NONPOSITIVELY CURVED, PIECEWISE EUCLIDEAN STRUCTURES ON HYPERBOLIC MANIFOLDS

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It is a well-known question as to whether any Riemannian manifold M of nonpositive sectional curvature admits a piecewise Euclidean metric which is nonpositively curved. Here "nonpositively curved" is in the sense of Aleksandrov and Gromov, i.e., it is defined by comparing small triangles in the space with triangles in the Euclidean plane, such triangles must satisfy the "CAT(0)-inequality". (See [BH] or [G] for the precise definition.) Our purpose in this paper is to describe a simple construction which gives an affirmative answer to the question in the case of constant sectional curvature.

The most naive approach to this problem does not work, at least not obviously. Namely, given a hyperbolic manifold, first find a triangulation of it by hyperbolic simplices. Next, replace each hyperbolic simplex by a Euclidean simplex with the same edge lengths. Finally, try to prove that the resulting metric is nonpositively curved. This approach works in dimension two. However, in higher dimensions there are at least two problems with it: 1) there are hyperbolic simplices which do not have the same set of edge lengths as a Euclidean simplex (consider a hyperbolic tetrahedron with one face a big triangle and fourth vertex almost on the plane of the triangle), and 2) even when the replacement process can be carried out, the dihedral angles in the Euclidean simplex can be smaller than the corresponding dihedral angles in the hyperbolic simplex, so the curvature can become positive. We shall take a different tack.

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We use the quadratic form model of hyperbolic n-space: $\mathbb{R}^{n,1}$ denotes an $(n+1)$-dimensional vector space with coordinates $(x_1, \ldots, x_{n+1})$ equipped with the indefinite symmetric bilinear form $\langle\ ,\ \rangle$ defined by $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$, and the associated quadratic form $q(x) = \langle x, x\rangle$. Hyperbolic space $\mathbb{H}^n$ is identified with the sheet of the hyperboloid $q(x) = -1$ defined by $x_{n+1} > 0$. If T is a k-dimensional linear subspace of $\mathbb{R}^{n,1}$, then there are three possibilities for the restriction of the bilinear form to T: either it is positive definite, positive semidefinite, or indefinte of signature $(k-1, 1)$. One says that T is, respectively, *spacelike, lightlike*, or *timelike*. If F is a k-dimensional convex subset of $\mathbb{R}^{n,1}$, then let T_F denote the k-dimensional linear subspace which is parallel to the affine span of F. We say that F is *spacelike, lightlike*, or *timelike* as T_F is.

Let V be a discrete subset of $\mathbb{H}^n$. The *Dirichlet region* D_v for V at a point v in V consists of the points in $\mathbb{H}^n$ which are at least as close to v as to $V - \{v\}$. Equivalently, $D_v = \{x \in \mathbb{H}^n \mid \langle x, v\rangle \geq \langle x, w\rangle \text{ for all } w \in V - \{v\}\}$. Any such Dirichlet region, being an intersection of hyperbolic half-spaces is a geodesically convex subset of $\mathbb{H}^n$. In fact, if the Dirichlet region is bounded, then it is an n-cell, i.e., a geodesically convex polytope if $\mathbb{H}^n$.

Let $\operatorname{Conv}(V)$ denote the closure of the convex hull of V in $\mathbb{R}^{n,1}$ and let $B(V)$ denote the boundary of $\operatorname{Conv}(V)$. Then every point in $B(V)$ is contained in a supporting hyperplane of $\operatorname{Conv}(V)$. (An affine hyperplane E is a *supporting hyperplane* if $E \cap \operatorname{Conv}(V) \neq \emptyset$ and $\operatorname{Conv}(V)$ lies in the closure of one component of $\mathbb{R}^{n,1} - E$.) We will show that, if every Dirichlet region is bounded, then B(V) has a natural piecewise Euclidean structure which is CAT(0). If $M = \mathbb{H}^n/\Gamma$ and V is a set of Γ-orbits, we obtain the desired piecewise Euclidean structure on M.

Let I denote the interior of the light cone, $I = \{z \in \mathbb{R}^{n,1} \mid q(z) < 0\,,\ z_{n+1} > 0\}$, and let $p\colon I \to \mathbb{H}^n$ be radial projection (defined by $p(z) = z/\sqrt{-q(x)}\,$).

Proposition 1. *Let V be a discrete subset of $\mathbb{H}^n$ such that each Dirichlet region for V is bounded. Then the following statements are true.*

(i) Each face of $B(V)$ is a spacelike cell (i.e., each face is compact and the restriction of the bilinear form to its tangent space is positive definite). Thus,

$B(V)$ is naturally a piecewise Euclidean cell complex.

(ii) Radial projection p takes $B(V)$ homeomorphically onto $\mathbb{H}^n$ and it takes each face of $B(V)$ onto a geodesically convex cell in $\mathbb{H}^n$.

The referee has pointed out that similar constructions appear in [EP] and [NP]. In [NP] Näätänen and Penner consider the same construction, but only in dimension two. In particular, when $n = 2$, they prove Proposition 1, as well as, Lemma 5 below. In [EP] Epstein and Penner use a similar construction to find piecewise Euclidean structures for certain noncompact hyperbolic manifolds; they take the convex hull of a set of points on the light cone associated to orbits of the cusps.

To prove the proposition we need three easy lemmas.

Lemma 2. *Let E be an affine hyperplane in $\mathbb{R}^{n,1}$ which has nonempty intersection with $\mathbb{H}^n$ and which is not tangent to $\mathbb{H}^n$. Let $\Sigma_E = E \cap \mathbb{H}^n$. If E is not spacelike, then the hypersurface Σ_E separates $\mathbb{H}^n$ into two unbounded regions. More precisely, the following statements are true.*

(i) If E is spacelike, then Σ_E is a sphere of some radius centered at some point in $\mathbb{H}^n$.

(ii) If E is lightlike, then Σ_E is a horosphere.

(iii) If E is timelike, then Σ_E is the "equidistant hypersurface" consisting of the points of some constant oriented distance from a hyperbolic hyperplane.

Proof. (i) Suppose T_E is spacelike. Its orthogonal complement is a timelike line which intersects $\mathbb{H}^n$ in a unique point x_0 and the function $x \to \langle x, x_0\rangle$ is constant on E. Since Σ_E is nonempty and not a singleton, the constant $c = \langle x, x_0\rangle$ is negative. Thus, Σ_E is the sphere of radius $\cosh^{-1}(-c)$ about x_0.

(ii) Suppose T_E is lightlike. Then there is a lightlike vector z, with $z_{n+1} > 0$, so that T_E is tangent to the light cone along $\mathbb{R}z$. Hence, $x \to \langle x, z\rangle$ is constant on E. Since $\Sigma_E \neq \emptyset$, this constant is negative and Σ_E is a horosphere.

(iii) Suppose T_E is timelike. Then Σ_E is an equidistant hypersurface to the hyperbolic hyperplane $T_E \cap \mathbb{H}^n$. $\square$

Lemma 3. *Let V be a discrete subset of $\mathbb{H}^n$. The following statements are equivalent.*

(i) Each Dirichlet region for V is bounded.

(ii) Each horoball in $\mathbb{H}^n$ has nonempty intersection with V.

Proof. (ii)$\Rightarrow$(i). Suppose that for some v in V the Dirichlet region D_v is unbounded. Then there is a geodesic ray $\gamma\colon [0,\infty) \to D_v$ starting at v. Let $\gamma(\infty)$ denote its "endpoint" on the sphere at infinity. Let H be the horoball centered at $\gamma(\infty)$ such that $v \in \partial H$, where ∂H denotes the corresponding horosphere. Let $\varphi\colon \mathbb{H}^n \to \mathbb{R}$ be the Busemann function determined by γ:

$$\varphi(x) = \lim_{t\to\infty} d(x, \gamma(t))) - t\,.$$

It is well-known that the horospheres centered at $\gamma(\infty)$ are level sets of φ. Moreover, $\partial H = \varphi^{-1}(0)$ and if H' is any smaller horoball centered at $\gamma(\infty)$, then $H' = \varphi^{-1}((-\infty, c])$ for some $c < 0$. If H' is any such smaller horoball, then $H' \cap V = \emptyset$. For suppose, to the contrary that $v' \in H' \cap V$. (Heuristically, v' is "closer to $\gamma(\infty)$" than is v.) Then for all sufficiently large t, $d(v', \gamma(t)) < t = d(v, \gamma(t))$, i.e., for such t, $\gamma(t)$ does not lie in D_v, a contradiction. Thus, the negation of (i) implies the negation of (ii).

(i)$\Rightarrow$(ii). Suppose there are horoballs which have empty intersections with V in their interiors. Choose a maximal such horoball H such that $\operatorname{int}(H) \cap V = \emptyset$ and $\partial H \cap V \neq \emptyset$. Let $v \in \partial H \cap V$ and let $\gamma\colon [0,\infty) \to \mathbb{H}^n$ be a ray with $\gamma(0) = v$ and $\gamma(\infty)$ the center of H. We claim that the image of γ is contained in D_v. For if $d(v', \gamma(t_0)) < d(v, \gamma(t_0,)) = t_0$ for some $v' \in V$ and some t_0, then for all $t \geq t_0$,

$$d(v', \gamma(t)) \leq d(v', \gamma(t_0)) + d(\gamma(t_0), \gamma(t)) < t_0 + (t - t_0) = t.$$

Thus, $\varphi(v') < 0$, so v' lies in the interior of H, contradicting the choice of H. Thus, the negation of (ii) implies the negation of (i). $\square$

Suppose that E is an affine hyperplane in $\mathbb{R}^{n,1}$ which has nonempty intersection with I (the interior of the light cone). Then, $I - E$ has two components. Call such a component *unbounded* if there is a ray, $t \to tz$, such that tz lies in that component for all t larger than some constant. Another way to phrase Lemma 2 is this: if E is timelike or lightlike, then both components of $I - E$ are unbounded, while if E is spacelike, then one component is unbounded and the other is not.

Lemma 4. *Let V be a discrete subset of $\mathbb{H}^n$ such that each Dirichlet region for V is bounded. Then each supporting hyperplane E of Conv(V) is spacelike and Conv(V) is contained in the closure of the unbounded component of $I - E$.*

Proof. This is immediate from Lemmas 2 and 3 since any unbounded component of $I - E$ contains a horoball.

Proof of Proposition 1. (i) By Lemma 4, each face of $B(V)$ is spacelike. Let F be such a face and E a supporting affine hyperplane containing F. Since the vertex set of F is a discrete subset of the sphere Σ_E, it is a finite set. Hence, F (being the convex hull of its vertex set) is compact.

(ii) We first remark that radial projection $p: I \to \mathbb{H}^n$ takes each convex subset of I onto a geodesically convex subset of $\mathbb{H}^n$. To show that $p\,|_{B(V)}: B(V) \to \mathbb{H}^n$ is a homeomorphism, it suffices to show it is one-to-one and onto. Both of these facts follow easily from Lemma 4. Indeed, suppose that $p\,|_{B(V)}$ is not onto. Let $x \in \mathbb{H}^n$ be a point which is not in the image. Let r_x denote the ray $t \to tx$. Then there exists a supporting hyperplane E of Conv(V) which separates r_x from Conv(V). The component of $I - E$ containing r_x is unbounded, contradicting Lemma 4. Hence, $p\,|_{B(V)}$ is onto. Similarly, suppose $p\,|_{B(V)}$ is not one-to-one. Then there is a point $x \in \mathbb{H}^n$ such that the ray r_x intersects $B(V)$ in two points, t_1x and t_2x, with $t_1 < t_2$. By convexity, the ray r_x then intersects Conv(V) in the line segment $[t_1x, t_2x]$. Let E be a supporting hyperplane of Conv(V) containing t_2x. Then r_x intersects E transversely (since E is spacelike) and $[t_1x, t_2x]$ lies on the bounded side of E. This again contradicts Lemma 4. $\square$

Lemma 5. *The cellulation of $B(V)$ by its faces is combinatorially dual to the cellulation of $\mathbb{H}^n$ by Dirichlet cells. Moreover, if σ is a Dirichlet cell and F_σ is its dual cell in $B(V)$, then the linear span of σ is the orthogonal complement of T_{F_σ} in $\mathbb{R}^{n,1}$.*

Proof. Suppose $x \in \mathbb{H}^n$ and $v \in V$. Let $E_{x,v}$ be the affine hyperplane:

$$E_{x,v} = \{z \in \mathbb{R}^{n,1} \mid \langle x, z \rangle = \langle x, v \rangle\}\,.$$

We first prove that $E_{x,v}$ is a supporting hyperplane for $\mathrm{Conv}\,(V)$ if and only if $x \in D_v$. To see this, note that the affine function $z \to \langle x, z - v\rangle$ does not change sign on the unbounded component of $I - E_{x,v}$. Furthermore, the sign must be negative (consider the restriction of the function to any ray). If $E_{x,v}$ is a supporting hyperplane of $\mathrm{Conv}\,(V)$, then by Lemma 4, the function $z \to \langle x, z - v\rangle$ is ≤ 0 on $\mathrm{Conv}\,(V)$. In particular, $\langle x, w\rangle \leq \langle x, v\rangle$ for all $w \in V$, i.e., $x \in D_v$. Similarly, if $x \in D_v$, then $\langle x, w\rangle \leq \langle x, v\rangle$ for all $w \in V$ and hence, $E_{x,v}$ is a supporting hyperplane. This proves the claim.

Next we show that the cellulation of $\mathbb{H}^n$ by Dirichlet cells is "dual" to the cellulation of $B(V)$ by faces. First suppose that σ is a k-face of some Dirichlet cell D_v. Let $V(\sigma) = \{w \in V \mid \sigma \subset D_w\}$. For each $x \in \sigma$, $\langle x, w\rangle \leq \langle x, v\rangle$ for all $w \in V$ with equality if and only if $w \in V(\sigma)$. The $(n-k)$-dimensional affine space $E_{\sigma,v} = \{z \in \mathbb{R}^{n,1} \mid \langle x, z - v\rangle = 0 \text{ for all } x \in \sigma\}$ contains $V(\sigma)$ and is an intersection of supporting hyperplanes (the $E_{x,v}$, $x \in \sigma$). Moreover, it is clear that the affine span of $V(\sigma)$ is $E_{\sigma,v}$. (There are at least $n-k$ linearly independent vectors of the form $w - v$, where $D_w \cap D_v$ is a codimension-one face containing σ.) Thus $F_\sigma = \mathrm{Conv}\,(V(\sigma))$ is an $(n-k)$-face of $B(V)$, called the *dual face* to σ. Note that the tangent space to F_σ is $T_{F_\sigma} = \{z \in \mathbb{R}^{n,1} \mid \langle x, z\rangle = 0 \text{ for all } x \in \sigma\}$. Thus, T_{F_σ} is the orthogonal complement of the linear span of σ. In fact, by dimension arguments, $\sigma = (T_{F_\sigma})^\perp \cap D_v$.

Conversely, suppose that F is an $(n-k)$-face of $B(V)$. Let V_F be the vertex set of F. Let

$$\sigma_F = \bigcap_{w \in V_F} D_w\,.$$

Fix an element v in V_F. Let $(T_F)^\perp$ denote the orthogonal complement of the tangent space to F. Thus, $(T_F)^\perp$ is a $(k+1)$-dimensional timelike subspace of $\mathbb{R}^{n,1}$. Then a point x in D_v lies in σ_F if and only if $\langle x, w - v\rangle = 0$ for all $w \in V_F$. In other words, $\sigma_F = (T_F)^\perp \cap D_v$. It follows that the supporting hyperplanes containing F

are precisely the planes $E_{x,v}$ with $x \in \sigma_F$. Thus, σ_F is a non-empty, k-dimensional face of D_v. Call σ_F the *dual face* to F.

It is clear that $\sigma \to F_\sigma$ and $F \to \sigma_F$ are inverse functions between the sets of faces of the two cellulations. (We remark that if F is an n-dimensional face of $B(V)$, and E is the affine hyperplane spanned by F, and if x is the corresponding dual vertex defined by $x = (T_F)^\perp \cap \mathbb{H}^n$, then x is the center of the sphere Σ_E.) $\square$

The cellulation by Dirichlet cells is known as the "Voronoi diagram" for V. Its dual is the "Delaunay tesselation". (See, for example, [R].) Thus, Lemma 5 show that $B(V)$ projects to the Delaunay tesselation associated to V.

Theorem 6. *The natural piecewise Euclidean metric on $B(V)$ is $CAT(0)$.*

Proof. As explained in [G] or [BH], to show that a piecewise Euclidean complex is CAT(0) it suffices to show that it is simply connected and that the link of each vertex is CAT(1). The link of a vertex v is the piecewise spherical cell complex whose cells are the "solid angles" in the Euclidean cells containing v. That is, if F is a Euclidean cell containing v, then the inward pointing unit tangent vectors to F at v form a spherical cell, which we denote by $\mathrm{link}\,(v, F)$. These cells are glued together in a natural fashion to get a piecewise spherical complex, denoted by $\mathrm{link}\,(v, B(V))$.

Since $B(V)$ is homeomorphic to $\mathbb{H}^n$, it is simply connected. It remains to show that links of vertices in $B(V)$ are CAT(1). For this we appeal to the main theorem in [CD]. As explained in [CD], associated to a geodesically convex n-cell X in $\mathbb{H}^n$ there is a piecewise spherical complex $P(X)$, called its "polar dual", defined as follows. If σ is a proper face of X, then let C_σ be the convex polyhedral cone in $\mathbb{R}^{n,1}$ consisting of all vectors which are orthogonal to σ and point outward from X, i.e., $C_\sigma = \{z \in \mathbb{R}^{n,1} \mid \langle z, y\rangle = 0, \forall y \in \sigma \text{ and } \langle z, x\rangle \leq 0, \forall x \in X\}$. This cone is spanned by the outward pointing normal vectors to the codimension one faces of X containing σ. Let $\mathbb{S}^n_1$ denote the unit pseudo-sphere in $\mathbb{R}_{n,1}$ consisting of all vectors z with $q(z) = 1$. Set $\check{\sigma} = \mathbb{S}^n_1 \cap C_\sigma$. Since C_σ spans a spacelike subspace of $\mathbb{R}^{n,1}$, $\check{\sigma}$ is naturally a spherical cell. By definition, $P(X)$ is the union of the $\check{\sigma}$, where σ

ranges over the proper faces of X. The main result of [CD], Theorem 4.1.1, asserts that $P(X)$ is CAT (1).

Fix a vertex v in $B(V)$. We claim that the link of v in $B(V)$ is the polar dual of the Dirichlet region D_v. If F is a face of $B(V)$ containing v, then the inward pointing tangent vectors to F at v form a polyhedral cone, C_F, spanned by the tangent vectors to the edges of F containing v. (That is, C_F consists of all positive linear combinations of these tangent vectors.) Thus, $\mathrm{link}\,(v, F) = C_F \cap \mathbb{S}^n_1$. By Lemma 5, the edges F_i of F containing v are dual to the codimension 1 faces σ_{F_i} of D_v containing σ_F, and moreover, the tangent vector to the F_i is the outward pointing normal to the face σ_{F_i}. Thus, $C_F = C_{\sigma_F}$ and $\mathrm{link}\,(v, F) = \check{\sigma}_F$. It follows that the link of v in $B(V)$ is the polar dual of D_v as claimed. □

In [CD] it is proved not only that the polar dual of a hyperbolic convex polytope X is CAT(1) (or "large") but that it is *extra large* in the sense that the length of every closed geodesic in $P(X)$ is strictly greater than 2π and similarly for links in $P(X)$. It follows that the links of cells in $B(V)$ are also extra large in this sense. This is what one expects in this situation: the negative curvature of $\mathbb{H}^n$ has not disappeared, it has been concentrated on the codimension two skeleton of $B(V)$. The property of having extra large links is interesting because it is an open condition, in that, it is stable under small perturbations of the piecewise Euclidean structure. (See Lemma 6.1.2 in [CD].) Without this additional property the condition of being CAT(0) is not stable under small perturbations.

Corollary 7. *Let M^n be a complete hyperbolic n-manifold. Then M^n admits a piecewise Euclidean metric which is nonpositively curved in the sense of [G] (i.e., M^n has a piecewise Euclidean structure which is locally CAT(0)). Moreover, the links of vertices in this structure are extra large.*

Proof of the Corollary. Let W be a discrete subset of M^n such that any point in M^n lies within a bounded distance of W. (Such W obviously exist. For example, if M^n is compact, then we can take W to be a singleton.) Identify the universal cover of M^n with $\mathbb{H}^n$ and $\pi_1(M^n)$ with a discrete subgroup of $O(n, 1)$. Let $\pi\colon \mathbb{H}^n \to M^n$ be the covering projection. Set $V = \pi^{-1}(W)$. By construction, V satisfies the hypothesis of the theorem; hence, $B(V)$ is CAT (0). Moreover, $\pi_1(M^n)$ acts freely on V and acts isometrically on $B(V)$; hence, $B(V)/\pi_1(M^n)$ gives a nonpositively curved, piecewise Euclidean structure on M^n with extra large links. □

Remark 8. The same proof shows that any complete hyperbolic orbifold admits a nonpositively curved piecewise Euclidean structure in the sense that its universal orbifold cover has a CAT(0) piecewise Euclidean structure on which the fundamental orbifold group acts isometrically.

We end with some remarks on generalizing Theorem 6. Let us say that a subset B of $\mathbb{R}^{n,1}$ is a *spacelike convex hypersurface* if it is the boundary of a closed convex set S and if each supporting hyperplane of S is spacelike. Let us also exclude the case where B consists of two parallel hyperplanes. Then it is not difficult to see that B is homeomorphic to $\mathbb{R}^n$. We say that B is *polyhedral* if S is the convex hull of a discrete set, and that B is *smooth* if it is a smooth submanifold. Theorem 6 can be generalized as follows.

Theorem 9. *If B is a polyhedral, spacelike convex hypersurface in $\mathbb{R}^{n,1}$, then the natural piecewise Euclidean structure on B is CAT(0) with extra large links.*

Proof. Suppose B is the boundary of the closed convex set S. Let F be an $(n-k)$-dimensional face of B. Identify $(T_F)^\perp$ with $\mathbb{R}^{k,1}$. Let C denote the polyhedral cone in $\mathbb{R}^{k,1}$ of all inward-pointing normal vectors to F. We can identify link (F,B) with $\partial C \cap \mathbb{S}^n_1$. Let C^* denote the dual cone of C, that is,

$$C^* = \{x \in \mathbb{R}^{k,1} \mid \langle x, y\rangle \le 0 \text{ for all } y \in C\}.$$

Since all supporting hyperplanes of C are spacelike, C^* intersects $\mathbb{H}^k$ in a convex hyperbolic polytope $X = C^* \cap \mathbb{H}^k$. As explained in [CD], the polar dual $P(X)$ of this polytope is $\partial C \cap \mathbb{S}^k_1$. Thus, the link of F in B is the polar dual of X. The theorem now follows from the main theorem of [CD]. □

A smooth convex hypersurface in $\mathbb{R}^{n,1}$ is spacelike if and only if the normal direction at each point is timelike. It follows from the Gauss equation (see page 107 in [O]), that any smooth spacelike convex hypersurface in $\mathbb{R}^{n,1}$ has sectional curvature ≤ 0 and hence, is CAT(0).

Now suppose we are given an arbitrary spacelike convex hypersurface B in $\mathbb{R}^{n,1}$ and suppose that $(B_i)_{i\in\mathbb{N}}$ is a sequence of spacelike convex hypersurfaces converging to B. If each B_i is either (a) polyhedral or (b) smooth, then each B_i is naturally a CAT(0) geodesic metric space. Moreover, it seems likely (although the details have not yet been written down) that in either case, the metrics on the B_i converge to a geodesic metric on B and that this metric is CAT(0). Since such approximations always exist, this would show that any spacelike convex hypersurface has a natural CAT(0) metric. In particular, if B is polyhedral, we obtain another proof that its natural metric is CAT(0) by approximating B by smooth convex hypersurfaces. (We thank B. Kleiner, C. Weber, and J.-M. Schlenker for pointing this out to us.) However, this argument does not show that links in B are extra large. Conversely, if B is smooth, we obtain a proof that its natural metric is CAT(0) without reference to the Gauss equation by approximating B by polyhedral hypersurfaces.

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