W\) is \(m\)-connected at infinity if and only if \(L - \sigma\) is \(m\)-connected for each \(\sigma \in S(L)\).
Thus, not only do the connectivity properties of the punctured links $L - \sigma$ determine the connectivity at infinity of $W$ (i.e., of $|W|$), the converse is also true. This leads to speculation that, in the general context of nonpositive curvature, similar asymptotic-to-local results might hold. For example, one might speculate that if $X$ is a nonpositively curved, finite Poincaré complex with, say, extendable geodesics, then the links of vertices in $X$ are forced to be generalized homology spheres (and hence, $X$ is a homology manifold). Similarly, it could be speculated that if $X$ is a nonpositively curved cubical complex (with extendable geodesics) and if $X$ is $m$-connected (resp., $m$-acyclic) at infinity, then the punctured links of vertices must be $m$-connected (resp., $m$-acyclic). The purpose of this note is to give some examples which set such speculations to rest: there are no general results of this nature. (For further examples illuminating the difficulty of getting asymptotic-to-local results, see [2].)

2. The Construction

The construction of our examples is essentially the same as the construction of [1]. We will show that by making minor modifications in the construction of $|W|$ one gets a model for $\mathbb{E}W$, $\mathfrak{M}$, with a CAT(0) cubical structure so that the connectivity properties at infinity do not descend to connectivity properties of links. In fact, in all of our examples $|W|$ will be a manifold while $\mathfrak{M}$ will not even be a homology manifold. We show that $|W|$ and $\mathfrak{M}$ are equivariantly proper homotopy equivalent, so if $\Gamma$ is any torsion-free subgroup of finite index in $W$, then $\mathfrak{M} / \Gamma$ is a nonpositively curved Poincaré complex that is not a homology manifold. For simplicity, we restrict our construction to right angled Coxeter groups, which we briefly review below.

Right Angled Coxeter Groups. A simplicial complex $L$ is a flag complex if any complete graph in the 1-skeleton of $L$ is actually the 1-skeleton of a simplex in $L$. The barycentric subdivision of any cell complex is a flag complex; hence, the condition of being a flag complex imposes no restriction on the topology of $L$ — it can be any polyhedron. The importance of flag complexes in CAT(0) geometry stems from the result of Gromov that the natural piecewise Euclidean metric on a cubical complex is nonpositively curved (= locally CAT(0) ) if and only if the link of each vertex is a flag complex [6, p. 122].

Suppose $L$ is a finite flag complex. For each integer $k \geq 0$, let $L^{(k)}$ denote the set of $k$-simplices in $L$ and as before let $\mathcal{S}(L)$ denote the poset of all simplices in $L$ (including the empty simplex).
Associated to $L$ there is a group $W$ defined as follows. For each $i \in L^{(0)}$ introduce a symbol $s_i$ and set $S = \{s_i\}_{i \in L^{(0)}}$. $W$ is defined by the presentation:

$$W = \langle S \mid s_i^2 = 1, s_is_j = s_js_i \text{ when } \{i,j\} \in L^{(1)} \rangle.$$ 

$(W, S)$ is called a right angled Coxeter system. Its nerve is $L$.

**The Cubical Complex $|W|$.** For each $\sigma \in S(L)$, let $W_\sigma$ denote the subgroup generated by the elements of $S$ which correspond to vertices of $\sigma$. Then $W_\sigma \simeq (\mathbb{Z}_2)^{\dim(\sigma)+1}$. Set

$$WS(L) = \prod_{\sigma \in S(L)} W/W_\sigma.$$ 

$WS(L)$ is called the poset of spherical cosets (the partial order is given by inclusion). The complex $|W|$ is defined to be the geometric realization of $WS(L)$. There is an obvious left $W$ action on $|W|$. The cubical structure on $|W|$ is defined as follows. There is one vertex of $|W|$ for each element of $W (= W/W_0)$. For each spherical coset $wW_\sigma$, we then fill in a Euclidean cube of dimension $\dim(\sigma) + 1$ with vertices corresponding to the elements of $wW_\sigma$. (Note that the elements of $W_\sigma$ can naturally be identified with the vertices of a cube of dimension $\dim(\sigma) + 1$.) The poset of cubes in $|W|$ is $WS(L)$ and the link of each vertex is $L$.

The geometric realization of the poset $S(L)$ is denoted $K$. The inclusion $S(L) \hookrightarrow WS(L)$ defined by $\sigma \mapsto W_\sigma$ induces an inclusion $K \hookrightarrow |W|$, and we identify $K$ with its image in $|W|$. Similarly, the orbit projection $WS(L) \twoheadrightarrow S(L)$ defined by $wW_\sigma \mapsto \sigma$ induces a projection $|W| \twoheadrightarrow K$ which factors through a homeomorphism $|W|/W \rightarrow K$. Thus, $K$ is a fundamental domain for the $W$-action on $|W|$ and the orbit projection $|W| \twoheadrightarrow K$ restricts to the identity on $K$.

The geometric realization of $S(L)_{\geq 0}$ can be identified with the barycentric subdivision $L'$ of $L$. Thus, $K$ is the cone on $L'$ (the empty set provides the cone point). For each $i \in L^{(0)}$, let $K_i$ denote the geometric realization of $S(L)_{\geq i}$, i.e., $K_i$ is the closed star of $i$ in $L'$. We call $K_i$ the mirror of $K$ of type $i$.

Here is another description of $|W|$. For each point $x \in K$ let $\sigma(x)$ be the simplex spanned by $\{i \in L^{(0)} \mid x \in K_i\}$. Then

$$|W| = (W \times K)/\sim$$

where the equivalence relation $\sim$ is defined by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{\sigma(x)}$. 
For a flag complex $L$ that can be decomposed as $L = L_1 \cup L_2$, we will construct a different CAT(0) cubical complex $\mathcal{M}$ on which the associated right angled Coxeter group $W$ acts as a cocompact reflection group. The complexes $\mathcal{M}$ and $|W|$ will have the same pro-homotopy type. However, the topology of the links of vertices in $\mathcal{M}$ can differ dramatically from that of the links of $|W|$.

**The construction of $\mathcal{M}$.** Suppose that a finite flag complex $L$ can be decomposed as the union of two full subcomplexes: $L = L_1 \cup L_2$. Set $L_0 = L_1 \cap L_2$. Since $L_0, L_1$ and $L_2$ are full subcomplexes of $L$ each of them is a flag complex.

For any simplicial complex $L$ and a point $z$ not in $L$, let $C_zL$ be the simplicial complex defined by taking the cone on $L$ with cone point $z$.

Let $x_1, x_2$ and $v$ be points that are not in $L$ and define new simplicial complexes:

$$\hat{L}_1 = L_1 \cup C_vL_0$$
$$\hat{L}_2 = L_2 \cup C_vL_0$$
$$\hat{K}_1 = C_{x_1}\hat{L}_1$$
$$\hat{K}_2 = C_{x_2}\hat{L}_2$$

Let $\hat{K}$ denote the result of gluing $\hat{K}_1$ to $\hat{K}_2$ along $C_vL_0$.

In Figure 1 we show a simple example that highlights the difference between $K$ and $\hat{K}$. The original simplicial complex $L$ is a circuit of length 8, and $K$ is the cone on this octagon. We let $L_0 \simeq S^0$ be two antipodal vertices (indicated by dots in the figure on the right), and let $L_1$ and $L_2$ be the two simplicial arcs in $L$ which are separated by $L_0$.

Returning to the case where $L$ is an arbitrary finite flag complex, we note that $L$ is a subcomplex of $\hat{K}$ (we think of it as the boundary of $\hat{K}$). Also, $\hat{K}$ is contractible (it is the union of two contractible pieces glued along a contractible subcomplex). The space $\mathcal{M}$ is defined by hollowing out each copy of $K$ in $|W|$ and replacing it with a copy of $\hat{K}$. Since $K$ and $\hat{K}$ are both contractible, $|W|$ and $\mathcal{M}$ are proper homotopy equivalent; hence, $\mathcal{M}$ is also contractible.

Here is a more precise description of $\mathcal{M}$. Recalling that for $i \in L^{(0)}$, $K_i$ is the closed star of $i$ in the barycentric subdivision of $L$ (which is a subspace of $\hat{K}$), we see that $K_i$ is identified with a subspace of $\hat{K}$. So, define $\hat{K}_i$ to be $K_i$. We then proceed as before. For each point $x \in \hat{K}$, let
Figure 1. The difference between \( K \) (left) and \( \hat{K} \) (right).

\( \sigma(x) \) be the simplex spanned by \( \{ i \in L^0 \mid x \in \hat{K}_i \} \) and let

\[ \mathfrak{W} = (W \times \hat{K})/\sim \]

where the equivalence relation \( \sim \) is defined as before.

It is not difficult to define the cubical structure on \( \mathfrak{W} \) and to see that it is CAT(0). The vertex set is \( Wx_1 \coprod Wx_2 \). For \( \alpha = 1, 2 \) and for a spherical coset \( wW_\sigma \in WS(L_0) \), the vertices \( wW_\sigma x_\alpha \) span a cube of dimension \( \dim(\sigma) + 1 \). Also, for each spherical coset \( wW_\sigma \in WS(L_0) \), we have a cube spanned by \( wW_\sigma x_1 \coprod wW_\sigma x_2 \). Its dimension is \( \dim(\sigma) + 2 \). (In particular, corresponding to the case where \( \sigma \) is empty, we have an edge from \( wx_1 \) to \( wx_2 \).) For \( \alpha = 1, 2 \), the link of \( x_\alpha \) in \( \mathfrak{W} \) is \( \hat{L}_\alpha \). Since \( \hat{L}_\alpha \) is a flag complex, the cubical structure is CAT(0). We also note that the punctured link \( \hat{L}_\alpha - v \) is homotopy equivalent to \( L_\alpha \).

**Remark.** In [1] the above construction was used only in the case where \( L \) is a homology sphere and \( L_0 \subset L \) is a homology sphere embedded in codimension one.

In the following examples we will always choose \( L \) to be a triangulation of an \( n \)-sphere and \( L_0 \) to be a codimension one submanifold triangulated as a full subcomplex. \( L_1 \) and \( L_2 \) will then be \( n \)-manifolds with boundary. (However, \( L_0, L_1 \) and \( L_2 \) need not be connected.) Since \( L \simeq S^n \), \( |W| \) is a contractible \((n + 1)\)-manifold; however, \( \mathfrak{W} \) need not be a manifold.

**Example 2.1.** Suppose that \( L \) is a 2-sphere, that \( L_1 \) is an annulus (a collared neighborhood of the equator) and that \( L_2 \) is the disjoint union of the two 2-disks (neighborhoods of the north and south poles). Then \( L_0 = L_1 \cap L_2 \) is the disjoint union of two circles, and \( \hat{L}_1 \) is an annulus.
with its boundary coned off. So, \( \hat{L}_1 \) is homeomorphic to a 2-sphere with two points identified. In particular, the link \( \hat{L}_1 \) is not simply connected \( (\pi_1(\hat{L}_1) \cong \mathbb{Z}) \). Similarly, \( \hat{L}_2 \) is the wedge of two 2-spheres. The punctured links \( L_1 \) and \( L_2 \) are also not simply connected. Nevertheless, the theorem quoted at the beginning implies that \( |W| \) (and hence \( \mathcal{W} \)) is simply connected at infinity, since the nerve \( L \) is a 2-sphere.

**Example 2.2.** Suppose \( L \) is an \( n \)-sphere and \( L_0 \) is a codimension one submanifold separating \( L \) into two pieces \( L_1 \) and \( L_2 \). Then \( \mathcal{W} \) is \( (n-1) \)-connected at infinity by the theorem quoted at the beginning. On the other hand, \( H_*(\hat{L}_1) \cong H_*(L_1, L_0) \) can be nonzero in any dimension \( < n \). Similarly, the homology of the punctured link \( L_1 \) is fairly arbitrary.

A true optimist might believe that these examples occur because there are two \( W \)-orbits of vertices, and that if \( W \) acts transitively on the 0-skeleton, then such examples disappear. The following modified version of our construction shows that this speculation is also false.

**A construction with only one vertex orbit.** Suppose \( L_0 \) is a subcomplex of \( L_1 \) and that \( t \) is a simplicial involution on \( L_0 \). Let \( L \) denote the result of gluing together two copies of \( L_1 \) along \( L_0 \) via the map \( t \). Call the two copies \( L_1 \) and \( L_2 \). Then \( t \) extends to an involution on \( L \) (also denoted \( t \)) that interchanges \( L_1 \) and \( L_2 \). Let \( W \) be the right angled Coxeter group associated to \( L \). Let \( G \) denote the semidirect product, \( G = W \times \mathbb{Z}_2 \). Here \( \mathbb{Z}_2 \) acts on the vertex set of \( L \) (the generating set of \( W \)) via \( t \). The \( W \)-action on \( \mathcal{W} \) extends to a \( G \)-action. Now there is only one \( G \)-orbit of vertices.

**Example 2.3.** Suppose that \( L_1 \) is the solid torus, \( L_1 = D^2 \times S^1 \) and that \( L_0 \) is its boundary, \( L_0 = S^1 \times S^1 \). Let \( t : S^1 \times S^1 \to S^1 \times S^1 \) be the involution which switches the factors. Then \( L = S^3 \) and \( \mathcal{W} \) is 2-connected at infinity.

The link of each vertex is isomorphic to \( \hat{L}_1 \). However, \( H_2(\hat{L}_1) \cong \mathbb{Z} \) and although \( \pi_1(\hat{L}_1) \cong \mathbb{Z} \), for the punctured link, \( L_1 \), we have \( \pi_1(L_1) \cong \mathbb{Z} \).

**Remark.** We note that there is a simple method of altering the local topology of \( |W| \) so that the connectivity of the links does not coincide with the connectivity at infinity: Form \( |W'| \) by attaching a copy of \([0,1]\) (or \([0,\infty)\)) to each vertex of \( |W| \). If one attaches unit intervals, then the resulting complex does not have extendable geodesics; if one attaches half lines, then the resulting complex is not cocompact. Further, while the complex \( |W'| \) deformation retracts onto \( |W| \), \( |W| \) does not sit as a retract inside \( \mathcal{W} \).
References