NOTES ON NONPOSITIVELY CURVED POLYHEDRA

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CONTENTS

Introduction.

At the turn of the century Poincaré showed how to associate to any topological space X a group, $\pi_1(X)$, called its fundamental group. Any group π is isomorphic to $\pi_1(X)$ for some space X. In fact, there is a particularly good choice of X, namely, the classifying space $B\pi$ of π. The space $B\pi$ is a CW complex and it is aspherical (meaning that its higher homotopy groups vanish) and, up to homotopy, it is characterized by these properties. Thus, the study of aspherical spaces lies at the center of the intersection of topology and group theory.

In an influential paper, [G1], Gromov focused attention on those groups which arise as the fundamental groups of nonpositively curved, compact manifolds or spaces. The nonpositive curvature condition has many group theoretical implications. For example, the word problem and conjugacy problem can be solved for such groups and any solvable subgroup of such a group must be virtually abelian (i.e., it must contain an abelian subgroup of finite index). The case where the curvature is required to be strictly negative is even more interesting. The negative curvature property can be defined in completely group theoretic terms and this leads to the notion of "word hyperbolic groups".

A second aspect of Gromov's paper was that he described many new techniques for constructing examples of nonpositively curved spaces, in particular, he gave many examples of nonpositively curved polyhedra. It is this second aspect which we shall focus on in these notes. For the most part we ignore the group theoretical implications of nonpositive curvature. Our focus is on the construction of examples.

In low dimensional topology aspherical manifolds and spaces play an important role. Aspherical 2-manifolds are the rule rather than the exception and here asphericity is always caused by nonpositive curvature. In dimension 3, it is a consequence of the Sphere Theorem that the prime 3-manifolds with infinite fundamental group are aspherical (with a few exceptions). The content of Thurston's Geometrization Conjecture is that these prime manifolds can further decomposed along tori into geometric pieces, the most significant type of which are negatively curved.

These notes reflect the material which was presented in lectures (by the first author) and recitations (by the second author) at the Turán Workshop on Low Dimensional Topology, Budapest, Hungary in August 1998. There is a substantial overlap of this material with [D6].

Chapter I: Nonpositively Curved Spaces.

1. Basic facts and definitions.

1.1 Aspherical CW complexes. Suppose that X is a connected topological space and that it is homotopy equivalent to a CW complex. Let $\tilde{X}$ denote its universal covering space. A basic fact of covering space theory is that the higher homotopy groups of X (that is, the groups $\pi_i(X)$ for $i > 1$) are equal to those of $\tilde{X}$. Hence, the following two conditions are equivalent:

(i) $\pi_i(X) = 0$ for all $i > 1$, and

(ii) $\tilde{X}$ is contractible.

If either condition holds, then X is said to be *aspherical.* (Sometimes we also say that X is a "$K(\pi, 1)$-space", where $\pi = \pi_1(X)$.)

For any group π there is a standard construction of an aspherical CW complex X with fundamental group π. Moreover, if X' is another such $K(\pi, 1)$-complex, then X and X' are homotopy equivalent. (However, in general it is not possible to choose X to be a finite CW complex or even a finite dimensional one.)

Examples of aspherical spaces. (All spaces here are assumed to be connected.)

• *Dimension 1.* Suppose X is a graph. Its universal cover $\tilde{X}$ is a tree, which is contractible. Hence, any graph is aspherical. Its fundamental group is a free group.

• *Dimension 2.* Suppose that X is a closed orientable surface of genus g where $g > 0$. It follows from the Uniformization Theorem of Riemann and Poincaré that X can be given a Riemannian metric so that its universal cover $\tilde{X}$ can be identified with either the Euclidean or hyperbolic plane (as $g = 1$ or $g > 1$, respectively). Hence, any such X is aspherical.

• *Dimension 3.* It follows from Papakyriakopoulos' Sphere Theorem that any closed, irreducible 3-manifold with infinite fundamental group is aspherical. ("Irreducible" means that every embedded 2-sphere bounds a 3-ball.)

• *Tori.* The n-dimensional torus $T^n (= \mathbb{R}^n/\mathbb{Z}^n)$ is aspherical, since its universal cover is $\mathbb{R}^n$. The same is true for all complete Euclidean manifolds.

• *Hyperbolic manifolds.* The universal cover of any complete hyperbolic n-manifold X can be identified with hyperbolic n-space $\mathbb{H}^n$. Since $\mathbb{H}^n$ is contractible, X is aspherical. (In other words, $X = \mathbb{H}^n/\Gamma$ where Γ is a discrete, torsion-free subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, the isometry group of $\mathbb{H}^n$).

• *Manifolds of nonpositive sectional curvature.* More generally, complete Riemannian manifolds of nonpositive sectional curvature are aspherical. This follows from the Cartan–Hadamard Theorem stated below. (The "sectional curvature" of a Riemannian manifold is a function, derived from the curvature tensor, which assigns a number to each tangent 2-plane. In dimension 2 it coincides with the usual Gaussian curvature. For the purpose of reading these notes it is not necessary to know much more about the definition.)

Cartan–Hadamard Theorem. *Suppose that M^n is a complete Riemannian manifold of nonpositive sectional curvature. Then for any $x \in M^n$ the exponential map,* $\exp : T_x M^n \to M^n$ *is a covering projection.*

In particular, since the vector space $T_x M^n$ is contractible, this theorem implies that M^n is aspherical.

Our initial goals in this chapter will be to broaden the definition of "nonpositive curvature" to include singular metric spaces, i.e., metric spaces which are not Riemannian manifolds (in fact, which need not even be topological manifolds) and then to give examples of some singular spaces with nonpositively curved polyhedral metrics.

1.2. Spaces of constant curvature.

• *Euclidean n-space, $\mathbb{E}^n$.* The *standard inner product* on $\mathbb{R}^n$ is defined by $x \cdot y = \Sigma x_i y_i$. *Euclidean n-space* is the affine space associated to $\mathbb{R}^n$ with distance d defined by $d(x, y) = |x - y| = [(x - y) \cdot (x - y)]^{\frac{1}{2}}$.

• *The n-sphere, $\mathbb{S}^n$.* Set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid x \cdot x = 1\}$. Given two vectors x and y in $\mathbb{S}^n$, define the angle θ between them by $\cos\theta = x \cdot y$. The (spherical) distance d between x and y is then defined by $d(x, y) = \theta = \cos^{-1}(x \cdot y)$.

• *Hyperbolic n-space, $\mathbb{H}^n$. Minkowski space* $\mathbb{R}^{n,1}$ is isomorphic as a vector space to $\mathbb{R}^{n+1}$ but it is equipped with an indefinite symmetric bilinear form of signature $(n, 1)$ (i.e., a "Lorentzian inner product"). If $(x_0, x_1, \dots, x_n)$ are coordinates for a point $x \in \mathbb{R}^{n,1}$, then the Lorentzian inner product is defined by

$$x \cdot y = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

Hyperbolic n-space is defined by $\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} \mid x \cdot x = -1 \text{ and } x_0 > 0\}$. (The hypersurface $x \cdot x = -1$ is a 2-sheeted hyperboloid, $\mathbb{H}^n$ is one of the sheets.)

Figure 1.2.1. Hyperbolic n-space

The formula for distance is given by $d(x, y) = \cosh^{-1}(-x \cdot y)$.

The spaces $\mathbb{E}^n$ and $\mathbb{S}^n$ are clearly Riemannian manifolds. Since the restriction of the Lorentzian inner product to $T_x\mathbb{H}^n$ is positive definite for any $x \in \mathbb{H}^n$, hyperbolic space also has a natural Riemannian structure (and $d(x, y) = \cosh^{-1}(-x \cdot y)$ is the associated metric). It turns out that in each case the sectional curvature is constant: for $\mathbb{H}^n, \mathbb{R}^n$ and $\mathbb{S}^n$ these constants are, respectively, $-1, 0$ and $+1$. If we rescale the metric on $\mathbb{S}^n$ by a factor of λ, then the new metric has curvature λ^{-2}. Similarly, rescaling by λ changes the metric on $\mathbb{H}^n$ to one of curvature $-\lambda^{-2}$. It is a classical theorem of Riemannian geometry that these spaces are the only simply connected complete Riemannian manifolds of constant sectional curvature, up to isometry.

Notation. For each real number ε and integer $n \geq 2$, let M^n_ε denote the complete simply connected Riemannian n-manifold of constant curvature ε. Thus, $M^n_0 = \mathbb{E}^n$ and for $\varepsilon \neq 0$, M^n_ε is equal to $\mathbb{H}^n$ or $\mathbb{S}^n$ with rescaled metric as ε is < 0 or > 0, respectively.

1.3. Geodesic spaces.

Suppose (X, d) is a metric space and that $\gamma : [a, b] \to X$ is a path. The *length* of γ, denoted by $\ell(\gamma)$, is defined by

$$\ell(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \right\}$$

where $(t_0, \dots, t_n)$ runs over all possible subdivisions of $[a, b]$ (i.e., $a = t_0 < t_1 < \cdots < t_n = b$). It follows immediately from the triangle inequality that $d(\gamma(a), \gamma(b)) \leq \ell(\gamma)$.

Definition 1.3.1. The path $\gamma : [a, b] \to X$ is a *geodesic* if it is an isometric embedding: $d(\gamma(s), \gamma(t)) = |s - t|$, for any $s, t \in [a, b]$. Similarly, a loop $\gamma : S^1 \to X$ is a *closed geodesic* if it is an isometric embedding. Here S^1 denotes the standard circle equipped with its arc metric (possibly rescaled so that its length can be arbitrary).

Remark. Suppose that $\gamma : [a, b] \to X$ is any path such that $\ell(\gamma) = d(\gamma(a), \gamma(b))$. It then follows that for any subinterval $[s, t] \subset [a, b]$, $\ell(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t))$. Hence, if we reparametrize γ by arc length we obtain a geodesic.

Definition 1.3.2. Suppose (X, d) is a metric space. Then X is a *length space* if for any $x, y \in X$, $d(x, y) = \inf\{\ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } y\}$. (Here we allow ∞ as a possible value of d.) X is a *geodesic space* if the above infimum is always realized and is $\neq \infty$, i.e., if there is a geodesic segment between any two points x and y in X.

Example 1.3.3. Suppose X is the boundary of a convex polytope in Euclidean space. For any path $\gamma : [a, b] \to X$ define its "piecewise Euclidean length", $\ell_{PE}(\gamma)$ by

$$\ell_{PE}(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_{i-1}) - \gamma(t_i)| \right\}$$

where the subdivisions $a = t_0 < \cdots < t_n = b$ are such that $\gamma([t_{i-1}, t_i])$ is always contained in some face of X. Then define a distance function on X by

$$d(x,y) = \inf\{\ell_{PE}(\gamma) \mid \gamma \text{ is a path from } x \text{ to } y\}.$$

It is easily checked that (X,d) is a geodesic space. Furthermore, the metric is compatible with the topology of X.

1.4. The $CAT(\varepsilon)$-inequality.

Suppose that (X,d) is a geodesic space, that ε is a real number and that M^2_ε is the 2-dimensional space of curvature ε, as in 1.2.

Definition 1.4.1. A *triangle* T in X is a configuration of three geodesic segments ("edges") connecting three points (its "vertices"). A *comparison triangle* for T is a triangle T^* in M^2_ε with the same edge lengths. For such a triangle to exist when $\varepsilon > 0$, it is necessary to assume $\ell(T) \le 2\pi/\sqrt{\varepsilon}$, where $\ell(T)$ is the sum of edge lengths of T. (The number $2\pi/\sqrt{\varepsilon}$ is the length of the equator in a 2-sphere of curvature ε). When $\varepsilon > 0$ we shall always make this assumption on $\ell(T)$.

If T^* is a comparison triangle for T, then for each edge of T we have a well defined isometry, denoted by $x \to x^*$, which takes the given edge of T isometrically onto the corresponding edge of T^*. With a slight abuse of language, we use the same notation for all three mappings.

Definition 1.4.2. Given a real number ε, a metric space X *satisfies* $CAT(\varepsilon)$ (or is a $CAT(\varepsilon)$-*space*), if the following two conditions hold:

(a) there exists a geodesic segment between any two points in X of distance less than $\pi/\sqrt{\varepsilon}$ apart if $\varepsilon > 0$, and between any two points if $\varepsilon \le 0$;

(b) the $CAT(\varepsilon)$-inequality: for any triangle T in X (with $\ell(T) < 2\pi/\sqrt{\varepsilon}$ if $\varepsilon > 0$) and for any two points x, y in T we have that

$$d(x,y) \le d^*(x^*, y^*)$$

where d^* denotes distance in M^2_ε and x^*, y^* are the corresponding points in the comparison triangle T^*.

Figure 1.4.1. Triangle comparison

Remark. Condition (a) says that X is a geodesic space if $\varepsilon \le 0$ (which in particular implies that X is connected). However, if $\varepsilon > 0$, then X need not be connected; only its connected components are required to be geodesic spaces. We shall see later that certain naturally arising $CAT(1)$-spaces may be disconnected, cf. Theorem 2.2.3.

Definition 1.4.3. Suppose that X is a geodesic space and γ_1, γ_2 are two arc length parametrized geodesic segments in X with a common initial point $x = \gamma_1(0) = \gamma_2(0)$. The *Aleksandrov angle* between γ_1 and γ_2 at x is defined as

$$\limsup_{s,t \to 0} \alpha(s,t)$$

where $\alpha(s,t)$ is the angle at the vertex corresponding to x of the comparison triangle in M_0^2 $(= \mathbb{E}^2)$ for the triangle with vertices at x, $\gamma_1(s)$ and $\gamma_2(t)$.

Figure 1.4.2. Aleksandrov angle

Remark. Using the exponential map it is easy to see that in the Riemannian case the Aleksandrov angle equals the usual infinitesimal concept of angle.

The following characterization of $CAT(\varepsilon)$-spaces in terms of angles is due to Aleksandrov. Its proof is left to the reader as an exercise in elementary geometry. Details may be found in Troyanov's article in [GH].

Proposition 1.4.4. *A metric space X satisfies $CAT(\varepsilon)$ if and only if condition (a) of Definition 1.4.2 and the following condition (b′) hold:*

(b′) The Aleksandrov angle between each pair of edges of any triangle in X (of perimeter $< 2\pi/\sqrt{\varepsilon}$ if $\varepsilon > 0$) does not exceed the corresponding angle of the comparison triangle in M_ε^2.

Remark. It turns out that in $CAT(\varepsilon)$ -spaces the function $\alpha(s,t)$ of Definition 1.4.3 is monotone non-decreasing, so its limsup is actually a limit. In fact, this property of $\alpha(s,t)$ may serve as another equivalent version of condition (b) of Definition 1.4.2, cf. [GH].

Definition 1.4.5. The space X has *curvature* $\leq \varepsilon$ (written as $\kappa(X) \leq \varepsilon$ or $\kappa \leq \varepsilon$) if $CAT(\varepsilon)$ holds locally in X.

The justification for the above definition is the following Comparison Theorem. (For a proof see the article by Troyanov in [GH].)

Theorem 1.4.6. (Aleksandrov, Toponogov) *A complete Riemannian manifold has sectional curvature $\leq \varepsilon$ if and only if the $CAT(\varepsilon)$-inequality holds locally.*

Remark. The reader may easily verify (using, say, Proposition 1.4.4 and the Laws of Cosines, cf. 1.6, below) that M_ε^2 satisfies $CAT(\delta)$ for all $\delta \geq \varepsilon$. Thus, $\kappa(X) \leq \varepsilon$ implies $\kappa(X) \leq \delta$ if $\delta \geq \varepsilon$, which justifies the notation.

Historical Remarks. In the 1940's and 50's Aleksandrov introduced the idea of a length metric (also called an "intrinsic" or "inner" metric) and the idea of curvature bounds on length spaces. He was primarily interested in nonnegative curvature, $\kappa \geq 0$ (defined by reversing the $CAT(0)$-inequality). His motivation was to prove the result that a complete geodesic metric on a 2-manifold has $\kappa \geq 0$ if and only if it is isometric to the boundary of a convex set in $\mathbb{E}^3$. He proved this is the case of polyhedral metrics and then used the technique of polyhedral approximation to get the same result for arbitrary geodesic metrics. About the same time, Busemann [Bu] was studying geodesic spaces satisfying a weaker version of the $CAT(0)$ condition (although he also required that geodesics be uniquely extendible). The subject of nonpositively curved geodesic spaces was repopularized by Gromov in the mid-1980's in [G1] in the context of providing examples of "coarse hyperbolic" metric spaces and "word hyperbolic groups".

Remark. Although much of the theory of $CAT(\varepsilon)$-spaces can be carried out in greater generality (cf. [BH]), we here make the blanket assumption that all metric spaces which we consider are locally compact and complete. This allows us the possibility of sketching simple proofs of some of the key theorems in Chapter I. Moreover, our examples and constructions will only involve such spaces.

1.5. Some consequences of the $CAT(\varepsilon)$ conditions.

Here we discuss some properties of any $CAT(\varepsilon)$-space X. We are mainly interested in the cases when $\varepsilon = 0$ or $\varepsilon = 1$.

• *There are no nondegenerate digons (of perimeter $< 2\pi/\sqrt{\varepsilon}$ if $\varepsilon > 0$).* By a *digon* in X we mean a configuration of two geodesic segments connecting two points in X. The digon is *nondegenerate* if the segments are distinct. Indeed, given any such digon (of perimeter $< 2\pi/\sqrt{\varepsilon}$ if $\varepsilon > 0$) by introducing a third vertex in the interior of one segment we obtain a triangle in X. Its comparison triangle is degenerate (i.e., it is a line segment); hence, the digon must also be degenerate. In particular,

• *If X is $CAT(1)$, then there is a unique geodesic between any two points of distance $< \pi$.* Similarly, the length of any closed geodesic in X must be $\geq 2\pi$.

• *If X is $CAT(0)$, then there is a unique geodesic between any two points.*

By the Arzelà–Ascoli Lemma about equicontinuous sequences of maps between compact metric spaces, these (unique) geodesic segments vary continuously with their endpoints. Thus, by moving points along geodesics to an arbitrarily chosen base point we obtain a (continuous) homotopy, the so-called *geodesic contraction.* As a consequence we have the following.

Theorem 1.5.1. *(i) Any $CAT(0)$-space is contractible.*

(ii) Similarly, if X is a $CAT(1)$-space, then the open ball of radius π about any point is contractible.

Remark. Complete geodesic spaces satisfying $CAT(0)$ are sometimes also called "Hadamard spaces" instead of $CAT(0)$ spaces.

A real-valued function f on a geodesic space is *convex* if its restriction to any geodesic segment is a convex function. If X is a geodesic space, then so is $X \times X$ (with the product metric). Moreover, geodesics in $X \times X$ are of the form $t \mapsto (\gamma_1(t), \gamma_2(t))$, where γ_1 and γ_2 are constant speed reparametrizations of two geodesics in X (allowing the possibility that one of γ_1 and γ_2 is constant). It follows that a function $f : X \times X \to \mathbb{R}$ is convex if and only if for any two (of constant, but not necessarily unit, speed) geodesics $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$, the function $[0,1] \to \mathbb{R}$ defined by $t \mapsto f(\gamma_1(t), \gamma_2(t))$ is a convex function. Another property of $CAT(0)$ spaces is the following (see Ballmann's article in [GH]).

- *If X is a $CAT(0)$ space, then the distance function $d : X \times X \to \mathbb{R}$ is convex.*

The proof of this statement is essentially a direct application of the $CAT(0)$-inequality to a configuration consisting of two triangles, see Figure 1.5.1. Details are left to the reader.

Figure 1.5.1. Convexity of metric

The following generalization of the Cartan–Hadamard Theorem was first stated by Gromov in [G1].

Theorem 1.5.2. (Gromov's version of the Cartan–Hadamard Theorem) *Suppose X is a locally compact and complete geodesic space with $\kappa(X) \leq \varepsilon$, where $\varepsilon \leq 0$. If X is simply connected, then it is a $CAT(\varepsilon)$ space.*

Corollary 1.5.3. *If X is a complete, nonpositively curved geodesic space, then it is aspherical.*

Proof of the Corollary. By Theorem 1.5.2, the universal cover of X is $CAT(0)$ and by Theorem 1.5.1, this implies that it is contractible. □

Sketch of proof of Theorem 1.5.2. (based on [BH] and Ballmann's article in [GH]) The first main step of the proof is existence and uniqueness of geodesics. Our technical tool in this direction is an adaptation of the so-called "Birkhoff curve-shortening process" which was classically designed to produce geodesics in Riemannian manifolds.

In a nonpositively curved metric space the Birkhoff process consists of iterated application of a construction that replaces a given rectifiable curve c with a broken geodesic $B(c)$ as indicated in Figure 1.5.2.

At the start we choose an equidistant subdivision of c which is fine enough so that nearby division points fall inside a common $CAT(0)$ patch. This ensures that the Birkhoff step comes together with a homotopy connecting c with $B(c)$. The Arzelà–Ascoli Lemma implies that the Birkhoff process converges, i.e., the sequence $c, B(c), B(B(c)), \ldots$ is uniformly convergent to a curve $\tilde{c}$. Obviously $B(\tilde{c}) = \tilde{c}$, which implies that $\tilde{c}$ is a geodesic when restricted to suitable neighborhoods of its points. (Such a curve is called a *local geodesic*.)

Figure 1.5.2. Birkhoff curve shortening

It follows from the above argument that in a nonpositively curved metric space each homotopy class of curves with given endpoints contains a local geodesic. Now we show that such a local geodesic is unique. Suppose that γ_0 and γ_1 are local geodesics homotopic relative to their common endpoints. The Birkhoff process applied to each stage of a homotopy (relative to endpoints) results in another homotopy (relative to endpoints) $\gamma_s (0 \leq s \leq 1)$ through local geodesics. Now, if $|s_1 - s_2|$ is sufficiently small, then (by an argument similar to the one preceding Theorem 1.5.2) the function $t \mapsto d(\gamma_{s_1}(t), \gamma_{s_2}(t))$ is convex. Hence, it is identically 0 which implies that $\gamma_0 = \gamma_1$.

In summary, we have proved that in a nonpositively curved metric space there exists a unique local geodesic in any homotopy class of paths with given endpoints. In particular, if the space is simply connected, then we have a unique local (and therefore, global) geodesic segment connecting any given pair of points.

Now suppose that X is simply connected with $\kappa(X) \leq \varepsilon \leq 0$. In order to prove the $CAT(\varepsilon)$-inequality for an arbitrary triangle T, first subdivide T in a way indicated in Figure 1.5.3 so that each of the small triangles is contained in some $CAT(\varepsilon)$ patch. (This is possible since uniqueness of geodesics implies continuous dependence of geodesics on their endpoints.)

Figure 1.5.3. Subdivision of T

We use induction to show that larger and larger subtriangles of T satisfy the angle comparison condition (b') of Proposition 1.4.4. First join together the small triangles in succession within each of the narrow triangles in Figure 1.5.3, starting at the vertex x, then join the narrow triangles together, one by one, eventually arriving at T. The inductive step in this process is the following Gluing Lemma of Aleksandrov.

Lemma 1.5.4. (Aleksandrov's Gluing Lemma) *Suppose that in a geodesic metric* *space a triangle* Δ *is subdivided into triangles* Δ_1 *and* Δ_2 *by a geodesic segment connecting a vertex of* Δ *with an interior point of the opposite edge. If the Aleksandrov angles of* Δ_1 *and* Δ_2 *do not exceed the corresponding angles of their comparison triangles in* M^2_ε*, then the same is true for* Δ.

The Gluing Lemma follows immediately when one applies the following observation (due to Aleksandrov) to the comparison triangles joined along their common edge. (The proof of this observation only involves elementary geometry of M^2_ε and is left to the reader as an exercise.)

Suppose that a quadrilateral in M^2_ε has a concave angle at vertex v. "Straighten out" the pair of sides joining up at v, keeping all side lengths unchanged. Then all three remaining angles increase.

Figure 1.5.4. Aleksandrov's Lemma

This completes our sketch of the proof of Theorem 1.5.2. □

Let us now turn to the situation where $\varepsilon > 0$. We need a local-to-global type statement analogous to Theorem 1.5.2. That is, we look for a property of metric spaces which, together with the condition $\kappa \leq \varepsilon$, implies $CAT(\varepsilon)$. The following examples show that simple connectivity is neither necessary nor sufficient here.

- A circle of length $\geq 2\pi$ is $CAT(1)$.

- The standard S^2 with the interior of a small open triangle removed is simply connected and is not $CAT(1)$ since the $CAT(1)$-inequality fails for the boundary triangle. On the other hand, it will follow immediately from results in Section 2.2 that this space has curvature ≤ 1. (By doubling along boundary we get an example of a metric of curvature ≤ 1 on the 2-sphere which is not $CAT(1)$.)

These examples suggest that the failure of $CAT(1)$ is caused by the presence of short closed geodesics. That this is indeed the case in general is the content of the following companion result to Theorem 1.5.2.

Theorem 1.5.5. (Gromov) *Suppose* X *is a compact geodesic space with* $\kappa(X) \leq \varepsilon$*, where* $\varepsilon > 0$*. Then* X *is* $CAT(\varepsilon)$ *if and only if every closed geodesic has length* $\geq 2\pi/\sqrt{\varepsilon}$.

Sketch of proof. Almost all steps of the proof of Theorem 1.5.2 can be adapted, with appropriate limitations on lengths of geodesics and sizes of triangles, to cover the case $\varepsilon > 0$. The only exception is the statement that relies on the convexity property of the metric (which is false if $\varepsilon > 0$), namely, uniqueness of local geodesics. In fact, the proof of Theorem 1.5.2 adapted to the case $\varepsilon > 0$ yields the following:

Suppose that X is a compact metric space with $\kappa(X) \leq \varepsilon$. Let $r(X)$ denote the supremum of real numbers d such that there exists a unique geodesic in X between any pair of points less than distance d apart. (This (positive) number $r(X)$ is sometimes called the *injectivity radius* of X.) Then Aleksandrov's angle comparison inequalities hold in X for all triangles of perimeter $< 2r(X)$.

Thus, the correct substitute for uniqueness of geodesics is the following lemma.

Lemma 1.5.6. *If $\kappa(X) \leq \varepsilon$ and $r(X) < \pi/\sqrt{\varepsilon}$, then X contains a closed geodesic of length $2r(X)$.*

Proof. Take a sequence of nondegenerate digons D_n whose perimeters $\ell(D_n)$ converge down to $2r(X)$. By compactness and the Arzelà–Ascoli Lemma we may assume that the sequence converges uniformly to a digon D. Let d_n denote the distance between midpoints of the two arcs of D_n. Then d_n cannot be less than $2r(X) - \frac{1}{2}\ell(D_n)$ (which converges to $r(X)$) since otherwise angle comparison would work for both triangles (of perimeter $< 2r(X)$) into which D_n is subdivided by the segment connecting the two midpoints; therefore, by Aleksandrov's Gluing Lemma, angle comparison would work for D_n which would imply that D_n is degenerate, a contradiction. The limit digon D is therefore nondegenerate, and a similar argument shows that the distance between any pair of opposite points of D equals $r(X)$. This means that D is a closed geodesic of length $2r(X)$. $\square$

Remark. From the point of view of topology, Corollary 1.5.3 is of great practical importance since it provides a method of proving asphericity. In fact, if a topological space is aspherical, then it is very often so because the space can be equipped with a complete nonpositively curved metric compatible with its topology. However, this is not always the case, as the following examples show. These examples are compact aspherical spaces that do not carry any nonpositively curved metric.

• Consider the "dunce hat" D which is obtained from a triangle by identifying edges as shown in Figure 1.5.5.

Figure 1.5.5. Dunce hat

Suppose that D admits a nonpositively curved metric. Then, since D is contractible, this metric is $CAT(0)$ by Theorem 1.5.2. Choose two points p and q whose distance equals the diameter of D. Then geodesic contraction of $D - \{q\}$ onto p shows that $D - \{q\}$ is contractible. But, by inspection, $D - \{q\}$ is never contractible.

• Consider a compact solvmanifold, i.e., a manifold M of the form $M = G/\Gamma$ where G is a noncommutative solvable Lie group and Γ is a cocompact discrete subgroup. Such examples exist in all dimensions above 2. For example, M can be any closed 3-manifold with Sol- or Nil- structure (see W. Neumann's notes in this volume). It is a well-known fact about groups acting on $CAT(0)$ spaces (see [BH]) that a solvable group can act discretely, isometrically and with compact quotient on a complete $CAT(0)$ space only if it has an abelian subgroup of finite index. It follows that such an M does not admit any metric of nonpositive curvature.

1.6. The cone on a $CAT(1)$ space.

If X is a topological space, then the *cone on* X, denoted CX, is defined to be the quotient space of $X \times [0, \infty)$ by the equivalence relation $\sim$ defined by $(x, s) \sim (y, t)$

if and only if $(x, s) = (y, t)$ or $s = t = 0$. The image of (x, s) in CX will be denoted by $[x, s]$. The *cone of radius* r is the image of $X \times [0, r]$ in CX.

If X is a metric space,then for $\varepsilon = +1, 0$, or -1 we will now define a metric d_ε on CX (when $\varepsilon = +1$ we will only consider the cone of radius $\pi/2$). The idea behind this definition is that when $X = \mathbb{S}^{n-1}$, by using "polar coordinates" and the exponential map, CX can be identified with $\mathbb{E}^n$ (if $\varepsilon = 0$) or $\mathbb{H}^n$ (if $\varepsilon = -1$). Similarly, the cone of radius $\pi/2$ can be identified with a hemisphere in $\mathbb{S}^n$ (if $\varepsilon = +1$). Transporting the constant curvature metric to $C\mathbb{S}^{n-1}$ we obtain a formula for d_ε on $C\mathbb{S}^{n-1}$. The same formula then defines a metric on CX for an arbitrary metric space (X, d).

To write this formula we first need to recall the Law of Cosines in $\mathbb{E}^2, \mathbb{S}^2$ or $\mathbb{H}^2$. Suppose we are given a triangle in M^2_ε with edge lengths s, t and d and angle θ between the first two sides.

Figure 1.6.1. A triangle in M^2_ε

Law of Cosines.

- in $\mathbb{R}^2$:

$$d^2 = s^2 + t^2 - 2st\cos\theta$$

- in $\mathbb{S}^2$:

$$\cos d = \cos s \cos t + \sin s \sin t \cos\theta$$

- in $\mathbb{H}^2$:

$$\cosh d = \cosh s \cosh t - \sinh s \sinh t \cos\theta.$$

Given two points x, y in X, put $\theta(x, y) = \min\{\pi, d(x, y)\}$. Given points $[x, s]$ and $[y, t]$ in CX define the metric d_0 by the formula

$$d_0([x, s], [y, t]) = (s^2 + t^2 - 2st\cos\theta(x, y))^{1/2}.$$

Similarly, define d_{-1} and d_{+1} so that the appropriate Law of Cosines holds with $d = d_\varepsilon$ and $\theta = \theta(x, y)$. Let $C_\varepsilon X$ denote CX equipped with the metric d_ε. (When $\varepsilon = 1$, for $C_1(X)$ we only consider the cone of radius $\frac{\pi}{2}$.) A proof of the following theorem can be found in [BH].

Theorem 1.6.1. *Suppose X is a complete geodesic space. Then*

(i) $C_\varepsilon X$ is a complete geodesic space.

(ii) $C_\varepsilon(X)$ is $CAT(\varepsilon)$ if and only if X is $CAT(1)$.

Definition 1.6.2. The *spherical suspension* of X, denoted by SX, is the union of two copies of C_1X glued together along $X \times \frac{\pi}{2}$. Equivalently, it can be defined as a $X \times [0, \pi]$ with both ends $X \times 0$ and $X \times \pi$ pinched to points and with metric defined by the same formula as for d_{+1}.

If X is $CAT(1)$, then so is SX.

2. Polyhedra of piecewise constant curvature.

2.1 Definitions and Examples.

As $\varepsilon = +1, 0$ or -1, M^n_ε stands for $\mathbb{S}^n, \mathbb{E}^n$ or $\mathbb{H}^n$. A *hyperplane* in $\mathbb{S}^n$ or $\mathbb{H}^n$ means the intersection of the model hypersurface with a linear hyperplane. A *hyperplane* in $\mathbb{E}^n$ is just an affine hyperplane. A *half-space* in M^n_ε is the closure of one of the two components bounded by a hyperplane.

Definition 2.1.1. A subset C of M^n_ε is a *convex cell* (or a "convex polytope") if

- C is compact, and
- C is the intersection of a finite number of half-spaces.

When $\varepsilon = 1$ and $C \subset \mathbb{S}^n$ we shall also require that C contain no pair of antipodal points.

Definition 2.1.2. An *M_ε-cell complex* is a cell complex which is constructed by gluing together convex cells in M^n_ε via isometries of their faces.

Terminology. Suppose X is an M_ε-cell complex. As $\varepsilon = +1, 0$, or -1 we shall say that X is *piecewise spherical* (abbreviated PS), *piecewise Euclidean* (abbreviated PE) or *piecewise hyperbolic* (abbreviated PH).

Next we shall give the definition of a length metric d which is naturally associated to an M_ε cell structure on X. Let d_ε denote the usual metric on M^n_ε. Given a path $\gamma : [a, b] \to X$ define its *piecewise M_ε-length*, $\ell_\varepsilon(\gamma)$, by the formula

$$\ell_\varepsilon(\gamma) = \sup \left\{ \sum_{i=1}^{k} d_\varepsilon(\gamma(t_{i-1}), \gamma(t_i)) \right\}$$

where $(t_0, \ldots, t_k)$ runs over all subdivisions of $[a, b]$ such that $\gamma(t_{i-1})$ and $\gamma(t_i)$ are contained in the same cell of X. Given two points x, y in X, $d(x, y)$ is then defined to be the infimum of $\ell_\varepsilon(\gamma)$ over all paths γ from x to y.

Henceforth, we shall make the blanket assumption that *any M_ε-complex is either a finite complex or a covering space (in the metric sense) of a finite complex.* In particular, any such complex is locally finite. (It is possible to get away with a weaker assumption such as only a finite number of isometry classes of cells occur.)

Lemma 2.1.3. *Suppose X is a M_ε-cell complex and d is the metric defined above. Then (X, d) is a complete geodesic space. Furthermore, the restriction of d to each cell of X is locally isometric to the original M_ε-metric of the cell.*

Question: When is $\kappa(X) \le \varepsilon$?

Before considering this question in detail let us consider some examples in dimension one and two.

One-dimensional examples. A 1-dimensional cell complex is a graph. Giving it an M_ε structure amounts to assigning a length to each edge. Let us call such a graph with assigned edge lengths a *metric graph.*

• A metric tree is $CAT(\varepsilon)$ for any ε. (Proof: a triangle in such a tree is a "tripod" which obviously satisfies $CAT(\varepsilon)$ for any ε.) Conversely, if $\varepsilon \leq 0$, then it follows from Theorem 1.5.1 *(i)* that any $CAT(\varepsilon)$ graph must be a tree.

• It follows that any metric graph X has $\kappa(X) \leq \varepsilon$ for any ε.

• In the case $\varepsilon = 1$, Theorem 1.5.5 implies that a metric graph X is $CAT(1)$ if and only if the length of every circuit in X is $\geq 2\pi$.

Two-dimensional examples.

• Suppose X is a $CAT(1)$ graph. Then, by Theorem 1.6.1 (ii), the cone $C_\varepsilon X$ is $CAT(\varepsilon)$ for any ε.

• Next suppose that Y is a 2-dimensional M_ε-cell complex and that $y \in Y$. Does y have a neighborhood which is $CAT(\varepsilon)$? This is automatic if y belongs to the interior of a 2-cell. It is also easy to see that the answer is yes if y is in the interior of an edge. However, when y is a vertex, we must examine the situation more carefully. In this case a sufficiently small neighborhood is isometric to a neighborhood of the cone point in $C_\varepsilon L$ where L is the so-called "link of y". That is to say, L is the graph with an edge for each pair (σ^2, y) where σ^2 is a 2-cell containing y and a vertex for each pair (σ^1, y) where σ^1 is an edge containing y. The edge corresponding to (σ^2, y) is assigned the length θ, the angle of σ^2 at its vertex y. Thus, Y is $CAT(\varepsilon)$ in a neighborhood of y if and only if $C_\varepsilon L$ is $CAT(\varepsilon)$, i.e., if and only if the metric graph L is $CAT(1)$. (This idea will be generalized to higher dimensions in the next section.)

• If a metric graph L is $CAT(1)$, then so is any subgraph L'. (If L has no circuits of length $\leq 2\pi$, then neither does L'.) It follows that if a 2-dimensional M_ε-cell complex Y has $\kappa \leq \varepsilon$, then so does any subcomplex Y'. (For $\varepsilon \leq 0$, this is a special case of Whitehead's Conjecture that a subcomplex of an aspherical 2-complex is aspherical.)

• Let us specialize further. Suppose that Y is surface with a M_ε-cell structure. Then $\kappa(Y) \leq \varepsilon$ if and only if at each vertex the sum of the angles is $\geq 2\pi$. For example, the natural PE cell structure on the surface of a cube is not locally $CAT(0)$ since at each vertex the sum of the angles is $\frac{3\pi}{2}$.

• Suppose Y is the product of two metric trees. Then Y is naturally a PE complex where each 2-cell is a rectangle. We claim that Y is $CAT(0)$. Indeed, it is a general fact that the product of two $CAT(0)$ spaces is $CAT(0)$, but in this case we can see this by the following local argument. A vertex in Y has the form (v_1, v_2) where v_1 an v_2 are vertices in the factors. The link of (v_1, v_2) is a complete bipartite graph (it is the join of the link of v_1 with the link of v_2). Hence, the minimum combinatorial length of any circuit is 4. Since each edge is assigned a length of $\frac{\pi}{2}$, the minimum length of any circuit is 2π; hence, Y is locally $CAT(0)$ and therefore, it is $CAT(0)$ since it is simply connected.

2.2 Links.

Suppose σ^n is a convex n-cell (= convex polytope) in M^n_ε and that v is a vertex of σ. Let S_v denote the unit sphere in the tangent space $T_v(M^n_\varepsilon)$. The *link of v in σ,* denoted $Lk(v,\sigma)$, is defined to be the set of unit tangent vectors in S_v which point into σ, i.e.,

$$Lk(v,\sigma) = \{x \in S_v \mid x \text{ points into } \sigma\}.$$

It is a spherical $(n-1)$-cell in S_v (which is isometric to $\mathbb{S}^{n-1}$). Similarly, if τ^k is a k-dimensional face of σ^n, then choose any point x in τ^k, and let S^{n-k-1}_x denote the set of unit vectors in $T_x M^n_\varepsilon$ which are orthogonal to τ^k. Let $Lk(\tau^k,\sigma^n) \subset S^{n-k-1}_x$ be the set of inward pointing unit vectors orthogonal to τ^k. Thus, $Lk(\tau^k,\sigma^n)$ is a spherical $(n-k-1)$-cell. Its isometry class obviously does not depend on the choice of the point x in τ^k.

Now suppose that X is an M_ε cell complex and that v is a vertex of X. If σ is a cell in X containing v, then we have a spherical cell $Lk(v,\sigma)$, well defined up to isometry; moreover, if τ is a face of σ and $v \in \tau$, then $Lk(v,\tau)$ is naturally identified with a face of $Lk(v,\sigma)$. Thus, if σ' is another such cell and τ is a common face of σ and σ', then $Lk(v,\tau)$ is naturally identified with a face of both $Lk(v,\sigma)$ and $Lk(v,\sigma')$. In other words, these spherical cells glue together to give a well-defined PS complex.

Definition 2.2.1. Given an M_ε cell complex X and a vertex v of X, the *link of v in X* is the PS cell complex defined by

$$Lk(v,X) = \bigcup_{\substack{\sigma \text{a cell} \\ \text{in} X}} Lk(v,\sigma).$$

For any cell τ in X one can define $Lk(\tau,X)$ in a similar fashion.

The following lemma is intuitively clear (although the proof requires some work).

Lemma 2.2.2. *Let v be a vertex in an M_ε-cell complex X. Then there is a neighborhood of v in X which is isometric to a neighborhood of the cone point in $C_\varepsilon(Lk(v,X))$.*

Combining this lemma with Theorem 1.6.1 we get the following result.

Theorem 2.2.3. *Suppose X is an M_ε-cell complex. Then the following statements are equivalent.*

(i) $\kappa(X) \le \varepsilon$.

(ii) For any cell τ of X, $Lk(\tau,X)$ is $CAT(1)$.

(iii) For any vertex v of X, $Lk(v,X)$ is $CAT(1)$.

Proof. To see the equivalence of (i) and (ii), first note that any point x in X lies in the relative interior of some cell τ. If $\dim\tau = k$, then a neighborhood of x in X is isometric to a neighborhood of the cone point in $C_\varepsilon(S^k Lk(\tau,X))$ where $S^k L$ denotes the iterated k-fold spherical suspension of L. To see the equivalence of (ii) and (iii), we note that if τ is a cell of X and v is a vertex of τ, then $Lk(\tau,X)$ can be identified with a link in $Lk(v,X)$, namely,

$$Lk(\tau, X) \cong Lk(Lk(v, \tau), Lk(v, X)). \qquad \square$$

Thus, the problem of deciding if $\kappa(X) \leq \varepsilon$ reduces to the following.

Question. When is a PS complex $CAT(1)$?

Using Theorem 2.2.3 and induction on dimension, Theorem 1.5.5 shows that the main issue in this question is to decide when every closed geodesic in a PS complex L has length $\geq 2\pi$.

2.3 Gromov's Lemma.

Definition 2.3.1. A cell complex L is a *flag complex* if

a) each cell in L is a simplex,

b) L is a simplicial complex (i.e., the intersection of two simplices is either empty or a common face of both), and

c) the following condition holds:

$(*)$ Given any subset $\{v_0, \dots, v_k\}$ of distinct vertices in L such that any two are connected by an edge, then $\{v_0, \dots, v_k\}$ spans a k-simplex in L.

Terminology. Gromov uses the phrase L satisfies the "no Δ condition" to mean that the simplicial complex L satisfies $(*)$. In [D2] and [M] we said L is "determined by its 1-skeleton" to mean that $(*)$ held.

Examples 2.3.2.

- An m-gon is a flag complex if and only if $m > 3$.

- Let $\mathcal{P}$ be a poset. A *chain* (or a "flag") in $\mathcal{P}$ is a linearly ordered subset. The *order complex* of $\mathcal{P}$ (also called its "derived complex"), denoted by $Ord(\mathcal{P})$, is the poset of finite chains in $\mathcal{P}$, partially ordered by inclusion. Then $Ord(\mathcal{P})$ is an abstract simplicial complex in the sense that it is isomorphic to the poset of simplices in some simplicial complex. Moreover, it is obviously a flag complex.

- Suppose that Y is a cell complex and that $Cell(Y)$ denotes the poset of cells in Y. Then $Ord(Cell(Y))$ can be identified with the barycentric subdivision of Y. Hence, *the barycentric subdivision of any cell complex is a flag complex.* We restate this observation as the following.

Proposition 2.3.3. *The condition that L be a flag complex does not restrict its topological type; it can be any polyhedron.*

Definition 2.3.4. A spherical $(n-1)$-simplex in $\mathbb{S}^{n-1}$ is *all right* if its vertices form an orthonormal basis for $\mathbb{R}^n$. Similarly a PS cell complex is *all right* if each of its cells is (isometric to) an all right simplex.

Examples 2.3.5.

- Let $\square^n$ denote the regular n-cube $[-1, 1]^n$ (of edge length 2) in $\mathbb{E}^n$. We note that if v is a vertex of $\square^n$, then $Lk(v, \square^n)$ is an all right $(n-1)$-simplex.

• By a *PE cubical complex* we mean an M_0 complex in which each cell is a regular cube. If X is a PE cubical complex, then for any vertex v, $Lk(v, X)$ is an all right PS complex.

Lemma 2.3.6. (Gromov's Lemma) *Suppose L is an all right PS cell complex. Then L is $CAT(1)$ if and only if it is a flag complex.*

Corollary 2.3.7. *A PE cubical complex X has $\kappa(X) \leq 0$ if and only if the link of each of its vertices is a flag complex.*

Proof of Lemma 2.3.6. The "only if" implication is easy and left for the reader as an exercise. The main content of the lemma is the "if" part. Note that the property being an all right PS flag complex is inherited by all links; therefore, in light of the last paragraph of Section 2.2, it suffices to prove the following statement:

If L is an all right PS flag complex, then the length of all closed geodesics in L is $\geq 2\pi$.

Let γ be a closed geodesic in L. Let v be a vertex of L and suppose that γ has nonempty intersection with the open star $O(v, L)$ of v in L. We then claim that the intersection of γ with the star $St(v, L)$ of v in L is an arc of γ, denoted by c, of length π.

To prove this claim we first observe that, due to the all right condition, $St(v, L)$ is isometrically isomorphic to $C_1(Lk(v, L))$, the spherical cone on the link of v in L, and we may identify $Lk(v, L)$ with the metric sphere of radius $\frac{\pi}{2}$ about v in L. If γ passes through v, then our claim is obvious. Suppose that γ misses v. Then consider the surface $S \subset St(v, L)$ defined as the union of all geodesic rays of length $\frac{\pi}{2}$ emanating from v and passing through points of γ. S is a finite union of isosceles spherical triangles of height $\frac{\pi}{2}$ with common apex at v, matched together in succession along γ. Develop S along γ in a locally isometric fashion into the standard metric 2-sphere $\mathbb{S}^2$. Assume that the image of v is the north pole. Under the developing map the curve c must be mapped onto a local geodesic, therefore, onto a segment of a great circle of $\mathbb{S}^2$. Since the image of c cuts inside the northern hemisphere, it must connect two points of the equator and must have length π. This proves our claim.

If we find two vertices, v and w, such that $O(v, L) \cap \gamma \neq \emptyset$, $O(w, L) \cap \gamma \neq \emptyset$ and $O(v, L) \cap O(w, L) = \emptyset$, then we are done. We show that there always exists such a pair of vertices. Put $V = \{v \in Vert(L) : O(v, L) \cap \gamma \neq \emptyset\}$, then γ is contained in the full subcomplex of L spanned by V. If $O(v, L) \cap O(w, L) \neq \emptyset$ held for all $v, w \in V$, then all pairs of vertices in V would be connected by an edge in L, therefore V would span a simplex in L, since L is a flag complex. But a simplex cannot contain a closed geodesic. □

Chapter II: Cubical Examples.

3. The complex X_L.

3.1 A cubical complex with prescribed links.

The point of this subsection is describe a simple construction which proves the theorem below. (The construction was first described explicitly by Danzer.)

Theorem 3.1.1. *Let L be a finite simplicial complex. Then there is a finite PE cubical complex X_L such that the link of each vertex of X_L is isomorphic to L.*

Observation 3.1.2. It follows from Gromov's Lemma that if L is a flag complex, then X_L is nonpositively curved.

The construction of X_L is closely related to a standard construction which shows that any simplicial complex L is a subcomplex of a simplex, namely, the full simplex on the vertex set of L.

Notation. For any set I, let $\mathbb{R}^I$ denote the Euclidean space consisting of all functions $x : I \to \mathbb{R}$. Let $\square^I = [-1, 1]^I$ denote the *standard cube* in $\mathbb{R}^I$. Suppose that $I = Vert(L)$, the vertex set of L. If σ is a simplex of L, then let $I(\sigma)$ denote the vertex set of σ. Let $\mathbb{R}^{I(\sigma)}$ denote the linear subspace of $\mathbb{R}^I$ consisting of all functions $I \to \mathbb{R}$ with support in $I(\sigma)$.

The Construction. X_L is the subcomplex of $\square^I$ consisting of all faces of $\square^I$ which are parallel to $\mathbb{R}^{I(\sigma)}$ for some $\sigma \in Simp(L)$, the set of simplices in L. In other words,

$$X_L = \bigcup_{\sigma \in Simp(L)} \left\{ \text{faces of } \square^I \text{ parallel to } \mathbb{R}^{I(v)} \right\}.$$

Note that $Vert(X_L) = \{\pm 1\}^I$ and that if $v \in Vert(X_L)$, then $Lk(v, X_L) \cong L$. This proves Theorem 3.1.1.

Examples.

- If L is the disjoint union of a 0-simplex and a 1-simplex, then X_L is the subcomplex of a 3-cube consisting of the top and bottom faces and the four vertical edges. Note that X_L is nonpositively curved.
- If L is a triangulation of an $(n-1)$-sphere, then X_L is an n-manifold.
- Suppose L is a simplicial graph. Then X_L is a 2-dimensional complex, each 2-cell is a square and the link of each vertex is L. Thus, X_L is nonpositively curved if and only if L contains no circuits of combinatorial length 3.

In the next subsection we analyze the universal cover and fundamental group of X_L.

3.2 Right-angled Coxeter groups.

We retain the notation $L, X_L, I, \square^I$ from the previous subsection.

Let $(e_i)_{i \in I}$ denote the standard basis of $\mathbb{R}^I$. For each $i \in I$, let $r_i : \mathbb{R}^I \to \mathbb{R}^I$ be the linear reflection which sends $e_i \mapsto -e_i$ and $e_j \mapsto e_j$ for $j \neq i$. The group G of linear transformations generated by $\{r_i\}_{i \in I}$ is isomorphic to $(\mathbb{Z}/2)^I$. The cube $\square^I$ is stable under G as is the subcomplex X_L.

The subspace $[0, 1]^I$ of $\square^I$ is a fundamental domain for the G-action in the strong sense that it projects homeomorphically onto the orbit space, i.e., $\square^I/G \cong [0, 1]^I$. Define K_L (or simply K) to be the intersection of X_L and $[0, 1]^I$. In other words, K is the union over all simplices σ of L of the cubical faces of the form $(\mathbb{R}^{I(\sigma)} + e) \cap [0, 1]^I$, where $e = (1, 1, \ldots, 1)$. We note that K is homeomorphic to the cone on L (the cone point is e). We shall call it the *cubical cone on* L. Moreover, $X_L/G \cong K$.

For each $i \in I$, put $K_i = K \cap \{x_i = 0\}$, that is to say, K_i is the intersection of K with the hyperplane fixed by r_i. We note that if i and j are distinct elements of I, then $K_i \cap K_j \neq \emptyset$ if and only if $\{i, j\} \in Edge(L)$ (i.e., if i and j span a 1-simplex in L). More generally, given a subset J of I, the intersection of the family $(K_j)_{j \in J}$ is nonempty if and only if $J = I(\sigma)$ for some simplex σ of L. As we shall see below, the fact that X_L can be nonsimply connected comes from the fact that intersections of the form $K_i \cap K_j$ can be empty.

Define a "right-angled Coxeter matrix" $(m_{ij}), (i, j) \in I \times I$, by the formula:

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } \{i, j\} \in Edge(L), \\ \infty & \text{otherwise.} \end{cases}$$

("Right-angled" refers to the fact that each off-diagonal entry is either 2 or ∞). For each $i \in I$, introduce a symbol s_i, and let $S = \{s_i\}_{i \in I}$. Let W be the group defined by the presentation:

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, \, (i, j) \in I \times I \rangle.$$

It turns out (cf. [B]) that the natural map $S \to W$ is injective; moreover, the order of each s_i in W is two and the order of $s_i s_j$ in W is m_{ij}. Hence, we can safely identify S with its image in W. The pair (W, S) is called a *right-angled Coxeter system.*

For each subset J of I, let $s(J) = \{s_j\}_{j \in J}$ and let W_J denote the subgroup generated by $s(J)$. ($W_\emptyset$ is the trivial subgroup.) Again it turns out that the situation is as nice as possible: each $(W_J, s(J))$ is a right-angled Coxeter system corresponding to the sub-Coxeter matrix $(m_{ij})_{(i,j) \in J \times J}$ (cf. page 20 in [B]).

A construction of Tits and Vinberg. Let Y be a space and $(Y_i)_{i \in I}$ a family of closed subspaces indexed by I (e.g., $Y = K$). For any point y in Y, let $I(y) = \{i \in I \mid y \in Y_i\}$ and put $W_y = W_{I(y)}$. (In the case $Y = K$, any nontrivial W_y is of the form $(\mathbb{Z}/2)^{I(\sigma)}$ for some simplex σ in L.) Give W the discrete topology and define $\mathcal{U}(W, Y)$ to be the quotient space of $W \times Y$ by an equivalence relation $\sim$, where $(w, y) \sim (w', y')$ if and only if $y = y'$ and $w^{-1} w' \in W_y$. Thus, $\mathcal{U}(W, Y)$ is the space formed by gluing together copies of Y one for each element of w, the copies $w \times Y$ and $ws_i \times Y$ being glued together along the subspaces $w \times Y_i$ and $ws_i \times Y_i$. The equivalence class of (w, y) in $\mathcal{U}(W, Y)$ is denoted by $[w, y]$. The natural action of W on $W \times Y$ by left translation descends to a W-action on $\mathcal{U}(W, Y)$. The map $Y \to \mathcal{U}(W, Y)$ defined by $y \to [1, y]$ is an embedding and henceforth, we identify Y with its image. Y is a strong fundamental domain in the sense that $\mathcal{U}(W, Y)/W \cong Y$. The W-space $\mathcal{U}(W, Y)$ has the following universal property: given any W-space Z and any map $f : Y \to Z$ such that $f(Y_i)$ is contained in the fixed point set of s_i on Z, then f extends to a W-equivariant map $\hat{f} : \mathcal{U}(W, Y) \to Z$. (The formula is : $\hat{f}([w, y]) = wf(y)$.) The proofs of all the above properties are easy and the reader should do them as exercises (the proofs can also be found in [V].) The proof of the next lemma is also left as an exercise for the reader.

Lemma 3.2.1. *The space $\mathcal{U}(W, Y)$ is connected if and only if the following two conditions hold:*

(i) Y is connected,

(ii) for each $i \in I$, Y_i is nonempty.

Lemma 3.2.2. *The space $\mathcal{U}(W,Y)$ is simply connected if and only if the following three conditions hold:*

(i) Y is simply connected,

(ii) for each $i \in I$, Y_i is nonempty and connected,

(iii) $Y_i \cap Y_j \neq \emptyset$ whenever $m_{ij} \neq \infty$.

Proof. Suppose conditions (i) through (iii) hold. Let $q : \tilde{\mathcal{U}} \to \mathcal{U}(W,Y)$ be the universal covering. By (i), the inclusion $Y \to \mathcal{U}(W,Y)$ lifts to $f : Y \to \tilde{\mathcal{U}}$. Lift each s_i to an involution $\tilde{s}_i : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ fixing a point in $f(Y_i)$. By (ii), $\tilde{s}_i$ must fix all of $f(Y_i)$. Then $\tilde{s}_i$ and $\tilde{s}_j$ both fix $f(Y_i \cap Y_j)$. By (iii), if $m_{ij} \neq \infty$, then the intersection is nonempty and hence, $\tilde{s}_i$ and $\tilde{s}_j$ commute (since they commute on an open neighborhood of a common fixed point.) Therefore, the W-action lifts to $\tilde{\mathcal{U}}$. By the universal property of $\mathcal{U}(W,Y)$, the map f extends to a map $\tilde{f} : \mathcal{U}(W,Y) \to \tilde{\mathcal{U}}$ which is clearly a section for q. Thus, $\tilde{\mathcal{U}} = \mathcal{U}(W,Y)$, i.e., $\mathcal{U}(W,Y)$ is simply connected. Conversely, if any one of conditions (i), (ii) or (iii) fails, then it is easy to construct a nontrivial covering space of $\mathcal{U}(W,Y)$. □

The universal cover of X_L. Now put

$$\mathcal{U}_L = \mathcal{U}(W,K).$$

Let $\varphi : W \to G, G = (\mathbb{Z}/2)^I$, be the homomorphism defined by $s_i \mapsto r_i$ and let Γ_L denote the kernel of φ. Define a map $p : \mathcal{U}_L \to X_L$ by $[w,x] \mapsto \varphi(w)x$. We shall leave it as an exercise for the reader to check that p is a covering projection. On the other hand, since K and each K_i are contractible (they are cones), we have the following corollary to Lemma 3.2.2.

Proposition 3.2.3. *$\mathcal{U}_L$ is the universal cover of X_L.*

In summary, we have established the following theorem.

Theorem 3.2.4. *Suppose L is a flag complex. Then X_L is a PE cubical complex of nonpositive curvature; its universal cover is $\mathcal{U}_L$ and its fundamental group is Γ_L.*

4. Applications.

4.1 Background on the Generalized Poincaré Conjecture.

Let C^n be a compact contractible manifold with boundary.

Question 1: Is C^n homeomorphic to the n-disk, D^n?

It follows from Poincaré duality and the exact sequence of the pair $(C^n, \partial C)$, that the pair has the same homology as (D^n, S^{n-1}) and that ∂C has the same homology as S^{n-1} (i.e., ∂C is a *homology sphere*), that is,

$$H_i(C^n, \partial C) = \begin{cases} 0 & \text{if} \quad i \neq n, \\ \mathbb{Z} & \text{if} \quad i = n; \end{cases}$$

$$H_i(\partial C) = \begin{cases} 0 & \text{if} \quad i \neq 0, n-1, \\ \mathbb{Z} & \text{if} \quad i = 0, n-1. \end{cases}$$

Is every homology sphere (of dimension > 1) simply connected? Poincaré provided a counterexample: the binary icosahedral group G (a finite group of order 120) acts freely on S^3 and S^3/G is a homology 3-sphere which is not simply connected. In fact, Kervaire [K] showed that in dimensions ≥ 5 any finitely presented group G satisfying $H_1(G) = H_2(G) = 0$, can be the fundamental group of a homology sphere. Recently, 3-dimensional homology spheres have been much studied in 3-dimensional topology since interesting invariants (such as the Casson invariant) can be associated to them.

It is also a fact that every homology $(n-1)$-sphere bounds a contractible topological n-manifold. (For $n > 4$, this is an easy result using surgery theory; for $n = 4$ it follows from the work of Freedman, [F].) Thus, the answer to Question 1 is no: for every $n \geq 4$, there are examples of compact, contractible C^n with ∂C not simply connected.

On the other hand, we have the following result.

Theorem 4.1.1. (The Generalized Poincaré Conjecture)

(i) If M^{n-1} is a simply connected homology sphere and $n-1 \neq 3$, then M^{n-1} is homeomorphic to S^{n-1}.

(ii) If C^n is a compact, contractible manifold with boundary, the boundary is simply connected and $n \neq 3, 4$, then C^n is homeomorphic to D^n.

The Generalized Poincaré Conjecture was proved by Smale in the case where M^{n-1} is smooth and of dimension ≥ 6. Subsequently, the result was extended to dimension 5 and to the PL and topological cases by others (notably Stallings and Newman). The result in dimension 4 is due to Freedman [F].

Next suppose that F^n is a contractible manifold without boundary.

Question 2: Is F^n homeomorphic to $\mathbb{R}^n$?

Suppose Y is a space which is not compact. A *neighborhood of* ∞ is the complement of a compact subset of Y. Y is *connected at* ∞ (or "1-ended") if every neighborhood of ∞ contains a connected neighborhood of ∞. Similarly, Y is *simply connected at* ∞ if every neighborhood of ∞ contains a simply connected neighborhood of ∞. For example, $\mathbb{R}^n$ is connected at ∞ for $n > 1$ and simply connected at ∞ for $n > 2$.

From the earlier discussion we see that the answer to Question 2 is no. Indeed, we could take $F^n = \overset{\circ}{C}{}^n$, the interior of C^n. It is then easy to see that F^n is not simply connected at ∞ if $\pi_1(\partial C) \neq 1$. Furthermore, it is known that the situation can be much worse than this: F^n need not be "tame" in the sense that it might not be homeomorphic to the interior of any compact manifold (because the fundamental groups of neighborhoods of ∞ might not "stabilize" to a finitely generated group). On the other hand we have the following theorem.

Theorem 4.1.2. (Stallings [Sta1] for $n \geq 5$ and Freedman [F] for $n = 4$) *For $n \geq 4$, if a contractible manifold F^n is simply connected at ∞, then it is homeomorphic to $\mathbb{R}^n$.*

Question 3: If F^n is the universal cover of a closed aspherical manifold, then is F^n homeomorphic to $\mathbb{R}^n$?

In view of the above, we see that the issue is not the existence of fake contractible manifolds (they exist), but rather if such an example can admit a discrete cocompact group of transformations.

In the next subsection we will show how the construction in Section 3 can be used to answer Question 3 in the negative.

4.2 Aspherical manifolds not covered by Euclidean space.

As in Section 3, L will denote a flag complex, X_L the associated nonpositively curved cubical complex and $\mathcal{U}_L$ its universal cover.

Theorem 4.2.1. *Suppose L^{n-1} is a homology $(n-1)$-sphere, with L not simply connected. Then $\mathcal{U}_L$ is not simply connected at ∞.*

This does not yet answer Question 3 in the negative. The problem is that since L is only required to be a homology sphere instead of the standard sphere, X_L might not be a manifold. However, X_L is very close to being a manifold, it is a "homology manifold", as defined below.

Definition 4.2.2. A space X is a *homology n-manifold* if it has the same local homology as an n-manifold, i.e., if for each $x \in X, H_*(X, X - x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$. (When X is a polyhedron this is equivalent to the condition $H_*(Lk(v, X)) \cong H_*(S^{n-1})$ for each vertex v of X.)

The idea is to modify the cubical complex X_L of Theorem 4.2.1 to be a manifold. Recall that $\mathcal{U}_L = \mathcal{U}(W, K)$ and that $X_L = \mathcal{U}_L/\Gamma_L$, where K is homeomorphic to the cone on L. As was mentioned in the previous subsection, since L is a homology $(n-1)$-sphere, it is always possible to find a contractible n-manifold $\hat{K}$ with $\partial\hat{K} = L$. For each $i \in I$ put $\hat{K}_i = K_i \subset \partial\hat{K}$ and set

$$\hat{X}_L = \mathcal{U}(W, \hat{K})/\Gamma_L.$$

It should now be clear that $\hat{X}_L$ is a manifold (the problem at the cone point has disappeared). We have a map $\hat{K} \to K$ which is the identity on $\partial\hat{K}$ and a homotopy equivalence rel $\partial\hat{K}$. Using the universal property of the $\mathcal{U}$ construction, we see that this map extends to a W-equivariant proper homotopy equivalence $\mathcal{U}(W, \hat{K}) \to \mathcal{U}(W, K)$. Thus, $\mathcal{U}(W, \hat{K})$ is also contractible.

Since the property of being simply connected at ∞ is a proper homotopy invariant, we see that $\mathcal{U}(W, \hat{K})$ is also not simply connected at ∞. Thus, $\mathcal{U}(W, \hat{K})$ is a contractible n-manifold which is not homeomorphic to $\mathbb{R}^n$. Since nonstandard homology $(n-1)$-spheres exists whenever $n - 1 \geq 3$, we get the following corollary to Theorem 4.2.1.

Corollary 4.2.3. ([D1]) *In each dimension $n \geq 4$, there exist closed aspherical n-manifolds which are not covered by $\mathbb{R}^n$.*

Next we sketch two slightly different proofs of Theorem 4.2.1. The idea in both cases is to calculate $\pi_1(\mathcal{U}_L - C_i)$ where $C_0 \subset C_1 \subset \cdots$ is an increasing family of compact subsets which exhaust $\mathcal{U}_L$ (i.e., $\mathcal{U}_L = \cup C_i$). First we sketch the approach of [D1].

Combinatorial picture. Order the elements of W: $w_0, w_1, w_2, \ldots$ so that $\ell(w_i) \leq \ell(w_{i+1})$ where $\ell(w)$ denotes the word length of w with respect to the generating set S. Put

$$C_i = w_0 K \cup \cdots \cup w_i K.$$

Using the combinatorial theory of Coxeter groups developed in [B], one shows that $w_i K \cap C_{i-1}$ is a union of subspaces of the form $(w_i K_j)_{j \in J}$, where W_J is finite. It follows that this intersection is homeomorphic to an $(n-1)$-disk in the boundary of $w_i K$. Thus, C_i is the "boundary connected sum" of copies of K and ∂C_i is the connected sum of $i+1$ copies of L. Furthermore, $\mathcal{U}_L - C_i$ is homotopy equivalent to ∂C_i. Thus, $\pi_1(\mathcal{U}_L - C_i)$ is the free product of $i+1$ copies of $\pi_1(L)$ and (with an appropriate choice of base points) the map $\pi_1(\mathcal{U}_L - C_{i+1}) \to \pi_1(\mathcal{U}_L - C_i)$ induced by the inclusion is the projection which sends the last factor to 1. It follows that $\mathcal{U}_L$ is not simply connected at ∞.

The second approach is that of [DJ].

$CAT(0)$ **picture.** Choose a base point x in $\mathcal{U}_L$ and let $S_x(r)$ denote the metric sphere of radius r about x. Let $\{v_0, \ldots, v_m\}$ be the set of vertices inside the ball, $B_x(r)$. (Assume that we have chosen r so that no vertices lie on $S_x(r)$.) It is shown in [DJ] that $S_x(r)$ is homeomorphic to the connected sum,

$$Lk(v_0) \# \cdots \# Lk(v_m) = L \# \cdots \# L.$$

Also, geodesic contraction provides a homotopy equivalence between $\mathcal{U}_L - B_x(r)$ and $S_x(r)$. Thus, if we choose an increasing sequence of real numbers, (r_i), with $\lim r_i = \infty$ and set $C_i = B_x(r_i)$, we reach the same conclusion as before.

The second approach suggests the following question which was first posed by Gromov.

Question 4: Is every $CAT(0)$ manifold homeomorphic to $\mathbb{R}^n$?

In Section 4.4 we will show that the answer to this is again no. As before, the idea is to modify the construction of X_L in Theorem 4.2.1 to make it a topological manifold while preserving its feature of nonpositive curvature.

4.3 Background on the Double Suspension Theorem.

Definition 4.3.1. A simplicial complex S is a PL *n-manifold* if the (combinatorial) link of each vertex is piecewise linearly homeomorphic to S^{n-1}.

It turns out that a simplicial complex X can be a topological manifold without satisfying the above definition.

Suppose that a simplicial complex L is a PL manifold and a homology sphere and that $\pi_1(L) \neq 1$. Then CL, the cone on L, is a homology manifold and a manifold except in a neighborhood of the cone point (since any deleted neighborhood of the cone point is not simply connected). Similarly, SL, the suspension of L, is not a manifold in neighborhoods of the suspension points. On the other hand, algebraic topology does not provide us with any reason why the double suspension $S^2L(= SSL)$ should not be a manifold. At a topology conference in 1963, Milnor listed this question as one of the seven most difficult problems in topology. In 1977 the question was settled in the affirmative by Edwards in many cases and in full generality by Cannon [Ca].

Theorem 4.3.2. (The Double Suspension Theorem of Edwards and Cannon) *Suppose that L^n is a homology n-sphere. Then S^2L is a manifold (and therefore, by the Generalized Poincaré Conjecture, it is homeomorphic to S^{n+2}).*

The definitive result in this direction is the following result of [E]. (A good place to find the proofs of these results is Daverman's book [Da].)

Theorem 4.3.3. (Edwards' Polyhedral Homology Manifold Recognition Theorem) *Suppose X^n is a polyhedral homology n-manifold, $n \geq 5$. Then X^n is a topological manifold if and only if the link of each vertex is simply connected.*

We will use this result in the next subsection to find the desired modification of X_L.

4.4 Nonpositively curved manifolds not covered by $\mathbb{R}^n$.

As in 4.2, L is a PL homology $(n-1)$-sphere and K is the cone on L. To answer Question 4 we want to replace K by a $CAT(0)$ manifold with boundary $\hat{K}$.

First we need the following lemma (a proof can be found in [DT].)

Lemma 4.4.1. *If $n-1 \geq 4$, then there is a PL submanifold M^{n-2} in L^{n-1} such that M^{n-2} is a homology sphere and such that $\pi_1(M^{n-2})$ normally generates $\pi_1(L^{n-1})$.*

Think of M^{n-2} as the "equator" of L^{n-1}. It separates L^{n-1} into two components N_1 and N_2, that is,

$$L = N_1 \bigcup_M N_2.$$

For $i = 1, 2$, put

$$\Lambda_i = N_i \bigcup_M CM.$$

Van Kampen's Theorem implies that Λ_i is simply connected. Furthermore, it is a homology $(n-1)$-manifold with the same homology as S^{n-1}.

Now triangulate L^{n-1} as a flag complex so that M^{n-2} is a full subcomplex. For $j = 1, 2$, let $\hat{K}^j$ denote the cubical cone on Λ_j and put

$$\hat{K} = \hat{K}^1 \bigcup_{CM} \hat{K}^2.$$

Then $\partial K = \partial \hat{K}$ so we can put $\hat{K}_i = K_i$, $i \in I$, and define

$$\hat{X}_L = \mathcal{U}(W, \hat{K})/\Gamma_L.$$

In the construction of $\hat{K}$ there are now two cone points, one with link Λ_1 and the other with link Λ_2. Since both of these are simply connected, Edwards' Theorem 4.3.3 implies that $\hat{K}$ is a contractible manifold with boundary and that $\mathcal{U}(W, \hat{K})$ is a manifold. The cubical structure on $\hat{K}$ extends to a cubical structure on $\mathcal{U}(W, \hat{K})$ in which the vertices are the translates of the two cone points. Since both links are flag complexes, this cubical structure is $CAT(0)$. Therefore we have proved the following.

Theorem 4.4.2. ([ADG]) *In every dimension $n \geq 5$, there exist nonpositively curved, PE n-manifolds with universal covers not homeomorphic to $\mathbb{R}^n$.*

We should note that the construction above does not work for $n = 4$. The reason is that Lemma 4.4.1 is not valid for $n = 4$: for then $\dim L = 3, \dim M = 2$, so M must be a 2-sphere (so its fundamental group cannot normally generate $\pi_1(L)$).

Thus, the following questions remains open in dimension 4. (Although it is proved in [Thp] that the answer is affirmative if the manifold has at least one "tame point".)

Question 5. Is every $CAT(0)$ 4-manifold homeomorphic to $\mathbb{R}^4$?

5. Hyperbolization.

5.1 The basic idea.

In [G1] Gromov described several functorial procedures for converting a cell complex into a nonpositively curved PE or PH cell complex. Further expositions of this idea are given in [DJ] and [CD2]. We denote such a procedure by $X \mapsto h(X)$ and call $h(X)$ a "hyperbolization" of the cell complex X. Since X is arbitrary and $h(X)$ is aspherical, $h(X)$ cannot be homeomorphic to X. The rough idea is that one constructs $h(X)$ by gluing together nonpositively curved manifolds with boundary in the same combinatorial pattern as the cells of X are glued together.

Next we describe some properties which such a procedure should have. Let $\mathcal{C}$ be a category, the objects of which are cell complexes (possibly with some additional requirements) and with morphisms embeddings onto a subcomplexes (i.e., a morphism is the composition of an isomorphism and an inclusion). Let $\mathcal{P}$ be a category, the objects of which are nonpositively curved polyhedra and the morphisms of which are isometric embeddings onto locally convex subpolyhedra.

Desired properties of $h(X)$.

1) (Functoriality) $h : \mathcal{C} \to \mathcal{P}$ is a functor.

2) (Preservation of local structure) If σ is a k-cell in X, then $h(\sigma)$ should be a k-manifold with boundary and $Lk(\sigma, X)$ should be PL homeomorphic to $Lk(h(\sigma), h(X))$.

3) If σ is a point, then $h(\sigma) = \sigma$.

4) (Orientability) For any cell σ, $h(\sigma)$ should be an orientable manifold with boundary.

Properties 1) and 2) imply that there is a map $f : h(X) \to X$ such that for each cell σ in X, f sends $h(\sigma)$ to σ. Moreover, f is unique up to a homotopy through such maps. Property 3) then implies that $f_* : H_*(h(X); \mathbb{Z}/2) \to H_*(X; \mathbb{Z}/2)$ is surjective. Moreover, if 4) holds, then f_* is surjective with integral coefficients. (Exercise: prove the statements in this paragraph.)

In the next subsection we shall describe a construction of Gromov which satisfies properties 1), 2) and 3). Gromov [G1] also described a hyperbolization procedure satisfying 4), but the construction is a little more complicated and we shall not discuss it here.

5.2 Gromov's Möbius band hyperbolization procedure.

Let $\mathcal{C}_n$ denote the category of cubical cell complexes of dimension $\leq n$ and let $\mathcal{P}_n$ denote the category of nonpositively curved PE cubical complexes of dimension $\leq n$. In both cases morphisms are as in the previous subsection. We shall now define a functor $\mathcal{C}_n \to \mathcal{P}_n$ by induction on n. If $\dim X = 0$, then if we want property 3) to hold there is only one possible definition $h(X) = X$. Now suppose by induction that we have defined $h(X)$ for $\dim X < n$. Next we want to define the hyperbolization of an n-cube, $\square^n$. Let $a : \square^n \to \square^n$ denote the antipodal map (i.e., the central symmetry). Define

$$h(\square^n) = (h(\partial \square^n) \times [-1, 1])/(\mathbb{Z}/2)$$

where $\mathbb{Z}/2$ acts by $h(a)$ on the first factor and by reflection on the second. Note that $\partial h(\square^n)$ is equal to $(h(\partial \square^n) \times \{-1, 1\})/(\mathbb{Z}/2)$ which is canonically identified with $h(\partial \square^n)$.

Now suppose that $\dim X = n$ and that X_{n-1} denotes its $(n-1)$-skeleton. We define $h(X)$ by attaching, for each n-cell $\square^n$ in X, a copy of $h(\square^n)$ to $h(X_{n-1})$ via the canonical identification of $\partial h(\square^n)$ with the subset $h(\partial \square^n)$ of $h(X_{n-1})$, i.e.,

$$h(X) = h(X_{n-1}) \cup \bigcup_{\square^n \subset X} h(\square^n).$$

We leave it as an exercise for the reader to check that (a) $h(X)$ is a nonpositively curved cubical complex and (b) $X \mapsto h(X)$ is a functor. (For (a) show that $h(X)$ has a natural cubical structure with the same vertex set as the cubical structure on X; moreover, the link of a vertex in $h(X)$ is the barycentric subdivision of its original link in X.)

Examples 5.2.1.

- h (1-cell)= 1-cell.
- If X is a graph, then $h(X) = X$.
- $h(\square^2) = (\partial \square^2 \times [-1, 1])/(\mathbb{Z}/2)$, which is a Möbius band (hence, our terminology).
- $h(\partial \square^3)$ is the result of replacing each of the six faces of a cube by a Möbius band, in other words, the surface $h(\partial \square^3)$ is the connected sum of six copies of $\mathbb{R}P^2$.

Here is an application of the above construction.

Theorem 5.2.2. (Gromov [G1]) *Any triangulable manifold is cobordant to an aspherical manifold.*

Proof. Given a manifold M^n, triangulate it and then form the cone CM. Thus, CM is a simplicial complex. Let c denote the cone point. Any simplicial complex can be subdivided into cubes as indicated in Figure 5.2.1 below.

Figure 5.2.1. Subdivision of a simplex into cubes in dimensions 2 and 3

Do this to CM and then apply the Möbius band hyperbolization procedure. Let $\mathrm{Star}(h(c))$ denote the union of all open cubes in $h(CM)$ which contain the vertex $h(c)$ in their closure. By property 2), $CM - \mathrm{Star}(h(c))$ is a manifold with two boundary components. One component is $h(M)$. The other component is $Lk(h(c), h(CM))$, which is just the barycentric subdivision of M. $\square$

5.3 Blow-ups of hyperplane arrangements in real projective space.

In this subsection we discuss some results from [DJS]. The point is that some manifolds constructed using the Möbius band hyperbolization procedure and similar constructions occur in "nature".

Suppose that N^{n-k} is a smooth submanifold of codimension k in a smooth manifold M^n. Let ν denote its normal bundle, $E(\nu)$ the total space of ν and $P(\nu)$ the associated projective space bundle over N. Thus, the fiber of $P(\nu) \to N$ is $\mathbb{R}P^{k-1}$. Let λ denote the canonical line bundle over $P(\nu)$ and $E(\lambda)$ its total space. Let $E_0(\lambda)$ and $E_0(\nu)$ denote the complements of the 0-sections in $E(\lambda)$ and $E(\nu)$, respectively. There is a natural identification of $E_0(\lambda)$ with $E_0(\nu)$.

The *real blow-up* $M_{\circledcirc}$ of M along N is defined as follows. As a set $M_{\circledcirc}$ is the disjoint union of $M - N$ and $P(\nu)$. $M_{\circledcirc}$ is given the structure of a smooth manifold as follows. By the Tubular Neighborhood Theorem an open neighborhood of N in M can be identified with $E(\nu)$. We then glue $M - N$ to $E(\lambda)$ along an open subset of both via the identification $E_0(\lambda) \cong E_0(\nu)$.

Now suppose that $\mathcal{H}$ is a finite collection of linear hyperplanes in $\mathbb{R}^{n+1}$ ($\mathcal{H}$ is a "hyperplane arrangement"). Let $\mathcal{M}$ be a collection of intersections of elements in $\mathcal{H}$. For each subspace $E \in \mathcal{M}$, let $P(E)$ denote the corresponding subprojective space of $\mathbb{R}P^n$. Let $\mathbb{R}P^n_{\circledcirc\mathcal{M}}$ denote the result of blowing up $\mathbb{R}P^n$ along the submanifolds $\{P(E)\}_{E\in\mathcal{M}}$. (There are some conditions that $\mathcal{M}$ must satisfy for this construction to be independent of the order of blowing-up.)

The hyperplanes in $\mathcal{H}$ divide S^n into spherical cells. This cellulation of S^n descends to a cellulation of $\mathbb{R}P^n$. One then has a dual cellulation of $\mathbb{R}P^n$ by "dual cells". (The i-cells in the dual cellulation are in bijective correspondence with the $(n-i)$-cells in the original cellulation.) The blowing-up process can be described by a procedure analogous to the Möbius band procedure (and which, in fact, sometimes coincides with the Möbius band procedure).

Example 5.3.1. Suppose $\mathcal{H}$ is the set of coordinate hyperplanes in $\mathbb{R}^{n+1}$, i.e., $\mathcal{H} = \{H_i\}_{1\le i\le n+1}$, where H_i is the hyperplane $x_i = 0$. The hyperplanes in $\mathcal{H}$ then divide $\mathbb{S}^n$ into all right spherical simplices, triangulating it as the boundary of the $(n+1)$-dimensional octahedron. The dual cellulation is as the boundary of an $(n+1)$-cube. The dual cellulation descends to the cellulation of $\mathbb{R}P^n$ as $(\partial\square^{n+1}/a$, where a is the antipodal map. Let $\mathcal{M}$ denote the set of all nonzero intersections of elements of $\mathcal{H}$. Then it is not hard to see the following theorem.

Theorem 5.3.2. ([DJS])

$$\mathbb{R}P^n_{\circledcirc\mathcal{M}} = h(\partial\square^{n+1}/a).$$

(Here $\mathcal{M}$ denotes the set of all non-zero subspaces determined by coordinate hyperplanes arrangement and h is Gromov's Möbius band procedure.)

Remarks. (i) The manifold $\mathbb{R}P^n_{\circledcirc\mathcal{M}}$ occurs in nature as the closure of a generic "torus" orbit in a flag manifold. To be precise consider the flag manifold $F = GL(n+1,\mathbb{R})/B$ where B is the subgroup of all upper triangular matrices. Let $H = (\mathbb{R}^*)^{n+1}$ denote the "torus" of all diagonal matrices in $GL(n+1,\mathbb{R})$. (Note that H is not connected, it is isomorphic to $(\mathbb{R}_+ \times \mathbb{Z}/2)^{n+1}$, where $\mathbb{R}_+$ denotes the positive reals). As shown in [DJS] the closure of the H-orbit on F of any point gB, where the eigenvalues of g are distinct, is diffeomorphic to $\mathbb{R}P^n_{\circledcirc\mathcal{M}}$. It follows from the results of the previous subsection (i.e., essentially from Gromov's Lemma) that these manifolds are nonpositively curved.

(ii) The blow-ups of many other hyperplane arrangements also give nonpositively curved cubical complexes. For example, if $\mathcal{H}$ is any arrangement in $\mathbb{R}^{n+1}$ and $\mathcal{M}$ is the maximal family of all nonzero intersections, then, as shown in [DJS], $\mathbb{R}P^n_{\circledcirc\mathcal{M}}$ is nonpositively curved. As another example, if $\mathcal{H}$ is a simplicial hyperplane arrangement in $\mathbb{R}^{n+1}$ which cannot be decomposed into 3 or more factors and if $\mathcal{M}_{\min}$ is the family of all "irreducible" intersections, then $\mathbb{R}P^n_{\circledcirc\mathcal{M}_{\min}}$ is again nonpositively curved. Furthermore, many of these blow-ups occur in nature, for example, as compactifications of certain moduli spaces.

Chapter III: Non-cubical examples.

6. Reflection groups.

6.1 Geometric reflection groups.

The first picture to understand is the action of the dihedral group D_m of order $2m$ on $\mathbb{R}^2$. Suppose that ℓ_1 and ℓ_2 are two lines through the origin in $\mathbb{R}^2$ making an angle of π/m, where m is an integer ≥ 2. Let s_1 and s_2 denote the orthogonal reflections across ℓ_1 and ℓ_2, respectively. Then s_1s_2 is a rotation through an angle of $2\pi/m$. Thus, s_1s_2 has order m and the *dihedral group* D_m generated by s_1 and s_2 is a group order $2m$. It is also not hard to see that a sector, bounded by rays starting at the origin in the directions ℓ_1 and ℓ_2, is a fundamental domain for the action (in the strong sense).

Next we move onto the general situation. Suppose that K is a convex polytope in M^n_ε, $n \geq 2$ and that $\{K_i\}_{i \in I}$ is the set of codimension-one faces of K (I is some index set). If $K_i \cap K_j$ is a codimension-two face, let us also assume the following "angle condition":

The dihedral angle along $K_i \cap K_j$ has the form π/m_{ij} where m_{ij} is an integer ≥ 2.

If $M^n_\varepsilon = \mathbb{S}^n$, then the condition that all the dihedral angles are $\leq \pi/2$ forces K to be a simplex. (This is an easy argument in linear algebra, see pages 77-78 in [B].) In the Euclidean and hyperbolic cases we can apply this observation to conclude that the link of each vertex is a simplex. (A polytope with this property is called *simple.*)

Let $\overline{s}_i$ denote reflection across the hyperplane in M^n_ε spanned by K_i. Let $\overline{W} = \langle \overline{s}_i \rangle$ be the subgroup of $\mathrm{Isom}(M^n_\varepsilon)$ generated by the $\overline{s}_i$. Our analysis of the dihedral group shows that the following relations hold in $\overline{W}$:

$$\overline{s}_i^2 = 1,\ (\overline{s}\overline{s}_j)^{m_{ij}} = 1.$$

Introduce symbols s_i, $i \in I$, and put $S = \{s_i\}_{i \in I}$. Define a group W by the presentation:

$$W = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $i \in I$ and (i, j) ranges over all pairs of distinct elements such that $K_i \cap K_j \neq \emptyset$. Let $\varphi : W \to \overline{W}$ be the homomorphism defined by $s_i \mapsto \overline{s}_i$.

We define the Tits-Vinberg construction $\mathcal{U}(W, K)$ just as in Section 3.2. Thus, for each $x \in K, I(x) = \{i \in I \mid x \in K_i\}$, W_x is the subgroup generated by the generators corresponding to $I(x)$ and $\mathcal{U}(W, K) = (W \times K)/\sim$, where $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_x$.

Theorem 6.1.1. *(i)* $W \cong \overline{W}$.

(ii) $\overline{W}$ *is a discrete subgroup of* $\mathrm{Isom}(M^n_\varepsilon)$.

(iii) M^n_ε *is tessellated by* $\{wK\}_{w \in \overline{W}}$.

Sketch of proof. Put $\mathcal{U} = \mathcal{U}(W, K)$. Define $p : \mathcal{U} \to M^n_\varepsilon$ by $[w, x] \mapsto \varphi(w)x$. First show that $\mathcal{U}$ can be given the structure of a M^n_ε-manifold so that p is a local isometry. Then show that $\mathcal{U}$ is complete and that p is a covering projection. Since M^n_ε is simply connected, one can conclude from this that p is an isometry. In particular, φ is an isomorphism so (i) holds, (ii) is immediate and since p is homeomorphism (iii) holds (since the same statement is clearly true for $\mathcal{U}$). □

In the spherical and Euclidean cases Coxeter classified all such geometric reflection groups. The main qualitative result is that in the spherical case, K is a spherical simplex while in the Euclidean case K must be (metrically) a product of Euclidean simplices. The possible dihedral angles are given by the "Coxeter diagram" of (W, S). This is a labeled graph with one vertex for each element of I. One connects distinct vertices i and j by an edge if $m_{ij} \neq 2$ and one labels the edge by m_{ij} if $m_{ij} \neq 3$. The possible spherical and Euclidean reflection groups are listed in the table below. (In both cases it suffices to list only the "irreducible" groups, that is, those groups for which the diagram is connected.

Table 6.1.1. Irreducible finite and affine Coxeter groups

Example 6.1.2. (Triangle groups) Suppose that K is a 2-simplex Δ and $I = \{1, 2, 3\}$. Given three integers m_{12}, m_{23} and m_{13} each ≥ 2, can we realize Δ as a 2-simplex in M^2_ε with angles π/m_{12}, π/m_{23} and π/m_{13}? As is well known, we can always do this either in $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ depending on whether the sum of the angles is $> \pi$, $= \pi$ or $< \pi$. Thus, any three such integers m_{12}, m_{23} and m_{13} determine a *triangle group* W and this group is spherical, Euclidean, or hyperbolic as the sum $(m_{12})^{-1} + (m_{23})^{-1} + (m_{13})^{-1}$ is > 1, $= 1$ or < 1, respectively.

Example 6.1.3. (Polygon groups) Now suppose K is a k-gon, $k \geq 4$. The question of whether K with given angles can be realized in $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ comes down to the question of whether the sum of the angles is $> \pi(k-2)$, $= \pi(k-2)$ or $< \pi(k-2)$. Since all the angles are $\leq \pi/2$ and $k \geq 4$, the spherical case does not occur. In fact, the only possible Euclidean case is the rectangle where all four angles are $\pi/2$. (This is the Coxeter group $\tilde{A}_1 \times \tilde{A}_1$ with diagram $\overset{\infty}{\circ\!-\!\circ}\ \overset{\infty}{\circ\!-\!\circ}$.) It is not hard to see that given any other assignment of angles π/m_i, one can realize K as a convex polygon in $\mathbb{H}^2$. (Proof: subdivide K into triangles.) The conclusion that we reach from this example and the previous one is that given any assignment of angles π/m_{ij} to an abstract polygon K, we can realize it as a fundamental chamber of a geometric reflection group; moreover in almost all cases this reflection group is hyperbolic. In the next subsection we will see that a similar result also holds in dimension three.

Example 6.1.4. (Simplicial Coxeter groups) The case where $K = \Delta^n$ is an n-simplex can also be analyzed in a relatively straightforward way. Put $I = \{1, \dots, n+1\}$ and suppose we are given integers $m_{ij} \geq 2$ for each unordered pair of distinct elements in I. Form the associated *cosine matrix* (c_{ij}). It is the symmetric $I \times I$ matrix defined by

$$c_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ -\cos(\pi/m_{ij}) & \text{if} \quad i \neq j. \end{cases}$$

If Δ^n can be realized as a geometric simplex in M^n_ε then (c_{ij}) is the Gram matrix of inner products $(u_i \cdot u_j)$ where u_i denotes the outward pointing unit normal vector to the face of Δ^n corresponding to i. If $\varepsilon = 1$ then the $(u_i)_{i \in I}$ are a basis for $\mathbb{R}^{n+1}$; hence, the matrix (c_{ij}) is positive definite. Similarly, if $\varepsilon = -1$ then the $(u_i)_{i \in I}$ are a basis for $\mathbb{R}^{n,1}$ and hence, (c_{ij}) is indefinite of type $(n, 1)$. Moreover, since the link of each vertex must be spherical it follows that each n by n minor of (c_{ij}) must be positive definite. Similarly, in the irreducible Euclidean case, (c_{ij}) must be semidefinite of type $(n, 0)$ and each n by n minor must be positive definite. It is also not difficult to establish the converse (see pages 100-101 [B]). Thus, the assignment $\{i, j\} \mapsto m_{ij}$ corresponds to a geometric reflection group with fundamental chamber an n-simplex if and only if each n by n minor of the associated cosine matrix (c_{ij}) is positive definite. Moreover, the reflection group is then spherical, Euclidean or hyperbolic as (c_{ij}) is positive definite, positive semidefinite of type $(n, 0)$, or indefinite of type $(n, 1)$, respectively. Coxeter's classification then amounts to listing the possibilities for when (c_{ij}) is positive semidefinite. It is also not difficult to classify the remaining hyperbolic reflection groups with fundamental chamber a simplex. It turns out that such examples can occur only in dimensions $2, 3$ and 4. The 2-dimensional examples are the hyperbolic triangle groups of Example 6.1.2. There are 9 more examples in dimension 3 and 5 more in dimension 4.

6.2 Andreev's Theorem.

Theorem 6.2.1. (Andreev [A]) *Let K^3 be a simple (combinatorial) 3-dimensional polytope different from a tetrahedron, let E be the set of edges in K and let $\Theta : E \to (0, \pi/2]$ be any function. Then K can be realized as a convex polytope in $\mathbb{H}^3$ with dihedral angles prescribed by Θ if and only if the following three conditions hold:*

(i) At each vertex, the three edges which meet there satisfy $\Theta(e_1) + \Theta(e_2) + \Theta(e_3) > \pi$.

(ii) If three faces intersect pairwise but do not have a common vertex then the angles at the edges of intersection satisfy $\Theta(e_1) + \Theta(e_2) + \Theta(e_3) < \pi$.

(iii) Four faces cannot intersect cyclically with all angles $\pi/2$ unless two of the opposite faces also intersect.

(iv) If K is a triangular prism, then the angles along base and top cannot all equal $\pi/2$.

Moreover, when the polytope is realizable it is unique up to an isometry of $\mathbb{H}^3$.

Corollary 6.2.2. *Suppose K^3 is a 3-dimensional combinatorial polytope and that e_{ij} denotes the edge $K_i \cap K_j$. Given an angle assignment $\Theta : E \to (0, \pi/2]$ of the form $\Theta(e_{ij}) = \pi/m_{ij}$, where m_{ij} is an integer ≥ 2, then K^3 can be realized as the fundamental chamber of a geometric reflection group W on $\mathbb{H}^3$ if and only if the $\Theta(e_{ij})$ satisfy Andreev's conditions. Moreover, W is unique up to conjugation in Isom$(\mathbb{H}^3)$.*

This result is actually a special case of Thurston's Geometrization Conjecture and it provided some of the first substantial evidence for its truth. The uniqueness statement in Corollary 6.2.2 is a special case of the Mostow Rigidity Theorem. We begin by giving topological reasons for the necessity of Andreev's conditions in Corollary 6.2.2. (This explanation is due to Thurston.)

Proof of the necessity of Andreev's conditions. Condition (i) in Theorem 6.2.1 is just the condition that the link of each vertex be a spherical simplex. It then insures that in Corollary 6.2.2 the isotropy subgroup at each vertex is a finite reflection group. Next suppose we have three faces, say K_1, K_2, K_3, which intersect pairwise but which don't have a common vertex. If $\pi/m_{12} + \pi/m_{23} + \pi/m_{13} > \pi$, then the subgroup F generated by s_1, s_2, s_3 is a spherical reflection group and hence, finite. This gives a decomposition of W as an amalgamated free product: $W = W' *_F W''$, where W' and W'' are the subgroups generated by the faces of K on the two sides of the triangle transverse to K_1, K_2 and K_3. In geometric terms, if Γ is a torsion-free subgroup of finite index in W, and if $M^3 = \mathbb{H}^3/\Gamma$, then a component of the inverse image of this triangle in M^3 gives a separating 2-sphere in M^3. Thus such an M^3 would be a nontrivial connected sum. Similarly, if $\pi/m_{12} + \pi/m_{23} + \pi/m_{13} = \pi$, then we get a Euclidean triangle group embedded in W (and hence, a copy of $\mathbb{Z} \times \mathbb{Z}$ in W) and an incompressible torus in M^3. If condition (iii) fails then we again get a Euclidean group (with fundamental chamber a square) embedded in W and an incompressible torus in M^3. The failure of condition (iv) leads to a decomposition $W = W' \times D_\infty$ and again (a copy of $\mathbb{Z} \times \mathbb{Z}$ and) an incompressible torus in M^3. $\square$

A proof of Andreev's Theorem can be found in [A] and a more geometric argument in Thurston's notes [Th]. Although we will not attempt to give its proof, let us indicate a suggestive dimension count.

Let M_K denote the space of convex polytopes in $\mathbb{H}^3$ of the same combinatorial type as K, modulo the action of Isom$(\mathbb{H}^3)$. A hyperplane (or half-space) in $\mathbb{H}^3$ is determined by its unit normal vector. The space of such unit normal vectors is a 3-dimensional submanifold of $\mathbb{R}^{3,1}$. Let f, e and v denote, respectively, the number of faces, edges and vertices of K. A convex polytope in $\mathbb{H}^3$, being an intersection of half-spaces in $\mathbb{H}^3$, is determined by its f outward pointing unit normal vectors. Since the polytope is simple these half-spaces intersect in general position; hence, any small perturbation of these normal vectors will yield a polytope of the same combinatorial type as K. It follows that M_K is a manifold of dimension $3f - 6$. (Isom$(\mathbb{H}^3)$ is a Lie group of dimension 6.) On the other hand, the space of angle assignments $\Theta : E \to (0, \pi/2 + \varepsilon)$ is an open subset of Euclidean space of dimension e. Since three edges meet at each vertex $3v = 2e$ and since the Euler characteristic of the 2-sphere is 2, $f - e + v = 2$; hence $3f - 6 = e$. Thus, M_K is a manifold of the same dimension e as the space of angle assignments. The content of Andreev's Theorem is then that the map from M_K to the space of allowable angle assignments is a homeomorphism.

There is no analog to Andreev's Theorem for hyperbolic reflection groups in higher dimensions. One way to see this is that the preceding dimension count fails in dimensions > 3. Indeed, Vinberg has shown that there do not exist cocompact hyperbolic reflection groups in dimensions > 30. In other words, there do not exist convex, compact polytopes $K^n \subset \mathbb{H}^3$ with all dihedral angles submultiples of π, if $n > 30$.

A dual formulation of Andreev's conditions.

Suppose L is the boundary of the dual polytope to K^3. Thus L is a triangulation of S^2 such that the i-faces of L correspond to the $(2-i)$-faces of ∂K, with inclusions reversed. The dual form of Andreev's conditions are then as follows.

(i)$'$ For any 2-simplex in L with edges e_{12}, e_{23} and e_{13}, $(m_{12})^{-1} + (m_{23})^{-1} + (m_{13})^{-1} > \pi$.

(ii)$'$ If e_{12}, e_{23}, e_{13} is any "empty" 3-circuit in L, then $(m_{12})^{-1} + (m_{23})^{-1} + (m_{13})^{-1} < 1$. ("Empty" means that it is not the boundary of a 2-simplex.)

(iii)$'$ There does not exist an "empty" 4-circuit $e_{12}, e_{23}, e_{34}, e_{14}$ with all $m_{ij} = 2$. (Here "empty" means that it is not the boundary of the union of two adjacent 2-simplices.

(iv)$'$ L is not the suspension of a triangle with all edges at the suspension points labeled 2.

Remark. There is also a version of Andreev's theorem for convex polytopes of finite volume in $\mathbb{H}^3$. Roughly speaking, one allows Euclidean reflection groups as subgroups but only at the cusps.

6.3 Coxeter groups.

The theory of abstract reflection groups was developed by Tits (who introduced the terminology "Coxeter groups" for these objects).

Let I be a set (usually a finite set.) A *Coxeter matrix* $M = (m_{ij})$ over I is a symmetric I by I matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ and $m_{ij} \geq 2$ whenever $i \neq j$. If J is any subset of I, then M_J denotes the J by J matrix formed by restricting M to J. We now repeat some of the material from 3.2. Introduce symbols $s_i, i \in I$, and put $S = \{s_i\}_{i \in I}$. Let $s : I \to S$ be the bijection $i \mapsto s_i$. The *Coxeter group* W (or $W(M)$) is the group defined by the presentation:

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1,\ (i, j) \in I \times I \rangle.$$

The pair (W, S) is a *Coxeter system.* For each subset J of I let W_J denote the subgroup generated by $s(J)$. It turns out that the natural map from $W(M_J)$ to W_J is an isomorphism, i.e., W_J is also a Coxeter group. The subset J is called *spherical* if W_J is a finite group.

Given a space X and a family of closed subspaces $\{X_i\}_{i \in I}$ one defines the Vinberg–Tits construction $\mathcal{U}(W, X)$ exactly as in 3.2 or 6.1.

Definition 6.3.1. A group action $G \times Y \to Y$ is a *reflection group* action if there is a Coxeter group W with generators indexed by I and a space X with closed subspaces $\{X_i\}_{i \in I}$ so that $G = W$ and Y is W-equivariantly homeomorphic to $\mathcal{U}(W, X)$.

Remark. It is more common to define a "reflection group" to be a group generated by "reflections" on a linear space or on a manifold and then to prove that (possibly after making some additional hypotheses) the group is a Coxeter group and the space has the form $\mathcal{U}(W, X)$. However, it seems simpler just to take this as the definition.

Let $\mathcal{S}^f$ denote the set of spherical subsets of I; it is partially ordered by inclusion. (Note: $\emptyset \in \mathcal{S}^f$).) The subposet $\mathcal{S}^f_{>\emptyset}$ consisting of the nonempty elements of $\mathcal{S}^f$ is an abstract simplicial complex which we denote by L and call the *nerve* of W (or of M). (Thus, I is the vertex set of L and a nonempty subset J of I spans a simplex if and only if it is spherical.)

The *poset of spherical cosets* is defined by

$$W\mathcal{S}^f = \coprod_{J \in \mathcal{S}^f} W/W_J.$$

Again, the partial ordering is inclusion.

6.4 The complex Σ.

The goal of the next four subsections will be to explain the proof of the following generalization of Theorem 3.2.4.

Theorem 6.4.1. ([M]) *For any finitely generated Coxeter group W, there is a $CAT(0)$, PE complex Σ on which W acts as an isometric reflection group with finite isotropy subgroups and with compact orbit space.*

The difficult part of this theorem is the proof that Σ is $CAT(0)$. Before giving the definition of Σ we need to make a few general remarks about the geometric realization of a poset.

Let $\mathcal{P}$ be a poset. Its order complex, $\mathrm{Ord}(\mathcal{P})$, is then a simplicial complex (see 2.3). The *geometric realization* of $\mathcal{P}$, denoted $\mathrm{geom}(\mathcal{P})$, is the underlying topological space of $\mathrm{Ord}(\mathcal{P})$. Given an element $p \in \mathcal{P}$ we define a subposet $\mathcal{P}_{\geq p}$ by $\mathcal{P}_{\geq p} = \{x \in \mathcal{P} \mid x \geq p\}$. The subposets $\mathcal{P}_{\leq p}$, $\mathcal{P}_{>p}$, $\mathcal{P}_{<p}$ are defined similarly. It follows that we have two decompositions of $\mathrm{geom}(\mathcal{P})$ into subcomplexes; both decompositions are indexed by $\mathcal{P}$. One decomposition is $\{\mathrm{geom}(\mathcal{P}_{\leq p})\}_{p \in \mathcal{P}}$ and the other is $\{\mathrm{geom}(\mathcal{P}_{\geq p})\}_{p \in \mathcal{P}}$. The subcomplexes $\mathrm{geom}(\mathcal{P}_{\leq p})$ are the *faces* of $\mathrm{geom}(\mathcal{P})$, while the $\mathrm{geom}(\mathcal{P}_{\geq p})$ are its *dual faces*.

Definitions 6.4.2.

$$\begin{aligned}
\Sigma &= \mathrm{geom}(W\mathcal{S}^f),\\
K &= \mathrm{geom}(\mathcal{S}^f),\\
K_i &= \mathrm{geom}(\mathcal{S}_{\geq \{i\}}),\ i \in I.
\end{aligned}$$

The natural inclusion $\mathcal{S}^f \to W\mathcal{S}^f$ defined by $J \to 1W_J$ induces an inclusion $K \to \Sigma$. The image of K is called the *fundamental chamber* of Σ. In fact, K is just the dual face $\mathrm{geom}(W\mathcal{S}^f_{\geq W_\emptyset})$. The group W acts naturally on $W\mathcal{S}^f$ and hence on Σ: the translates of K are also called *chambers*. The subcomplex K_i is a *mirror* of K. The proof of the next lemma is left as an exercise for the reader.

Lemma 6.4.3. *The natural map $\mathcal{U}(W, K) \to \Sigma$ is a W-equivariant homeomorphism. Thus, W is a reflection group on Σ.*

In the next two subsections we show that the faces of Σ are naturally identified with certain convex polytopes in Euclidean space. (In the right-angled case these polytopes are cubes.) This will then define the PE structure on Σ.

6.5 Finite Coxeter groups.

In this subsection, we suppose that $\mathrm{Card}(I) = n$. As we indicated already in Example 6.1.4, there is the following lemma.

Lemma 6.5.1. *The following statements are equivalent.*

(i) The group W is finite;

(ii) W is an orthogonal linear reflection group on $\mathbb{R}^n$;

(iii) W is a geometric reflection group on $\mathbb{S}^{n-1}$;

(iv) The cosine matrix (c_{ij}), defined by $c_{ij} = -\cos(\pi/m_{ij})$, is positive definite.

Proof. (ii) and (iii) are equivalent since $\mathrm{Isom}\ \mathbb{S}^{n-1} = O(n)$. The equivalence of (iii) and (iv) is indicated in Example 6.1.4. To see that (i) $\Leftrightarrow$ (iv), one first shows, as in [B], that W acts on $\mathbb{R}^n$ preserving the bilinear form corresponding to (c_{ij}). If (iv) holds, then the group of all linear transformations preserving this form is isomorphic to the compact group $O(n)$ and since W is discrete in this group it must be finite. In the converse direction one shows that if W is irreducible then, up to scalar multiplication, there is a unique W-invariant symmetric bilinear form on $\mathbb{R}^n$. (cf. page 66 in [B]). If W is finite, we can always find an invariant inner product, i.e., we can find such a form which is positive definite. Thus, (i) $\Rightarrow$ (iv). $\square$

Definition 6.5.2. If σ^{n-1} is a spherical $(n-1)$-simplex in $\mathbb{S}^{n-1}$ then its it dual simplex σ^* is the simplex in $\mathbb{S}^{n-1}$ spanned by the outward pointing unit normal vectors to σ.

In particular, if Δ denotes a fundamental simplex for a finite reflection group W on $\mathbb{S}^{n-1}$, then Δ^* is a simplex with the property the Gram matrix $(u_i \cdot u_j)$ of its vertex set is just the cosine matrix associated to (m_{ij}).

Now suppose W is a finite reflection group on $\mathbb{R}^n$. Choose a point x in the interior of a fundamental chamber (such an x is determined by specifying its distance to each of the bounding hyperplanes, i.e., by an element of $(0, \infty)^I$).

Definition 6.5.3. The *Coxeter cell* C associated to W is the convex hull of Wx, the W-orbit of x.

Examples 6.5.4. (i) If $W = D_m$ is the dihedral group of order $2m$, then C is a $2m$-gon. (It is a regular $2m$-gon if x is equidistant from the two bounding rays.

(ii) If $W = (\mathbb{Z}/2)^n$ and x is equidistant from the bounding hyperplanes, then C is a regular n-cube.

(iii) If $W = A_3$, the symmetric group on 4 elements, then C is the “permutohedron” pictured below.

The proof of the following lemma is left as an exercise for the reader.

Figure 6.5.1. Permutohedron

Lemma 6.5.5. *Suppose W is finite and C is a Coxeter cell for it. Let Face(C) denote the poset of faces of C. Then the correspondence $w \mapsto wx$ induces an isomorphism of posets $W\mathcal{S}^f \cong$ Face(C). (In other words a subset of W is the vertex set of a face of C if and only if it is a coset of W_J for some $J \subset I$.)*

It follows that the order complex of $W\mathcal{S}^f$ is naturally identified with the barycentric subdivision of C. Thus, when W is finite we define a PE structure on Σ by setting $C = \Sigma$. We note that we then have $Lk(v, \Sigma) = Lk(v, C) = \Delta^*$.

6.6 The PE cell structure on Σ.

We return to the situation where the Coxeter group W is arbitrary. Choose a function $f : I \to (0, \infty)$ which will be used to determine the shape of Coxeter cells. For each spherical subset J of I, let C_J denote the Coxeter cell corresponding to W_J and $f|_J$. Lemma 6.5.5 shows that for any spherical coset $\alpha = wW_J \in W\mathcal{S}^f$, we have that the poset $W\mathcal{S}^f_{\leq\alpha}$ is naturally isomorphic to Face(C_J). In other words the faces of $W\mathcal{S}^f$ are naturally convex polytopes. This defines a PE structure on Σ: we identify each geom$(W\mathcal{S}^f_{\leq\alpha})$, $\alpha = wW_J$, with C_J. The W-action on Σ is then clearly by isometries.

We now come to the crucial issue of showing that Σ is $CAT(0)$. An argument similar to the proof of Lemma 3.2.2 shows that Σ is simply connected. So, by Theorems 1.5.2 and 2.2.3, the problem reduces to showing that the link of each vertex is $CAT(1)$. What are the vertices of Σ? They correspond to cosets of the trivial subgroup $W_\emptyset$, i.e., they correspond to elements of W. Since W acts transitively on the vertices of Σ, it suffices to consider the vertex v corresponding to $1 \cdot W_\emptyset$. There is a cell of Σ properly containing v for each element of $\mathcal{S}^f_{>\emptyset}$, i.e., $Lk(v, \Sigma)$ is simplicially isomorphic to the complex L. What is the PS structure on L? For each $J \in \mathcal{S}^f$, we have that $Lk(v, C_J) = \Delta^*_J$, the dual simplex to a fundamental simplex for W_J. So, Theorem 6.4.1 comes down to showing that this PS structure on L is $CAT(1)$.

6.7 A generalization of Gromov's Lemma.

Let $\sigma^{n-1} \subset \mathbb{S}^{n-1}$ be a spherical $n-1$ simplex with vertex set $\{v_1, \ldots, v_n\}$. Set

$$\ell_{ij} = d(v_i, v_j) = \cos^{-1}(v_1 \cdot v_j) \quad \text{and}$$
$$c_{ij} = \cos(\ell_{ij}) = v_1 \cdot v_j \,,$$

where $d(v_i, v_j)$ denotes the length of the circular arc from v_i to v_j. The cosine matrix (c_{ij}) is then positive definite since $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$. Conversely, given a symmetric matrix (ℓ_{ij}) with entries in $(0, \pi)$, we can define a cosine matrix (c_{ij}) by the above formula and if (c_{ij}) is positive definite then (ℓ_{ij}) is the matrix of edge lengths of a spherical simplex. (Proof: choose a basis $(v_1, \dots, v_n)$ for $\mathbb{R}^n$ with Gram matrix (c_{ij}).)

Definition 6.7.1. A spherical simplex has *size* $\geq \pi/2$ if the length of each of its edges is $\geq \pi/2$ (or equivalently, if each off-diagonal entry of (c_{ij}) is nonpositive).

Suppose that L is a PS cell complex and that each cell is a simplex of size $\geq \pi/2$.

Definition 6.7.2. L is a metric flag complex if it is a simplicial complex and if the following condition $(*)$ holds.

$(*)$ Suppose $\{v_0, \dots, v_k\}$ is a set of vertices of L any two of which are connected by an edge. Put $c_{ij} = \cos^{-1}(d(v_i, v_j))$. Then $\{v_0, \dots, v_k\}$ spans a simplex in L if and only if (c_{ij}) is positive definite.

In other words L is a metric flag complex if and only if L is "metrically determined by its 1-skeleton."

Example 6.7.3. Suppose L is the nerve of a Coxeter group with PS structure described above. Suppose u_i, u_j are vertices of L which are connected by an edge. Since $d(u_i, u_j) = \pi - \pi/m_{ij}$ which is $\geq \pi/2$, we see that each simplex in L has size $\geq \pi/2$. Furthermore, by Lemma 6.5.1 a subset J of I is spherical if and only if the associated cosine matrix is positive definite. Thus, L is a metric flag complex.

The following result of [M] generalizes Gromov's Lemma.

Lemma 6.7.4. ([M]) *Suppose L is a PS cell complex in which all the cells are simplices of size $\geq \pi/2$. Then L is $CAT(1)$ if and only if it is a metric flag complex.*

Corollary 6.7.5. ([M]) *Σ is $CAT(0)$.*

Proof of Lemma 6.7.4. Just as in the case of Gromov's Lemma 2.3.6 the "only if" implication is easy and left as an exercise for the reader. We sketch a simple proof of the other implication found recently by Krammer [Kra]. It is based on the following characterization of $CAT(1)$ spaces given by Bowditch. We say that a closed rectifiable curve in a metric space is shrinkable if it can be homotoped through a family of closed rectifiable curves in a way that the lengths of these curves do not increase and do not remain constant during the homotopy. (Note that a shrinkable closed curve in a compact space is always shrinkable to a constant curve.) Bowditch in [Bow] uses the Birkhoff curve-shortening process to prove the following.

Lemma 6.7.6. (Bowditch) *Let X be a compact metric space of curvature ≤ 1. Suppose that every nonconstant closed rectifiable curve of length $< 2\pi$ is shrinkable in X. Then X satisfies $CAT(1)$.*

Let now L denote a metric flag complex in which all simplices have size $\geq \pi/2$. These properties are inherited by all links; therefore, by induction on dimension we may assume that L has curvature ≤ 1. Suppose that L is not $CAT(1)$. Then by Bowditch's Lemma there exists a nonconstant, nonshrinkable, closed curve of length $< 2\pi$ in L. Choose a shortest possible among such curves, then this curve, denoted by γ, is a nonshrinkable closed geodesic of length $< 2\pi$. As L consists of simplices of size $\geq \pi/2$, for each vertex v of L the closed $\pi/2$-ball B_v about v is isometric to the spherical cone $C_1(Lk(v, L))$ and is contained in the star $St(v, L)$. It is easy to see that L is covered by the interiors of balls B_v, $v \in Vert(L)$. As in the proof of Gromov's Lemma we see that $\gamma \cap B_v$ is an arc of length π whenever γ meets the interior of B_v.

Suppose that v is a vertex of L such that γ meets the interior of B_v and does not pass through v. Then let us perform the following modification on γ: replace the segment $\gamma \cap B_v$ with the union of the two geodesic segments from v to $\gamma \cap \partial B_v$. This modification can clearly be achieved via a homotopy through curves that have constant length. It follows that the modified closed curve continues to be a nonshrinkable closed geodesic of same length.

Repeated application of such modifications results in a nonshrinkable closed geodesic that passes through the maximum number of vertices. Such a curve is necessarily contained in the 1-skeleton of L. Since all edges have length $\geq \pi/2$, the curve is a 3-circuit in the 1-skeleton. But in a metric flag complex every 3-circuit with edges of total length $< 2\pi$ is the boundary of a 2-simplex; therefore, it cannot be a closed geodesic, a contradiction. □

6.8 When is Σ $CAT(-1)$?

Sometimes the phrase "large space" is used synonymously with "$CAT(1)$ space". Call a $CAT(1)$ space L *extra large* if there is a constant $\delta > 0$ so that the length of every closed geodesic in L is $\geq 2\pi + \delta$.

In this subsection we shall state (without proofs) some results from [M].

Lemma 6.8.1. ([M]) *Suppose X is a PE cell complex with $\kappa(X) \leq 0$. Then the metric can be deformed to a PH structure with $\kappa(X) \leq -1$ if and only if the link of each vertex is extra large.*

Lemma 6.8.2. ([M]) *Suppose that $L(= L(W))$ is the nerve of a Coxeter group W. Then L is extra large if and only for any subset J of I the following two conditions hold:*

- *$W_J \neq W_{J_1} \times W_{J_2}$ where both factors are infinite.*
- *W_J is not a Euclidean reflection group with $Card(J) \geq 3$.*

Corollary 6.8.3. ([M]) *The following statements are equivalent:*

(i) W satisfies the condition in Lemma 6.8.2;

(ii) Σ can be given a PH structure which is $CAT(-1)$;

(iii) W is "word hyperbolic" in the sense of Gromov, [G1];

(iv) W does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

The logic for the proof of this corollary is as follows. Lemma 6.8.1 and 6.8.2 show that (i) $\Rightarrow$ (ii). If a group acts properly, isometrically and cocompactly on a connected metric space, then the group and the space are in a certain sense "quasi-isometric", a $CAT(-1)$ space is certainly coarsely hyperbolic (= negatively curved); so a group acting on such a space is word hyperbolic; hence, (ii) $\Rightarrow$ (iii). (For definitions and discussion of terms in the previous sentence, see [G1].) Word hyperbolic groups do not contain subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (see [GH]). Thus, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

7. Convex hypersurfaces in Minkowski space.

7.1 Minkowski space.

Minkowski space $\mathbb{R}^{n,1}$ is the vector space $\mathbb{R}^{n+1}$ with coordinates $x = (x_0, x_1, \dots, x_n)$ but equipped with the indefinite symmetric bilinear form, denoted by $(x, y) \mapsto x \cdot y$, and defined by

$$x \cdot y = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ denote the corresponding quadratic form, i.e., $\varphi(x) = x \cdot x$. There are three special hypersurfaces in $\mathbb{R}^{n,1}$:

$$\begin{aligned} \mathbb{H}^n &= \varphi^{-1}(-1) \cap \{x_0 > 0\}, \\ \mathbb{S}^{n-1}_1 &= \varphi^{-1}(1), \\ \mathbb{L}^n &= \varphi^{-1}(0). \end{aligned}$$

We have already discussed hyperbolic n-space $\mathbb{H}^n$ in 1.2. $\mathbb{S}^{n-1}_1$ is called the *de Sitter sphere*; it is a Lorentzian manifold in the sense that its tangent space, equipped with the induced bilinear form, is isometric to $\mathbb{R}^{n-1,1}$. $\mathbb{L}^n$ is the *light cone.*

Let E be a $(k+1)$-dimensional linear subspace of $\mathbb{R}^{n,1}$, then E is *spacelike* if $\varphi|_E$ is positive definite; it is *timelike* if $\varphi|_E$ is of type $(k, 1)$; and it is *lightlike* if $\varphi|_E$ is degenerate. Similarly, a nonzero vector $v \in \mathbb{R}^{n,1}$ is *spacelike, timelike* or *lightlike* as $v \cdot v$ is > 0, < 0, or $= 0$, respectively.

7.2 The Gauss Equation.

Suppose that M^n is a smooth hypersurface in an $(n+1)$-dimensional affine space. Then M^n is *convex* if it lies on one side of each of its tangent hyperplanes. The following is one of the classical results of differential geometry.

Theorem 7.2.1. (Hadamard) *Suppose M^n is a smooth complete hypersurface in $\mathbb{E}^{n+1}$. Then the sectional curvature of M^n (with the induced Riemannian metric) is ≥ 0 if and only if M^n is convex.*

Sketch of proof. Using a unit normal vector field one defines the "second fundamental form" $\ell : T_xM \times T_xM \to \mathbb{R}$. One shows that ℓ is essentially the Hessian of

Figure 7.1.1. $\mathbf{R}^{n,1}$

a locally defined function $f : \mathbb{R}^n \to \mathbb{R}$ such that near x, M^n is equal to the graph of f. It follows that M^n is a convex hypersurface if and only if ℓ is positive semidefinite or negative semidefinite (depending on the two possible choices of the unit normal field). The following equation (the "Gauss Equation") is the main content of Gauss' Theorema Egregium:

$$\kappa(u, v) = (n \cdot n)\big(\ell(u, u)\ell(v, v) - \ell(u, v)^2\big),$$

where n is the unit normal vector to $T_x M^n$, u and v are two orthogonal unit vectors in $T_x M^n$ and $\kappa(u, v)$ is the sectional curvature of the 2-plane spanned by u and v. Hence, ℓ is semidefinite if and only if $\kappa(u, v) \geq 0$. $\square$

Now suppose that M^n is a smooth hypersurface in $\mathbb{R}^{n,1}$. Given $x \in M^n$, $T_x M^n$ is spacelike if and only if each nonzero normal vector n_x is timelike (i.e., $n_x \cdot n_x < 0$). We shall say that M^n is *spacelike* if $T_x M^n$ is spacelike for each $x \in M^n$. If this is the case, then the restriction of φ to $T_x M^n$ gives M^n a Riemannian structure. As before, M^n is convex if and only if ℓ is semidefinite. The Gauss Equation remains valid (see page 101 in [O]) except that now the term $(n \cdot n)$ becomes -1 instead of $+1$. Thus, we have the following result.

Theorem 7.2.2. *Let M^n be a smooth spacelike hypersurface in $\mathbb{R}^{n,1}$. Then the sectional curvature of M^n is nonpositive if and only if M^n is convex.*

Example 7.2.3. $\mathbb{H}^n$ is a spacelike convex hypersurface in $\mathbb{R}^{n,1}$.

Similarly, if M^{n-1} is a spacelike convex hypersurface in $\mathbb{S}_1^{n-1}$, then its sectional curvature is ≤ 1.

7.3 Convex polyhedral hypersurfaces in $\mathbb{R}^{n,1}$.

Suppose that V is a finite dimensional vector space and that X is a closed convex subset of V with nonempty interior. An affine hyperplane P in V is a *supporting hyperplane* of X if X lies in one of the half-spaces bounded by P and $P \cap X \neq \emptyset$. The *boundary* ∂X of X consists of those points of X which lie on some supporting hyperplane. ∂X is called a *convex hypersurface* in V.

The convex set X is a *convex polyhedral set* (abbreviated *cps*) if it is the intersection of a collection of closed half-spaces whose boundary hyperplanes form a locally finite family in V. The *faces* of a cps X are subsets of the form $X \cap P$ where P is a supporting hyperplane of X. Each face is a cps when viewed as a subset of its affine span (called its *tangent space*). A cps X is a *convex polyhedral cone* (abbreviated *cpc*) if, in addition, there is a point x which lies in each of its supporting hyperplanes. (Note that in this case X is the intersection of a finite number of half-spaces.) If X contains no line, then the cone X is "pointed", i.e., the point x is unique. It is called the vertex of the cpc; usually we will take the vertex to be the origin. If X is a cps, then ∂X is a *convex polyhedral hypersurface* (abbreviated *cph*).

We now take $V = \mathbb{R}^{n,1}$. A convex hypersurface $Y = \partial X$ in $\mathbb{R}^{n,1}$ is *spacelike* if each supporting hyperplane is spacelike. If we exclude the case where X is a region bounded by two parallel hyperplanes, then Y is homeomorphic to $\mathbb{R}^n$. If a cph Y is spacelike, then the tangent spaces to each of its faces are spacelike. Thus each face of such a Y is naturally identified with a convex set in a Euclidean space, that is to say, Y has a PE structure.

Similarly, suppose that X is a cpc (with vertex at the origin) and that $Y = \partial X$ is spacelike. Then the intersection of each face of Y with $\mathbb{S}_1^{n-1}$ is naturally a spherical cell. Thus, $L = Y \cap \mathbb{S}_1^{n-1}$ is naturally a PS cell complex. Also note that in this case Y is isometric to the Euclidean cone $C_0 L$, defined in 1.6.

Theorem 7.3.1. ([CD4]) *(i) Suppose $L = Y \cap \mathbb{S}_1^{n-1}$, where Y is the spacelike boundary of a pointed cpc, as above. Then L is an extra large PS complex.*

(ii) Suppose that Y is a spacelike cph in $\mathbb{R}^{n,1}$. Then Y is a $CAT(0)$, PE space (not necessarily complete). Moreover, the link of each vertex is extra large.

We omit the proof but simply observe that by Theorem 1.6.1 *(ii)* statements (i) and (ii) are equivalent.

Remark. The second author has generalized this theorem to arbitrary spacelike convex hypersurfaces.

Corollary 7.3.2. ([CDM]) *Any complete hyperbolic manifold admits a nonpositively curved PE cell structure with extra large links.*

Sketch of proof. A complete hyperbolic manifold M^n can be written as $M^n = \mathbb{H}^n/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. We note that $\mathrm{Isom}(\mathbb{H}^n)$ is a subgroup of index two in $O(n,1)$, the group of linear transformations of $\mathbb{R}^{n,1}$ which preserve φ. Thus, Γ can be considered as a subgroup of $O(n,1)$. Let $p : \mathbb{H}^n \to M^n$ be the covering projection. Choose a net V in M^n (i.e., $V \subset M^n$ is discrete and there is a constant c such that $d(x,V) \le c$ for all $x \in M^n$). Put $\tilde{V} = p^{-1}(V)$. Let X be the convex hull of $\tilde{V}$ in $\mathbb{R}^{n,1}$ and $Y = \partial X$. Since the subset $\tilde{V}$ is stable under Γ, Γ acts on X and hence, on Y. Radial projection provides a Γ-equivariant homeomorphism $\mathbb{H}^n \to Y$. It is not hard to see (cf. [CDM]) that the fact that V is a net guarantees that Y is a cps and each face of Y is spacelike. Thus, Y/Γ is the desired PE structure on M^n. □

This corollary raises the following two questions.

Question 1: Suppose M^n is a nonpositively curved Riemannian manifold. Does M^n admit a nonpositively curved PE cell structure?

Question 2: Suppose M^n is a Riemannian manifold and that it is strictly negatively curved, i.e., $\kappa(M^n) \le \delta < 0$. Does M^n admit a nonpositively curved cell structure? With extra large links?

Leeb showed in [Le] that the answer to Question 1 is no in general. A similar result is proved in [DOZ]. Leeb generalized Gromov's Rigidity Theorem to geodesic spaces (with extendible geodesics): Suppose M^n is a compact, locally symmetric Riemannian manifold of noncompact type. (This means that M^n is covered by G/K where G is a noncompact semisimple Lie group and K is a maximal compact subgroup.) Suppose further that G/K is irreducible and of rank ≥ 2. Let X be a compact, nonpositively curved geodesic space with extendible geodesics (in Gromov's version one assumes that X is a nonpositively curved Riemannian manifold). The result of [Le] states that if $\pi_1(M) \cong \pi_1(X)$, then X and M are isometric. In particular, this shows that the metric on X cannot come from a PE structure, since G/K is not locally isometric to Euclidean space.

The answer to Question 2 is not known but is probably no. For example, it seems unlikely that complex hyperbolic or quaternionic hyperbolic manifolds admit nonpositively curved polyhedral metrics.

7.4 Dimension 2: uniqueness.

In the case of nonnegative curvature there is the following classical result.

Theorem 7.4.1. (Aleksandrov, Pogorelov) *Suppose that X denotes the 2-sphere equipped with a nonnegatively curved length metric (i.e., X satisfies the reverse $CAT(0)$-inequality). Then X is isometric to the boundary of a convex body in $\mathbb{E}^3$ (allowing the possibility that X is the double of a convex 2-disk). Moreover, this convex body is unique up to an isometry of $\mathbb{E}^3$.*

In the case of curvature bounded from above the corresponding results are much more recent.

Theorem 7.4.2. (Rivin [RH]) *Suppose that X is a PS cell complex homeomorphic to the 2-sphere and that X is $CAT(1)$ and extra large. Then X is isometric to a spacelike convex polyhedral hypersurface in $\mathbb{S}^2_1$ (as in Theorem 7.3.1. (i)). Moreover, the hypersurface is unique up to an isometry of $\mathbb{R}^{3,1}$.*

The corresponding result for Riemannian metrics was proved by Schlenker [Sc1]. This theorem has recently been extended to general length metrics on the 2-sphere by the second author by using Rivin's Theorem and ideas of Aleksandrov.

Next suppose that M^2 is a closed orientable surface of genus ≥ 2 equipped with a Riemannian metric. The Uniformization Theorem implies that the metric on M^2

is conformally equivalent to an essentially unique hyperbolic metric. That is to say the conformal structure on M^2 gives an embedding of $\Gamma = \pi_1(M^2)$ into $O(2,1)$, well-defined up to conjugation.

Theorem 7.4.3. (Schlenker [Sc2]). *Suppose that M^2 is a closed Riemann surface with $\kappa(M^2) < 0$. Then, with notation as above, its universal cover $\tilde{M}^2$ is Γ-equivariantly isometric to a smooth, spacelike convex hypersurface in $\mathbb{R}^{2,1}$. Furthermore, this hypersurface is unique up to an isometry of $\mathbb{R}^{2,1}$.*

Remark. I. Iskhakov has made progress on a similar result for nonpositively curved PE structures on surfaces.

7.5 Some open questions in dimension 3.

The uniqueness results of the previous subsection will not hold in higher dimensions; however, Thurston's Geometrization Conjecture suggests that something should be true in dimension 3.

Suppose that M^3 is a closed, nonpositively curved, PE 3-manifold with extra large links of vertices.

Question 3: Is M^3 homeomorphic to a hyperbolic 3-manifold?

Remarks. Nonpositive curvature implies that the universal cover $\tilde{M}^3$ is homeomorphic to $\mathbb{R}^3$ and hence, that M^3 is irreducible. The condition that the links are extra large implies that $\tilde{M}^3$ does not admit an isometric embedding of $\mathbb{R}^2$. Then a result of Gromov implies that $\pi_1(M^3)$ is word hyperbolic (and in particular, that $\pi_1(M^3)$ does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$). Thus, the Geometrization Conjecture implies that the answer to Question 3 is yes.

One approach to Question 3 is through the following.

Question 4: Let M^3 be as above. Can the metric on M^3 be deformed (say, through nonpositively curved PE metrics) so that $\tilde{M}^3$ is isometric to a spacelike cph in $\mathbb{R}^{3,1}$?

Casson has also suggested a direct approach to the Geometrization Conjecture along the above lines. He suggested that one put an arbitrary PH simplicial metric on a 3-manifold and then try to deform it, through PH metrics to be smooth (by making the link of each edge have length 2π).

Question 5: (Casson) Suppose M^3 is an irreducible 3-manifold and that $\mathbb{Z} \times \mathbb{Z}$ is not contained in $\pi_1(M^3)$. Can every PH or PE metric be deformed to a (smooth) hyperbolic metric on M^3?

Chapter IV: L_2-homology and Euler Characteristics.

8. The Euler characteristic of a closed aspherical manifold.

8.1 The Euler Characteristic Conjecture.

In this section we shall discuss various formulations of the following conjecture.

Conjecture 8.1.1. (Chern, Hopf, Thurston) *Suppose M^{2n} is a closed, aspherical $2n$-dimensional manifold. Then $(-1)^n\chi(M^{2n}) \geq 0$. (Here χ denotes the Euler characteristic.)*

Remark 8.1.2. There is no odd dimensional analog of the above conjecture, since if M^{2n+1} is any odd dimensional closed manifold then it follows from Poincaré duality that $\chi(M^{2n+1}) = 0$.

Suppose that Y is a cell complex and that G is a discrete group acting cellularly on Y. Throughout this chapter we will make the following assumptions:

- Y/G is compact;
- the stabilizers of cells are finite, that is, for any cell σ, the subgroup $G_\sigma = \{g \in G \mid g\sigma = \sigma\}$ is finite.

The quotient space Y/G is called an "orbihedron" or an "orbifold" in the case where Y is a manifold. We shall not attempt a complete definition of these terms here except to say that the word "orbihedron" encompasses more structure than just the topological space Y/G; in particular, it includes the local isomorphism type of the action of the isotropy subgroup G_y on a G_y-stable neighborhood of y, for any $y \in Y$. For example, if H a normal subgroup of G which acts freely on Y and $Z = Y/H$, then $Z/(G/H)$ and Y/G are the same orbihedron; furthermore, Z is an "orbihedral covering space" of Y/G. If Y can be chosen to be contractible, then Y/G is an *aspherical orbihedron.*

Definition 8.1.3. The *orbihedral Euler characteristic* of Y/G is the rational number defined by the formula

$$\chi^{orb}(Y/G) = \sum_\sigma \frac{(-1)^{\dim\sigma}}{|G_\sigma|},$$

where σ ranges over a set of representatives for the G-orbits of cells.

Suppose H is a subgroup of finite index in G. The main property of the orbihedral Euler characteristic is that it is multiplicative with respect to coverings, i.e.,

$$\chi^{orb}(Y/H) = [G:H]\,\chi^{orb}(Y/G).$$

We might as well expand Conjecture 8.1.1 to the following:

Conjecture 8.1.4. *Suppose X^{2n} is a closed, aspherical orbifold of dimension $2n$. Then $(-1)^n\chi^{orb}(X^{2n}) \geq 0$.*

Example 8.1.5. Suppose $G = W_L$, the right-angled Coxeter group with nerve L, and that $Y = \mathcal{U}_L$, the $CAT(0)$ cubical complex of 3.2. There is one W_L-orbit of cubes for each element J in $\mathcal{S}^f$. Furthermore, the dimension of a corresponding cube is $\mathrm{Card}(J)$ and its stabilizer is isomorphic to $(\mathbb{Z}/2)^J$. Hence,

$$\begin{aligned}\chi^{orb}(\mathcal{U}/W_L) &= \sum_{J\in\mathcal{S}^f} \frac{(-1)^{\mathrm{Card}(J)}}{2^{\mathrm{Card}(J)}}\\ &= 1 + \sum_\sigma \frac{(-1)^{\dim\sigma+1}}{2^{\dim\sigma+1}}\\ &= 1 + \sum_{i=0}^{\dim L} (-\frac{1}{2})^{i+1} f_i\,,\end{aligned}$$

where f_i denotes the number of i-simplices in L. The focus of the remainder of this chapter will be on Conjecture 8.1.4 in the special case where L is a triangulation of the $(2n-1)$-sphere (so that $\mathcal{U}_L$ is a $2n$-manifold).

8.2 The Chern-Gauss-Bonnet Theorem.

Theorem 8.2.1. (Chern, Gauss, Bonnet) *Suppose M^{2n} is a closed, $2n$-dimensional Riemannian manifold. Then*

$$\chi(M^{2n}) = \int \kappa$$

where κ is a certain $2n$-form on M^{2n} called the "Euler form". (κ is a constant multiple of the Pfaffian of the curvature).

The theorem was proved in dimension two by Gauss and Bonnet; in this case κ is just the Gaussian curvature (times $1/2\pi$). Versions of the higher dimensional result were proved by Poincaré, Hopf and Allendoerfer–Weil. The "correct" differential geometric proof in higher dimensions is due to Chern. On the basis of Theorem 8.2.1, Hopf (as reported by Chern in [C]) asked if the following special case of Conjecture 8.1.1 holds.

Conjecture 8.2.2. *Suppose M^{2n} is a closed, Riemannian manifold of dimension $2n$ of nonpositive sectional curvature. Then $(-1)^n\chi(M^{2n}) \geq 0$.*

Later Thurston asked about the same conjecture under the weaker assumption that M^{2n} is aspherical so we have added his name to Conjecture 8.1.1.

Remark 8.2.3. The naive idea for proving Conjecture 8.2.2 is to show that the condition that the sectional curvature be ≤ 0 at each point forces the Chern-Gauss-Bonnet integrand κ to have the correct sign, i.e., $(-1)^n\kappa \geq 0$. In dimension 2, κ is just the Gaussian curvature (up to a positive constant), so this naive idea works. As shown by Chern [C] (who attributes the result to Milnor) the naive idea also works in dimension 4. Later Geroch [Ge] showed that in dimensions ≥ 6 the naive idea does not work (that is, the condition that the sectional curvature be ≤ 0 at a point does not imply that κ has the correct sign.)

8.3 The Flag Complex Conjecture.

We begin this subsection by discussing a combinatorial version of the Chern-Gauss-Bonnet Theorem.

Let σ^{n-1} be a convex polytope in $\mathbb{S}^{n-1}$. If $u_1, \dots, u_k$ are the outward pointing unit normal vectors to the codimension-one faces of σ, then the convex hull of $\{u_1, \dots u_k\}$ is a convex polytope in $\mathbb{S}^{n-1}$, denoted by σ^*, called the *dual cell* to σ. The *angle* determined by σ, denoted $a(\sigma)$, is its $(n-1)$-dimensional volume, normalized so that the volume of $\mathbb{S}^{n-1}$ is 1, i.e., $a(\sigma) = vol(\sigma)/vol(\mathbb{S}^{n-1})$. Its *exterior angle*, $a^*(\sigma)$, is defined by $a^*(\sigma) = a(\sigma^*)$.

Now suppose that L is a finite PS cell complex. Define $\kappa(L)$ by the formula

$$\kappa(L) = 1 + \sum_{\sigma} (-1)^{\dim\sigma+1} a^*(\sigma).$$

Example 8.3.1. Suppose σ is an all right, spherical $(n-1)$-simplex. Since $\mathbb{S}^{n-1}$ is tessellated by 2^n copies of σ we have that $a(\sigma) = (\frac{1}{2})^n$. Since σ^* is isometric to σ, $a^*(\sigma)$ is also equal to $(\frac{1}{2})^n$. Hence, if L is an all right PS simplicial complex, then

$$\kappa(L) = 1 + \sum_{i=0}^{\dim L} (-\frac{1}{2})^{i+1} f_i$$

where f_i denotes the number of i-simplices in L. (Compare this to the formula in Example 8.1.5.)

Now suppose that X is a PE cell complex. For each vertex v of X put $\kappa_v = \kappa(Lk(v, X))$.

Theorem 8.3.2. (The Combinatorial Gauss-Bonnet Theorem, [CMS]). *If X is a finite PE cell complex, then*

$$\chi(X) = \sum_{v \in Vert(X)} \kappa_v .$$

The proof is based on the geometrically obvious fact that for any Euclidean convex polytope C, $\sum a^*(Lk(v, C)) = 1$, where the summation is over all vertices of C. The proof of the formula in Theorem 8.3.2 is then left as an exercise for the reader.

In contrast to Geroch's result, it is conjectured in [CD3] that for PE manifolds, Conjecture 8.1.1 should hold for local reasons. That is to say, we have the following:

Conjecture 8.3.3. *Suppose that L^{2n-1} is a PS cell structure on the $(2n-1)$-sphere. If L is $CAT(1)$, then $(-1)^n \kappa(L) \geq 0$.*

Again, this conjecture is obviously true if L is a circle, for then $\kappa(L) = 1 - (2\pi)^{-1}\ell(L)$. Hence, if $\kappa(L)$ is $CAT(1)$, then $\ell(L) \geq 2\pi$ and so, $\kappa(L) \leq 0$.

The special case of this conjecture where L is an all right, PS simplicial cell complex will here be called the "Flag Complex Conjecture".

Conjecture 8.3.4. (The Flag Complex Conjecture of [CD3]). *Suppose that L^{2n-1} is an all right, PS simplicial cell structure on S^{2n-1}. If L is a flag complex, then $(-1)^n \kappa(L) \geq 0$, where as before,*

$$\kappa(L) = 1 + \sum_{i=0}^{2n-1} (-\frac{1}{2})^{i+1} f_i \, .$$

Theorem 8.3.5. ([CD3]). *The Flag Complex Conjecture is equivalent to the Euler Characteristic Conjecture (Conjecture 8.1.1) for nonpositively curved, PE cubical manifolds.*

Proof. It follows immediately from the Combinatorial Gauss-Bonnet Theorem, that the Flag Complex Conjecture implies the Euler Characteristic Conjecture. Conversely, suppose that L^{2n-1} is a triangulation of S^{2n-1} as a flag complex. Let X_L

be the PE cubical manifold constructed in 3.1. Then $\chi(X_L) = 2^{f_0}\kappa(L)$, (where f_0 denotes the number of vertices in L). Thus, $\kappa(L)$ and $\chi(X_L)$ have the same sign, so the Euler Characteristic Conjecture implies the Flag Complex Conjecture. $\square$

The easiest way to find a flag triangulation of S^{2n-1} is to take the barycentric subdivision of the boundary of a convex polytope. In this situation, Babson observed that a result of Stanley [St] implies the conjecture.

Theorem 8.3.6. (Babson, Stanley). *The Flag Complex Conjecture holds for L the barycentric subdivision of the boundary of a convex polytope.*

For further details on these conjectures the reader is referred to [CD3].

In the sequel we will discuss an approach to the Flag Complex Conjecture due to the first author and B. Okun. In particular, this will yield a proof of the conjecture in dimension three.

9. L_2-homology.

9.1 Basic definitions.

Suppose G is a countable discrete group. We denote by $\ell_2(G)$ the vector space of real-valued L_2 functions on G, i.e.,

$$\ell_2(G) = \left\{ f : G \to \mathbb{R} \mid \sum_{g\in G} f(g)^2 < \infty \right\}.$$

Define an inner product $\langle\ ,\ \rangle : \ell_2(G) \times \ell_2(G) \to \mathbb{R}$ by

$$\langle f_1, f_2\rangle = \sum_{g\in G} f_1(g) f_2(g).$$

Thus, $\ell_2(G)$ is a Hilbert space. The group ring $\mathbb{R}G$ can be identified with the dense subspace of $\ell_2(G)$ consisting of all functions of finite support. (N.B. $\mathbb{R}G$ is a ring; however, its multiplication does not extend to $\ell_2(G)$.) For any $h \in G$, let $e_h : G \to \mathbb{R}$ be the characteristic function of $\{h\}$ (defined by $e_h(g) = \delta(h, g)$ where $\delta(\ ,\ )$ is the Kronecker delta). Then $(e_h)_{h\in G}$ is a basis for $\mathbb{R}G$ and an orthonormal basis for $\ell_2(G)$ (in the Hilbert space sense). There are two actions of G on $\ell_2(G)$ (by left translation and by right translation). Both actions are by isometries.

Now suppose that Y is a CW complex equipped with a cellular G-action (a G-CW complex, for short). As usual we assume that the action is cocompact (i.e., Y/G is compact) and proper (i.e., cell stabilizers are finite). Define $C_i^{(2)}(Y)$ to be the vector space of (infinite) cellular i-chains on Y with square summable coefficients. Equivalently,

$$\begin{aligned} C_i^{(2)}(Y) &= C_i(Y) \otimes_{\mathbb{Z}G} \ell_2(G) \\ &= C_i(Y; \ell_2(G)). \end{aligned}$$

where $C_i(Y)$ denotes the ordinary i-chains on Y. As a $\mathbb{Z}G$-module, $C_i(Y)$ can be decomposed as

$$C_i(Y) = \sum_{\substack{G\text{-orbits} \\ \text{of } i\text{-cells } \sigma}} \mathbb{Z}(G/G_\sigma)$$

where $\mathbb{Z}(G/G_\sigma)$ denotes the permutation module $\mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} \mathbb{Z}$. Similarly, $C_i^{(2)}(Y)$ decomposes as $\sum \ell_2(G/G_\sigma)$, where $\ell_2(G/G_\sigma) = \ell_2(G) \otimes_{\mathbb{R}G_\sigma} \mathbb{R}$ can be thought of as the Hilbert space of all L_2 functions on G/G_σ.

Since dual Hilbert spaces are canonically isomorphic, there is a canonical isomorphism between L_2-chains and L_2-cochains: $C_i^{(2)}(Y) \cong C^i_{(2)}(Y)$. Define the *boundary map*, $d_i : C_i^{(2)}(Y) \to C_{i-1}^{(2)}(Y)$, by the usual formula. Its adjoint $\delta_{i-1} = d_i^* : C_{i-1}^{(2)}(Y) \to C_i^{(2)}(Y)$ is the *coboundary* map. Both d_1 and δ_i are bounded linear operators. Define subspaces of $C_i^{(2)}(Y)$:

$$\begin{aligned} Z_i^{(2)}(Y) &= Ker\, d_i, & Z^i_{(2)}(Y) &= Ker\, \delta_i, \\ B_i^{(2)}(Y) &= Im\, d_{i+1}, & B^i_{(2)}(Y) &= Im\, \delta_{i+1}, \end{aligned}$$

called the L_2-cycles, -cocycles, -boundaries and -coboundaries, respectively. The corresponding quotient spaces

$$H_i^{(2)}(Y) = Z_i^{(2)}(Y)/B_i^{(2)}(Y) \qquad \text{and} \qquad H^i_{(2)}(Y) = Z^i_{(2)}(Y)/B^i_{(2)}(Y)$$

are the (unreduced) L_2-homology and cohomology groups. Here an important distinction from the finite dimensional case occurs, the subspaces $B_i^{(2)}(Y)$ and $B_i^{(2)}(Y)$ of $C_i^{(2)}(Y)$ need not be closed. Thus, the unreduced L_2-homology and cohomology groups need not be Hilbert spaces. This is remedied by the following:

Definition 9.1.1. The *reduced* L_2 homology of Y is defined by

$$\mathcal{H}_i(Y) = Z_i^{(2)}(Y)/\overline{B}_i^{(2)}(Y)$$

(where $\overline{B}_i^{(2)}$ denotes the closure of $B_i^{(2)}$).

Harmonic cycles. (the Hodge–de Rham decomposition). To simplify notation, we omit "Y" and instead of $C_i^{(2)}(Y)$, $Z_i^{(2)}(Y)$, we write $C_i^{(2)}$, $Z_i^{(2)}$, etc. We have the orthogonal direct sum decomposition:

$$\begin{aligned} C_i^{(2)} &= \overline{Im\, d_{i+1}} \oplus (Im\, d_{i+1})^{\perp} \\ &= \overline{B}_i^{(2)} \oplus Ker\, d_{i+1}^* \\ &= \overline{B}_i^{(2)} \oplus Z^i_{(2)}. \end{aligned}$$

Similarly,

$$C_i^{(2)} = \overline{B}^i_{(2)} \oplus Z_i^{(2)}.$$

Since $\langle \delta_{i-1}x, d_{i+1}y\rangle = \langle x, d_i d_{i+1} y\rangle = 0$, we see that the subspaces $\overline{B}^i_{(2)}$ and $\overline{B}_i^{(2)}$ are orthogonal. Thus,

$$C_i^{(2)} = \overline{B}_i^{(2)} \oplus \overline{B}^i_{(2)} \oplus (Z_i^{(2)} \cap Z^i_{(2)}).$$

An element of $Z_i^{(2)} \cap Z^i_{(2)}$ is called a *harmonic cycle*; it is a chain (=cochain) which is simultaneously both a cycle and a cocycle. It follows from the above formula that the closed subspace $Z_i^{(2)} \cap Z^i_{(2)}$ maps isomorphically onto the reduced L_2-homology (or cohomology) $\mathcal{H}_i$. Henceforth, we identify $Z_i^{(2)} \cap Z^i_{(2)}$ with $\mathcal{H}_i$.

Examples 9.1.2.

• Suppose $Y = \mathbb{R}$, cellulated as the union of intervals $[n, n+1], n \in \mathbb{Z}$. A 0-chain is just an L_2 function on the set of vertices, i.e., an element of $\ell_2(\mathbb{Z})$. If it is a 0-cocycle, it must be constant and since it is square summable, the constant must be 0. Thus, $H^0_{(2)}(\mathbb{R}) = 0 = \mathcal{H}_0(\mathbb{R})$. Similarly, any 1-cycle must be a constant function on the set of edges; hence, $H_1^{(2)}(\mathbb{R}) = 0 = \mathcal{H}_1(\mathbb{R})$. (On the other hand, the unreduced 0^{th} homology and 1^{st} cohomology groups are not zero.)

• More generally, the same reasoning shows that whenever the vertex set of Y is infinite and Y is connected, then $\mathcal{H}_0(Y) = 0$.

• Suppose Y is the infinite regular trivalent tree. Let α be the 1-cycle pictured below. (This figure is copied from [G2].)

Figure 9.1.1. L_2 1-cycle

We have

$$|\alpha|^2 = 1 + 4(\frac{1}{2})^2 + 8(\frac{1}{4})^2 + 16(\frac{1}{8})^2 + \cdots$$

which converges, so α is L_2. Since there are no 2-cells, α is automatically a cocycle and therefore, harmonic. Thus, $\mathcal{H}_1(Y) \neq 0$.

9.2 Von Neumann dimension and L_2 Betti numbers.

The *Von Neumann algebra* $N(G)$ associated to $\ell_2(G)$ is the set of all bounded linear operators from $\ell_2(G)$ to itself which are equivariant with respect to the left G-action. The right action of $\mathbb{R}G$ on $\ell_2(G)$ shows that $\mathbb{R}G \subset N(G)$. Moreover, $N(G)$ is naturally embedded in $\ell_2(G)$ by the map which sends $\varphi \in N(G)$ to $\varphi(e_1)$. We define a trace function $tr_G : N(G) \to \mathbb{R}$ by taking the "coefficient of e_1," i.e., by the formula

$$tr_G(\varphi) = \langle \varphi(e_1), e_1 \rangle.$$

If A is a G-equivariant bounded linear endomorphism of $\ell_2(G)^n$, $n \in \mathbb{N}$, then we can represent A as an n by n matrix (A_{ij}) with coefficients in $N(G)$. Define

$$tr_G(A) = \sum_{i=1}^{n} tr_G(A_{ii}).$$

Now suppose that that V is a closed G-invariant subspace of $\ell_2(G)^n$. Let $p_V : \ell_2(G)^n \to \ell_2(G)^n$ denote the orthogonal projection onto V. The *Von Neumann dimension* of V is defined by

$$\dim_G V = tr_G(p_V).$$

Next we list some basic properties of $\dim_G$ which are fairly easy to verify.

Properties of Von Neumann dimension.

- $\dim_G V$ is a nonnegative real number.
- $\dim_G V = 0$ if and only if $V = 0$.
- $\dim_G(V_1 \oplus V_2) = \dim_G V_1 + \dim_G V_2$.
- $\dim_G(\ell_2(G)) = 1$.
- If H is a subgroup of finite index m in G, then $\dim_H V = m \dim_G V$.
- Suppose H is a subgroup of G (possibly of infinite index) and that E is a closed H-invariant subspace of $\ell_2(H)^n$. Then $\ell_2(G) \otimes_{\mathbb{R}H} E$ is naturally a closed G-invariant subspace of $\ell_2(G)^n$ and $\dim_G(\ell_2(G) \otimes_{\mathbb{R}H} E) = \dim_H E$.
- If H is a finite group and E is $\mathbb{R}$ with trivial H-action, then $\dim_H E = 1/|H|$.

(This follows from the two previous properties.)

Definition 9.2.1. The i^{th} *L_2-Betti number* of Y is defined by

$$b_i^{(2)}(Y; G) = \dim_G \mathcal{H}_i(Y).$$

We list some basic properties of L_2-Betti numbers which follow easily from the above properties of $\dim_G$.

Properties of L_2-Betti numbers.

- $b_i^{(2)}(Y;G) = 0$ if and only if $\mathcal{H}_i(Y) = 0$.
- If H is of finite index m in G, then $b_i^{(2)}(Y;H) = mb_i^{(2)}(Y;G)$.
- Suppose X is a CW complex with a cocompact, proper H-action where H is a subgroup of G. Let $G \times_H X$ denote the quotient space of $G \times X$ by the H-action defined by $h \cdot (g, x) = (gh^{-1}, hx)$. Then $G \times_H X$ is a G-CW complex (it is a disjoint union of copies of X) and

$$b_i^{(2)}(G \times_H X; G) = b_i^{(2)}(X; H).$$

9.3 Basic facts about L_2-homology.

As usual, we deal only with G-CW complexes which are proper and cocompact. We list some basic properties of L_2-homology.

1) *(Homotopy invariance)* Suppose $f : Y_1 \to Y_2$ is an equivariant cellular map of G-CW complexes. If f is a homotopy equivalence, then $\mathcal{H}_*(Y_1)$ and $\mathcal{H}_*(Y_2)$ are isomorphic.

2) *(Atiyah's Theorem):*

$$\chi^{orb}(Y/G) = \Sigma(-1)^i b_i(Y;G).$$

3) *(The Künneth Formula):* Suppose Y_i is a G_1-CW complex and Y_2 is a G_2-CW complex. Then $Y_1 \times Y_2$ is a $G_1 \times G_2$-CW complex and

$$b_m(Y_1 \times Y_2; G_1 \times G_2) = \sum_{i=0}^{m} b_i(Y_1; G_1) b_{m-i}(Y_2, G_2).$$

4) *(The exact sequence of a pair):* Suppose that X is a G-invariant subcomplex of Y. Then $\mathcal{H}_i(Y, X)$ is defined as usual (as the relative harmonic cycles which vanish on X) and the following sequence is weak exact

$$\to \mathcal{H}_i(X) \to \mathcal{H}_i(Y) \to \mathcal{H}_i(Y, X) \to \mathcal{H}_{i-1}(X) \to .$$

("Weak exactness" means that the image of one map is dense in the kernel of the next.)

5) *(The Mayer-Vietoris sequence):* Suppose that a G-CW complex U can be written as $U = Y_1 \cup Y_2$ with $X = Y_1 \cap Y_2$, where Y_1, Y_2 and X are all G-CW complexes. Then the following sequence is weak exact:

$$\to \mathcal{H}_i(X) \to \mathcal{H}_i(Y_1) \oplus \mathcal{H}_i(Y_2) \to \mathcal{H}_i(U) \to .$$

6) *(Poincaré duality):* Suppose the G-CW complex is an n-manifold M^n. Then

$$\mathcal{H}_i(M^n) \cong \mathcal{H}_{n-i}(M^n).$$

Similarly, if M^n is a manifold with boundary, then

$$\mathcal{H}_i(M, \partial M) \cong \mathcal{H}_{n-i}(M).$$

7) *(Hyperbolic space):* The reduced L_2-homology of hyperbolic space, $\mathcal{H}_i(\mathbb{H}^n)$ vanishes, unless $n = 2k$ and $i = k$ (in which case it is not zero).

Before stating the last two properties let use make one observation using homotopy invariance. If G is capable of acting properly and cocompactly on a contractible CW complex Y, then by 1), $\mathcal{H}_i(Y)$ is independent of the choice of Y. This leads to the following notation.

Notation. Suppose G is capable of acting properly and cocompactly on a contractible CW complex Y. Then put

$$\begin{aligned} \mathcal{H}_i^{(2)}(G) &= \mathcal{H}_i(Y) \quad \text{and} \\ b_i^{(2)}(G) &= b_i^{(2)}(Y;G). \end{aligned}$$

8) *(Bounded geometry,* [CG1]*):* Suppose M is a complete, contractible Riemannian manifold and that G is a discrete group of isometries of M. Further assume that M has bounded geometry, i.e., $|\kappa(Y)| \leq$ constant (bounded sectional curvature) and $InjRad(M) \geq$ constant > 0 (the injectivity radius is bounded away from 0). If $Vol(M/G) < \infty$, then

$$\begin{aligned} \mathcal{H}_i(G) &= \mathcal{H}_i(M) \quad \text{and} \\ b_i^{(2)}(G) &= \dim_G \mathcal{H}_i(M). \end{aligned}$$

9) *(Amenable groups,* [CG2]*):* If G is an amenable group with $K(G,1)$ a finite CW complex, then $b_i^{(2)}(G) = 0$ for all i.

In the next theorem we list a few well-known consequences of some of the above properties.

Theorem 9.3.1.

(i) $\mathcal{H}_i(\mathbb{E}^n) = 0$ *for all* i.

(ii) $b_i^{(2)}(\mathbb{Z}^n) = 0$ *for all* i.

(iii) If $\mathbb{H}^n/G$ *is a complete hyperbolic manifold or orbifold of finite volume, then*

$$b_i^{(2)}(G) = \begin{cases} 0 & \text{if} \quad i \neq \frac{n}{2} \\ (-1)^k \chi(G) & \text{if} \quad n = 2k \text{ and } i = k. \end{cases}$$

Proof. Statement (i) follows from the Künneth formula and the calculation in Example 9.1.2. Statement (ii) is equivalent to (i) ((ii) is also implied by 9)). Consider statement (iii). By 7) and 8), $b_i^{(2)}(G)$ vanishes unless $n = 2k$ and $i = k$. The formula $b^{(2)}(G) = (-1)^k \chi(G)$ follows from Atiyah's Theorem (2). (Here $\chi(G) = \chi^{orb}(\mathbb{H}^n/G)$.) Hence, (iii) holds. □

We end this subsection with a proof of Atiyah's Theorem which we have taken from [Ec].

Proof of Atiyah's Theorem. To simplify the notation let us assume that G acts freely so that Y/G is again a CW complex. Let a_i denote the number of i-cells in Y/G. Thus, $a_i = \dim_G C_i^{(2)}(Y)$. Therefore,

$$\begin{aligned}\chi(Y/G) &= \Sigma(-1)^i a_i \\ &= \Sigma(-1)^i \dim_G C_i^{(2)}(Y).\end{aligned}$$

The Hodge–de Rham decomposition gives $C_i^{(2)} = \overline{B}_i^{(2)} \oplus \overline{B}^i_{(2)} \oplus \mathcal{H}_i$. The orthogonal complement of $Z^{i-1}_{(2)}$ in $C^{(2)}_{i-1}$ is $\overline{B}^{(2)}_{i-1}$ and δ_{i-1} takes it isomorphically onto $B^i_{(2)}$. This means that $\overline{B}^{(2)}_{i-1}$ and $\overline{B}^i_{(2)}$ are "weakly isomorphic", which implies that they are isomorphic, cf. [Ec]. Therefore, $\dim_G \overline{B}^{(2)}_{i-1} = \dim_G \overline{B}^i_{(2)}$. Thus,

$$\begin{aligned}\sum(-1)^i \dim_G C_i^{(2)} &= \sum(-1)^i \dim_G \mathcal{H}_i \\ &\quad + \sum(-1)^i \dim_G \overline{B}_i^{(2)} + \sum(-1)^i \overline{B}^i_{(2)} \\ &= \sum(-1) b_i^{(2)}(Y;G),\end{aligned}$$

since the terms in the last two summations cancel. □

Since $\chi^{orb}(Y/G)$ is always a rational number, Atiyah's Theorem suggests the following well-known conjecture.

Conjecture 9.3.2. (Atiyah) *The L_2-Betti numbers $b_i^{(2)}(Y;G)$ are rational.*

A elementary reference for most of the material in this subsection is [Ec]. For a better overall picture the reader is referred to [G2] and [CG2].

9.4 Singer's Conjecture.

Suppose M^n is a closed aspherical manifold (or orbifold) with fundamental group π and with universal cover $\tilde{M}$. In the late seventies Singer asked if the following conjecture was true.

Conjecture 9.4.1. (Singer) $\mathcal{H}_i(\tilde{M}^n) = 0$, *except possibly in the case $n = 2k$ and $i = k$. In other words,*

(i) if $n = 2k+1$ is odd, then $b_i^{(2)}(\pi) = 0$ for all i, while

(ii) if $n = 2k$ is even, then

$$b_i^{(2)}(\pi) = \begin{cases} 0 & \text{if} \quad i \neq k, \\ (-1)^k \chi^{orb}(M^{2k}) & \text{if} \quad i = k. \end{cases}$$

Since L_2 Betti numbers are nonnegative, we have that Singer's Conjecture implies the Euler Characteristic Conjecture (Conjecture 8.1.1 or 8.1.4).

As evidence let us consider some examples where Conjecture 9.4.1 is true.

Examples 9.4.2.

- $\dim M = 1$. Then $M = S^1$, $\pi_1(M) = \mathbb{Z}$, $\tilde{M} = \mathbb{R}$, and as we have seen already, $b_0^{(2)}(\mathbb{Z}) = b_1^{(2)}(\mathbb{Z}) = 0$. Similarly, if $\pi = D_\infty$, the infinite dihedral group, then $b_0^{(2)}(D_\infty) = b_1^{(2)}(D_\infty) = 0$.
- $\dim M = 2$. Suppose M_g^2 is the orientable surface of genus g, $g > 0$. Then

$$\begin{aligned} b_0^{(2)}(\pi) &= 0 \qquad \text{(since } \pi \text{ is infinite)}, \\ b_2^{(2)}(\pi) &= 0 \qquad \text{(by Poincaré duality)}, \\ b_1^{(2)}(\pi) &= \chi(M_g^2) = 2 - 2g \quad \text{(by Atiyah's Theorem)}. \end{aligned}$$

Similarly, if X^2 is an aspherical 2-dimensional orbifold, then $b_0^{(2)}$ and $b_2^{(2)}$ vanish and $b_1^{(2)}(\pi) = \chi^{orb}(X^2)$.

- $\dim M = n$. If M^n is flat or hyperbolic (i.e., if it is Riemannian and covered by $\mathbb{E}^n$ or $\mathbb{H}^n$), then Singer's Conjecture holds by Theorem 9.3.1.

In the last subsection we will discuss some recent work of the first author and B. Okun, which gives a proof of the following theorem.

Theorem 9.4.3. ([DO]) *Suppose $M^n = X_L$, where L is a flag triangulation of S^{n-1} and X_L is the PE cubical manifold constructed in 3.1. Then Singer's Conjecture holds for M^n for $n = 3$ or 4.*

10. L_2-homology of right-angled Coxeter groups.

10.1 Notation and calculations.

As in Chapter II, L is a flag complex, W_L is the associated right-angled Coxeter group and Σ_L $(= \mathcal{U}_L)$ is the $CAT(0)$, PE cubical complex on which W_L acts.

Recall that a subcomplex A of a simplicial complex J is said to be *full* if whenever $v_0, \dots, v_k$ are vertices in A such that $\{v_, \dots, v_k\}$ spans a simplex in J, then that simplex lies in A.

If A is a full subcomplex of a flag complex L, then A is also a flag complex and its associated Coxeter group W_A is naturally a subgroup of W_L. (The point is that two fundamental generators s_1 and s_2 of W_A are connected by a relation for W_A if and only if it is also a relation for W_L.) It follows that Σ_A is naturally a subcomplex of Σ_L and that the translates of Σ_A in Σ_L give an embedding $W_L \times_{W_A} \Sigma_A \subset \Sigma_L$. (In fact, it is not difficult to see that Σ_A is a locally convex subcomplex of Σ_L.) In this way we have associated to each flag complex L a space Σ_L with W_L-action and to each full subcomplex $A \subset L$ a subspace $W_L \times_{W_A} \Sigma_A$. We can then try to calculate the L_2-homology of these spaces.

Notation.

$$\begin{aligned} h_i(L) &= \mathcal{H}_i(\Sigma_L) \\ \beta_i(L) &= b_i^{(2)}(W_L) \\ h_i(L,A) &= \mathcal{H}_i(\Sigma_L, W_L \times_{W_A} \Sigma_A) \\ \beta_i(L,A) &= \dim_{W_L} h_i(L,A) \end{aligned}$$

Calculations.

a) Suppose L is a single point. Then $W_L = \mathbb{Z}/2$, $\beta_0(L) = \frac{1}{2}$ and $\beta_i(L) = 0$ for $i > 0$.

b) Suppose $L = S^0$ (2 points). Then $W_L = \mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$, $\Sigma_L = \mathbb{R}$, and $\beta_i(L) = 0$ for all i.

c) Suppose L is the disjoint union of 3 points. Then $W_L = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ and Σ_L is a trivalent tree. Moreover, $\chi(W_L) = \chi^{orb}(\Sigma_L/W_L) = 1 - 3/2 = -\frac{1}{2}$ (see Example 8.1.5). Since W_L is infinite, $\beta_0(L) = 0$, and since Σ_L is 1-dimensional, $\beta_i(L) = 0$ for $i > 1$. Hence, $\beta_1(L) = \frac{1}{2}$ by Atiyah's Theorem.

d) More generally, if L is the disjoint union of m points, $m > 1$, then $\beta_1(L) = (m-2)/2$ and $\beta_i(L) = 0$ for $i \neq 1$.

e) If L is a k-simplex, then $W_L = (\mathbb{Z}/2)^{k+1}, \Sigma_L = \square^{k+1}, \beta_0(L) = (\frac{1}{2})^{k+1}$ and $\beta_i(L) = 0$ for $i > 0$.

f) If l is the join of two flag complexes L_1 and L_2, then $W_L = W_{L_1} \times W_{L_2}$ and $\Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2}$. Hence, the Künneth Formula gives $\beta_n(L) = \Sigma\beta_i(L_1)\beta_{m-i}(L)$.

g) In particular, if L is the join of two copies of the complex consisting of three points, then $\beta_2(L) = \frac{1}{4}$ and $\beta_i(L) = 0, i \neq 2$.

h) If L is an m-gon, $m > 3$, then Σ_L is a $CAT(0)$ 2-manifold and $\chi(W_L) = 1 - m/2 + m/4 = 1 - m/4$. Hence, $\beta_0(L) = 0, \beta_2(L) = 0$ (Poincaré duality) and $\beta_1(L) = (m-4)/4$. Thus, Singer's Conjecture holds in this case.

i) Let CL (the cone on L) denote the join of L with a point. Then $W_{CL} = W_L \times \mathbb{Z}/2$. Hence, the Künneth Formula gives $\beta_i(CL) = \frac{1}{2}\beta_i(L)$. (This also follows from the fact that W_L is index 2 in W_{CL}.) We also have that $\Sigma_{CL} = \Sigma_L \times \square^1$ and that the involution s corresponding to the cone point acts by reflection on the interval $\square^1$. It follows that the sequence of the pair breaks into short exact sequences $0 \to h_{i+1}(CL, L) \to h_i(L) \to h_i(CL) \to 0$. (In fact, $h_i(CL)$ and $h_i(CL, L)$ can be identified with the $+1$ and -1 eigenspaces of s on $h_i(L)$.) Thus, we also have $\beta_{i+1}(CL, L) = \frac{1}{2}\beta_i(L)$.

j) Let SL (the suspension of L) denote the join of L with 2 points. Then $W_{SL} = W_L \times D_\infty$, so by the Künneth Formula, $\beta_i(SL) = 0$ for all i.

10.2 Variations on Singer's Conjecture.

From now on we suppose that L is a triangulation of S^{n-1} as a flag complex and that A is any full subcomplex of L. In this subsection we are going to consider several statements, $I(n)$, $II(n)$, $III(n)$ and $III'(n)$ which depend on the dimension n of the manifold Σ_L. The first one is simply a restatement of Singer's Conjecture for Σ_L.

$I(n)$: If $n = 2k$ is even, then $\beta_i(L) = 0$, for all i, with $i \neq k$. If $n = 2k+1$ is odd, then $\beta_i(L) = 0$, for all i.

$II(n)$: If $n = 2k$, then $\beta_i(L, A) = 0$ for all i, with $i \geq k+1$. If $n = 2k+1$, then $\beta_i(A) = 0$ for all i, with $i \geq k+1$.

Lemma 10.2.1. *$II(n)$ implies $I(n)$.*

Proof. If $n = 2k$, take $A = \emptyset$. Then $II(2k)$ implies $\beta_i(L) = 0$ for $i \geq k+1$ and then by Poincaré duality, $\beta_i(L) = 0$ for $i \leq k-1$. If $n = 2k+1$, take $A = L$. Again, $II(2k+1)$ means $\beta_i(L) = 0$ for $i \geq k+1$ and then by Poincaré duality, $\beta_i(L) = 0$. for all i. $\square$

If v is a vertex of L, then let L_v be the combinatorial link of v in L and let D_v denote the complement of the open star of v (i.e., D_v is the full subcomplex of L spanned by $Vert(L) - v$). Thus, (D_v, L_v) is a triangulation of (D^{n-1}, S^{n-2}).

$III(2k+1)$: For any vertex v, in the exact sequence of the pair (L, L_v), the map $h_k(L_v) \to h_k(L)$ is the zero map.

$III'(2k+1)$: For any vertex v, in the Mayer-Vietoris sequence for $L = D_v \cup CL_v$, the map

$$H_k(L_v) \to H_k(D_v) \oplus H_k(CL_v)$$

is injective.

Lemma 10.2.2. *(i) $I(2k+1)$ implies $III(2k+1)$.*

(ii) Statements $III(2k+1)$ and $III'(2k+1)$ are equivalent.

Proof. (i) If $I(2k+1)$ holds, then $h_*(L)$ vanishes so $h_k(L_v) \to h_k(L)$ is zero.

(ii) There is an excision isomorphism $h_*(L, L_v) \cong h_*(D_v, L_v) \oplus h_*(CL_v, L_v)$. It then follows from Poincaré duality that the sequence of the pair (L, L_v) and the Mayer-Vietoris sequence are dual sequences:

$$\begin{array}{ccccccc}
\to h_k(L_v) & \xrightarrow{\varphi} & h_k(D_v) & \bigoplus & h_k(CL_v) & \to h_k(L) \to \\
\| \wr & & & \| \wr & & \| \wr \\
\leftarrow h_k(L_v) & \xleftarrow{\theta} & h_{k+1}(D_v, L_v) & \bigoplus & h_{k+1}(CL_v, L_v) & \leftarrow h_{k+1}(L) \leftarrow
\end{array}$$

The vertical isomorphisms are given by Poincaré duality. Hence, φ is injective if and only if θ is weakly surjective (i.e., has dense image) if and only if $h_k(L_v) \to h_k(L)$ is the zero map. $\square$

10.3 Inductive arguments.

The idea is to try to prove $I(n)$ by induction on dimension n. To this end we have the following.

Lemma 10.3.1. *(i) $II(2k-1)$ implies $II(2k)$.*

(ii) Statements $I(2k)$ and $III'(2k+1)$ imply $II(2k+1)$.

If we examine Lemmas 10.2.1, 10.2.2 and 10.3.1 we see that we have reduced the problem to proving $III(2k{+}1)$. Indeed, after proving the above lemma, we will have basically shown that the odd-dimensional case implies the result in the next higher dimension. Thus, if we know that $I(n)$ holds for $n < 2k$, then we have $I(2k-2)$ and $I(2k-1) \Rightarrow I(2k-2)$ and $III'(2k-1) \Rightarrow II(2k-1) \Rightarrow II(2k) \Rightarrow I(2k)$. On the other hand, if we know $I(n)$ holds for $n < 2k+1$, then we still need to prove $III(2k+1)$ in order to conclude $I(2k+1)$.

Proof of Lemma 10.3.1. (i) Assume $II(2k-1)$ holds. We want to prove $II(2k)$: that $h_i(L, A)$ vanishes for $i \geq k+1$ and A any full subcomplex of L. The proof is by induction on the number of vertices in $L - A$. Now suppose by induction that $II(2k)$ holds for a full subcomplex B and let v be a vertex of B. Let A the the full subcomplex spanned by the vertices in $B - v$. So, $B = A \cup CA_0$, where $A_0 = Lk(v, B)$. Consider the exact sequence of the triple (L, B, A),

$$\to h_i(B, A) \to h_i(L, A) \to h_i(L, B) \to$$

where $i \geq k+1$. By excision, $h_i(B, A) = h_i(CA_0, A_0)$. Also, $\beta_i(CA_0, A_0) = \frac{1}{2}\beta_{i-1}(A_0)$, which is 0, by $II(2k-1)$ (since $i-1 \geq k$). By the inductive hypothesis, $h_i(L, B) = 0$. Therefore, $h_i(L, A) = 0$.

(ii) Assume $I(2k)$ and $III'(2k+1)$ hold. We want to prove $II(2k+1)$: that $h_i(A) = 0$ for $i \geq k+1$. This time we use induction on the number of vertices in A. It is clearly true for $A = \emptyset$. We assume inductively that it holds for A and we try to prove it for B, the full subcomplex spanned by A and another vertex v. As before, $B = A \cup CA_0$. We compare the Mayer-Vietoris sequences for B and $L = D_v \cup CL_v$.

$$\begin{array}{cccccc}
 & h_{k+1}(L_v, A_0) & & & & \\
 & \downarrow & & & & \\
0 \to h_{k+1}(B) & \to h_k(A_0) & \xrightarrow{e} h_k(A) & \bigoplus & h_k(CA_0) \\
 & \downarrow f & & \downarrow & & \\
 & h_k(L_v) & \xrightarrow{g} h_k(D_v) & \bigoplus & h_k(cL_v)
\end{array}$$

By $I(2k)$, $h_{k+1}(L_v, A_0) = 0$; hence, f is injective. By $III'(2k+1)$, g is injective; hence, e is injective. Therefore, $h_{k+1}(B) = 0$. The Mayer-Vietoris sequence also shows that $h_i(B) = 0$ for $i > k+1$. $\square$

Lemma 10.3.2. $I(3)$ *is true.*

Proof. Basically this follows from Andreev's Theorem. Let L be a flag triangulation of the 2-sphere. Consider Andreev's condition $(*)$ below:

$(*)$ Every 4-circuit in L is either the link of a vertex or the boundary of the union of two adjacent 2-simplices. Furthermore, L is not the suspension of a polygon.

Suppose L satisfies $(*)$. Let $\hat{L}$ denote the full subcomplex of L spanned by the vertices whose links are circuits of length > 4. By Andreev's Theorem, $\hat{L}$ is the dual of a right-angled polytope of finite volume in $\mathbb{H}^3$ (the ideal points correspond to the missing vertices). By Theorem 9.3.1 (iii), $h_*(\hat{L})$ vanishes. It then follows from the Mayer-Vietoris sequence (since $h_*(S^0 * S^0) = 0$), that $h_*(L) = 0$.

If L does not satisfy $(*)$, then we can find a 4-circuit T which separates it into two pieces $\overline{L}_1$ and $\overline{L}_2$. Put $L_i = \overline{L}_i \cup CT$. Since L_i has fewer vertices than L, we may assume by induction that $h_*(L_i) = 0$. Then by a Mayer-Vietoris sequence $h_*(L) = 0$, as before. □

An immediate corollary of this lemma is Theorem 9.4.3 which we restate as the following.

Corollary 10.3.3. *$I(n)$ holds for $n \leq 4$.*

Corollary 10.3.4. *The Flag Complex Conjecture (8.3.4) holds for triangulations of the 3-sphere. Hence, the Euler Characteristic Conjecture (8.1.1) holds for 4-dimensional cubical manifolds.*

The above arguments also show that in fact $II(n)$ holds for $n \leq 4$. For $n = 3$ we state this as follows.

Corollary 10.3.5. *Suppose L^2 is a flag triangulation of the 2-sphere and that A is a full subcomplex. Then $h_2(A) = 0$.*

Corollary 10.3.6. *Suppose that A is a simplicial graph without 3-circuits. If A is a planar graph, then $h_2(A) = 0$.*

Proof. If A is planar, then we can embed it as a full subcomplex of some flag triangulation of the 2-sphere. □

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